EFFECTIVE THEORY OF FLUCTUATING HYDRODYNAMICS FROM HOLOGRAPHY

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions. The work was done under the guidance of Prof. R Loganayagam, at the International Centre for Theoretical Sciences (ICTS), Tata Institute of Fundamental Research (TIFR), Bengaluru.

(Akhil Sivakumar)
Date: 16 Sep 2022

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

(R Loganayagam)
Date: 16 Sep 2022

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Contents

Publications

Publications relevant to the thesis:

- Jewel K. Ghosh, R. Loganayagam, Siddharth G. Prabhu, Mukund Rangamani, Akhil Sivakumar, V. Vishal, *"Effective field theory of stochastic diffusion from gravity"*, *JHEP* **05** (2021) 130, arXiv: 2012.03999 [hep-th]

- Temple He, R. Loganayagam, Mukund Rangamani, Akhil Sivakumar, Julio Virrueta, *"The timbre of Hawking gravitons: an effective description of energy transport from holography"*, *JHEP* **09** (2022) 092, arXiv: 2202.04079 [hep-th]

Other publications:

- Bidisha Chakrabarty, Joydeep Chakravarty, Soumyadeep Chaudhuri, Chandan Jana, R. Loganayagam, Akhil Sivakumar, *"Non-linear Langevin equation via holography"*, *JHEP* **01** (2020) 165, arXiv: 1906.07762 [hep-th]

- R. Loganayagam, Krishnendu Ray, Akhil Sivakumar, *"Fermionic open EFT from holography"*, arXiv: 2011.07039 [hep-th]

- R. Loganayagam, Krishnendu Ray, Shivam Sharma, Akhil Sivakumar, *"Holographic KMS relations at finite density "*, *JHEP* **03** (2021) 233, arXiv: 2011.08173 [hep-th]

Chapter 1

Introduction

Fluid dynamics is the earliest example of field theories studied in physics. The paradigm of effective field theories aptly explains the ubiquity of fluid like behaviour observed in nature – fluid dynamics should be understood as the effective description of long distance, collective excitations within a microscopic theory [1]. The specifics of the underlying microscopic description is mostly unimportant to the investigation of this so called hydrodynamic limit of the system.[1] The hydrodynamics of a generic field theory is dominated by gapless excitations such as Goldstone bosons (from broken symmetries), conserved charge densities (global charges and energy-momentum) etc. As an effective field theory, hydrodynamics is an important component in a physicist's toolkit as its application spans a wide range of topics such as heavy-ion-collisions [2–5], condensed matter physics [6–9], meteorology[10, 11] and cosmology[12–15].

The advent of AdS/CFT correspondence or holographic duality opened up yet another avenue for hydrodynamics based investigations. The duality predicts that certain strongly coupled CFTs admit an alternate description as weakly coupled gravitational theories [16–18]. More specifically, thermal states of d dimensional CFTs get mapped to black holes in AdS$_{d+1}$ via the duality. Furthermore, it suggests that hydrodynamic excitations of a planar CFT can be identified with the long wavelength deformations of AdS black branes which indeed display gapless spectrum [19–21].[2] This identification can be made more rigorous to derive a corollary of the duality, the so called fluid/gravity correspondence – the large scale deformations of black branes in AdS spacetimes are dual to the configurations of dissipative conformal fluids evolving via a conformal generalisation of the Navier-Stokes equation [22, 23]. In this sense the holographic duality makes precise the long-standing idea of black hole membrane paradigm [24]. The black brane deformations can now be thought of as representing the hydrodynamics of a precise microscopic

[1] The hydrodynamics emerges at time scales much larger than the mean free-path of the system.

[2] Due to the absence of intrinsic scales, the hydrodynamic scale in a conformal theory depends only on its temperature. Therefore conformal hydrodynamics should be an effective theory at scales sufficiently above the inverse temperature β.

theory defined by the holographic CFT.

It is well understood that dissipative physical systems should admit completions as stochastically driven systems. The reason is that dissipation is the manifestation of energy transfer to underlying microscopic degrees of freedom, which in turn inevitably source fluctuations in the system. The stochastic completion of the dissipative dynamics is not arbitrary either – the noise distribution is constrained by the transport characteristics, which is the essence of fluctuation-dissipation theorems [25]. In the context of dissipative fluids this implies that a more realistic description should accommodate thermal fluctuations of the hydrodynamic or long-lived modes themselves. For example, dissipative fluid equations cannot correctly capture hydrodynamical correlation functions as they fail to account for thermally excited fluctuations of hydrodynamic modes themselves (see [26–28] for the discussion of 'long-time tails' in hydrodynamic correlation functions).

At the level of linearised hydrodynamics, a remedy to account for such effects is to supplement the hydrodynamic constitutive relations with noise terms [29] (see [30] for a textbook treatment of non-relativistic fluctuating hydrodynamics).[3] However, it is not straightforward to extend such a simplistic approach to a non-linear theory where the effects of hydrodynamic fluctuations turn out to be rather severe – they generate non-analytic (in gradient/frequency expansion) corrections to the effective transport coefficients. Such non-analytic contributions manifest as long-time power-law tails in the hydrodynamic correlations mentioned above. As these effects originate from large wavelength fluctuations, they are particularly stark in low dimensions.[4] For example, in $2+1$ dimensions the non-analyticities are severe enough to affect the shear viscosity of the fluid [34], while in $3+1$ dimensions, they are important only at the level of second order hydrodynamics [35] and so on. This is a serious problem as any phenomenologically accurate model would involve higher order hydrodynamics.[5] In short, conventional hydrodynamics is not equipped to systematically study the effects of hydrodynamic fluctuations, for long-lived fluctuations are in tension with one of its central tenets, that fluctuations die out much before the hydrodynamic time scale [38, 39].

There are multiple challenges we need to address in order to build a framework for *fluctuating hydrodynamics*. How can we relax the assumption of short-lived fluctuations and allow the fluid degrees of freedom to simmer? How should we distill out hydrodynamic fluctuations from short-lived ones which decay safely within the characteristic time scale of the fluid? Though it seems plausible that these questions can be answered starting from the underlying microscopic theory, in practice such an exercise is not feasible – deriving the hydrodynamic limit involves a complicated rewriting of the collective degrees

[3]See [31–33] for a generalisation of this approach to relativistic boost-invariant fluid flows.

[4]This is expected as the phase space volume of IR modes is progressively suppressed in higher dimensions.

[5]Moreover, second order hydrodynamics is also necessary to prevent acausality and instabilities in the hydrodynamic evolution [36, 37].

of freedom, which is beyond the scope of standard analytical techniques.

However, we argue that within AdS/CFT, this problem can be dealt with in an analytic fashion. The preceding discussion on the stochastic completion of dissipative systems has a natural counterpart in the gravitational description. While black branes admit dissipative perturbations as quasi-normal modes [40], they also source fluctuations, both stimulated and spontaneous, as Hawking radiation [41]. This is required for consistency of the semi-classical field theory on the black brane. Situated within the context of fluid/gravity correspondence, this suggests that a theory of fluctuating hydrodynamics can be constructed via holography – one only needs to account for Hawking effect on the black brane, specifically with respect to the gapless sector of its deformations, which from the perspective of the CFT, causes its hydrodynamic degrees of freedom to fluctuate. Further, as we will show, the underlying gauge theoretical structure suggests a natural prescription to separate out the short-lived and long-lived excitations of the fluid, which we refer to as *Markovian* and *non-Markovian* excitations respectively .

The central proposition of our work is that fluctuating hydrodynamics should be envisaged as an open effective field theory which ought to be dealt with in a Wilsonian fashion. This perspective is motivated from the following question – what is the effective theory that describes the evolution of a probe coupled to a fluctuating fluid? The answer to this question depends on the details of the probe-fluid coupling. If the probe couples to Markovian or fast excitations in the fluid, it is expected that over the hydrodynamical times scale its evolution should be local, allowing us to completely integrate out Markovian degrees of freedom from the effective theory. But how should we treat probes that couple to non-Markovian excitations which show long-term memory?

Clearly, integrating out non-Markovian excitations is undesirable as it would transfer the long-term memory to the dynamics of the probe, or in other words make its evolution non-local. The intuitive remedy is to not integrate out the non-Markovian degrees of freedom, instead, to retain them in the effective description. This motivates us to think of the combined probe-fluid system as a whole probing the underlying bath of fast Markovian modes. We term the resulting description a Wilsonian open effective theory which describes the local evolution of the probe-fluid system, but includes the *local influence* of the forgotten Markovian degrees of freedom. The key difference here from the Wilsonian paradigm of unitary quantum field theories is that, hydrodynamic modes could be inherently dissipative and hence have no counterpart in the unitary description (for example, diffusive hydrodynamic modes satisfy the dispersion relation $\omega = -iDk^2 + \ldots$).[6] Therefore the Wilsonian programme now ought to be carried out within an open field theory set up from the outset.

Working within the Schwinger-Keldysh formalism for open field theories, we show that

[6]The hydrodynamic spectrum also includes propagating modes which are defined even in the absence of dissipation and hence are similar to ordinary particles.

the Wilsonian effective field theory of the probe-fluid system is characterized by a gadget we term the Wilsonian influence functional (WIF). The WIF should be understood as an object of mixed character – it serves as the correlation generating functional for the Markovian fields and effective action for the non-Markovian fields. The open effective action of a general probe-fluid system can be obtained by identifying the Markovian sources in the WIF with appropriate probe operators and turning on desired interactions with the non-Markovian fields.

We are specifically interested in the WIF dual to metric and gauge field perturbations of a Schwarzschild-AdS$_{d+1}$ black brane which describes the evolution of a probe coupled to a neutral plasma in the dual CFT. As mentioned before, guided by the gauge invariance of the gravitational theory, we establish a formalism to derive the WIF of this holographic fluctuating hydrodynamic system by cleanly separating out the Markovian and non-Markovian dynamics. The dynamics of the Markovian and diffusive subset of the non-Markovian modes was first studied in [42] in a unified manner in terms of certain auxiliary scalar probes of the black brane. The corresponding analysis of the non-Markovian mode describing energy-momentum transport was originally reported in [43].

We conclude this introduction by briefly commenting on other approaches to fluctuating hydrodynamics developed in the recent literature. The theory of fluctuating hydrodynamics in low dimensional systems have enjoyed significant attention due to recognition of its aforementioned sickness. Several authors have attempted to systematically derive a description of varied weakly coupled systems [44, 45] – these techniques typically coarse grain the dynamics within a chosen scale and thereby implement a *projection* on to the hydrodynamical variables. The details of such hydrodynamic projections are often model specific, and in particular, the treatment of fluctuations is often carried out in a bottom-up fashion using insights from fluctuation-dissipation relations. While these approaches provide significant insight into the dynamics of specific models, it is not immediately clear how to generalize them to other models including higher dimensional systems, or how to systematically include non-linear corrections.

There have also been parallel efforts to characterize the universal features of fluctuating hydrodynamic systems from the perspective of Schwinger-Keldysh effective descriptions (see [39, 46–55]). The Schwinger-Keldysh formalism is well suited for investigations of this nature as it packages the constraints from microscopic unitarity and thermality of the system in a unified manner. However, the constraints on such effective actions are mostly motivated from intuitions gained from working with simple, weakly coupled models where explicit analysis is possible. It behooves us to check the validity of these predictions in the less understood regime of strongly coupled systems.

Holography provides a variety of toy models where these checks can be performed [56, 57]. Their success in capturing the universal features of strongly coupled systems suggests that open effective theories generated from holography should also yield qualitatively relevant predictions for strongly coupled fluctuating systems. As a particularly pertinent example, we refer to the result of [58, 59] which derived a non-linear generalization of the Langevin equation. Their analysis demonstrated that the non-linear corrections in the Langevin equation and non-Gaussianities in the fluctuations are tightly constrained.[7] These predictions were validated from an independent holographic approach in [60] (see also [61] for a generalization to scalar fields).

Our approach to fluctuating hydrodynamics in a way can be considered the generalization of these holographic models to the case of fluid/gravity correspondence and therefore open quantum systems with memory. In doing so, we expect to perform an unbiased and independent check of various ideas developed in the context of relativistic fluctuating hydrodynamics [62]. Note that, in this thesis we report only a clean characterisation of the gravitational dynamics and do not attempt to repackage this data in the language of hydrodynamic effective theories. As we explain later, from gravity we find promising evidence in favour of the effective theory structures anticipated in [62].

1.1 Outline of the thesis

In the rest of this introductory chapter, we summarise essential lessons from the technology of real-time path integrals. We have tried to be self-contained in our discussion. In §1.2 we begin by reviewing Schwinger-Keldysh path integral, motivating and highlighting its features in a pedagogical manner. Its various symmetry properties, KMS conditions, etc. are discussed. In §1.2.1 we explain how the Schwinger-Keldysh approach provides a natural pathway towards construction and study of open field theories.

In §1.3 we introduce the gravitational Schwinger-Keldysh geometry dual to the Schwinger-Keldysh path integrals of the field theory. We have tried to limit our discussion to essential features of this geometry. The reader interested in a more detailed discussion of its construction may consult [61].

In chapter 2 we present the constructions of an effective theory of the diffusive modes in a holographic plasma. We begin by further elaborating the idea of holographic fluctuating hydrodynamics and open quantum systems with memory in §2.1. Then we summarise the main results from the chapter in §2.1.2. A detailed outline of the chapter's organization and its associated appendices is provided in §2.1.3.

In chapter 3 we discuss the holographic construction of the effective theory describing sound propagation in the boundary plasma. The chapter begins by taking stock of the essential results from the previous chapter and a detailed preview of our analysis in §3.1. The outline of this chapter and corresponding appendices is described in §3.1.1.

Finally in chapter 4 we conclude with a summary of salient results and outlook of the

[7] This followed from the well motivated assumption that the bath satisfies a time reversal symmetry.

thesis, and provide a brief discussion of interesting future directions.

1.2 Review of Schwinger-Keldysh formalism

Quantum systems are typically modelled as closed systems whose states are defined to be vectors drawn from a Hilbert space. The primary question we seek to understand about such systems is – how do they evolve from a given initial state to another. We could further ask how is this evolution modified under external influences, say if we drive the system with a time dependent force. The answers to these questions, at least theoretically, are well understood – given an initial state ψ_0, it evolves unitarily via the map

$$\psi_0 \to \psi_t \equiv \mathsf{U}[J]\psi_0 \ , \qquad \mathsf{U}[J] = \mathbb{T}\exp\left[-i\int_0^t \mathrm{d}t \ \mathcal{H}[J]\right] \ , \tag{1.1}$$

where J denotes external forces acting on the system, $\mathcal{H}[J]$ is the instantaneous Hamiltonian of the system and $\mathbb{T}$ is the time-ordering symbol. All the above statements are well familiar from a standard undergraduate course in quantum mechanics and we recount them here to remind the reader of the following point. For closed systems, unitarity and locality of evolution imply the existence of an instantaneous Hamiltonian. This in turn, inevitably brings in the notion of time-ordering with the evolution operator. Therefore, the evolution of a unitary system is completely characterized by the *time-ordered correlation functions* of operators acting on its Hilbert space. The Feynman path integral formalism is tailored to yield such time-ordered correlation functions of the system and hence are ideal computational devices to study unitary quantum systems. The unitarity is implicitly assumed within the path integral formalism as one performs a weighted summation over only those paths which match the specific initial and final states of the system.

However, as outlined before, we wish to study the holographic fluctuating hydrodynamic systems which are open systems interacting with the microscopic constituents of the holographic fluid. In general, open systems are obtained by integrating out a selected number of degrees of freedom, the *environment*, from a microscopic theory. The degrees of freedom that are retained form the *system*, which evolves non-unitarily under the influence of the environment. The corresponding quantum states are obtained by tracing out the environment variables from the states of the parent theory and are therefore not pure states, but mixed states defined by density operators. Therefore, in order to study the dynamics of open systems, we require a technology which describes the evolution of mixed states.

Clearly, conventional path integral techniques are not designed to address such questions. More specifically, the evolution of an open system is not completely determined by time-ordered correlation functions of its operators. However, it is possible to adapt the path integral formalism to deal with such systems – the resulting technology called the Schwinger-Keldysh path integral directly describes the evolution of elements of a general density operator [63–65]. We will now briefly summarize this formalism. The reader may refer to [25, 66, 67] for a textbook discussion of the same.

It is simpler to understand the Schwinger-Keldysh formalism first in the case of closed systems. We start by recalling the standard path integral representation of time evolving states in a field theory. Consider the state ψ_0 at time $t = 0$. Its image under the unitary evolution till time t is given by

$$\psi_t[\phi] = \int D[\iota]\ \psi_0[\iota] \int \mathcal{D}[\varphi] \exp\{iS[\varphi]\}\ . \tag{1.2}$$

Here we used the notation $\psi_0[\iota] \equiv \langle\iota|\,\psi_0$ to denote the projection of ψ_0 on to a field coherent state $|\iota\rangle$ with eigenvalue ι (and similarly for $\psi_t[\phi]$). The path integral $\int \mathcal{D}[\varphi]$ is defined over paths satisfying

$$\varphi(t) = \phi\ , \qquad \varphi(0) = \iota\ , \tag{1.3}$$

and $S[\varphi]$ denotes the corresponding action. In the case of unitary systems with initial states defined via some density operator ρ_0, we generalize (1.2) to get

$$\rho_t[\phi_{\rm R}|\phi_{\rm L}] = \int D[\iota_{\rm R,L}]\ \rho_0[\iota_{\rm R}|\iota_{\rm L}] \int \mathcal{D}[\varphi_{\rm L,R}] \exp\{iS[\varphi_{\rm R}] - iS[\varphi_{\rm L}]\}\ , \tag{1.4}$$

where $\rho_0[\iota_{\rm R}|\iota_{\rm L}] \equiv \langle\iota_{\rm R}|\,\rho_0\,|\iota_{\rm L}\rangle$ (and similarly for $\rho_t[\phi_{\rm R}|\phi_{\rm L}]$). The path integral $\int \mathcal{D}[\varphi_{\rm R,L}]$ is defined over paths satisfying

$$\varphi_{\rm R/L}(t) = \phi_{\rm R/L}\ , \qquad \varphi_{\rm R/L}(0) = \iota_{\rm R/L}\ . \tag{1.5}$$

Evidently, (1.4) is a joined path integral for evolutions of ket and bra co-ordinates of the density operator. We use subscripts R and L to indicate ket and bra copies of variables respectively. It is easy to see that the integral over paths of $\varphi_{\rm R}$ in (1.4) is identical to that of paths of φ in (1.2). All we have done here is to include an additional *independent, complex conjugated* factor of path integral to express evolution of the bra co-ordinate of ρ_0. Next we define a sourced version of ρ_t, where we imagine perturbing the ket and bra evolutions *independently* with external sources. To wit,

$$\rho_t[\phi_{\rm R}; J_{\rm R}|\phi_{\rm L}; J_{\rm L}] \equiv \int D[\iota_{\rm R,L}]\ \rho_0[\iota_{\rm R}|\iota_{\rm L}] \int \mathcal{D}[\varphi_{\rm R,L}] \exp\{iS[\varphi_{\rm R}, J_{\rm R}] - iS[\varphi_{\rm L}, J_{\rm L}]\}\ , \tag{1.6}$$

where $S[\varphi, J]$ denotes, as in usual path integrals, the action in the presence of sources.

The Schwinger-Keldysh generating functional corresponding to the state ρ_0 is defined as

$$\begin{aligned}\mathcal{Z}_{\text{SK}}[J_{\text{R}}, J_{\text{L}}] &\equiv \text{Tr}\left\{\mathsf{U}[J_{\text{R}}]\; \rho_0\; \mathsf{U}^\dagger[J_{\text{L}}]\right\} \ , \\ &= \lim_{t\to\infty}\int D[\phi]\; \rho_t[\phi; J_{\text{R}}|\phi; J_{\text{L}}] \ , \\ &= \int D[\iota_{\text{R,L}}]\, \rho_0[\iota_{\text{R}}|\iota_{\text{L}}] \int \mathcal{D}[\varphi_{\text{R,L}}] \exp\left\{iS[\varphi_{\text{R}}, J_{\text{R}}] - iS[\varphi_{\text{L}}, J_{\text{L}}]\right\} \ .\end{aligned} \tag{1.7}$$

Here $\mathsf{U}[J]$ denotes unitary evolution under the influence of source J. Notice that in the second line of (1.7) we have implemented the trace operation via identifying the ket and bra co-ordinates of the density operator in the far future. In the path integral expression of third line, this identification is implicit.

As usual, differentiating $\mathcal{Z}_{\text{SK}}[J_{\text{R}}, J_{\text{L}}]$ with source functions $J_{\text{R,L}}$ and setting them to vanish yield correlations functions of the corresponding operators $\mathcal{O}_{\text{R,L}}$ (they denote R and L copies of the operator $\mathcal{O}$). It is clear from the first line of (1.7) that R operators act on the density operator from its left (ket side) and L operators act on it from the right (bra side). Correlation functions of $\mathcal{O}_{\text{R}}$ and $\mathcal{O}_{\text{L}}$ generated this way turn out to be time-ordered and anti-time-ordered respectively. While the former follows from arguments familiar from the case of usual path integrals, the latter can be argued from the complex conjugated nature of the φ_{L} path integral in (1.6). In addition to this, the future trace operation which identifies φ_{R} and φ_{L} degrees of freedom causes $\mathcal{O}_{\text{R}}$ and $\mathcal{O}_{\text{L}}$ to be non-trivially correlated with each other. Using the cyclicity property of the trace, we can always recast such off-diagonal R-L correlations in terms of correlations of the original operator $\mathcal{O}$ where all operators with L labels act from the left of operators with R labels. Therefore, all the Schwinger-Keldysh correlation functions can be expressed in terms of correlations of $\mathcal{O}$. To wit,

$$\langle \mathcal{O}_{\sigma_1}(t_1)\mathcal{O}_{\sigma_2}(t_2)\cdots\mathcal{O}_{\sigma_n}(t_n)\rangle = \text{Tr}\left\{\rho_0\left(\widetilde{\mathbb{T}}\prod_{i=1,\ \sigma_i=\text{L}}^{n}\mathcal{O}(t_i)\right)\left(\mathbb{T}\prod_{i=1,\ \sigma_i=\text{R}}^{n}\mathcal{O}(t_i)\right)\right\} \ , \tag{1.8}$$

where $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ stand for time-ordering and anti-time-ordering symbols respectively.[8] In the particular case of 2-pt functions we get

$$\begin{aligned}&\langle \mathcal{O}_{\text{R}}(t_1)\mathcal{O}_{\text{R}}(t_2)\rangle = \langle \mathbb{T}\ \mathcal{O}(t_1)\mathcal{O}(t_2)\rangle_{\rho_0}, \qquad &&\langle \mathcal{O}_{\text{R}}(t_1)\mathcal{O}_{\text{L}}(t_2)\rangle = \langle \mathcal{O}(t_2)\mathcal{O}(t_1)\rangle_{\rho_0} \\ &\langle \mathcal{O}_{\text{L}}(t_1)\mathcal{O}_{\text{R}}(t_2)\rangle = \langle \mathcal{O}(t_1)\mathcal{O}(t_2)\rangle_{\rho_0} \ , &&\langle \mathcal{O}_{\text{L}}(t_1)\mathcal{O}_{\text{L}}(t_2)\rangle = \langle \widetilde{\mathbb{T}}\ \mathcal{O}(t_1)\mathcal{O}(t_2)\rangle_{\rho_0} \ ,\end{aligned} \tag{1.9}$$

where $\langle\cdots\rangle_{\rho_0} \equiv \text{Tr}\{\rho_0\cdots\}$ denotes that correlations are evaluated in the state ρ_0.

Schwinger-Keldysh time contours: We may in fact take the point of view that Schwinger-Keldysh correlation functions are defined via (1.8). The operator ordering

[8] We will assume that all operators under consideration are bosonic. The reader interested in a more general discussion including fermionic operators is directed to [25].

implied by the right hand side of (1.8) motivates an alternate perspective on Schwinger-Keldysh path integrals as follows. First, notice that correlations of the L operators are anti-time ordered and their insertions are arranged to act from the left of insertions of R operators. Using the property that path integrals give rise to contour ordered correlation functions, we argue that both of the above conditions can be implemented by unifying the path integrals over R and L fields into a single integral, but along a folded time contour as in Fig. 1.1. We will refer to this contour as the *Schwinger-Keldysh time contour*.

Figure 1.1: The Schwinger-Keldysh time contour. The initial condition for the path integral over fields $\varphi_{R/L}$ is fixed by the density operator $\rho_0 = |\alpha_R\rangle \langle\alpha_L|$. The branch of the contour evolving forward (backward) in time is labelled C_R (C_L).

It is useful to think of this contour as comprised of a forward evolving branch C_R and a backward evolving branch C_L which are slightly offset in the complex time plane along the imaginary direction. They are smoothly glued at the time-fold in the future, which we remind, replaces the trace operation vital to our previous discussion. R and L fields are exclusively supported on C_R and C_L respectively. The two boundaries of the Schwinger-Keldysh contour naturally host the initial state boundary conditions on the path integral.

We wish to briefly comment on an interesting generalization of the above construction. First, let us introduce the concept of *time-ordering violation* in a correlation function. A correlation function is said to have an instance of time-ordering violation for every triplet of operators in it which are placed adjacent to each other, but are not ordered according to increasing or decreasing values of their time co-ordinate. In particular, time-ordered and anti-time ordered correlation functions do not violate time ordering. As (1.8) groups operator insertions into two strings which are time-ordered and anti-time ordered, it is straightforward to check that this class of correlation functions have at most one violation of time-ordering. A take away from our discussion so far is that, in order to study the evolution of any mixed state, it is sufficient to limit ourselves to Schwinger-Keldysh correlation functions.

However, the Schwinger-Keldysh construction can be generalized to include correlations with multiple violations of time-ordering by considering more exotic time contours with multiple time-folds [68]. The import of these constructions is that, to generate correlation functions with at most $2n+1$ time-ordering violations, it is sufficient to consider path integral contours with $2n+1$ time-folds – $n+1$ future time-folds and n past time-folds. Such general *out-of-time-ordered correlation functions* (OTOCs) have proved to be very interesting in recent years as diagnostics of quantum chaos [69, 70]. We will not discuss these generalized Schwinger-Keldysh constructions any further.

Keldysh basis and response functions: In order to understand the physics encoded in Schwinger-Keldysh correlation functions (1.8), it is often useful to adopt the so called Keldysh basis or the *average-difference* basis of operators [25]. To wit,

$$\mathcal{O}_a \equiv \frac{1}{2}\left(\mathcal{O}_{\text{R}} + \mathcal{O}_{\text{L}}\right)\ , \qquad \mathcal{O}_d \equiv \mathcal{O}_{\text{R}} - \mathcal{O}_{\text{L}}\ . \tag{1.10}$$

It turns out that when expressed in the Keldysh basis, the number of independent correlation functions manifestly diminish. In addition, correlations expressed in these variables are more easily relatable to physically meaningful quantities such as response functions. For example, in the case of two-point functions, we get

$$\begin{aligned}
\langle \mathcal{O}_a(t_1)\mathcal{O}_a(t_2)\rangle &= \frac{1}{2}\left\langle\{\mathcal{O}(t_1),\mathcal{O}(t_2)\}\right\rangle_{\rho_0} &&\equiv \langle \mathcal{O}(t_1)\mathcal{O}(t_2)\rangle^{\text{Kel}}\\
\langle \mathcal{O}_a(t_1)\mathcal{O}_d(t_2)\rangle &= \theta(t_1-t_2)\left\langle[\mathcal{O}(t_1),\mathcal{O}(t_2)]\right\rangle_{\rho_0} &&\equiv \langle \mathcal{O}(t_1)\mathcal{O}(t_2)\rangle^{\text{Ret}}\\
\langle \mathcal{O}_d(t_1)\mathcal{O}_a(t_2)\rangle &= \theta(t_2-t_1)\left\langle[\mathcal{O}(t_2),\mathcal{O}(t_1)]\right\rangle_{\rho_0} &&\equiv \langle \mathcal{O}(t_1)\mathcal{O}(t_2)\rangle^{\text{Adv}}\\
\langle \mathcal{O}_d(t_1)\mathcal{O}_d(t_2)\rangle &= 0\ ,
\end{aligned} \tag{1.11}$$

where $\theta(t)$ denotes the Heaviside theta function. In the above we have indicated that correlation functions $\langle\mathcal{O}_a\mathcal{O}_d\rangle$ and $\langle\mathcal{O}_d\mathcal{O}_a\rangle$ correspond to advanced and retarded Green's functions respectively, thereby making the Keldysh basis physically more intuitive. Its simplicity is enhanced by the fact that the 2-pt function of $\mathcal{O}_d$ operator vanishes. Further, notice that any 2-pt correlation function with an $\mathcal{O}_d$ inserted at the latest time vanishes. This property is in fact true for correlation functions with arbitrary number of insertions and is called the *largest-time equation* – it is a direct consequence of the unitarity of the theory [71, 72].

We also define average-difference combinations of sources as

$$J_a \equiv \frac{1}{2}\left(J_{\text{R}} + J_{\text{L}}\right)\ , \qquad J_d \equiv J_{\text{R}} - J_{\text{L}}\ . \tag{1.12}$$

Notice that when translated to the average-difference basis, source terms in (1.7) take the form

$$J_a\mathcal{O}_d + J_d\mathcal{O}_a = J_{\text{R}}\mathcal{O}_{\text{R}} - J_{\text{L}}\mathcal{O}_{\text{L}}\ , \tag{1.13}$$

indicating that the average source J_a couples to the difference operator $\mathcal{O}_d$ and vice versa.

Constraints from microscopic unitarity: We now discuss some important consistency conditions satisfied by the generating functional $\mathcal{Z}_{\text{SK}}$. The first of these follows from the unitarity of the system. Recall that unitary evolutions of a density operator preserve its trace. Therefore, if we identify the R and L sources used to define $\mathcal{Z}_{\text{SK}}$ in

(1.7), we get

$$\mathcal{Z}_{\text{SK}}[J_{\text{R}}, J_{\text{L}}]\Big|_{J_{\text{R}}=J_{\text{L}}} = 1 \ , \qquad \text{or equivalently} \qquad \mathcal{Z}_{\text{SK}}[J_a, J_d]\Big|_{J_d=0} = 1 \ , \tag{1.14}$$

where in the latter expression we have slightly abused the notation to depict the dependence of $\mathcal{Z}_{\text{SK}}$ on sources written in Keldysh basis. We will refer to this property as the *Schwinger-Keldysh collapse rule.* It says that in the limit the difference source J_d vanishes, the generating functional $\mathcal{Z}_{\text{SK}}$ trivializes. Together with the off-diagonal coupling of sources and operators in the Keldysh basis, the collapse rule implies that correlators with insertions of only $\mathcal{O}_d$ identically vanish for they are obtained by differentiating $\mathcal{Z}_{\text{SK}}$ with J_a (this is a special case of the largest-time equation).

A second consistency condition follows from the observation that physical density operators are Hermitian, and satisfy $\rho_0^\dagger = \rho_0$. Therefore taking a complex conjugate of $\mathcal{Z}_{\text{SK}}$ in (1.7) we obtain

$$\begin{aligned} \left(\mathcal{Z}_{\text{SK}}[J_{\text{R}}, J_{\text{L}}]\right)^* &= \text{Tr}\left\{\mathsf{U}[J_{\text{L}}]\ \rho_0^\dagger\ \mathsf{U}^\dagger[J_{\text{R}}]\right\} \ , \\ &= \mathcal{Z}_{\text{SK}}[J_{\text{L}}, J_{\text{R}}] \ . \end{aligned} \tag{1.15}$$

In the average-difference basis, this *reality condition* translates to

$$\left(\mathcal{Z}_{\text{SK}}[J_a, J_d]\right)^* = \mathcal{Z}_{\text{SK}}[J_a, -J_d] \ . \tag{1.16}$$

Thermal state and KMS conditions: As we are particularly interested in the fluctuating hydrodynamical system close to equilibrium configuration, we will from now on concentrate on thermal initial states defined by

$$\rho_0 = e^{-\beta\mathcal{H}} \ . \tag{1.17}$$

It is well known that thermal correlation functions in a quantum field theory can be studied using the Matsubara formalism [73, 74], where one considers evolution of the system along periodically identified Euclidean time. The technology of preparing thermal states using Euclidean path integrals can be adapted straightforwardly to Schwinger-Keldysh formalism where its evolution further along the real-time direction can be studied [25, 75]. Compared to the Schwinger-Keldysh contour discussed previously, such *thermal Schwinger-Keldysh contours* involve an additional Euclidean segment C_{E} which corresponds to preparation of the thermal initial state. The resulting path integrals are therefore defined over *closed time contours* like the one illustrated in Fig. 1.2. The correlation functions generated from path integrals over such contours completely characterize the real-time evolution of the quantum field theory at finite temperature.[9]

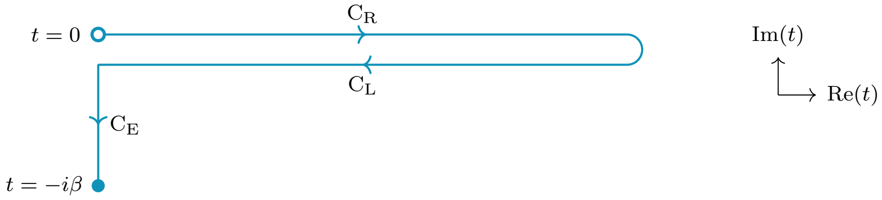

Figure 1.2: The thermal Schwinger-Keldysh time contour with inverse temperature β. The marked points at times $t = 0$ and $t = -i\beta$ are identified to close the contour. In addition to branches $C_{R/L}$ present in Fig. 1.1, here we also have the branch C_E denoting a Euclidean time evolution. C_E represents the thermal circle which has been cut and evolved in real-time along $C_{R/L}$.

Thermal correlation functions are known to satisfy the Kubo-Martin-Schwinger (KMS) conditions [76, 77]. To recap quickly, these are relations among correlation functions with different operator ordering, but when considered as analytic functions of their time arguments. They arise from the same principle that is behind the Matsubara formalism, namely – thermal density operators are identical to evolution operators along imaginary time direction. This observation, together with the cyclicity of the trace operation, allows us to cyclically rearrange the operator insertions in a correlation function, provided we simultaneously translate them along the imaginary time direction. In order to avoid complications arising from considering operator insertions at complex times, we will instead choose to write down KMS relations directly in the Fourier space where their content is less ambiguous.[10] Therefore, we write

$$\left\langle \mathcal{O}(\omega_1)\, \mathcal{O}(\omega_2) \cdots \mathcal{O}(\omega_n) \right\rangle_{\rho_0} = e^{\beta\omega_1} \left\langle \mathcal{O}(\omega_2) \cdots \mathcal{O}(\omega_n)\, \mathcal{O}(\omega_1) \right\rangle_{\rho_0} . \tag{1.18}$$

In the above, we have transferred the left most operator $\mathcal{O}(\omega_1)$ to the far right at the cost of an additional Boltzmann like factor.[11]

Clearly, for a fixed choice of operators and their frequency dependence, such KMS relations constrain the space of their independent $n-$pt correlation functions. For example, in the case of 2-pt functions, KMS relations can be used to derive the relation

$$\langle \mathcal{O}(-\omega)\, \mathcal{O}(\omega) \rangle^{\mathrm{Kel}} = \frac{1}{2} \coth\left(\frac{\beta\omega}{2}\right) \mathrm{Re}\Big[\langle \mathcal{O}(-\omega)\, \mathcal{O}(\omega) \rangle^{\mathrm{Ret}} \Big] . \tag{1.19}$$

In writing the above, we have incorporated the result that for systems that are time-translation invariant, the only non-trivial correlation functions are those in which the

[9] Here we only mean that Schwinger-Keldysh correlators completely characterise evolution of the state. However, there are more complicated features of the dynamics such as those related to quantum chaos which are encoded in generalised Schwinger-Keldysh correlators alluded to previously.

[10] We follow the standard convention used for Fourier transform, viz., $\mathcal{O}(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \mathcal{O}(\omega) e^{-i\omega t}$.

[11] Again we assume all operators satisfy bosonic statistics. For fermionic operators this generalizes straightforwardly with additional statistical phase factors.

sum of frequencies of all operators vanishes.

Since we are interested in constructing the generating functional $\mathcal{Z}_{\text{SK}}$ for these correlators, we ask – what restrictions do $\mathcal{Z}_{\text{SK}}$ incur from KMS relations obeyed by the correlation functions? In order to answer this question it is useful to work with yet another basis for the source variables we refer to as the *retarded-advanced* (RA) variables [78].[12] They are defined as

$$\begin{aligned} J_{\bar{\text{F}}}(\omega, \mathbf{k}) &\equiv -\Big[(1+n_B)J_{\text{R}}(\omega, \mathbf{k}) - n_B J_{\text{L}}(\omega, \mathbf{k})\Big] , \\ J_{\bar{\text{P}}}(\omega, \mathbf{k}) &\equiv -n_B\Big[J_{\text{R}}(\omega, \mathbf{k}) - J_{\text{L}}(\omega, \mathbf{k})\Big] , \end{aligned} \tag{1.20}$$

where

$$n_B \equiv \frac{1}{e^{\beta\omega}-1} , \tag{1.21}$$

is the Bose-Einstein factor. When the source functions are expressed in RA basis, the Schwinger-Keldysh collapse rule discussed before and the KMS relations are equivalent to the relations [78]

$$\mathcal{Z}_{\text{SK}}[J_{\bar{\text{P}}}, J_{\bar{\text{F}}}]\Big|_{J_{\bar{\text{F}}}=0} = \mathcal{Z}_{\text{SK}}[J_{\bar{\text{P}}}, J_{\bar{\text{F}}}]\Big|_{J_{\bar{\text{P}}}=0} = 1 . \tag{1.22}$$

Here we have again taken the liberty to slightly abuse the notation in expressing the arguments of $\mathcal{Z}_{\text{SK}}$. These relations can be thought of as the thermal version of the Schwinger-Keldysh collapse rule.

1.2.1 Open systems from Schwinger-Keldysh path integrals

As outlined before, open quantum systems are by definition systems which undergo non-autonomous evolution under the influence of their *environment*. Open dynamics of quantum mechanical systems is a quite well understood subject with applications spanning across topics such as quantum optics, physics of cold atoms and decoherence theory. See [79, 80] for textbook level introduction to the subject.

However, the wider class of open quantum systems represented by open quantum field theories have not received its due attention. Such systems show up in a variety of contexts including heavy-ion-physics[81], cosmology [82, 83], non-equilibrium systems [84] and hydrodynamics. Further, they are highly pertinent to the physics of black holes (and in general, spacetimes with horizons) as the semi-classical field theory outside the horizon naturally fits into this description .

As field theories are most conveniently treated within the path integral formalism, the language of Schwinger-Keldysh path integrals introduced in the previous subsections

[12] The choice of RA variables in [78] is non-standard (see [72, 75]) but was shown to simplify spectral representations of general OTOCs.

provide an efficient toolkit to study open field theories. We will now briefly summarize the treatment of open field theories within the Schwinger-Keldysh formalism by sketching the derivation of such a theory.

We begin by considering a microscopic theory with a predetermined bi-partition of its degrees of freedom into that of the *system* and the *environment*. The dynamics of the system could be intrinsically distinct from that of the environment, for example, in the case of external probes introduced into the environment. Alternatively, the distinction between system and environment could originate from some kind of coarse-graining of a parent system, as is the case in hydrodynamics. We will not distinguish between these scenarios and in either case, in accordance with the philosophy motivated in the introduction, we will consider the system as probing the environment. Suppose that the combined unitary dynamics of the probe-environment variables is governed by the action

$$\begin{aligned} S[\mathcal{P},\mathcal{X}] &= S_{\text{p}}[\mathcal{P}] + S_{\text{e}}[\mathcal{X}] + S_{\text{p-e}}[\mathcal{P},\mathcal{X}] \ , \\ S_{\text{p-e}}[\mathcal{P},\mathcal{X}] &= \int \mathrm{d}x \ \mathcal{O}_{\mathcal{P}}(x)\, \mathcal{O}_{\mathcal{X}}(x) \ . \end{aligned} \tag{1.23}$$

Here we have collectively denoted the probe (environment) fields with $\mathcal{P}$ ($\mathcal{X}$). The action of the total system, S splits into three pieces; S_{p} and S_{e} respectively denote autonomous evolution of $\mathcal{P}$ and $\mathcal{X}$, and $S_{\text{p-e}}$ encodes their interaction. $S_{\text{p-e}}$ is taken to be a local functional of the fields where the interaction is mediated via *local operators* $\mathcal{O}_{\mathcal{P}}$ and $\mathcal{O}_{\mathcal{X}}$ of probe and environment fields respectively.[13]

To derive the open effective dynamics of $\mathcal{P}$ fields, we first lift the action (1.23) to its Schwinger-Keldysh avatar. To wit,

$$S_{\text{SK}}[\mathcal{P}_{\text{R}},\mathcal{P}_{\text{L}},\mathcal{X}_{\text{R}},\mathcal{X}_{\text{L}}] = S[\mathcal{P}_{\text{R}},\mathcal{X}_{\text{R}}] - S[\mathcal{P}_{\text{L}},\mathcal{X}_{\text{L}}] \ . \tag{1.24}$$

For simplicity, the initial density operator is taken to be factorised, viz.,

$$\rho_0[\text{P}_{\text{R}},\text{X}_{\text{R}}|\text{P}_{\text{L}},\text{X}_{\text{L}}] = \rho_{\text{in}}[\text{P}_{\text{R}}|\text{P}_{\text{L}}] \times \rho_{\text{in}}[\text{X}_{\text{R}}|\text{X}_{\text{L}}] \ . \tag{1.25}$$

Once we adopt the Schwinger-Keldysh formalism, it is straightforward to integrate out the subset of degrees of freedom we deem as the environment and study the evolution of the reduced density operator of fields $\mathcal{P}$. Since we are interested in the dynamics of only $\mathcal{P}$, following (1.7), we define the generating functional of their Schwinger-Keldysh

[13]For simplicity, we consider only one channel of interaction between $\mathcal{P}$ and $\mathcal{X}$ fields. The generalisation to multiple channels is easy to obtain by considering $S_{\text{p}-\text{e}} = \int \lambda_{m,n}\, \mathcal{O}_{\mathcal{P}}^m\, \mathcal{O}_{\mathcal{X}}^n$.

correlations via

$$\begin{aligned}\mathcal{Z}_{\rm p}[J_{\rm R},J_{\rm L}] = &\int D[{\rm P}_{\rm R,L},{\rm X}_{\rm R,L}]\,\rho_0[{\rm P}_{\rm R},{\rm X}_{\rm R}|{\rm P}_{\rm L},{\rm X}_{\rm L}]\\ &\int\mathcal{D}[\mathcal{P}_{\rm R,L},\mathcal{X}_{\rm R,L}]\exp\left\{i\,S_{\rm SK}[\mathcal{P}_{\rm R},\mathcal{P}_{\rm L},\mathcal{X}_{\rm R},\mathcal{X}_{\rm L}]+i\int {\rm d}x\,\left(J_{\rm R}\mathcal{P}_{\rm R}-J_{\rm L}\mathcal{P}_{\rm L}\right)\right\}\ .\end{aligned}\tag{1.26}$$

In order to obtain an effective description of $\mathcal{Z}_{\rm p}$ involving only fields $\mathcal{P}$, we imagine (1.26) as being evaluated in two steps. First, we *freeze the path integral over* $\mathcal{P}$, and selectively integrate out environment fields $\mathcal{X}$. This gives,

$$\begin{aligned}\mathcal{Z}_{\rm p}[J_{\rm R},J_{\rm L}] = &\int D[{\rm P}_{\rm R,L}]\,\rho_0[{\rm P}_{\rm R}|{\rm P}_{\rm L}]\\ &\int\mathcal{D}[\mathcal{P}_{\rm R,L}]\exp\left\{i\,\mathcal{S}_{\rm SK}[\mathcal{P}_{\rm R},\mathcal{P}_{\rm L}]+i\int {\rm d}x\,\left(J_{\rm R}\mathcal{P}_{\rm R}-J_{\rm L}\mathcal{P}_{\rm L}\right)\right\}\ ,\end{aligned}\tag{1.27}$$

where $\mathcal{S}_{\rm SK}[\mathcal{P}_{\rm R},\mathcal{P}_{\rm L}]$ is the *Schwinger-Keldysh effective action* of probe fields $\mathcal{P}$ given by

$$\mathcal{S}_{\rm SK}[\mathcal{P}_{\rm R},\mathcal{P}_{\rm L}] = S_{\rm p}[\mathcal{P}_{\rm R}]-S_{\rm p}[\mathcal{P}_{\rm L}]+S_{\rm IF}[(\mathcal{O}_{\mathcal{P}})_{\rm R},(\mathcal{O}_{\mathcal{P}})_{\rm L}]\ .\tag{1.28}$$

The path integral over $\mathcal{P}$ is performed only at the second step, but using the effective action $\mathcal{S}_{\rm SK}$. Here $S_{\rm IF}$ is the *influence functional* that encodes the effect of environment variables $\mathcal{X}$ in the evolution of $\mathcal{P}$ [65]. It is useful to think of $S_{\rm IF}$ as an independently defined gadget viz.,

$$\begin{aligned}S_{\rm IF}[\bar{J}_{\rm R},\bar{J}_{\rm L}] \equiv &-i\log\left(\mathcal{Z}_{{\rm e},\,\mathcal{P}=0}[\bar{J}_{\rm R},\bar{J}_{\rm L}]\right)\ ,\\ \mathcal{Z}_{{\rm e},\,\mathcal{P}=0}[\bar{J}_{\rm R},\bar{J}_{\rm L}] \equiv &\int D[{\rm X}_{\rm R,L}]\,\rho_{\rm in}[{\rm X}_{\rm R}|{\rm X}_{\rm L}]\\ &\int\mathcal{D}[\mathcal{X}_{\rm R,L}]\exp\left\{i\,S_{\rm e}[\mathcal{X}_R]-i\,S_{\rm e}[\mathcal{X}_R]+i\int {\rm d}x\,\left(\bar{J}_{\rm R}\,(\mathcal{O}_{\mathcal{X}})_{\rm R}-\bar{J}_{\rm L}\,(\mathcal{O}_{\mathcal{X}})_{\rm L}\right)\right\}.\end{aligned}\tag{1.29}$$

Let us elaborate. $\mathcal{Z}_{{\rm e},\,\mathcal{P}=0}$ denotes the Schwinger-Keldysh correlation generating functional for the environment operators $\mathcal{O}_{\mathcal{X}}$, but in a theory where probe fields $\mathcal{P}$ have been *switched off by setting* $\mathcal{P}=0$. The influence functional $S_{\rm IF}$ is defined as the logarithm of $\mathcal{Z}_{{\rm e},\,\mathcal{P}=0}$ (or in the parlance of Feynman diagrammatics, its connected component). The import of equations (1.27), (1.28) and (1.29) is that the influence of the environment on probes can be accounted for within an autonomous Schwinger-Keldysh evolution of the latter, but using the effective action $\mathcal{S}_{\rm SK}$. Moreover, $\mathcal{S}_{\rm SK}$ takes a specific form depending on the nature of operators mediating the interaction between probes and environment.

Recall that for unitary systems the Schwinger-Keldysh evolution factorised into that of R and L fields – the action $S_{\rm SK}$ in (1.23) does not include any direct interaction between two species of Schwinger-Keldysh fields.[14] In contrast, once we integrate out a

[14]However, the R and L fields are entangled with each other due to the future boundary condition.

subset of fields, the additional interactions supplemented by the influence functional S_{IF} generically couple the evolution of R and L fields. This is the fundamental difference between unitary and open quantum systems. The interaction between the R and L fields encodes the entanglement of probes with the forgotten environment degrees of freedom.

We conclude this section by noting that S_{IF} and thereby $\mathcal{S}_{\text{SK}}$ inherits various consistency conditions from $\mathcal{Z}_{\text{e},\mathcal{P}=0}$. It is straightforward to check that (1.14), (1.15) and (1.22), imply the following:

- The Schwinger-Keldysh collapse rule

$$\mathcal{S}_{\text{SK}}[\mathcal{P}_{\text{R}}, \mathcal{P}_{\text{L}}]\Big|_{\mathcal{P}_{\text{R}}=\mathcal{P}_{\text{L}}} = 0. \tag{1.30}$$

  In the average-difference basis this implies that every term in $\mathcal{S}_{\text{SK}}$ contains at least one factor of the difference field $\mathcal{P}_d$.

- The reality condition[15]

$$\left(\mathcal{S}_{\text{SK}}[\mathcal{P}_{\text{R}}, \mathcal{P}_L]\right)^* = -\mathcal{S}_{\text{SK}}[\mathcal{P}_{\text{L}}, \mathcal{P}_R] \ . \tag{1.31}$$

- For environments initially in thermal equilibrium,

$$S_{\text{IF}}[J_{\bar{\text{P}}}, J_{\bar{\text{F}}}]\Big|_{J_{\bar{\text{F}}}=0} = S_{\text{IF}}[J_{\bar{\text{P}}}, J_{\bar{\text{F}}}]\Big|_{J_{\bar{\text{P}}}=0} = 0 \ , \tag{1.32}$$

  which encodes both the Schwinger-Keldysh collapse rule and KMS conditions. This condition disallows any term exclusively made out of $J_{\bar{\text{P}}}$ or $J_{\bar{\text{F}}}$ in S_{IF}.

Verifying that open effective actions we derive from holography satisfy the above conditions will attest to the consistency of our results. We will now review the construction of geometries which describe the gravitational dual of Schwinger-Keldysh path integrals of a holographic CFT.

1.3 Review of grSK geometry

The holographic duality relates field theory path integrals evaluated on a manifold B and gravitational path integrals evaluated on the manifold M such that B is the boundary of M. Practically, gravitational computations can be performed only when the corresponding path integral settles down to a classical saddle configuration or in other words a background geometry. Semi-classical computations can then be performed on this background by systematically allowing the fields (including the metric) to fluctuate about the

[15] We assume fields $\mathcal{P}$ to be real and the generalisation to more complicated fields is straightforward.

saddle. Such perturbative computations can be easily book-kept using Feynman-Witten diagrams [17].

When situated within AdS/CFT, the Schwinger-Keldysh formalism suggests the existence of a semi-classical geometry such that the gravitational path integral around it is dual to the Schwinger-Keldysh path integral of the CFT. Many authors have discussed the construction of such a saddle of the gravitational path integral [85–87]. However the more recent proposal [88] by Glorioso, Crossley and Liu has proved to be particularly useful to understand its geometry. Following [61], we refer to the saddle conjectured by them as the *gravitational Schwinger-Keldysh geometry or grSK geometry.*[16] We now briefly summarize its construction.

Recall that we are interested in the hydrodynamics of a d dimensional neutral plasma. Therefore the dual gravitational dynamics should describe the deformations and Hawking fluctuations around a Schwarzschild-AdS$_{d+1}$ black brane. Not surprisingly, the prescription of [88] envisages the grSK geometry as a particular analytical continuation of the Schwarzschild-AdS$_{d+1}$ black brane.[17] This geometry is characterized by the metric (we mostly follow conventions introduced in [61])[18]

$$\mathrm{d}s^2 = -r^2\, f(br)\, \mathrm{d}v^2 + i\,\beta\, r^2\, f(br)\, \mathrm{d}v\, \mathrm{d}\zeta + r^2\, \mathrm{d}\mathbf{x}^2\,, \qquad f(\xi) = 1 - \xi^{-d}\,. \tag{1.33}$$

Here ζ is the *mock tortoise coordinate*, which parameterizes a curve in the complexified radial plane as we illustrate in Fig. 1.3. The coordinate ζ is defined by the differential relation (β is the inverse temperature)

$$\frac{\mathrm{d}r}{\mathrm{d}\zeta} = \frac{i\,\beta}{2}\, r^2\, f(br)\,, \qquad \beta = \frac{4\pi b}{d} \equiv \frac{4\pi}{d\, r_h}\,, \tag{1.34}$$

subject to the following boundary conditions at the cut-off surface $r = r_c$

$$\zeta(r_c + i\,\varepsilon) = 0\,, \qquad \zeta(r_c - i\,\varepsilon) = 1\,. \tag{1.35}$$

To understand this geometry, consider the familiar planar-Schwarzchild-AdS$_{d+1}$ geometry in ingoing Eddington-Finkelstein coordinates:

$$\mathrm{d}s^2 = -r^2\, f(br)\, \mathrm{d}v^2 + 2\, \mathrm{d}v\, \mathrm{d}r + r^2\, \mathrm{d}\mathbf{x}^2\,. \tag{1.36}$$

[16] See [61] for a description of how this geometry differs from the double sided eternal black hole geometry. The import of their discussion is that the double sided black hole geometry corresponds to the thermofield double (TFD) cousin of the thermal Schwinger-Keldysh contour.

[17] The grSK construction for charged and rotating black holes can be found in [89, 90].

[18] A useful identity which helps in various simplifications is $x\, \frac{\mathrm{d}}{\mathrm{d}x} f(x) = d\,(1 - f(x))$. This also explains why we choose to parameterize the horizon radius $r_h = \frac{1}{b}$ as in [22].

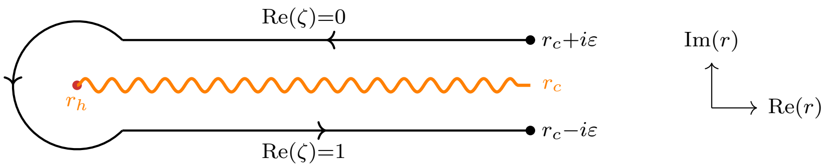

Figure 1.3: The complex r plane with the locations of the two boundaries and the horizon marked. The grSK contour is a codimension-1 surface in this plane (drawn at fixed v). As indicated the direction of the contour is counter-clockwise and it encircles the branch point at the horizon.

The asymptotic timelike boundary has a time coordinate v, which for the Schwinger-Keldysh path integral contour should be viewed as a curve in the complex time domain. The grSK spacetime fills in this Schwinger-Keldysh contour with a metric of constant negative curvature and provides a saddle point for the real-time gravitational path integral (see Fig. 1.4).[19] The metric (1.33) is only a re-writing of the above metric using ζ. The spacetime as such comprises of two copies of the planar black hole: one at $\text{Re}(\zeta) = 0$ and another at $\text{Re}(\zeta) = 1$ glued smoothly across a horizon-cap. In what follows we will find the following non-dimensional radial coordinates helpful to simplify expressions:

$$\xi \equiv b\,r\,, \qquad \rho \equiv \left(\frac{1}{b\,r}\right)^d = \xi^{-d} \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\zeta} = -2\pi i\,\rho^{1-\frac{1}{d}}\,(1-\rho)\,\frac{\mathrm{d}}{\mathrm{d}\rho}\,. \tag{1.37}$$

We can find a closed form expression for $\zeta(r)$ in terms of incomplete Beta function $\mathrm{B}\,(p,q;z)$, see [92, section 8.17] (alternatively we may express this as a hypergeometric function as done in [61]). We have the formal solution

$$\zeta(r) = \frac{d}{2\pi i}\int_{\infty+i0}^{\xi}\frac{y^{d-2}\,\mathrm{d}y}{y^d-1} = -\frac{1}{2\pi i}\;\mathrm{B}\left(\frac{1}{d},0;\left(\frac{1}{\xi}\right)^d\right) \equiv -\frac{1}{2\pi i}\;\mathrm{B}\left(\mathsf{s},0;\rho\right)\,, \tag{1.38}$$

where we have introduced a useful shorthand

$$\mathsf{s} \equiv \frac{1}{d}\,. \tag{1.39}$$

Time reversal isometry on grSK: In order to streamline the ensuing discussion, it will be convenient to introduce certain derivative operators that will appear extensively in our analysis. We define:

$$\mathbb{D}_\pm = r^2\,f(r)\,\frac{\partial}{\partial r} \pm \frac{\partial}{\partial v}\,, \qquad \mathbb{D}_\pm = r^2\,f(r)\,\frac{\partial}{\partial r} \mp i\,\omega\,, \tag{1.40}$$

[19] To be clear, here we only refer to the fact that given the asymptotic boundary conditions the grSK geometry is a smooth spacetime satisfying the Einstein's equations with the prescribed boundary conditions. It is an open question whether the geometry is the unique or dominant saddle, but the fact that it gives results consistent with field theory expectations [60, 61, 88] strongly argues in its favour. For further comments about real-time gravitational saddles see [91].

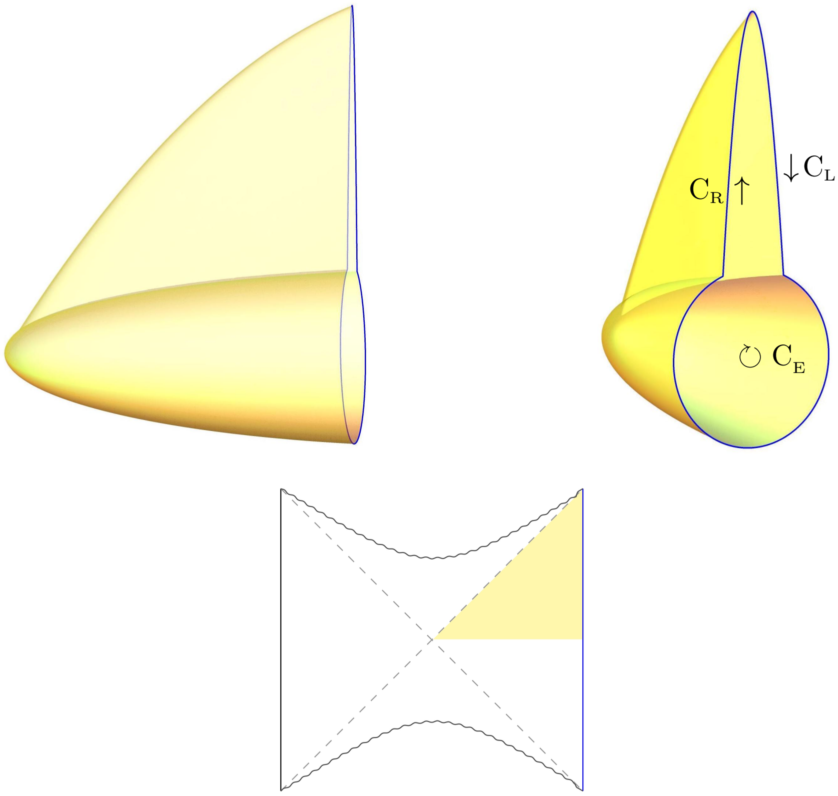

Figure 1.4: A schematic illustration of the grSK geometry from different perspectives. The view on the top left panel depicts two identical sections of the Lorentzian black brane glued along their future horizons and further to a Euclidean black brane (the cigar) in the past. In the bottom panel we display the subsection of the Lorentzian Schwarzschild-AdS$_{d+1}$ Penrose diagram which is retained in the grSK geometry. The top right panel highlights how this geometry asymptotically approaches the thermal Schwinger-Keldysh contour of the field theory.

in the time and Fourier domain, respectively.

While the choice of ingoing coordinates breaks the explicit time-reversal invariance, the geometry retains a time-reversal $\mathbb{Z}_2$ isometry as the transformation

$$v \mapsto i\beta\zeta - v \tag{1.41}$$

preserves the form of the metric. The operator $\mathbb{D}_+$ is naturally covariant under this isometry as can be seen from the identity

$$\begin{aligned} \mathrm{d}r\,\partial_r + \mathrm{d}v\,\partial_v + \mathrm{d}x^i\,\partial_i &= \frac{\mathrm{d}r}{r^2 f}\,\mathbb{D}_+ + \left(\mathrm{d}v - \frac{\mathrm{d}r}{r^2 f}\right)\partial_v + \mathrm{d}x^i\,\partial_i \\ &= \frac{i\beta}{2}\mathrm{d}\zeta\,\mathbb{D}_+ + \left(\mathrm{d}v - \frac{i\beta}{2}\mathrm{d}\zeta\right)\partial_v + \mathrm{d}x^i\partial_i\ , \end{aligned} \tag{1.42}$$

where we have used the real domain expression $\mathbb{D}_+ = r^2 f\partial_r + \partial_v$. The 1-forms that appear in r.h.s., $\{\frac{\mathrm{d}r}{r^2 f}, \mathrm{d}v - \frac{\mathrm{d}r}{r^2 f}, \mathrm{d}x^i\}$ furnish a basis of cotangent space that is covariant under the time-reversal $\mathbb{Z}_2$. It follows that the dual derivative operators $\{\mathbb{D}_+, \partial_v, \partial_i\}$ furnish a natural basis of the bulk tangent space covariant under time-reversal.

An important implication of the time-reversal isometry is that, given an ingoing field configuration on the grSK geometry, via time-reversal it can be mapped to an outgoing configuration [60, 61]. In particular, this implies that given an ingoing solution of the field equations, we may generate a Hawking solution by time reversing it. More explicitly, the action of the time reversal isometry on an ingoing scalar mode $\varphi^{\text{in}}(r, v, \mathbf{x})$ is given by[20]

$$\varphi^{\text{in}}(r, v, \mathbf{x}) \mapsto \varphi^{\text{in}}(r, i\beta\zeta - v, \mathbf{x}) \equiv \varphi^{\text{out}}(r, v, \mathbf{x}) \ . \tag{1.43}$$

We introduce the corresponding Fourier domain fields via the inverse Fourier transform

$$\varphi(r, v, \mathbf{x}) = \int_k \varphi(r, \omega, \mathbf{k})\, e^{-i\omega v + i\mathbf{k}.\mathbf{x}} \ , \tag{1.44}$$

where we have defined the shorthand

$$\int_k \equiv \int \frac{d\omega}{2\pi} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \ . \tag{1.45}$$

In terms of the Fourier fields, (1.43) takes a more illuminating form given by

$$\varphi^{\text{out}}(r, \omega, \mathbf{k}) = e^{-\beta\omega\zeta} \varphi^{\text{in}}(r, -\omega, \mathbf{k}) \ . \tag{1.46}$$

The action of the time-reversal covariant derivative $\mathbb{D}_+$ is given by

$$\begin{aligned} \mathbb{D}_+ \varphi^{\text{out}}(r, v, \mathbf{x}) &= \Big[\mathbb{D}_+ \varphi^{\text{in}}(r, v, \mathbf{x})\Big]_{v \to i\beta\zeta - v} \ , \\ \mathbb{D}_+ \varphi^{\text{out}}(r, \omega, \mathbf{k}) &= e^{-\beta\omega\zeta} \Big[\mathbb{D}_+ \varphi^{\text{in}}(r, \omega, \mathbf{k})\Big]_{\omega \to -\omega} \ , \end{aligned} \tag{1.47}$$

where we have used the real and frequency space representations of $\mathbb{D}_+$ respectively.

The relation (1.46) shows that outgoing solutions on the grSK contour involve explicit dressing with ζ dependent factors. Therefore, unlike ingoing solutions which are analytic functions of the radial co-ordinate r, Hawking solutions are non-analytic functions with branch-cut singularities and their accompanying monodromies. This distinction in the analytical behaviours of field modes is a salient ingredient of the grSK construction. On studying the field solutions on grSK geometry we will find that the non-trivial monodromy exhibited by Hawking solutions across the two branches of the spacetime is precisely what relates them to the noise observed in the dual plasma – Hawking solutions manifest as fields (or their sources) which differ across the L and R copies of the plasma.

[20] It is straightforward to generalise the action of the time-reversal to fields of integer spins. The case of fields with half-integer spins is more subtle and was worked out in [93]. In addition, even for scalar fields one needs to account for phases due to intrinsic time-reversal parity of the field.

A word on our conventions: Uppercase Latin indices are used for the bulk AdS spacetime, with lowercase Greek indices reserved for the timelike boundary. Spatial directions along the boundary are further indexed by lowercase mid-alphabet Latin indices. g_{AB} is the bulk metric, $\gamma_{\mu\nu}$ the induced boundary metric, and n^A the unit normal to the boundary.

Chapter 2

Effective theory of stochastic diffusion

This chapter is based on [42] written by the author in collaboration with Jewel K. Ghosh, R. Loganayagam, Siddharth G. Prabhu, Mukund Rangamani and V. Vishal.

2.1 Remembrance of a black hole's past

Black holes when disturbed settle down after classically ringing in quasinormal modes [94, 95]. In addition, they undergo simulated radiation in Hawking modes [41] (which may also be spontaneous owing to vacuum fluctuations). As in any quantum statistical system these are two intimately related features of a thermodynamic ensemble of states. The quasinormal modes signify dissipation into a medium and the Hawking modes correspond to the attendant quantum statistical fluctuations. They are related through fluctuation-dissipation relations. The timescales for the ring-down are typically short for non-extremal black holes with compact horizon topology. Typically, black holes carry no long-term memory.

However, black holes in negatively curved anti-de Sitter (AdS) spacetimes with non-compact horizons exhibit long-lived quasinormal modes. Quasinormal modes of AdS black holes correspond to thermalization rates of the dual field theory [40]. Massless spin-1 and spin-2 fields in planar AdS black holes have long-lived quasinormal modes with dispersions that are characteristic of hydrodynamic fluctuations [19–21]. These long lived modes correspond to the charge and momentum diffusion, and (attenuated) sound waves in the dual field theory plasma. One can trace their origins to the underlying gauge invariance of massless spin-1 and spin-2 fields manifested as global conservation laws for charge currents and energy-momentum tensor. For field theories on $\mathbb{R}^{d-1,1}$ these conserved currents decay slowly.[1] These results originally derived for linearized perturbations also

[1]While these modes are present for planar AdS black holes with non-compact horizon topology, as

hold at the non-linear level as manifested by the fluid/gravity correspondence [22, 23].

While the study of the dissipative dynamics of long-lived quasinormal modes of AdS black holes has a rich and storied history (cf., [97, 98] for comprehensive reviews), the corresponding discussion for the fluctuations of the long-lived modes (i.e., long-lived Hawking modes) is less well developed. Our goal in this work is to provide a unified treatment of the diffusive modes of AdS black branes and obtain an effective action governing their dynamics.

2.1.1 Open quantum systems with memory

As outlined in chapter 1, we find it helpful to phrase the problem in the language of open quantum systems following [61]. It was demonstrated in this work that holography is a natural arena to study open quantum field theories by allowing one to compute real-time QFT observables without the limitations of perturbation theory. Consider a holographic field theory at finite temperature, e.g., $\mathcal{N} = 4$ SYM, coupled to an external probe. The probe which evolves as an open quantum system, carries in its effective dynamics (obtained after integrating out the holographic system), an imprint of the thermal medium's correlations. Thus, the probe's real-time dynamics should encode both the dissipation in the plasma and the fluctuations it experiences. Consequently, such a probe, if coupled to the CFT's conserved currents, should be able to record the long memory of AdS black holes. We seek to understand how open quantum systems work in this non-Markovian regime where long-term memory is retained.

The simplest examples of such probes are quantum mechanical ($0 + 1$ dimensional probes). Indeed, it was demonstrated in [99, 100] that external point particle probes (quarks) exhibit stochastic Brownian motion in the plasma. More recently, building on the proposal of [88] for computing real-time (Schwinger-Keldysh) observables in AdS/CFT (see [85–87, 101, 102] for important earlier works on the subject), [60] were able to derive the non-linear fluctuation-dissipation relations for the Brownian particle. Crucial in this regard was the use of time reversal isometry that gives a clean construction of the Hawking fluctuations from the in-falling dissipative modes.

Quantum field theoretic probes however are more interesting as the problem of constructing local open effective field theories remains a challenge using standard techniques (see [61] for a brief commentary on this issue). In [61] it was shown that holographic engineering of local open EFTs is quite straightforward. They considered a scalar probe Ψ coupled to a single-trace bosonic operator $\mathcal{O}$ in the holographic field theory.[2] As explained in §1.2.1, the Ψ effective action has coefficients that are determined by the real-time correlators of $\mathcal{O}$. The latter can be computed using the gravity dual to the

explained in [96] one expects nearly long-lived modes in large AdS black holes.

[2]This discussion was generalized to fermionic open EFTs in [89, 93] and to rotating backgrounds in [90].

field theory Schwinger-Keldysh contour, the grSK geometry discussed in §1.3. For operators which have short-lived quasinormal modes, the correlations decay quite fast. The thermal plasma viewed as a bath/environment for the probe field Ψ has a very short memory. We will refer to such probes as Markovian; they are forgetful and do not keep a detailed record of the black hole state. Indeed, one of the reasons for the success in the holographic modelling of Markovian open EFTs in [61] may be attributed to the fact that holographic thermal baths scramble information optimally.

But there are other physical questions about open EFTs where there is a bath degree of freedom which exhibits long-time correlations and has to be retained in the effective description even as the rest of the bath is integrated out. The non-Markovian behaviour of this kind often happens for a physical reason: either because of a Goldstone mode emerging out of a spontaneous breaking of a continuous symmetry (e.g., holographic superfluids), or because of the effect of Fermi surface with long-lived quasiparticles, or because there is a conserved charge/energy/momentum whose relaxation happens at macroscopic time scales as described above. It is interesting to ask what holography can teach us about the structure of open EFTs in such situations.

Unfortunately, the most straightforward models which incorporate the kind of physics described above are technically involved. They typically involve systems whose standard description in the holographic geometry exhibits gauge invariance with its usual companions, Gauss constraints and Bianchi identities. Thus, study of the physics of Hawking modes and fluctuations in such systems runs into two issues: one conceptual and another technical. The conceptual issue is to come up with a way to think about the long-lived part of the Hawking radiation (dual to hydrodynamic fluctuations) by decoupling it from the short-lived part without spoiling gauge invariance. This should be distinguished from the technical issue of making suitable choice of gauge to solve for the Hawking fluctuations. For instance, much of the AdS/CFT literature uses a radial gauge, which is a poor choice in this context [88]. Physically, one may understand its inefficacy by the fact that the time-reversal isometry used to directly construct the Hawking modes in [60, 61] does not preserve the radial gauge.

We aim to address both the conceptual and the technical issues, though it will prove convenient to first decouple the two and address them independently. To this end, it will prove helpful to first build intuition regarding the key physical aspects of non-Markovian probes of black holes. We shall do so by analyzing a probe scalar field with suitably dressed gravitational interactions. This allows us to decouple the questions of long-time correlations from those of gauge invariance.

Inspired by the above, we introduce a class of models of scalar probes coupled to a thermal holographic bath. These probes will be characterized by a single parameter $\mathcal{M}$ which we will refer to as the *Markovianity index*. Probes with $\mathcal{M} > -1$ will have short-lived memory and behave analogous to the massive scalar probes studied in [61].

Probes with $\mathcal{M} < -1$, however, retain long-term memory.[3] These capture the essence of the non-Markovian physics we wish to analyze. In the gravity dual we will model such probes using a dilatonic coupling. The heuristic intuition is that one wants to amplify the coupling of the field near the horizon where the thermal atmosphere of the black hole is the strongest, thus amplifying the low-lying IR modes. Relatedly, we want to suppress the coupling to the UV modes since the dynamics of the long-lived modes such as those that appear in hydrodynamics are largely universal, and insensitive to the detailed microscopic description. We will study thus a class of designer scalar probes where the dilaton, which governs the coupling of the probe to the geometry, is a simple function of the radial holographic coordinate and is characterized by $\mathcal{M}$. The class of models we study has been previously examined in the holographic literature in [103–105] in the context of holographic RG flows and applications to AdS/CMT.[4]

While the designer scalar parameterized by the Markovianity index is a convenient proxy for analyzing the dynamics of non-Markovian probes, it also allows us to capture the physics of gauge invariant degrees of freedom. Specifically, we will demonstrate that the dynamics of holographic diffusion, be it charge or momentum, can be mapped to the study of such scalars. This can be engineered by organizing the perturbation of spin-1 gauge fields and spin-2 gravitons in gauge invariant combinations, e.g., as carried out in [106, 107] in the study of black hole perturbations. The designer scalar actions (along with variational boundary terms and boundary counterterms) naturally descend from the underlying Maxwell or Einstein-Hilbert dynamics for spin-1 and spin-2 fields respectively. In other words, the scalar probes we study are effectively coupling to particular polarizations of the boundary charge current or energy-momentum tensor. The Markovianity index for charge diffusion is $\mathcal{M} = -(d-3)$ while that for momentum diffusion is $\mathcal{M} = -(d-1)$.

This gauge agnostic treatment has several advantages compared to earlier studies. In the fluid gravity correspondence literature one treats the radial Gauss constraint(s) distinctly from the radial evolution equations. Solving the radial evolution equations usually takes into account the effects of the fast Markovian modes whereas the hydrodynamics of the long-lived non-Markovian modes is encoded within the conservation laws inherent in radial Gauss constraints. For example, in the gravitational problem the diffeomorphism Gauss constraints on the radial hyper-surface become the Navier-Stokes equations for the dual CFT plasma. In contrast, we will solve all the bulk equations of motion including the radial Gauss constraints in our discussion below. We will still keep the non-Markovian fluid modes off-shell by turning on appropriate non-normalizable modes.

There is another way to understand the inefficacy of the radial gauge for gauge and

[3] The zero-point here has been chosen so that a minimally coupled scalar in AdS_{d+1} has $\mathcal{M} = d-1$. Our choice is engineered for a simple analytic continuation rule from Markovian to non-Markovian fields.

[4] We will also demonstrate that the models we consider may equivalently be viewed in terms of the dynamics of scalar probes in thermal hyperscaling violating backgrounds, cf., §2.2.1.

gravitational dynamics. The grSK geometry has two timelike boundaries which correspond to the L and R (or bra and ket) components of the boundary Schwinger-Keldysh contour. However, these are connected through the AdS bulk – one therefore finds only a single radial gauge constraint. This automatically puts the difference Schwinger-Keldysh fields on-shell and thus imposes the conservation of the difference currents. This is a problem: if we attempt to derive a generating function for current correlators we end up missing a degree of freedom from the difference current. One way to proceed is to attempt to take this difference current off-shell. This is the choice made by [88, 108], who do so by imposing an explicit source for the difference fields at the horizon. However, it makes more physical sense to let the gravitational dynamics lead the way and impose all the bulk equations of motion. Then one has a more natural parameterization in terms of the boundary expectation values.

Having motivated the study of the designer scalar system, and its connection to the dynamics of conserved currents of a holographic thermal baths, let us now describe what should one compute, given the dynamics. For the Markovian sector with $\mathcal{M} > -1$ it is clear that one can follow as in [61] the usual rules of the AdS/CFT correspondence, now uplifted to the grSK geometry, and compute the generating function of real-time correlation functions (with Schwinger-Keldysh time ordering). For non-Markovian probes such a generating function is guaranteed to be non-local – one is integrating out long-lived modes. While this would be a fine approach to take, it is more conducive in the spirit of EFT to ask if there is an alternative that allows us to derive a local effective action for non-Markovian probes. We will show that there indeed is one, which is obtained by a Legendre transformation of the generating function of connected correlators into a Wilsonian Schwinger-Keldysh action parameterized by the non-Markovian operators (or their expectation values in AdS/CFT parlance). We will demonstrate that such a Wilsonian influence functional for real-time non-Markovian observables is well defined and obtain an expression for the same at the quadratic order (in amplitudes). From this functional one can of course obtain the real-time correlators for the non-Markovian fields. We will indeed recover the known physics of diffusion as well as the stochastic noise associated thereto.

2.1.2 Synopsis of salient results

To aid the reader in navigating this chapter, let us quickly record some of the salient results we derive in the discussion below:

- We obtain the expected diffusive dynamics of charge and momentum recovering the expected form of the retarded Green's functions of charge and energy-momentum current. Specifically, for momentum diffusion we find the dispersion:

$$0 = i\,\omega - \frac{\beta}{4\pi}\,k^2 + \frac{\beta}{4\pi}\,\text{Har}\left(\frac{2}{d} - 2\right)\omega^2 + \cdots \tag{2.1}$$

which gives the famous shear diffusion pole.[5] The coefficient of the ω^2 contains contributions from two hydrodynamic transport coefficients k_R and k_σ in the notation of [62]. As generalizations, we compute the next order (cubic in gradients) term in the retarded Green's function and also obtain results that pertain to probes of hyperscaling violating backgrounds at finite temperature, cf., (2.44) for the general result parameterized by the Markovianity index $\mathcal{M}$.

- One may interpret our results for the retarded Green's function in terms of the greybody factors of the planar Schwarzchild-AdS$_{d+1}$ black holes for photons and gravitons, accurate to cubic order in a (boundary) gradient expansion. The relevant expressions are given in (2.115) and (2.125), for photons and gravitons respectively. Our results match with earlier derivations from holography [21, 109, 110] for energy momentum and charge correlators. Explicit expressions for the stress tensor correlators of thermal $\mathcal{N} = 4$ SYM plasma are given in (2.135) and (2.140), respectively.

- In addition we also derive the fluctuations associated with the dissipative modes using the Schwinger-Keldysh formalism. We obtain the leading contribution (at quadratic order) to the Hawking noise correlations associated with thermal photons and gravitons. We package this information in the Wilsonian influence functional as mentioned above. Explicit expressions for stress tensor correlators for $\mathcal{N} = 4$ SYM plasma can be found in (2.136) and (2.141), respectively.

- One of the key technical insights of the discussion is a clean separation in systems with gauge invariance of the Markovian and non-Markovian degrees of freedom. For instance, for stress tensor dynamics we show that the non-Markovian transverse momentum diffusion modes can be cleanly separated from the Markovian transverse tensor polarizations. This separation uses a classic gauge invariant decomposition of linearized perturbations (cf., [106]) which has been employed in various studies of gravitational problems over the years.

- Finally, a corollary of our work is a derivation of an effective action for diffusive modes. In particular, we verify the structural form of the class L action for non-dissipative coefficients conjectured from fluid/gravity transport data in [62].[6]

[5] Here Har(z) denotes the harmonic number function defined as $\text{Har}(z) = \int_0^1 \frac{1-x^z}{1-x}\text{d}x$.

[6] In the classification of hydrodynamic actions in [62], class L fluids are those which admit a Lagrangian description for their non-dissipative dynamics. In other words, their stress-energy densities, modulo dissipative terms, can be derived via varying the corresponding class L action. See (2.143) for the class L action they predict for holographic fluids.

While we are not the first to ask these questions our treatment of the problem will be quite different from the discussion in the literature hitherto for the most part. For instance, the semi-holographic models of [111] provided a template for analyzing the effective dynamics of holographic systems with long-lived excitations. This motivated [112] to provide a framework for understanding holographic liquids. Furthermore, [113, 114] attempted to derive effective actions for hydrodynamics using holography (primarily in the non-dissipative sector at the ideal fluid level). Closely related to our present discussion is the work of [88, 115] who used the grSK geometry to study effective dynamics of charge diffusion. Our treatment here while having some elements of commonality, differs substantially in that our primary focus is on the physics of outgoing Hawking modes.[7] However, for the most part we will carefully analyze the physics of infalling quanta. Once we understand this in a gauge agnostic manner, by exploiting the time-reversal isometry discussed before, we can extract the outgoing modes efficiently.

In other words, we seek a treatment of the problem, that respects time-reversal properties, has a sensible local gradient expansion amenable to effective field theory analysis, and captures the physics of long-term black hole memory. The fact that all of these can be done self-consistently is a central thesis of our work.

2.1.3 Outline of the chapter

The outline of this chapter is as follows: In §2.2 we introduce the class of designer fields which we use to characterize Markovian and non-Markovian dynamics. The reader will find a succinct summary here of the connection to probes coupled to conserved currents. The central thesis of our discussion of memory is given in §2.3. Here using the auxiliary scalar system we explain the salient differences between Markovian and non-Markovian probes, and explain how we should encode effectively the physics of probes which are sensitive to the long-term memory of the black hole. §2.4 re-purposes the discussion of [61] to describe Markovian probes, while §2.5 deals with non-Markovian probes. As we describe in detail there, we do not resolve the dynamics of the non-Markovian probes, but rather, exploit an analytic continuation in the Markovianity index to extract the necessary physical information. In §2.6 we put together the intuition gleaned from the designer scalar analysis and compute the two-point real-time correlation functions for both Markovian and non-Markovian probes. In §2.7 we finally turn to probe gauge fields and demonstrate that a suitable gauge invariant decomposition allows us to map their dynamics to the designer scalars, and use this intuition to extract the physics of current

[7] On the field theory side there have been many constructions of the Schwinger-Keldysh effective action for the hydrodynamic modes, see eg., [39, 47–55]. At the quadratic order, our effective action agrees trivially with all these constructions. Since the non-trivial aspects of these constructions appear only at the interacting level, a detailed comparison to these formalisms is not possible at the current stage of our programme.

correlators in the holographic field theory. In §2.8 we turn to the dynamics of transverse tensor and vector gravitons and show that they too can be mapped to our designer scalars. One of our central results which we describe in this section is the quadratic fluctuations (including stochastic noise) of gravitons in the black brane background.

There are several supplemental appendices to the chapter which contain some of the technical details. Appendix A contains various details related to our designer scalar system which we draw upon in §2.4 and §2.5. Some of the details relating to gauge field probes can be found in Appendix B, where we also describe the physics in the (suboptimal) radial gauge. The reduction of gravitational dynamics onto our designer systems is explained in Appendix C. In Appendix D we give the expressions for the conserved currents in terms of the designer scalar fields, which may be employed to directly compute the current correlation functions.

The appendices Appendix E and Appendix F contain salient details of incomplete beta functions and plane wave harmonics, that we employ in our analysis throughout.

2.2 Designing gravitational probes with memory

The Schwarzschild-AdS$_{d+1}$ provides for us a holographic thermal bath which we wish to probe. Our primary interest is in understanding probes that couple to the conserved currents, global charge current or energy-momentum current, of the dual field theory which exhibit long-lived diffusive behaviour. It will transpire that one can give a unified presentation in terms of a scalar probe of the grSK background, albeit one that not only couples to the gravitational background, but also to an auxiliary dilaton. We will demonstrate below that the conserved current dynamics can be mapped directly on this scalar system for a specific choice of the dilaton profile which depends on the nature of the current.

2.2.1 Designer scalar and gauge probes

Our prototypical holographic probe field is what we call a designer scalar. It is a massless Klein-Gordon field $\varphi_{\mathcal{M}}$ coupled minimally to gravity and to a dilaton χ_s. Specifically, consider an action of the form

$$S_{ds} = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\; e^{\chi_s}\,\nabla^A\varphi_{\mathcal{M}}\nabla_A\varphi_{\mathcal{M}} + S_{bdy}\,, \qquad e^{\chi_s} \equiv r^{\mathcal{M}+1-d} \tag{2.2}$$

The auxiliary dilaton which depends on a designer parameter $\mathcal{M}$ serves the purpose of modulating the coupling of our probe as a function of the energy scale.[8] Depending on the sign of $\mathcal{M}$ the coupling of the scalar to the geometry can be modulated across the

[8]Choosing $\mathcal{M} = d-1$ gives us a massless minimally coupled scalar, which was studied in [61].

radial direction. For $\mathcal{M}$ sufficiently negative the scalar is weakly coupled deep in the bulk relative to its dynamics in the asymptotic region. This mimics the modes of the dual field theory which, say like the hydrodynamic modes, represent a small number of UV degrees of freedom, but a large number of IR degrees of freedom. This heuristic is based on the UV/IR dictionary of the AdS/CFT duality [116] along with the idea that bulk couplings roughly correspond inversely to the number of CFT degrees of freedom. We will later demonstrate that, quite amusingly, this simple family of scalar actions describe a varied class of probe fields in the gravitational theory, which is our original motivation to introduce it.

In Fourier domain, the action (2.2) gives rise to an equation of motion of the form[9]

$$\frac{1}{r^{\mathcal{M}}}\mathbb{D}_+\left(r^{\mathcal{M}}\,\mathbb{D}_+\varphi_{_\mathcal{M}}\right)+\left(\omega^2-k^2 f\right)\varphi_{_\mathcal{M}}=0 \tag{2.3}$$

We will argue that for sufficiently negative $\mathcal{M}$, this system does indeed exhibit long-lived correlations and hence gives a simple toy model for a bath with a memory. While this might seem like an oversimplified model with no gauge invariance etc., we will see that this system indeed captures the essence of many realistic gauge systems.

A second probe which we will examine in detail is a bulk Maxwell system, also dilatonically coupled, designed with an action of the form

$$S_{dv}=-\frac{1}{4}\int d^{d+1}x\,\sqrt{-g}\;e^{\chi_v}\,\mathcal{C}^{AB}\mathcal{C}_{AB}\,,\qquad e^{\chi_v}\equiv r^2\,e^{\chi_s}=r^{\mathcal{M}+3-d} \tag{2.4}$$

where $\mathcal{C}_{AB}=2\,\nabla_{[A}\mathcal{V}_{B]}$ denotes the Maxwell field strength and $\mathcal{V}$ the 1-form potential. We will demonstrate how this system with gauge invariance also exhibits long-lived correlations associated with the physics of charge diffusion in the dual CFT. In particular, we note that setting $\mathcal{M}=d-3$ corresponds to the standard Maxwell system. Furthermore, in a precise sense, we will exhibit how this gauge system can be reduced to the scalar problem (2.2). This explains why we continue to employ the same symbol $\mathcal{M}$ to characterize the Markovianity index for the designer vector.

Both the scalar and the gauge field will be taken to probe the grSK saddle geometry (1.33). On this two sheeted geometry the Lagrangian density for the systems is integrated over the grSK contour depicted in Fig. 1.3. Effectively, this means that the radial integral is morphed into a contour integral $\int dr \mapsto \oint d\zeta$ for the real-time part of the evolution. We won't write this out at the moment, since for the most part our analysis can be done on a single sheet and thence upgraded onto the two-sheeted grSK contour.

[9]We have yet to formulate the variational principle, which involves specifying the boundary terms for (2.2). In addition we will also have to introduce boundary counterterms. We will address these terms in due course.

Designer probes and hyperscaling violation: There is a useful metaphor that helps to understand our designer scalar system and sharpens the intuition that the parameter $\mathcal{M}$ allows for tuning the relative coupling of low and high energy modes of the field to the geometry. Our designer scalar can be mapped onto the dynamics of a massless scalar probing a thermal state in a Lorentzian hyperscaling violating background with $z = 1$. More specifically, consider the following Einstein-dilaton action in $\bar{d}+1$ dimensions which has been analyzed extensively in [104] (see also [103]):

$$\begin{aligned} S_{ED} &= \frac{1}{16\pi G_N} \int d^{\bar{d}+1}x\, \sqrt{-\bar{g}} \left(R + V_0\, e^{\bar{b}\,\chi} - \frac{1}{2}\, \bar{g}^{AB}\, \partial_A\chi\, \partial_B\chi \right) \\ \text{with} \qquad & V_0 = (\bar{d}+1-\theta)(\bar{d}-\theta)\,, \qquad \bar{b}^2 = \frac{2\theta}{\bar{d}(\theta-\bar{d})}\,. \end{aligned} \tag{2.5}$$

which has a black hole solution

$$\begin{aligned} d\bar{s}^2 &= r^{-2\frac{\theta}{\bar{d}}} \left(-f(br)\, r^2\, dv^2 + 2\, dv\, dr + r^2\, d\mathbf{x}_{\bar{d}}^2 \right) \\ e^{\bar{b}\,\chi} &= r^{-2\frac{\theta}{\bar{d}-1}}\,, \qquad f(br) = 1 - \left(\frac{1}{br}\right)^{\bar{d}+1-\theta}\,. \end{aligned} \tag{2.6}$$

A massless scalar $\varphi_{\mathcal{M}}$ probing the geometry (2.6) with dynamics

$$\int d^{\bar{d}+1}x\, \sqrt{-\bar{g}}\, e^{\bar{\alpha}\,\bar{b}\,\chi}\, \bar{g}^{AB}\, \partial_A\varphi_{\mathcal{M}}\, \partial_B\varphi_{\mathcal{M}}\,, \tag{2.7}$$

would map to the action (2.2) with the identifications:

$$\bar{d} - \theta = d\,, \qquad \mathcal{M} = \bar{d} - 1 - \theta\left(1 + \frac{2\,\bar{\alpha}}{\bar{d}-1}\right) = d - 1 - \frac{2\,\bar{\alpha}\,\theta}{\theta + d - 1}\,. \tag{2.8}$$

As discussed extensively in the AdS/CMT literature (cf., [56] for an excellent review) the Lorentzian hyperscaling violating geometries can be interpreted as RG flows arising when we turn on dangerously irrelevant primaries. The IR dynamics in these examples strongly deviates from the UV leading to emergent scaling dynamics at low energies. The construction of designer probes is also reminiscent of attempts to understand holographic duals of non-conformal branes [117] and Ricci-flat holography [118] using generalized dimensional reduction [119].

2.2.2 Origins of the designer fields

Before we proceed, let us explain the rationale behind our choice of the designer fields. The designer scalar and gauge field arise quite naturally when we consider the dynamics of Maxwell fields and gravitons in the Schwarzschild-AdS$_{d+1}$ background. We give here a quick synopsis of the salient statements deferring the details to subsequent sections.

We give a quick reference summary of the values of $\mathcal{M}$ obtained from the study of

massless field	Scalar sector	Vector sector	Tensor sector
Scalar	$d-1$	–	–
Gauge field	$-(d-3)$	$d-3$	–
Metric	$-(d-3)$	$-(d-1)$	$d-1$
Nambu-Goto (linearized)	2		

Table 2.1: Values of $\mathcal{M}$ obtained for linearized perturbations of various fields of interest in Schwarzschild-AdS$_{d+1}$ background. Note that both positive and negative values of $\mathcal{M}$ are thus obtained.

various massless fields in the Schwarzschild-AdS$_{d+1}$ background in Table 2.1.[10] More specifically, the following statements hold, as we will demonstrate in due course:

1. The field $\int_k \varphi_{d-1}\, \mathbb{S}$ solves the massless Klein-Gordon equation. Here $\mathbb{S} = e^{-i\omega v + i\,\mathbf{k}\cdot\mathbf{x}}$ is the plane wave on $\mathbb{R}^{d-1,1}$. In particular, its definition includes the $e^{-i\omega v}$ frequency dependence.[11]

2. The 1-form $\mathcal{V}$ parameterized as

$$\mathcal{V}_A\, dx^A = \int_k \left[\frac{1}{r^{d-3}}\left(dv\ \mathbb{D}_+ - dr\,\frac{\mathrm{d}}{\mathrm{d}r}\right)\varphi_{3-d}\,\mathbb{S} + \sum_{\alpha=1}^{N_V}\varphi^{\alpha}_{d-3}\ \mathbb{V}^{\alpha}_i dx^i\right] \tag{2.9}$$

   solves the Maxwell equations §2.7.

   Here $\mathbb{V}^{\alpha}_i$ denote the $N_V = (d-2)$ transverse planar vector waves with $k_i\mathbb{V}_i = 0$ on $\mathbb{R}^{d-1,1}$. They transform in the spin-1 representation of SO$(d-2)$ transverse to $\mathbf{k}$ and represent electromagnetic waves polarized along black brane which quickly fall into the brane. On the other hand, φ_{3-d} corresponds to the radially polarized electromagnetic wave grazing the black brane. The latter is a long-lived (non-Markovian) mode dual to the charge diffusion mode in the CFT.

3. Finally, the spin-2 symmetric tensor combination

$$\delta g_{AB}\, dx^A dx^B = r^2\int_k\left[2\,\frac{dx^i}{r^{d-1}}\left(dv\,\mathbb{D}_+ - dr\,\frac{\mathrm{d}}{\mathrm{d}r}\right)\sum_{\alpha=1}^{N_V}\varphi^{\alpha}_{1-d}\ \mathbb{V}^{\alpha}_i + \sum_{\sigma=1}^{N_T}\varphi^{\sigma}_{d-1}\ \mathbb{T}^{\sigma}_{ij}\, dx^i\, dx^j + \dots\right] \tag{2.10}$$

   solves the linearized Einstein equations §2.8. The ellipses denotes the scalar polarizations dual to the CFT sound mode, which we will discuss in chapter 3.

   We have introduced $\mathbb{T}^{\sigma}_{ij}$ which are the transverse, symmetric, trace-free, tensor plane waves on $\mathbb{R}^{d-1,1}$ with $k_i\,\mathbb{T}^{\sigma}_{ij} = 0$ and $\mathbb{T}^{\sigma}_{ii} = 0$. They form a spin-two representation of SO$(d-2)$ transverse to $\mathbf{k}$. There are $N_T = \frac{1}{2}d(d-3)$ such linearly independent tensor

[10] Our statement that $\mathcal{M} = -(d-3)$ in the scalar sector of gravity requires careful interpretation, as it primarily refers to the relative fall-off between modes with $k \neq 0$. We will explore this sector which has qualitatively different physics in chapter 3.

[11] Our conventions for the planar harmonics are described in Appendix F.

waves and they represent the in-falling graviton modes which decay very quickly. In contrast, φ^{α}_{1-d} corresponds to the graviton modes grazing the black brane which fall in slowly. They are therefore long-lived (non-Markovian) and are dual to the momentum diffusion/shear modes in the CFT.

4. A corollary of points 2 and 3 above is that the gravitational perturbations in the tensor and vector sector can be expressed in terms of a Klein-Gordon scalar and designer vector with $e^{\chi_s} = 1$ and $e^{\chi_v} = r^2$, respectively.

The proof of these statements will be given in §§2.7 and 2.8 and Appendices B and C. The reader familiar with the study of linearized gravity in terms of the diffeomorphism invariant combinations starting from the early work of Regge-Wheeler-Vishweshwara-Zerelli [94, 120–122] and the more recent analysis of Kodama-Ishibashi [106, 107] will find our parameterization natural. These have been adopted in the AdS/CFT literature in the course of the study of quasinormal modes and linearized hydrodynamics [98, 123–126]. We will outline the connection with these works when we explain the derivation of the designer system. While the discussion in the main text will be in terms of gauge invariant variables, we have chosen to write the gauge field and metric in (2.9) and (2.10) in a particularly convenient gauge fixed form (in a Debye gauge).[12]

We note that the perturbations (2.9) and (2.10) are written in terms of the time-reversal covariant derivations introduced in §1.3, see (1.42). In particular, the derivative operator $dv\, \mathbb{D}_+ - dr\, \partial_r$ is the natural operator built of $\mathbb{Z}_2$ covariant 1-forms and $\mathbb{Z}_2$ covariant derivative operators. This operator is odd under the time-reversal $\mathbb{Z}_2$ isometry $v \mapsto i\beta\zeta - v$. This fact will play a crucial role in our analysis.

The field equations for the designer fields under discussion are also invariant under this isometry. More precisely, one gets an action of this $\mathbb{Z}_2$ on the space of solutions as follows: if $\varphi_{\mathcal{M}}(\zeta, \omega, \mathbf{k})$ is a solution of (2.3), then it can be shown that $e^{-\beta\omega\zeta}\, \varphi_{\mathcal{M}}(\zeta, -\omega, \mathbf{k})$ is also a solution as mentioned in §1.3. Depending on the physical problem under study, the field $\varphi_{\mathcal{M}}$ can also have an *intrinsic time-reversal parity* which means that we will sometimes define the time-reversed solution with an extra minus sign, i.e., we take $-e^{-\beta\omega\zeta}\, \varphi_{\mathcal{M}}(\zeta, -\omega, \mathbf{k})$ to be the time reversed solution. In the example above, given that $dv\, \mathbb{D}_+ - dr\, \partial_r$ is odd under time-reversal, the intrinsic time-reversal parity of φ_{3-d} and φ_{d-3} are opposite to each other. Similarly, φ_{1-d} and φ_{d-1} fields above should also have opposite intrinsic time-reversal parity.

Having introduced our two designer systems, in the following section we will examine the central features of each of them. In particular we will demonstrate how seemingly identical designer models capture drastically different physics based on the value of $\mathcal{M}$.

[12] Similar statements hold in global AdS_{d+1} where the different gauge and gravitational perturbations can be reduced to scalar fields with different Ricci-shifted mass terms [127] (cf., [106, 107]). Note that therefore our decomposition is background dependent; the dynamical content captured in our designer scalars should be viewed as being tailored to the specific background of interest.

2.3 Markovianity and lack thereof: memories lost and regained

As already mentioned, it will transpire that the designer gauge fields can be mapped with a suitable parameterization onto designer scalars. Therefore it will suffice for us to explore the designer scalar dynamics in some detail, and thus realise lessons valid for both classes of designer systems at a go. In this section we will begin exploring some general features of the designer scalar, emphasizing the key differences between models with $\mathcal{M} > -1$ and those with $\mathcal{M} < -1$. We will argue that the former correspond to *Markovian probes* while the latter are *Non-Markovian probes* which carry non-trivial memory far into the future.

Since our aim here is to delineate the salient, universal features of the model, we will analyze the wave equation (2.3) in the Schwarzschild-AdS$_{d+1}$ background directly in the standard radial coordinate parameterization. Translation to the grSK saddle will be straightforward once we understand the central features of the solutions. To this end, we work with the wave equation written out more explicitly in the form:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\mathcal{M}+2}\, f\, \frac{\mathrm{d}\varphi_{\mathcal{M}}}{\mathrm{d}r}\right) - i\omega\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\mathcal{M}}\varphi_{\mathcal{M}}\right) + r^{\mathcal{M}}\frac{\mathrm{d}\varphi_{\mathcal{M}}}{\mathrm{d}r}\right] - k^2\, r^{\mathcal{M}-2}\varphi_{\mathcal{M}} = 0\,. \tag{2.11}$$

2.3.1 Analytic versus Monodromy modes and their interpretation

Let us first characterize the solutions of the differential equation (2.11). To begin with, set $\omega, \mathbf{k}$ to zero and look for solutions that are constant along the boundary. We have then a homogeneous second order ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\mathcal{M}+2} f \frac{\mathrm{d}}{\mathrm{d}r}\varphi_{\mathcal{M}}\right) = 0\,, \tag{2.12}$$

whose general solution is of the form

$$\begin{aligned}\varphi^{(0)}_{\mathcal{M}} &= c_a + c_m \int_{\xi_c+i0}^{\xi} \frac{y^{d-2}\, dy}{y^{\mathcal{M}}(y^d-1)}\,,\\ &= c_a + \tilde{c}_m \int_{\rho_c+i0}^{\rho} \frac{\bar{\rho}^{\,\mathsf{s}(\mathcal{M}+1)}\, d\bar{\rho}}{1-\bar{\rho}}\end{aligned} \tag{2.13}$$

where $r_c \gg 1$ is the radial cutoff chosen to regulate the UV region (ξ_c and ρ_c similarly defined).

Let us note some crucial features of this solution:

- When $c_m = 0$, the solution is analytic and ingoing at the future horizon. When uplifted to the grSK geometry by replacing $r(\zeta)$ this solution is smooth. In particular, it does not have a branch cut – it is identical on both sheets of the grSK saddle.

- When $c_m \neq 0$, the solution has a logarithmic branch cut at the horizon $r = \frac{1}{b}$. This is evident from the change of variables to the dimensionless ρ coordinate. If we start off with a value of (c_a, c_m) on one sheet of the grSK saddle, then we pick up a monodromy from the logarithmic branch-cut as we cross over to the other sheet. Specifically, c_a gets shifted by an amount proportional to c_m, while c_m is unchanged, under this crossing, i.e.,

$$J_{\mathrm{R}} = J_{\mathrm{L}} + \frac{2\pi i}{d}\, c_m\,, \qquad c_a(\mathrm{R}) = J_{\mathrm{R}}\,,\ \ c_a(\mathrm{L}) = J_{\mathrm{L}}\,. \tag{2.14}$$

Hence we will refer to the two modes multiplying c_a and c_m as the analytic and monodromy modes, respectively.

Based on the above, we can equivalently parameterize the general solution of the homogeneous equation (2.12) in terms of the boundary values J_{L} and J_{R} as

$$\varphi^{(0)}_{\mathcal{M}} = J_{\mathrm{L}} + \frac{d}{2\pi i}\left(J_{\mathrm{R}} - J_{\mathrm{L}}\right) \int_{\xi_c + i0}^{\xi} \frac{y^{d-2-\mathcal{M}}\, dy}{y^d - 1}\ . \tag{2.15}$$

We note that the function multiplying $J_{\mathrm{R}} - J_{\mathrm{L}}$ behaves as $r^{-\mathcal{M}-1}$ near conformal boundary of AdS. Therefore, turning on $J_{\mathrm{R}} - J_{\mathrm{L}}$ is equivalent to 'dressing' the original analytic solution by adding a solution which behaves as $r^{-\mathcal{M}-1}$ near infinity. Clearly, the nature of the solution depends on whether the dressing decays or grows at large r. This leads us to the two advertised cases of Markovian and non-Markovian dynamics, respectively.

2.3.1.1 The Markovian case: $\mathcal{M} + 1 > 0$

Let us first examine the case where $\mathcal{M} + 1 > 0$, which we term to be a Markovian probe, for reasons that will become clear soon. For such a designer scalar turning on $J_{\mathrm{R}} - J_{\mathrm{L}}$ is equivalent to turning on a normalizable mode. The monodromy mode sourced by $J_{\mathrm{R}} - J_{\mathrm{L}}$ is sub-dominant at large r relative to the analytic mode. We furthermore recall from [61] which analyzed a massless Klein-Gordon field, a particular exemplar with $\mathcal{M} = d-1 > 0$, that turning on $J_{\mathrm{R}} - J_{\mathrm{L}}$ turns on Hawking radiation in the bulk which is ultimately a normalizable mode in the black brane background.

Let us understand the implications of this statement for the grSK saddle. An immediate consequence of the normalizability is that on the grSK geometry, a double Dirichlet boundary condition problem, which specifies the coefficient of non-normalizable mode near both the left and right boundaries, is thus well-posed. Specification of J_{R} and J_{L} uniquely determines the bulk solution. This should be viewed as the SK version of the standard boundary condition in AdS/CFT. Moreover, inspection of the wave equation (2.11) makes clear that this conclusion holds even when $\omega, k \neq 0$ and thus the solutions to the differential equation can be obtained order by order in a long wavelength, low frequency, expansion (as illustrated in [61] for massive Klein-Gordon fields).

The classical picture from the study of the wave equation can easily be upgraded to the quantum realm. The designer system can be quantized by performing a path integral over the normalizable modes, taking them off-shell in the process, whilst leaving the left and right non-normalizable modes (the sources) frozen at the respective boundaries. Since the normalizable modes, by definition, have a finite action (with the addition of suitable counterterms as usual), they contribute to the semiclassical path integral with an amplitude determined by the said action. Summing over all the off-shell modes leads to an answer which is a functional purely of the non-normalizable sources. This is the standard GKPW [17, 18] dictionary of AdS/CFT applied to the grSK saddle, as indeed argued in [61] for $\mathcal{M} = d - 1$.

2.3.1.2 The non-Markovian case: $\mathcal{M} + 1 \leq 0$

Let us turn to the case $\mathcal{M} + 1 \leq 0$. Realize that turning on $J_{\text{R}} - J_{\text{L}}$ is now tantamount to turning on a non-normalizable mode that *grows* at infinity as $r^{|\mathcal{M}|-1}$. In the most general solution at zero derivative order, the coefficient of this growing solution (c_m in (2.13)) is the same on both left and the right branches of the grSK geometry. Thus, *imposing double boundary conditions on non-normalizable modes has no solution at the zero derivative order*! Again for reasons that will become clear shortly, we will term such a designer field with $\mathcal{M} + 1 \leq 0$ where this phenomenon happens as a non-Markovian probe.

How should one proceed in this case? One may wish to change the boundary condition to freeze the growing mode. This would clearly work and would involve changing the variational principle to keep the mode parameterized by c_m to be frozen. Operationally this is a Legendre transformation on the space of boundary conditions, akin to the choice made for highly relevant operators [128] or multi-trace deformations in AdS/CFT [129]. Indeed, explicit computations reveal that the non-normalizable modes start differing on the two branches as we go to higher orders. This would however have to be done at the expense of abandoning the gradient expansion and letting c_m have non-analytic dependence on ω and k. With these changes we would be able to write down a solution with double boundary conditions on the non-normalizable modes. With these modifications one could come up with an 'alternate' GKPW dictionary for such probes, enabling us thereby compute their Schwinger-Keldysh correlation functions.

It is however worth reflecting on the physics prior to abandoning the gradient expansion altogether. The failure of the long wavelength, low frequency expansion in the bulk, is indicative of the fact that there are slow propagating degrees of freedom in the probe. These prevent us from approximating the correlators by contact terms at long distances and times. In other words the system retains memory of its origins, justifying the characterization of such designer probes as *non-Markovian*.[13] Attempting to solve the bulk

[13] Analogously one justifies the characterization of the Markovian probes discussed in §2.3.1.1 – such probes retain no memory and have no lasting effect on the system.

problem with double boundary conditions on non-normalizable modes is tantamount to integrating out these slow degrees of freedom by going beyond the derivative expansion.

But what if we did not want to integrate out these modes? We might want to freeze the slow modes and integrate out only the fast modes (as in the derivation of the Born-Oppenheimer effective potential for nuclei). The latter can be tackled in a gradient expansion. With this motivation, we ask what is the class of solutions in the bulk SK geometry where a derivative expansion is possible? As we will explain now, there is indeed such a class of solutions.

2.3.2 The well of memory: hydrodynamic moduli space

As we have seen the non-Markovian probes are characterized by the presence of long-lived, low lying, Goldstone type modes which if integrated out lead to a non-local functional of the boundary sources (the generator of connected correlators). We will now give a precise characterization of how to deal with such modes and use them to define a low energy moduli space. Since such modes appear in the study of designer probes arising from gauge or gravitational perturbations, which in the dual field theory, corresponds to the dynamics of conserved currents, we will refer to the low energy space as the *hydrodynamic moduli space*.

2.3.2.1 Analytic continuation into the hydrodynamic moduli space

To begin with, a mathematical characterization of solution to the non-Markovian probe's equation of motion, may be obtained via an analytic continuation in the designer exponent $\mathcal{M}$ from positive to negative values. When $\mathcal{M}$ is in the Markovian regime as argued in §2.3.1.1 we can set-up double Dirichlet boundary conditions on the (analytic) non-normalizable modes and a solution may be obtained order by order in the gradient expansion. Furthermore, as in the fluid/gravity correspondence [22, 23], given slowly varying non-normalizable data on the two boundaries, call them J_{R} and J_{L}, respectively, as in (2.15), one can write down a solution of the grSK geometry which admits a local series expansion in derivatives along the boundary. The main point to note here is that the normalizable modes are fixed completely at each order in this gradient expansion in terms of the pair of non-normalizable data $J_{\text{R}}, J_{\text{L}}$. For the non-Markovian probe we propose to take the Markovian solution for a given $\mathcal{M} > 1$ and analytically continue it to a non-Markovian solution with $\mathcal{M} < -1$.[14]

[14] For the most part of our discussion we will focus on $\mathcal{M} < -1$ non-Markovian fields, omitting a detailed analysis of the marginal case $\mathcal{M} = -1$. The issue in this case is that the asymptotic behaviour is analogous to minimally coupled massive scalars at the Breitenlohner-Freedman bound, viz., solutions with an admixture of a logarithmic mode. Note $\mathcal{M} = -1$ is physically realized for a probe Maxwell field in Schwarzschild-AdS$_5$ background.

The analytic continuation in the exponent $\mathcal{M}$ from the Markovian to non-Markovian regime, roughly speaking, flips the normalizable and the non-normalizable pieces. More precisely, the normalizable modes on the Markovian side analytically continue to non-normalizable modes on the non-Markovian side. On the other hand, as we will argue now, the non-normalizable data, J_{R} and J_{L} on the Markovian side analytically continue into normalizable *Wilsonian Schwinger-Keldysh effective fields* $\breve{\Phi}_{\text{R}}$ and $\breve{\Phi}_{\text{L}}$, respectively, on the non-Markovian side.

To see this, note that by analytic continuation we do indeed obtain a space of solutions for the non-Markovian system parameterized by a pair of boundary data, $\{\breve{\Phi}_{\text{R}}, \breve{\Phi}_{\text{L}}\}$. Moreover, the solutions are, by construction, obtained in a gradient expansion in these boundary parameters. This space of solutions constitutes the hydrodynamic moduli space parameterized by the effective fields $\{\breve{\Phi}_{\text{R}}, \breve{\Phi}_{\text{L}}\}$. In any such solution we can determine the doubled non-normalizable data in terms of these hydrodynamic moduli. This is equivalent to determining the hydrodynamic equations of motion for $\{\breve{\Phi}_{\text{R}}, \breve{\Phi}_{\text{L}}\}$.

In fact, the non-normalizable mode is determined as a derivative operator acting on $\breve{\Phi}_{\text{R,L}}$. This, in turn, means that if we want to set the non-normalizable modes to zero (or any fixed value for that matter), these fields should satisfy appropriate differential equations. At a linearized level, in the Fourier domain, setting the derivative operator to zero results in a *dispersion relation* for these fields parameterizing the boundary degrees of freedom of the non-Markovian field. We will see that the dispersion relation for the designer scalar takes the form of a diffusive mode, of the form:

$$\omega = -i\,\frac{d}{4\pi\left(|\mathcal{M}|+1\right)T}k^2 + \cdots \tag{2.16}$$

leading to a diffusion constant $\mathcal{D} = \frac{d}{4\pi\,(|\mathcal{M}|+1)\,T}$. For gravitational perturbations in the vector sector which give rise to the momentum diffusion mode, using $|\mathcal{M}| = d-1$, from Table 2.1 we find $\mathcal{D} = \frac{1}{4\pi T}$ which is a rewriting of the famous relation for the shear viscosity since $\mathcal{D} = \frac{\eta}{\epsilon+P} = \frac{\eta}{T\,s}$ [19]. We will actually demonstrate a more remarkable fact: the non-Markovian dispersion relation can be obtained by analytically continuing the retarded Green's function of the corresponding Markovian field, cf., §2.5.3.

The Schwinger-Keldysh Green's function for the non-Markovian probe can be then obtained by solving the hydrodynamic equations for $\breve{\Phi}_{\text{R,L}}$ and inverting these moduli in terms of the doubled non-normalizable data. In this approach, at the first step one obtains the bulk solution as a local expression in terms of the fields $\breve{\Phi}_{\text{R,L}}$. The non-locality of the SK correlators appears only in the second step where we solve for $\breve{\Phi}_{\text{R,L}}$ in terms of the appropriate non-normalizable sources. These two steps then have a very natural Wilsonian interpretation: the first step involves integrating out the fast modes and parameterizing the solution in terms of the state (because these are normalizable data) of the slow modes. In the second step, we then solve for the slow modes in terms of their sources which give rise to correlations over long distances and times.

These facts justify why the intermediate objects $\breve{\Phi}_{\text{R,L}}$ can be interpreted as the effective fields on the Schwinger-Keldysh contour. Our strategy may be summarized as starting in the forgetful sector and analytically continuing into the hydrodynamic moduli space, regaining memory when we finally solve for the dispersion of the long-lived, low momentum, hydrodynamic modes. This picture characterizes, quite universally, the essential physics of diffusive dynamics.

Equivalently, one might be interested in the local Wilsonian Schwinger-Keldysh effective action which yields the above hydrodynamic equations for non-Markovian probes. This can be done by deriving the Legendre transform of the on-shell action parameterized by non-normalizable modes. As we will argue later, freezing non-normalizable modes at the AdS boundary requires one to quantize the non-Markovian probes with Neumann boundary conditions. In turn, one requires a variational boundary term to the free designer scalar action (2.2) to impose such a Neumann boundary condition. Fortunately, the Legendre transform we are after cancels the aforementioned variational boundary term. Thus, a direct on-shell evaluation of the free designer scalar action (2.2) in terms of normalizable modes $\{\breve{\Phi}_{\text{R}}, \breve{\Phi}_{\text{L}}\}$ will give us the required local Wilsonian Schwinger-Keldysh effective action.[15] As we elucidate below, this Schwinger-Keldysh effective action is the *Wilsonian influence functional* advertised at the onset in chapter 1.

2.3.2.2 Observables on hydrodynamic moduli space

A general system would have both Markovian and non-Markovian modes interacting with each other. As a prototypical example, one can consider dynamics of energy-momentum tensor in a CFT, which is dual to gravitational Einstein-Hilbert dynamics. The tensor sector of the energy-momentum dynamics is Markovian, but the vector and scalar sectors are non-Markovian (see Table 2.1). These sectors interact with each other (beyond the quadratic order) due to the non-linearities of gravity. We would like to characterize the observables in such a system.

We can parameterize the Markovian field data with non-normalizable sources $\{J_{\text{R}}, J_{\text{L}}\}$. The non-Markovian sector we continue to parameterize with the right/left (normalizable) hydrodynamic moduli $\{\breve{\Phi}_{\text{R}}, \breve{\Phi}_{\text{L}}\}$ introduced above. The full bulk solution can be parameterized in terms of this data in a boundary gradient expansion.

Evaluating the on-shell action of this bulk solution we will get a local action, viz.,

[15] For a free massless scalar one sees that the change from the Dirichlet to Neumann boundary conditions involves a variational boundary term $\phi\partial\phi$, which is the product of the field and its conjugate momentum. The Legendre transformation switching between the generating function of correlators and the Wilsonian effective action parameterized by the normalizable modes requires an addition of $-\phi\partial\phi$, which exhibits the fortunate cancellation alluded to above.

something which takes the form:

$$\mathcal{S}_{\text{WIF}}\left[J_{\text{R}}, J_{\text{L}}, \breve{\Phi}_{\text{R}}, \breve{\Phi}_{\text{L}}\right] = \int d^d x \; \mathcal{L}_{\text{WIF}}\left[J_{\text{R}}, J_{\text{L}}, \breve{\Phi}_{\text{R}}, \breve{\Phi}_{\text{L}}\right], \tag{2.17}$$

where $\mathcal{S}_{\text{WIF}}$ is the Wilsonian influence functional for the hydrodynamic moduli. We can add non-Markovian sources, $\{\breve{J}_{\text{R}}, \breve{J}_{\text{L}}\}$, to this action and obtain the hydrodynamic equations for $\{\breve{\Phi}_{\text{R}}, \breve{\Phi}_{\text{L}}\}$ in the presence of these sources. Alternately, as explained in §2.3.2.1, we may obtain these hydrodynamic equations by looking at the non-normalizable (source) data for the non-Markovian sector in the full bulk solution. Furthermore, the normalizable data for the non-Markovian sector yield the expectation values of the hydrodynamic moduli, i.e., for the dual gauge invariant boundary operators (single trace primaries).

Our procedure is similar in spirit to the holographic Wilsonian renormalization group [130, 131] and semi-holographic models [111]. There, one first solves for the bulk dynamics with Dirichlet boundary conditions at some intermediate radial position (corresponding to mixed boundary conditions at the AdS boundary) and thereafter integrates over the chosen Dirichlet data on this fiducial surface. Closer in spirit is the seminal discussion of [112] who were interested in deriving the universal low energy dynamics of holographic liquids.

Our main goal in the rest of this chapter is to explicitly construct the space of solutions advertised above, order by order in a gradient expansion, for the designer scalars and gauge fields, and establish a clear link to the advertised physics of charge and momentum diffusion. We will obtain from our analysis the Wilsonian influence functional (2.17) in these models, which serves as our input to the integral over the hydrodynamic moduli space. In the following sections we will implement this exercise at the quadratic order for probe scalars, gauge fields, and gravitons.

2.4 Time-reversal invariant scalar system 1: Markovian dynamics

We now turn to a detailed analysis of the designer scalar system introduced in §2.2. We first elaborate on the construction of the ingoing solution which is analytic at the horizon. We have already explained that there is a unique analytic solution once we fix the overall normalization, albeit one that has a very different interpretation depending on whether we are dealing with a Markovian ($\mathcal{M}+1>0$) or a non-Markovian ($\mathcal{M}+1<0$) probe.

For the Markovian fields as explained in §2.3.1.1 an asymptotic Dirichlet boundary condition serves to uniquely pick out the analytic Green's function with fixed normalization. Once we have the ingoing Green's function we can determine the outgoing Green's function by exploiting the time-reversal isometry (1.42) of the geometry.

With this understanding, we will now describe the explicit solutions for the ingoing Green's functions in a gradient expansion along the boundary for Markovian scalars $\mathcal{M} > -1$. We work in the Fourier domain and denote by $G^{\text{in}}_{\mathcal{M}}(\omega, r, \mathbf{k})$ the solution that is analytic at the horizon. In the Markovian sector, this is the retarded boundary to bulk Green's function which encodes the infalling quasinormal modes of $\varphi_{\mathcal{M}}$.

2.4.1 Ingoing solution in derivative expansion

We parameterize $G^{\text{in}}_{\mathcal{M}}$ in a derivative expansion as follows:[16]

$$G^{\text{in}}_{\mathcal{M}} = e^{-i\,\mathfrak{w} F(\mathcal{M},\xi)} \left[1 - \mathfrak{q}^2\, H_k(\mathcal{M},\xi) - \mathfrak{w}^2\, H_\omega(\mathcal{M},\xi) + i\,\mathfrak{w}\,\mathfrak{q}^2\, I_k(\mathcal{M},\xi) + i\mathfrak{w}^3\, I_\omega(\mathcal{M},\xi) + \cdots\right], \tag{2.18}$$

where we have introduced dimensionless variables,

$$\mathfrak{w} = b\omega = \frac{d}{4\pi}\,\beta\omega\,, \qquad \mathfrak{q} = bk = \frac{d}{4\pi}\,\beta k\,, \qquad \xi = br\ . \tag{2.19}$$

In writing the expansion we have exploited the fact that the spatial reflection symmetry of the background guarantees the absence of terms odd in the momenta $\mathbf{k}$ and have restricted attention to the first three orders in the gradient expansion. The choice of parameterization above, where we separate out the factor $e^{-i\,\mathfrak{w}\,F(\mathcal{M},\xi)}$ is made to simplify the structure of the gradient expansion. Our parameterization is largely inspired by the fluid/gravity literature especially [132].

We employ the following normalization convention for our ingoing Green's function, demanding,

$$\lim_{\xi\to\infty} G^{\text{in}}_{\mathcal{M}}(\omega, r, \mathbf{k}) = 1\,, \tag{2.20}$$

and that $G^{\text{in}}_{\mathcal{M}}(\omega, r, \mathbf{k})$ be analytic everywhere. This implies that in the Markovian case of $\mathcal{M} > -1$ we will simply require the functions F, H_k, H_ω and other higher order gradient terms to vanish asymptotically. Thus we seek purely normalizable solutions to the wave equation (2.11) at higher orders in the derivative expansion. The non-Markovian results will be obtained by analytically continuing in $\mathcal{M}$.

Plugging in our ansatz (2.18) into (2.11), we get a hierarchy of radial second order ODEs of the form:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\mathcal{M}+2}\, f\, \frac{\mathrm{d}\mathfrak{F}(r)}{\mathrm{d}r}\right) = \mathfrak{J}(r). \tag{2.21}$$

where $\mathfrak{F}$ is a placeholder for a function that appears at some order in the gradient ex-

[16] The expressions can be made much more compact by passing to the dimensionless ρ coordinate (1.37). We however stick to the radial variable as it is more familiar in the black hole context.

pansion and $\mathfrak{J}$ refers to the 'source' constructed out of lower order terms in the gradient expansion. We therefore need to invert the differential operator $\frac{d}{dr}\left(r^{\mathcal{M}+2}\, f\, \frac{d}{dr}\right)$ which bears a close similarity to the fluid/gravity discussion of [22].

Moreover, we already know the two homogeneous solutions of this differential operator from (2.13). The constant mode is analytic while the monodromy mode is singular at the horizon. Hence the solution to the (2.21) is unique at any order in the derivative expansion.

The strategy to find this unique solution is then straightforward: we start with a particular solution to the equation and subtract from its derivative a piece that is a multiple of $\frac{1}{r^{\mathcal{M}+2}}$, so as to make it analytic, and then integrate it back up. In the final integral we impose the vanishing of the functions at infinity. Schematically, we have (we use the dimensionless variable ξ in the expressions below):

$$\mathfrak{F}(\xi) = \int_{\infty}^{\xi} \frac{dy}{y^{\mathcal{M}+2}\, f(y)} \int_1^y \mathfrak{J}(y')\, dy' \tag{2.22}$$

Since the inner integral usually involves the difference of function evaluated at some radial position from its value at the horizon, we will introduce a shorthand notation for this combination by decorating the symbol with a hat, viz., for any $\mathfrak{F}(\mathcal{M},\xi)$ we define

$$\widehat{\mathfrak{F}}(\mathcal{M},\xi) \equiv \mathfrak{F}(\mathcal{M},\xi) - \mathfrak{F}(\mathcal{M},1)\,. \tag{2.23}$$

It is amusing to write the iterated integral expression in (2.22) as follows in the ρ coordinate of (1.37):

$$\mathfrak{F}(\rho) = \int_0^{\rho} d\tilde{\rho}\, \frac{\tilde{\rho}^{\,\mathsf{s}(1+\mathcal{M})-1}}{1-\tilde{\rho}} \int_1^{\tilde{\rho}} \mathfrak{J}(\rho')\, d\rho'\,, \qquad \mathsf{s} = \frac{1}{d} \tag{2.24}$$

This is a Bose-Einstein integral. (For more direct verification introduce a further change of variables $\rho = e^{-z}$). This suggests a formal analogy between the solutions to the wave equation in the black hole background to the Chapman-Enskog and Grad moment methods for solving the Boltzmann equation by perturbing around the equilibrium Maxwell-Boltzmann distribution (cf., [133]). The really curious fact, though one we are quite used to from AdS black holes, is that the wave equation immediately is aware of the statistical distribution functions. In a certain sense the above form suggests that the black hole solution itself should be viewed as a condensation of gravitons distributed according to bosonic statistics, with the radial direction playing the role of an energy scale.

2.4.2 Explicit parameterization of ingoing Green's function

Proceeding in a manner described above, we reduce the problem to the following set of equations for the functions $\{F, H_k, H_\omega, I_k, I_\omega\}$ that appear at the first three orders in the gradient expansion. The first two of these satisfy a simple first order equation after integrating up once, see Appendix A.1.

The differential equations above can be solved using incomplete Beta functions $\mathrm{B}\,(a,0;\rho)$ with one of its arguments being zero.[17] To keep our expressions compact we will write the solutions using the shorthand $\mathsf{s}=\frac{1}{d}$ and use the dimensionless ρ coordinate (1.37). For the functions F and H_k we immediately find:

$$\begin{aligned} F(\mathcal{M},\xi) &= \mathsf{s}\,\mathrm{B}\,(\mathsf{s},0;\rho) - \mathsf{s}\,\mathrm{B}\,(\mathsf{s}(\mathcal{M}+1),0;\rho)\ , \\ H_k(\mathcal{M},\xi) &= \frac{\mathsf{s}}{\mathcal{M}-1}\left[\mathrm{B}\,(2\mathsf{s},0;\rho) - \mathrm{B}\,(\mathsf{s}(\mathcal{M}+1),0;\rho)\right] . \end{aligned} \tag{2.25}$$

They satisfy the defining ODE obtained from (2.11) at $\mathcal{O}(\omega)$ and $\mathcal{O}(k^2)$, respectively. The differential equations themselves can be found in (A.9).

The solution for H_ω can be also written down directly, but it is helpful to first define a new function $\Delta(\mathcal{M},\rho)$ via

$$\Delta(\mathcal{M},\xi) \equiv \mathsf{s}\,\mathrm{B}\,(\mathsf{s}(1-\mathcal{M}),0;\rho) - \mathsf{s}\,\mathrm{B}\,(\mathsf{s}(1+\mathcal{M}),0;\rho)\,, \tag{2.26}$$

This combination is antisymmetric in $\mathcal{M}\to-\mathcal{M}$ and is introduced to simplify aspects of the analytic continuation from the Markovian to the non-Markovian case. It too satisfies a simple ODE (A.11). We parameterize the solution for the function H_ω in terms of Δ as

$$\begin{aligned} H_\omega(\mathcal{M},\xi) &= \mathsf{s}\left[\frac{1}{2}\,\Delta(\mathcal{M},\rho) - \Delta(\mathcal{M},1)\right]\mathrm{B}\,(\mathsf{s}(1+\mathcal{M}),0;\rho) \\ &\quad + \frac{\mathsf{s}^2}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n+\mathsf{s}\,(1-\mathcal{M})} - \frac{1}{n+\mathsf{s}(1+\mathcal{M})}\right)\mathrm{B}\,(n+2\,\mathsf{s},0;\rho)\ . \end{aligned} \tag{2.27}$$

One can check that this function satisfies (A.10) and the boundary conditions (2.20).

A similar analysis applies for the third order functions I_k and I_ω which are sourced by $\widehat{H}_k$ and $\widehat{H}_\omega$, respectively. The reader can find the details in Appendix A.1, where we solve the equations using similar tricks; for instance we introduce new functions Δ_k and Δ_ω in parallel with Δ to simplify the analytic continuation and extraction of asymptotics.

All of the functions $\{F,H_\omega,H_k,I_\omega,I_k\}$ vanish as a power law for Markovian probes ($\mathcal{M}>-1$) as we approach the asymptotic boundary. In particular, we have up to the second order:

$$F(\mathcal{M},\xi)\sim\mathcal{O}\big(\xi^{-1}\big)\,,\quad H_\omega(\mathcal{M},\xi)\sim\mathcal{O}\big(\xi^{-2}\big)\,,\quad H_k(\mathcal{M},\xi)\sim\mathcal{O}\big(\xi^{-2}\big)\,,\qquad \text{as}\ \ \xi\to\infty\,. \tag{2.28}$$

The complete series solution is in fact easily read off using the defining series represen-

[17]Some basic facts about this subclass of incomplete Beta functions are collected in Appendix E.

tation of the incomplete beta function (E.1). The non-trivial behaviour is that of the auxiliary function $\Delta(\mathcal{M},\xi)$ which one can check asymptotes as (for $\mathcal{M} > -1$)

$$\Delta(\mathcal{M},\xi) \sim \mathcal{O}\left(\xi^{\mathcal{M}-1}\right), \qquad \text{as } \xi \to \infty \,. \tag{2.29}$$

This is in fact one reason for introducing the function (and others of its kind at higher orders) – one can use it to isolate the modes that grow rapidly as we approach the boundary.[18]

With this data the ingoing Green's function for the Markovian problem can be written down given the parameterization in (2.18). As noted above there are no subtleties with the asymptotic behaviour since these modes satisfy the standard Dirichlet boundary conditions at the AdS boundary (and the source has been fixed to unity). To extract the boundary correlators we need to supply some counterterms as function of the boundary sources as usual. We turn to these issues next.

2.4.3 Counterterms and boundary correlators: Markovian scalar

To wrap up the discussion, and to obtain the boundary observables, we will give a quick summary of the canonical conjugate momentum of the designer scalar system (2.2). For the Markovian probe the canonical conjugate is simply the normalizable mode, whose large r expansion yields the expectation value of the dual single trace primary of the boundary CFT.[19]

For the action (2.2) the momentum conjugate to radial evolution is given by the normal derivative. Letting n^A be the unit spacelike normal to the fixed r hypersurface, and $\gamma_{\mu\nu}$ the induced timelike metric on the hypersurface (see §1.3), we have the projected derivative

$$\partial_n \varphi_{\mathcal{M}} \equiv n^A \nabla_A \varphi_{\mathcal{M}} = \frac{1}{r\sqrt{f}}\, \mathbb{D}_+ \varphi_{\mathcal{M}} \,, \tag{2.30}$$

in terms of which the canonical momentum density is given by

$$\pi_{\mathcal{M}} = -\sqrt{-\gamma}\, e^{\chi_s}\, \partial_n \varphi_{\mathcal{M}} = -r^{\mathcal{M}}\, \mathbb{D}_+ \varphi_{\mathcal{M}} \,, \tag{2.31}$$

where we used $e^{\chi_s} = r^{\mathcal{M}+1-d}$.

As is often the case in AdS/CFT we are interested in the renormalized value of this canonical conjugate density evaluated on the ingoing solution, given in a derivative expansion. An explicit computation gives up to the third order in gradient expansion the

[18] There is another more important reason: the analytic continuation to $\mathcal{M} < -1$, which we will describe in §2.5.

[19] In the non-Markovian case one will instead be dealing with the source of the dual field theory (at a given point in the hydrodynamic moduli space).

following:

$$
\begin{aligned}
r^{\mathcal{M}}\,\mathbb{D}_+ G^{\text{in}}_{\mathcal{M}} = \frac{r^{\mathcal{M}-1}}{\mathcal{M}-1}\,(k^2-\omega^2)\,G^{\text{in}}_{\mathcal{M}} + \cdots \\
+ \frac{1}{b^{\mathcal{M}+1}}\Bigg\{ - i\,\mathfrak{w} - \frac{\mathfrak{q}^2}{\mathcal{M}-1} + \mathfrak{w}^2\left(\Delta(\mathcal{M},br) + \frac{(br)^{\mathcal{M}-1}}{\mathcal{M}-1} - \Delta(\mathcal{M},1)\right) \\
- 2i\,\mathfrak{w}\left[\mathfrak{q}^2\,H_k(\mathcal{M},y) + \mathfrak{w}^2\,H_\omega(\mathcal{M},y)\right]_{y=1}^{y=br} + \cdots \Bigg\}\,G^{\text{in}}_{\mathcal{M}}\,.
\end{aligned}
\tag{2.32}
$$

We have organized the result to isolate the terms that are pure UV effects from the point of view of the boundary CFT and those that have non-trivial knowledge of the black hole (and thus IR physics in the CFT). In the first line we have collected a set of terms where a temperature-independent operator acts on $G^{\text{in}}_{\mathcal{M}}$. In the Markovian case with $-1 < \mathcal{M} < d$, these vacuum UV contributions, which diverge as $r \to \infty$, whereas the last two lines have a finite limit. In making this statement, we have used the asymptotic expansions given in (2.28) and (2.29).

To remove the vacuum contribution, we add, as usual, counterterms to our original action. The correct counterterm we need to add can be inferred from the above expansion

$$
S_{ds}[\varphi_{\mathcal{M}}] = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\;e^{\chi_s}\,\nabla^A\varphi_{\mathcal{M}}\nabla_A\varphi_{\mathcal{M}} - \frac{c^{(2)}_{\varphi}}{2}\int d^dx\,\sqrt{-\gamma}\;e^{\chi_s}\,\gamma^{\mu\nu}\partial_\mu\varphi_{\mathcal{M}}\partial_\nu\varphi_{\mathcal{M}}\,.
\tag{2.33}
$$

One can check that the desired counterterm coefficient is fixed by the asymptotic behaviour of the solution to be

$$
c^{(2)}_{\varphi} = -\frac{1}{\mathcal{M}-1}\,.
\tag{2.34}
$$

Including this boundary counterterm, the renormalized canonical conjugate density to $\varphi_{\mathcal{M}}$ is

$$
\pi_{\mathcal{M}}\Big|_{\text{ren}} = -r^{\mathcal{M}}\mathbb{D}_+\varphi_{\mathcal{M}} - \frac{1}{\mathcal{M}-1}\sqrt{f}\;r^{\mathcal{M}-1}\left(\partial_i\partial_i - \frac{1}{f}\partial_v^2\right)\varphi_{\mathcal{M}}\,.
\tag{2.35}
$$

Note that we have only retained the counterterms till quadratic order in the gradient expansion. Higher derivative counterterms may be necessary depending on $\mathcal{M}$ and the spacetime dimension.

To cubic order, we can finally give the expression for the renormalized retarded boundary two-point function, $K^{\text{in}}_{\mathcal{M}}(\omega,k)$, of the operator dual to the field $\varphi_{\mathcal{M}}$. We find taking the asymptotic limit:

$$
\begin{aligned}
K^{\text{in}}_{\mathcal{M}}(\omega,k) &\equiv -\lim_{r\to\infty} \pi_{\mathcal{M}}\Big|_{\text{ren}} \\
&= \frac{1}{b^{\mathcal{M}+1}}\left\{-i\,\mathfrak{w} - \frac{\mathfrak{q}^2}{\mathcal{M}-1} - \mathfrak{w}^2\Delta(\mathcal{M},1) + 2i\,\mathfrak{w}\left[\mathfrak{q}^2 H_k(\mathcal{M},1) + \mathfrak{w}^2\, H_\omega(\mathcal{M},1)\right] + \cdots\right\}
\end{aligned}
\tag{2.36}
$$

The explicit values of the functions appearing in the gradient expansion at the horizon are given in (A.25) and (A.26) using which we can write down an explicit formula for the retarded boundary correlator for a Markovian scalar with Markovianity index $\mathcal{M}$ dual to a field theory operator in d spacetime dimensions. This is the central result for the Markovian sector. We will use it later to compute the full set of thermal Schwinger-Keldysh correlators from the grSK geometry in §2.6.

2.5 Time reversal invariant scalar system 2: non-Markovian dynamics

We have understood the Markovian designer scalar as a generalization of the massive scalar probes studied in [61] and now are in a position to tackle the non-Markovian probe. As noted in §2.3.1.2 we are not allowed to simply impose Dirichlet boundary conditions in the non-Markovian case. The analytic solution is normalizable at leading order in the gradient expansion to begin with. As we proceed in the gradient expansion we would need to additionally turn on appropriate non-normalizable modes to support the normalizable mode. This is a tedious way to proceed. Fortunately, as explained in §2.3.1.2 we can sidestep the issue by demanding that the non-Markovian solution for a given $\mathcal{M} < -1$ be related to a Markovian solution for a corresponding value $-\mathcal{M} > 1$.[20] Per se, this is merely a convention for parameterizing the solutions: any other parameterization can be related to our prescription by a field redefinition of the non-Markovian modes, which is always permitted.

Before we embark on our analysis let us pause to make a note regarding our terminology. Our use of the adjectives 'normalizable' and 'non-normalizable' refers a-priori to the particulars of the fall-off behaviour of the mode functions. A mode that grows as $r \to \infty$ will be classified as non-normalizable if its near-boundary expansion starts off with a non-negative exponent. These are modes we imagine having to freeze at the AdS boundary and taking them to specify the boundary conditions for radial evolution. How we treat such modes and construct a classical phase space and thence the Hilbert space by imposing canonical commutation relations depends on boundary conditions,

[20] For technical reasons we will refrain from considering the special values $\mathcal{M} = \pm 1$ in our discussion. This is an interesting degenerate class that should be dealt with separately.

counterterms etc., which dictate what the inner product ought to be. We will of course see in a while that our adjectives are indeed appropriate, as we will exhibit a well defined variational problem for the non-Markovian fields with Neumann boundary conditions.

2.5.1 Parameterization of the ingoing solution

We will start with the explicit solutions for the ingoing solution in a gradient expansion along the boundary for our non-Markovian scalars $\mathcal{M} < -1$. We will continue to work in the Fourier domain and denote the inverse Green's function which is analytic at the horizon by $G^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k})$. This function is no longer the retarded boundary to bulk Green's function describing the infalling quasinormal modes of $\varphi_{-\mathcal{M}}$. Rather it corresponds to the inverse Green's function, which gives the sources in terms of field expectation values. In other words, this is the derivative operator that defines the hydrodynamic equations.

We parameterize $G^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k})$ in analogy with (2.18)

$$\begin{aligned} G^{\text{in}}_{-\mathcal{M}} = e^{-i\,\mathfrak{w}\,F(-\mathcal{M},\xi)}\Big[1 &- \mathfrak{q}^2\,H_k(-\mathcal{M},\xi) - \mathfrak{w}^2\,H_\omega(-\mathcal{M},\xi) \\ &+ i\,\mathfrak{w}\mathfrak{q}^2\,I_k(-\mathcal{M},\xi) + i\,\mathfrak{w}^3\,I_\omega(-\mathcal{M},\xi) + \cdots\Big]. \end{aligned} \tag{2.37}$$

We may once again follow the steps outlined in §2.4.2 and solve for the functions order by order in a gradient expansion.

Fortunately, we have already done the heavy lifting. We can now reveal the rationale behind the function $\Delta(\mathcal{M},\xi)$ which we had introduced in (2.26). We recall that it was antisymmetric in $\mathcal{M} \to -\mathcal{M}$. As a result it allows us to find the analytic continuation of $\{F, H_k, H_\omega\}$ from positive to negative values of $\mathcal{M}$. It is straightforward to verify that:

$$\begin{aligned} F(-\mathcal{M},\xi) &= F(\mathcal{M},\xi) - \Delta(\mathcal{M},\xi)\,, \\ H_k(-\mathcal{M},\xi) &= -\frac{\mathcal{M}-1}{\mathcal{M}+1}\,H_k(\mathcal{M},\xi) + \frac{1}{\mathcal{M}+1}\,\Delta(\mathcal{M},\xi)\,, \\ H_\omega(-\mathcal{M},\xi) &= -H_\omega(\mathcal{M},\xi) + \Delta(\mathcal{M},1)\,\Delta(\mathcal{M},\xi) - \frac{1}{2}\,\Delta(\mathcal{M},\xi)^2\,. \end{aligned} \tag{2.38}$$

For $\mathcal{M} < -1$ (the non-Markovian case) we use these definitions for the analytic continuations. Expressions at higher orders may similarly be derived, cf., (A.15) which are relevant at the third order. We will need these for the construction of boundary observables. This completes for us the specification of the ingoing inverse Green's function at the first few orders in the gradient expansion.

2.5.2 Inverse Green's function and dispersion relations

Armed with $G^{\text{in}}_{-\mathcal{M}}$ we can proceed as suggested in §2.3.2. The primary novelty in our discussion appears in the treatment of non-Markovian fields, the Markovian sector being a simple extension of the analysis in [61] as discussed hitherto in §2.4.3. We will proceed to delineate non-Markovian observables, elaborate on some of the points made in §2.3.2 and extract the dispersion relation for these long-lived modes.

Let us examine more closely the derivative expansion for the non-Markovian ingoing inverse Green's function $G^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k})$. We use the parameterization (2.37) which we write out explicitly using (2.38), as

$$\begin{aligned} G^{\text{in}}_{-\mathcal{M}} &= 1 - i\,\mathfrak{w}\,F(-\mathcal{M},\xi) - \frac{\mathfrak{w}^2}{2}\,F(-\mathcal{M},\xi)^2 - \mathfrak{q}^2\,H_k(-\mathcal{M},\xi) - \mathfrak{w}^2\,H_\omega(-\mathcal{M},\xi) + \cdots \\ &= 1 - i\,\mathfrak{w}\,F(\mathcal{M},\xi) - \frac{\mathfrak{w}^2}{2}\,F(\mathcal{M},\xi)^2 + \frac{\mathcal{M}-1}{\mathcal{M}+1}\,\mathfrak{q}^2\,H_k(\mathcal{M},\xi) + \mathfrak{w}^2\,H_\omega(\mathcal{M},\xi) \\ &\qquad - \left[-i\,\mathfrak{w} + \frac{\mathfrak{q}^2}{\mathcal{M}+1} + \mathfrak{w}^2\,\Delta(\mathcal{M},1) - \mathfrak{w}^2\,F(\mathcal{M},\xi)\right]\Delta(\mathcal{M},\xi) + \cdots . \end{aligned} \tag{2.39}$$

As discussed in (2.28) for positive $\mathcal{M}$, the functions $\{F, H_k, H_\omega\}$ vanish at infinity whereas (2.29) shows that $\Delta(\mathcal{M},\xi) \sim \xi^{\mathcal{M}-1}$ at large r. It follows that the asymptotic behaviour of $G^{\text{in}}_{-\mathcal{M}}$ is given up to second order in boundary gradients by[21]

$$G^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k}) \sim \frac{(br)^{\mathcal{M}-1}}{\mathcal{M}-1}\left[-i\,\mathfrak{w} + \frac{\mathfrak{q}^2}{\mathcal{M}+1} + \mathfrak{w}^2\,\Delta(\mathcal{M},1) + \cdots\right] . \tag{2.40}$$

This demonstrates explicitly our earlier claim: in the non-Markovian case turning on a normalizable ingoing solution inevitably turns on the non-normalizable mode at higher orders in derivative expansion (at generic $\{\omega, \mathbf{k}\}$). There is however a subset of $\{\omega, \mathbf{k}\}$ defined by a dispersion relation where this non-normalizable mode vanishes. At these loci alone can one have a purely normalizable ingoing solution.

To extract this more efficiently, let us parameterize the non-Markovian inverse Green's function in a particularly convenient form: We write:

$$G^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k}) = \widetilde{G}^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k})\left[1 - \frac{1}{b^{\mathcal{M}-1}}\,K^{\text{in}}_{-\mathcal{M}}(\omega, \mathbf{k})\,\Xi_{\text{nn}}(\omega, r, \mathbf{k})\right] \tag{2.41}$$

The three pieces introduced above in the parameterization are the following:

- $K^{\text{in}}_{-\mathcal{M}}(\omega, \mathbf{k})$ is a dispersion function whose vanishing locus parameterizes a hyperspace

[21] Explicit results, accurate to third order in the gradients, are given in Appendix A.2.

of the $(\omega, \mathbf{k})$ space where normalizable modes exist. Explicitly, we find:

$$K^{\text{in}}_{-\mathcal{M}}(\omega, \mathbf{k}) = b^{\mathcal{M}-1}\left[-i\,\mathfrak{w} + \frac{\mathfrak{q}^2}{\mathcal{M}+1} + \mathfrak{w}^2\,\Delta(\mathcal{M},1) + \cdots\right] \tag{2.42}$$

up to the second order in the gradient expansion. At third order the relevant expression can be found in (A.20).

- $\Xi_{\text{nn}}(\omega, \mathbf{k})$ is the non-normalizable mode function that is generically turned on at higher orders in the gradient expansion, as anticipated previously. An explicit expression accurate to third order can be found in (A.21).
- Finally, $\widetilde{G}^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k})$ is a purely normalizable component of the Green's function given in (A.22) up to third order in gradients. We note that $\widetilde{G}^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k})$ is very closely related to $G^{\text{in}}_{\mathcal{M}}(\omega, r, \mathbf{k})$, the ingoing boundary to bulk Green's function of the Markovian case obtained earlier in (2.18) (there are small differences in the scaling of the various functions, see Appendix A.2).

Let us examine the locus where the ingoing non-Markovian Green's function is purely normalizable. The hypersurface is parameterized in the form of a dispersion relation which from (2.42) takes the form:

$$0 = i\,\mathfrak{w} - \frac{\mathfrak{q}^2}{\mathcal{M}+1} - \mathfrak{w}^2\Delta(\mathcal{M},1) + \cdots \tag{2.43}$$

We recognize this as the dispersion relation of a diffusion mode with the diffusion constant $\frac{b}{\mathcal{M}+1}$ as advertised in (2.16), along with some higher order corrections. Thus the moduli space of purely normalizable ingoing solutions in non-Markovian case is identical to the moduli space of solutions of a generalized diffusion equation. This should be contrasted against the Markovian case where there is no purely normalizable ingoing solution and the corresponding moduli space is an empty set.

Using the results of Appendices A.2 and A.3 we can write down the dispersion relation to the third order in the gradient expansion for a d dimensional boundary field theory as:

$$\begin{aligned}
0 &= i\,\mathfrak{w} - \frac{\mathfrak{q}^2}{\mathcal{M}+1} - \mathfrak{w}^2\Delta(\mathcal{M},1) + 2i\,\mathfrak{w}\,\mathfrak{q}^2\frac{\mathcal{M}-1}{\mathcal{M}+1}\,H_k(\mathcal{M},1) + 2i\,\mathfrak{w}^3\,H_\omega(\mathcal{M},1) + \cdots\,, \\
&= i\,\mathfrak{w} - \frac{\mathfrak{q}^2}{\mathcal{M}+1} - \mathfrak{w}^2\frac{\psi\left(\frac{\mathcal{M}+1}{d}\right) - \psi\left(\frac{1-\mathcal{M}}{d}\right)}{d} + 2i\,\mathfrak{w}\,\mathfrak{q}^2\frac{\psi\left(\frac{\mathcal{M}+1}{d}\right) - \psi\left(\frac{2}{d}\right)}{d(\mathcal{M}+1)} \\
&\quad + i\,\mathfrak{w}^3\left\{\frac{\psi\left(\frac{\mathcal{M}+1}{d}\right)\left[\psi\left(\frac{\mathcal{M}+1}{d}\right) - \psi\left(\frac{1-\mathcal{M}}{d}\right)\right]}{d^2} + \sum_{n=0}^{\infty}\left[\frac{1}{n+\frac{1+\mathcal{M}}{d}} - \frac{1}{n+\frac{1-\mathcal{M}}{d}}\right]\psi\left(n+\frac{2}{d}\right)\right\} + \cdots\,,
\end{aligned} \tag{2.44}$$

where $\psi(x)$ is the digamma function.

2.5.3 Two observations about non-Markovian scalars

Before we proceed to discuss further details about the Markovian scalars, let us make note of two observations that while empirical at this point, suggest a deeper principle in operation.

The dispersion function & the Markovian Green's function: We had deliberately denoted the renormalized Markovian retarded Green function as $K^{\text{in}}_{\mathcal{M}}(\omega, \mathbf{k})$ in (2.36). As can be readily checked, this function is the analytic continuation of the non-Markovian dispersion function $K^{\text{in}}_{-\mathcal{M}}(\omega, \mathbf{k})$ obtained in (2.42) (for the third order expression see (A.20)). We thus come to a remarkable conclusion: not only are the solutions in the Markovian and non-Markovian bulk-boundary ingoing Green's functions related by analytic continuation, but more interestingly, *the dispersion function of the non-Markovian field can also be obtained by analytically continuing the retarded boundary Green's function of the Markovian field.* While we have not given a general argument why this should be the case, the fact that this holds till third order in derivative expansion is a strong evidence for this statement.[22] If this statement is assumed to be true, it gives the simplest way to compute the dispersion function on the non-Markovian side without ever having to solve the non-Markovian problem.[23]

A field redefinition to diffusion: We had mentioned in §2.3.2, with a suitable field redefinition, the dispersion relation obtained above can be made into an explicit diffusion dispersion. Let us describe now in detail how this could be achieved. Consider a rescaling of the non-Markovian inverse Green's function by a factor, viz.,

$$\begin{aligned}
\check{G}^{\text{in}}_{-\mathcal{M}} &\equiv \left(1 - i\,\mathfrak{w}\,\Delta(\mathcal{M},1) + \frac{\mathfrak{q}^2}{\mathcal{M}+1}\Delta(\mathcal{M},1) + \cdots\right) G^{\text{in}}_{-\mathcal{M}} \\
&= e^{-i\,\mathfrak{w}\,\Delta(\mathcal{M},1)} \times e^{-i\,\mathfrak{w} F(-\mathcal{M},\xi)} \left[1 - \mathfrak{q}^2\left(H_k(-\mathcal{M},\xi) - \frac{\Delta(\mathcal{M},1)}{\mathcal{M}+1}\right)\right. \\
&\qquad\qquad \left. - \mathfrak{w}^2\left(H_\omega(-\mathcal{M},\xi) - \frac{1}{2}\Delta(\mathcal{M},1)^2\right) + \cdots\right]
\end{aligned} \tag{2.45}$$

where we have distributed the constant shift by $\Delta(\mathcal{M}, 1)$ into the various functions appearing in the parameterization. By construction, $\check{G}^{\text{in}}_{-\mathcal{M}}$ also solves the generalized KG

[22] Notice that in the particular case of the Maxwell fields in $d = 3$, one can attribute this feature to the self-duality of the 4 dimensional Maxwell Lagrangian which exchanges electric and magnetic excitations and hence the scalar and vector photons.

[23] This does not imply that every non-Markovian mode in a system appears in coalition with a Markovian mode. For example, there is no Markovian mode associated with the sound mode discussed in chapter 3.

equation with the exponent $\mathcal{M}$. It has an asymptotic behaviour

$$\check{G}^{\text{in}}_{-\mathcal{M}} \sim \frac{(br)^{\mathcal{M}-1}}{\mathcal{M}-1}\left(-i\,\mathfrak{w} + \frac{\mathfrak{q}^2}{\mathcal{M}+1}\right) + \cdots . \tag{2.46}$$

which as presaged gives us the correct diffusive dispersion relation, accurate to second order.

2.5.4 Counterterms and boundary correlators: non-Markovian scalar

By direct computation we obtain the canonical conjugate momentum of the non-Markovian scalar to be

$$\begin{aligned}
\pi_{-\mathcal{M}} &= -r^{-\mathcal{M}}\,\mathbb{D}_+ G^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k}) \\
&= -r^{-\mathcal{M}}\,\mathbb{D}_+ \widetilde{G}^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k}) + \frac{r^{-\mathcal{M}}}{b^{\mathcal{M}-1}}\,K^{\text{in}}_{-\mathcal{M}}(\omega, \mathbf{k})\,\mathbb{D}_+\left[\widetilde{G}^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k})\,\Xi_{\text{nn}}(\omega, r, \mathbf{k})\right] \\
&= -K^{\text{in}}_{-\mathcal{M}}(\omega, \mathbf{k}) + \text{subleading}
\end{aligned} \tag{2.47}$$

In deriving the above we have used the asymptotic behaviour obtained in (A.23) and (A.24). This can also be directly obtained from direct differentiation of (2.40).

What we see here is that the canonical momentum conjugate to the non-Markovian designer scalar picks out the non-normalizable mode of the field. Moreover, since the asymptotic behaviour of $\pi_{-\mathcal{M}}$ is finite, one needs no counterterms to regulate it.[24] These facts are a-priori quite counter-intuitive. The standard AdS/CFT dictionary relates the non-normalizable mode to the asymptotic field value, and not to the conjugate momentum (this is indeed the case for the Markovian scalar, see §2.4.3). The non-Markovian field is very unconventional in this regard. One can discern that the dilatonic coupling in (2.2) is highly damped in the near AdS-boundary region and is clearly responsible for this unconventional behaviour.

Armed with the observation above, we can ask how does one quantize the non-Markovian designer field with AdS asymptotic boundary conditions. The usual rules of AdS/CFT tell us that the modes that grow asymptotically are to be frozen. We see here that $\pi_{-\mathcal{M}}$ is the non-normalizable mode which we need to freeze. The standard AdS boundary conditions freeze the field, but freezing the conjugate momentum is easy to do. Instead of quantizing $\varphi_{-\mathcal{M}}$ with Dirichlet boundary conditions we impose Neumann boundary conditions on the non-Markovian field.

In practice implementing Neumann boundary conditions is simple: one simply starts

[24] In making this statement we are restricting attention to the Gaussian (free) non-Markovian designer scalar. If there are bulk self-interactions then one would perhaps need additional counterterms to account for renormalization of bulk Witten diagrams (the analog of source renormalization in the grSK contour discovered in [61]). We will leave it for future work to deduce such effects, if present.

with the usual Dirichlet boundary conditions where the asymptotic field value is frozen and then Legendre transforms to the Neumann boundary condition where the momentum is frozen instead. This is the usual story for the alternate and multi-trace boundary conditions in AdS/CFT [128, 129]. To wit, the variational problem of the non-Markovian field, which defines the classical phase space, and subsequently is used to compute the generating function of connected correlators, requires one Legendre transform to impose Neumann boundary conditions. One would define the action for the non-Markovian scalar, completing the discussion around (2.2), as

$$\begin{aligned} S_{ds}[\varphi_{_{-\mathcal{M}}}] &= -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\; e^{\chi_s}\,\nabla^A\varphi_{_{-\mathcal{M}}}\nabla_A\varphi_{_{-\mathcal{M}}} + \int d^dx\,\sqrt{-\gamma}\,e^{\chi_s}\,\varphi_{_{-\mathcal{M}}}\,n^A\nabla_A\varphi_{_{-\mathcal{M}}} + S_{\text{ct}}\,, \\ &= -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\; e^{\chi_s}\,\nabla^A\varphi_{_{-\mathcal{M}}}\nabla_A\varphi_{_{-\mathcal{M}}} - \int d^dx\,\pi_{_{-\mathcal{M}}}\,\varphi_{_{-\mathcal{M}}} + S_{\text{ct}}\,. \end{aligned} \tag{2.48}$$

Here we note that $e^{\chi_s} = r^{-\mathcal{M}+1-d}$ since we are discussing the non-Markovian scalar and used (2.31). In writing the above we have also acknowledged that there might yet be additional boundary counterterms necessary to compute the boundary generating function

$$\mathcal{Z}_{SK}[\hat{J}_{\text{R}}, \hat{J}_{\text{L}}] = \int [D\varphi_{_{-\mathcal{M}}}]\; e^{iS_{ds}[\varphi_{_{-\mathcal{M}}}]}\,, \tag{2.49}$$

with the bulk functional integral being assumed to be carried out on the grSK saddle geometry with non-Markovian sources $\hat{J}_{\text{R,L}}$.

However, as we discussed hitherto in §2.3, we compute first the Wilsonian influence functional which is parameterized, in the current parlance, in terms of normalizable modes (the hydrodynamic moduli). From the Wilsonian Influence Functional we can obtain the generating function of connected correlators by Legendre transforming the hydrodynamic moduli onto the non-Markovian sources.

By a happy circumstance (or clever design depending on one's perspective), these two Legendre transformations however cancel each other out! We conclude that *for the non-Markovian scalars, the standard Klein-Gordon action with no additional variational boundary terms automatically computes the effective action for the non-Markovian fields.* Thus we may write:

$$\begin{aligned} \mathcal{L}_{\text{WIF}}[\breve{\Phi}_{\text{R}}, \breve{\Phi}_{\text{L}}] &= \int [D\varphi_{_{-\mathcal{M}}}]\; e^{i\breve{S}_{ds}[\varphi_{_{-\mathcal{M}}}]}\,, \\ \breve{S}_{ds}[\varphi_{_{-\mathcal{M}}}] &= -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\; e^{\chi_s}\,\nabla^A\varphi_{_{-\mathcal{M}}}\nabla_A\varphi_{_{-\mathcal{M}}} + S_{\text{ct}}\,. \end{aligned} \tag{2.50}$$

Consequently, we are in a fortuitous circumstance where the computation of the on-shell action involves no variational boundary terms. This story strongly parallels the Markovian case, which makes the analysis quite straightforward, despite the seeming complications.

There will however be boundary counterterms necessary as indicated in S_{ct}. While the non-normalizable modes do not need any counterterms for their definition, the normalizable mode and the on-shell action require appropriate counterterms made of non-normalizable modes to cancel the large r divergences. Since the canonical momentum $\pi_{_{-\mathcal{M}}}$ corresponds to the non-normalizable mode, its canonical conjugate i.e., the field $\varphi_{_{-\mathcal{M}}}$ itself with appropriate counterterms should give the normalizable mode.

The allowed counterterms should be built out of the non-normalizable modes or what we would want to call the 'CFT sources' themselves viz., $\partial_n\varphi_{_{-\mathcal{M}}}$ following (2.30). To make this explicit, we will first rewrite $G^{\text{in}}_{_{-\mathcal{M}}}$ in terms of $\mathbb{D}_+ G^{\text{in}}_{_{-\mathcal{M}}}$ to extract the non-normalizable pieces explicitly. With a bit of algebra, we find

$$\begin{aligned}
G^{\text{in}}_{_{-\mathcal{M}}} &= \frac{1}{r(\mathcal{M}-1)}\left(1-\frac{k^2-\omega^2}{r^2\,(\mathcal{M}-1)\,(\mathcal{M}-3)}\right)\mathbb{D}_+ G^{\text{in}}_{_{-\mathcal{M}}} \\
&\quad + e^{-i\,\mathfrak{w}\,F(\mathcal{M},\xi)}\times\left[1-\frac{K^{\text{in}}_{_{-\mathcal{M}}}}{b^{\mathcal{M}-1}}\,\Xi^{\text{R}}_{\text{nn}}(\mathcal{M},\xi)\right]\times\left[1+\frac{k^2}{r^2(\mathcal{M}^2-1)}+\frac{\mathcal{M}^2-1}{\mathcal{M}+1}\mathfrak{q}^2\,H_k(\mathcal{M},\xi)\right. \\
&\qquad\qquad \left. +\,\mathfrak{w}^2 H_\omega(\mathcal{M},\xi)-i\,\mathfrak{w}\,\mathfrak{q}^2\,\frac{\mathcal{M}-1}{\mathcal{M}+1}\,I_k(\mathcal{M},\xi)-i\,\mathfrak{w}^3\,I_\omega(\mathcal{M},\xi)+\cdots\right].
\end{aligned} \tag{2.51}$$

We have isolated the large r divergences in the first line. We see that this is a vacuum contribution from the b-independence of the prefactor. We also introduced $\Xi^{\text{R}}_{\text{nn}}(\mathcal{M},\xi)$ – this is the 'renormalized' non-normalizable mode function which is engineered to vanish as $r\to\infty$. It is obtained from $\Xi_{\text{nn}}(\mathcal{M},\xi)$ by a minimal subtraction scheme to remove the divergent pieces. An explicit parameterization of this function can be given in terms of data described in Appendix A.1, especially the functions $\{\Delta(\mathcal{M},\xi),\Delta_k(\mathcal{M},\xi),\Delta_\omega(\mathcal{M},\xi)\}$ introduced there. We find:

$$\begin{aligned}
\Xi^{\text{R}}_{\text{nn}}(\mathcal{M},\xi) &\equiv \Delta(\mathcal{M},\xi)+\frac{\xi^{\mathcal{M}-1}}{\mathcal{M}-1} \\
&\quad -2\left[\frac{\mathcal{M}-1}{\mathcal{M}+1}\mathfrak{q}^2\,H_k(\mathcal{M},\xi)+\mathfrak{w}^2\,H_\omega(\mathcal{M},\xi)\right]\left(\Delta(\mathcal{M},\xi)+\frac{\xi^{\mathcal{M}-1}}{\mathcal{M}-1}-\Delta(\mathcal{M},1)\right) \\
&\quad +\left(\frac{\mathfrak{q}^2}{\mathcal{M}+1}-\frac{\mathfrak{w}^2}{2(\mathcal{M}-1)}\right)\left(\Delta_k(\mathcal{M},\xi)-2\frac{\xi^{\mathcal{M}-3}}{(\mathcal{M}-1)(\mathcal{M}-3)}\right)+\mathfrak{w}^2\,\Delta_\omega(\mathcal{M},\xi)+\cdots
\end{aligned} \tag{2.52}$$

Using (2.29) and (A.17) we can see that $\Xi^{\text{R}}_{\text{nn}}$ actually vanishes as we take $\xi\to\infty$ limit.

Having extracted out the vacuum divergences, we can cancel them by adding appropriate counterterms made of $\partial_n\varphi_{_{-\mathcal{M}}}$ and its derivatives to our original action. Adding the most general counterterms admissible up to the third order in boundary derivatives, we obtain

$$
\begin{aligned}
\breve{S}_{ds}[\varphi_{_{-\mathcal{M}}}] &= S_{\text{bulk}} + S_{\text{ct}} \\
S_{\text{bulk}} &= -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\,r^{-\mathcal{M}+1-d}\;\nabla^A\varphi_{_{-\mathcal{M}}}\,\nabla_A\varphi_{_{-\mathcal{M}}} \\
S_{\text{ct}} &= \frac{1}{2}\int d^dx\,\sqrt{-\gamma}\,r^{-\mathcal{M}+1-d}\;\left[c^{(0)}_{_\pi}\,(\partial_n\varphi_{_{-\mathcal{M}}})^2 - c^{(2)}_{_\pi}\,\gamma^{\mu\nu}(\partial_\mu\partial_n\varphi_{_{-\mathcal{M}}})(\partial_\nu\partial_n\varphi_{_{-\mathcal{M}}})\right]
\end{aligned}
\tag{2.53}
$$

From the asymptotic behaviour of the solution we can fix the coefficients of the counterterms. We find:[25]

$$
c^{(0)}_{_\pi} = -\frac{1}{\mathcal{M}-1}\,, \qquad c^{(2)}_{_\pi} = -\frac{1}{(\mathcal{M}-1)^2\,(\mathcal{M}-3)}\,. \tag{2.54}
$$

Let us check how these counterterms work to give us a finite result for the boundary observables. Let us first check the canonical pairs in the classical phase space prior to addition of counterterms. The variation of the bulk action δS_{bulk} in (2.53) gives a boundary term $-\sqrt{-\gamma}\;r^{-\mathcal{M}+1-d}\;\partial_n\varphi_{_{-\mathcal{M}}}\;\delta\varphi_{_{-\mathcal{M}}}$. The variational boundary term required to impose Neumann boundary conditions, further shifts the boundary variation to

$$
\begin{aligned}
\delta S_{\text{bulk}}\Big|_{\text{Neumann}} &\propto \int d^dx\,\varphi_{_{-\mathcal{M}}}\;\delta\left(\sqrt{-\gamma}\;r^{-\mathcal{M}+1-d}\;\partial_n\varphi_{_{-\mathcal{M}}}\right) \\
&= \int d^dx\,\varphi_{_{-\mathcal{M}}}\,\delta\left(r^{-\mathcal{M}}\,\mathbb{D}_+\varphi_{_{-\mathcal{M}}}\right) = -\int d^dx\,\varphi_{_{-\mathcal{M}}}\delta\pi_{_{-\mathcal{M}}}
\end{aligned}
\tag{2.55}
$$

This shows that the canonical conjugate of $-\pi_{_{-\mathcal{M}}}$ is the field $\varphi_{_{-\mathcal{M}}}$ as we had intuitively anticipated earlier and justifies our choices made hitherto.

We can account for the addition of counterterms, and learn that the regulated statement is that $-\pi_{_{-\mathcal{M}}}$ is conjugate to:

$$
\begin{aligned}
-\pi_{_{-\mathcal{M}}} &\longleftrightarrow \varphi_{_{-\mathcal{M}}} + c^{(0)}_{_\pi}\,\partial_n\varphi_{_{-\mathcal{M}}} + c^{(2)}_{_\pi}\,r^{-2}\left(\partial_i\partial_i - \frac{1}{f}\partial_v^2\right)\partial_n\varphi_{_{-\mathcal{M}}}\,, \\
-\pi_{_{-\mathcal{M}}} &\longleftrightarrow \varphi_{_{-\mathcal{M}}} - \frac{1}{\sqrt{f}}\left[-\frac{c^{(0)}_{_\pi}}{r} + \frac{c^{(2)}_{_\pi}}{r^3}\left(k^2 - \frac{1}{f}\omega^2\right)\right]\mathbb{D}_+\varphi_{_{-\mathcal{M}}}.
\end{aligned}
\tag{2.56}
$$

In the second line we have written out the normal derivative in terms of the derivation $\mathbb{D}_+$ introduced in (1.40) and passed into the Fourier domain. In the $r \to \infty$ limit, we then find the following:

$$
\lim_{r\to\infty}\left\{G^{\text{in}}_{_{-\mathcal{M}}} - \frac{1}{\sqrt{f}}\left[\frac{1}{r(\mathcal{M}-1)} - \frac{1}{r^3}\,\frac{1}{(\mathcal{M}-1)^2(\mathcal{M}-3)}\left(k^2 - \frac{1}{f}\omega^2\right)\right]\mathbb{D}_+G^{\text{in}}_{_{-\mathcal{M}}} + \cdots\right\} = 1\,. \tag{2.57}
$$

[25] We will see in subsequent sections that these coefficients coincide with those for the Einstein-Hilbert action (with standard Dirichlet boundary conditions) in the special case $\mathcal{M} = d - 1$.

This shows that, with the choices described above, $G^{\text{in}}_{-\mathcal{M}}$ describes the state of the non-Markovian scalar which is dual to a CFT state with a unit renormalized vacuum expectation value for the CFT single trace primary (along with a CFT source that is needed to maintain this expectation value).

2.6 Solution and on-shell action on grSK geometry

Having constructed the retarded boundary to bulk Green's functions, we will now begin by constructing the solution on grSK geometry satisfying the appropriate SK boundary conditions. Once the solution is constructed, we can evaluate the on-shell bulk action on the solution to obtain the Wilsonian influence phase in the dual CFT defined in (2.17).

As described in §2.3, our goal will be to integrate out the fast Markovian degrees of freedom while freezing the slow non-Markovian modes and getting a Wilsonian influence phase in terms of the doubled Markovian sources (denoted by $\{J_{\text{R}}, J_{\text{L}}\}$) as well as non-Markovian effective fields (denoted by $\{\breve{\Phi}_{\text{R}}, \breve{\Phi}_{\text{L}}\}$). Furthermore, as we explained in §2.5.4 such a Wilsonian influence phase is computed by evaluating the on-shell action on the gravity side without including the variational counterterm that implements Neumann boundary-condition in the non-Markovian sector.

We will begin our discussion by generating the solutions that describe outgoing Hawking modes. This is most efficiently done by exploiting the $\mathbb{Z}_2$ time-reversal isometry $v \mapsto i\beta\zeta - v$ of the black brane background [60, 61] which was described in §1.3. For functions in Fourier domain, this amounts to the map $\omega \mapsto -\omega$ followed by a multiplication with a factor of $e^{-\beta\omega\zeta}$ to go from the ingoing Green's functions to the outgoing Green's functions. We define the time-reversed Green's function

$$G^{\text{rev}}_{\mathcal{M}}(\omega, \mathbf{k}) \equiv G^{\text{in}}_{\mathcal{M}}(-\omega, \mathbf{k}) \ , \tag{2.58}$$

so that the outgoing Green's function has the form

$$G^{\text{out}}_{\mathcal{M}}(\omega, \mathbf{k}) \equiv G^{\text{rev}}_{\mathcal{M}}(\omega, \mathbf{k})\, e^{-\beta\omega\zeta} \equiv G^{\text{in}}_{\mathcal{M}}(-\omega, \mathbf{k})\, e^{-\beta\omega\zeta} \,. \tag{2.59}$$

Given that all our Green's functions come with a phase factor $e^{-i\mathfrak{w}F(\mathcal{M},\xi)}$, we conclude that the outgoing Green's functions are obtained by reversing the ω's and performing a shift

$$F(\mathcal{M}, \xi) \mapsto F(\mathcal{M}, \xi) + i\frac{\beta}{b}\zeta = F(\mathcal{M}, \xi) + \frac{4\pi i}{d}\zeta \ , \tag{2.60}$$

where we have used the relation (1.34) to relate the inverse Hawking temperature and the inverse horizon radius of the Schwarzschild-AdS$_{d+1}$ black hole.

We have hitherto observed in §2.4 and §2.5 that counterterms are needed for the definition of normalizable modes within the ingoing solution. We will now see that the *same* counterterms render finite the on-shell action evaluated on grSK geometry which, in addition, incorporates the effects of outgoing Hawking modes. In particular, while the effective theory we derive would be non-unitary, only the unitary counterterms of the microscopic theory are needed to get finite answers.

2.6.1 Markovian probes

Let us begin with the Markovian sector where the analysis parallels that of [61]. The most general solution on grSK geometry (1.33) is given by

$$\varphi^{\text{SK}}_{\mathcal{M}}(\omega,\zeta,\mathbf{k}) = \mathfrak{c}^{\text{in}}_{\mathcal{M}}\, G^{\text{in}}_{\mathcal{M}}(\omega,\zeta,\mathbf{k}) + \mathfrak{c}^{\text{rev}}_{\mathcal{M}}\, G^{\text{rev}}_{\mathcal{M}}(\omega,\zeta,\mathbf{k})\, e^{-\beta\omega\zeta}\,. \tag{2.61}$$

We relate the non-normalizable modes of the field $\varphi_{\mathcal{M}}$ to the CFT sources at the left and the right boundaries, i.e., at $r = \infty \pm i0$ where ζ takes the values 0 and 1 respectively:

$$\lim_{r\to\infty+i0} \varphi^{\text{SK}}_{\mathcal{M}} = J_{\text{L}}\,, \qquad \lim_{r\to\infty-i0} \varphi^{\text{SK}}_{\mathcal{M}} = J_{\text{R}}\,. \tag{2.62}$$

Our normalization of the ingoing Green's function (2.20) and the action of time-reversal (2.58) together imply that the coefficients $\mathfrak{c}^{\text{in}}_{\mathcal{M}}$ and $\mathfrak{c}^{\text{rev}}_{\mathcal{M}}$ are given by

$$\mathfrak{c}^{\text{in}}_{\mathcal{M}} + \mathfrak{c}^{\text{rev}}_{\mathcal{M}} = J_L\,, \qquad \mathfrak{c}^{\text{in}}_{\mathcal{M}} + e^{-\beta\omega}\mathfrak{c}^{\text{rev}}_{\mathcal{M}} = J_R\,. \tag{2.63}$$

Fixing the constants with the above boundary conditions we see that the solution of the designer scalar on the grSK geometry is then given by the following [60, 61][26]

$$\begin{aligned}\varphi^{\text{SK}}_{\mathcal{M}}(\omega,\zeta,\mathbf{k}) &= G^{\text{in}}_{\mathcal{M}}\left[(n_B+1)\,J_{\text{R}} - n_B\,J_{\text{L}}\right] - G^{\text{rev}}_{\mathcal{M}}\, n_B\,(J_{\text{R}} - J_{\text{L}})\, e^{\beta\omega(1-\zeta)} \\ &= G^{\text{in}}_{\mathcal{M}}\, J_a + \left[\left(n_B + \frac{1}{2}\right) G^{\text{in}}_{\mathcal{M}} - n_B\, e^{\beta\omega(1-\zeta)} G^{\text{rev}}_{\mathcal{M}}\right] J_d\,.\end{aligned} \tag{2.64}$$

where

$$n_B \equiv \frac{1}{e^{\beta\omega}-1}\,, \tag{2.65}$$

is the Bose-Einstein factor. Notice that the amplitude of ingoing and outgoing contributions to the solution (2.64) is controlled by the RA basis sources introduced in (1.20). In the last line of (2.64), we re-write the solution in terms of the average-difference or Keldysh basis sources (1.12).

It can be explicitly checked that if $G^{\text{in}}_{\mathcal{M}}$ has a derivative expansion such that all odd powers at least have one factor of ω, then $\varphi^{\text{SK}}_{\mathcal{M}}$ can also be written in a derivative expansion [61]. More precisely, if $G^{\text{in}}_{\mathcal{M}}$ is known till n^{th} order in derivative expansion, $\varphi^{\text{SK}}_{\mathcal{M}}$ can be

[26] An earlier incarnation of this expression was obtained in [100] on a single copy of the bulk by analogy with thermal Green's functions in field theory.

determined to $(n-1)^{\text{th}}$ order in derivative expansion. As indicated at several points in our discussion these statements should not be surprising in the Markovian designer scalar context. The general structure largely parallels the massive scalar probes studied in the aforementioned reference.

We are now ready to compute the one-point function in the presence of sources, by examining the right/left normalizable modes. This, as presaged, requires the inclusion of the appropriate counterterms determined hitherto. Putting all the pieces computed in §2.4.3 we obtain

$$\left\langle \mathcal{O}_{\text{L/R}} \right\rangle = \lim_{r\to\infty\pm i0} \left[-r^{\mathcal{M}} \mathbb{D}_+ \varphi_{_\mathcal{M}} + c^{(2)}_{\varphi} \sqrt{f}\, r^{\mathcal{M}-1} \left(\partial_i \partial_i - \frac{1}{f} \partial_v^2 \right) \varphi_{_\mathcal{M}} + \ldots \right]. \tag{2.66}$$

This yields the expressions for the right and left one-point functions to be

$$\begin{aligned} \langle \mathcal{O}_{\text{R}}(\omega, \mathbf{k}) \rangle &= -K^{\text{in}}_{\mathcal{M}}(\omega, \mathbf{k}) \left[(n_{_B}+1)\, J_{\text{R}} - n_{_B}\, J_{\text{L}} \right] + n_{_B}\, K^{\text{rev}}_{\mathcal{M}}(\omega, \mathbf{k}) \left[J_{\text{R}} - J_{\text{L}} \right] \;, \\ \langle \mathcal{O}_{\text{L}}(\omega, \mathbf{k}) \rangle &= -K^{\text{in}}_{\mathcal{M}}(\omega, \mathbf{k}) \left[(n_{_B}+1)\, J_{\text{R}} - n_{_B}\, J_{\text{L}} \right] + (n_{_B}+1) K^{\text{rev}}_{\mathcal{M}}(\omega, \mathbf{k}) \left[J_{\text{R}} - J_{\text{L}} \right] \;. \end{aligned} \tag{2.67}$$

Here, we have used the Bose-Einstein identity $n_{_B}\, e^{\beta\omega} = n_{_B} + 1$ and defined the renormalized reversed Green's function

$$K^{\text{rev}}_{\mathcal{M}}(\omega, \mathbf{k}) \equiv K^{\text{in}}_{\mathcal{M}}(-\omega, \mathbf{k}) \,. \tag{2.68}$$

In carrying out the computations it is helpful to note that the divergence and counterterm structures in this case only involve even powers of ω. Consequently, the counterterms that cancel the divergences of $G^{\text{in}}_{\mathcal{M}}$ also cancel the divergences in $G^{\text{rev}}_{\mathcal{M}}$ using the observations of [61] mentioned above. A useful relation in this regard is

$$\mathbb{D}_+ \left[G^{\text{in}}_{\mathcal{M}}(-\omega, \mathbf{k}) e^{-\beta\omega\zeta} \right] = \left[\mathbb{D}_+ G^{\text{in}}_{\mathcal{M}}(\omega, \mathbf{k}) \right]_{\omega\to-\omega} e^{-\beta\omega\zeta} \,. \tag{2.69}$$

Let us also record the analog of (2.67) in the average-difference basis. Taking a linear combination we end up with

$$\begin{aligned} \langle \mathcal{O}_a(\omega, \mathbf{k}) \rangle &= -K^{\text{in}}_{\mathcal{M}}\, J_a - \left(n_{_B} + \frac{1}{2} \right) \left[K^{\text{in}}_{\mathcal{M}} - K^{\text{rev}}_{\mathcal{M}} \right] J_d \;, \\ \langle \mathcal{O}_d(\omega, \mathbf{k}) \rangle &= -K^{\text{rev}}_{\mathcal{M}}\, J_d \,. \end{aligned} \tag{2.70}$$

We see here the characteristic upper triangular structure of two point functions in the average-difference basis with the $\langle \mathcal{O}_a\, \mathcal{O}_d \rangle$ and $\langle \mathcal{O}_d \mathcal{O}_a \rangle$ corresponding to retarded and advanced Green's functions, respectively. The Keldysh Green's function $\langle \mathcal{O}_a \mathcal{O}_a \rangle$ is an even function of the frequency whose derivative expansion is given by

$$\left(n_{_B}+\frac{1}{2}\right)\left[K^{\text{in}}_{\mathcal{M}}-K^{\text{rev}}_{\mathcal{M}}\right]=\frac{d}{2\pi i}\left(\frac{4\pi}{d\,\beta}\right)^{\mathcal{M}+1}\left(1+\left(\frac{4\pi}{d}\right)^2\frac{\mathfrak{w}^2}{12}\right)\\ \times\left(1-2\,\mathfrak{q}^2\,H_k(\mathcal{M},1)-2\,\mathfrak{w}^2\,H_\omega(\mathcal{M},1)+\cdots\right). \tag{2.71}$$

We have used the explicit form of the retarded Green's function (2.36) to derive the above. We see that in the Markovian case, the one-point functions are given by local expressions, i.e., the CFT one-point functions at a CFT spacetime point depend on the value of the CFT source and its derivatives at that point.

Alternatively, these one-point functions can also be computed by varying the CFT influence phase with respect to the CFT sources. The computation of this influence phase proceeds by generalizing the GKPW method of evaluating the on-shell action to grSK geometry as described in [61]. The generalized Klein-Gordon action with the boundary counterterms $S[\varphi_{\mathcal{M}}]$ defined in (2.33) evaluates on-shell to a pure boundary term (cf., the discussion in Section 5.1 of [61]). Evaluating this in the boundary Fourier domain we find the on-shell action:[27]

$$S[\varphi_{\mathcal{M}}]\Big|_{\text{on-shell}}=-\frac{1}{2}\lim_{r_c\to\infty}\int_k\left[\varphi^{\dagger}_{\mathcal{M}}\left\{r^{\mathcal{M}}\mathbb{D}_+ + c^{(2)}_{\varphi}\,r^{\mathcal{M}-1}\sqrt{f}\left(k^2-\frac{1}{f}\omega^2\right)+\cdots\right\}\varphi_{\mathcal{M}}\right]^{r=r_c-i0}_{r=r_c+i0}. \tag{2.72}$$

Using the explicit solution this can be shown to be

$$\begin{split} S[\varphi_{\mathcal{M}}]\Big|_{\text{on-shell}} &= -\frac{1}{2}\int_k (J_{\text{R}}-J_{\text{L}})^{\dagger}\,K^{\text{in}}_{\mathcal{M}}\Big((n_{_B}+1)\,J_{\text{R}}-n_{_B}\,J_{\text{L}}\Big)\\ &\quad+\frac{1}{2}\int_k\Big(n_{_B}J_{\text{R}}-(n_{_B}+1)J_{\text{L}}\Big)^{\dagger}K^{\text{rev}}_{\mathcal{M}}\Big(J_{\text{R}}-J_{\text{L}}\Big)\\ &=-\int_k (J_{\text{R}}-J_{\text{L}})^{\dagger}\,K^{\text{in}}_{\mathcal{M}}\Big((n_{_B}+1)\,J_{\text{R}}-n_{_B}\,J_{\text{L}}\Big). \end{split} \tag{2.73}$$

In the last line, we have redefined $\omega\to-\omega$ in the second integral and have used the identity $1+n_{_B}(\omega)+n_{_B}(-\omega)=0$. The last line represents the quadratic influence phase written in the advanced-retarded (RA) basis. In the average-difference or Keldysh basis, we get

$$\begin{split} S[\varphi_{\mathcal{M}}]\Big|_{\text{on-shell}} &= -\int_k J_d{}^{\dagger}K^{\text{in}}_{\mathcal{M}}\left[J_a+\left(n_{_B}+\frac{1}{2}\right)\,J_d\right]\\ &=-\int_k\left[J_d{}^{\dagger}K^{\text{in}}_{\mathcal{M}}J_a+\frac{1}{2}\left(n_{_B}+\frac{1}{2}\right)\,J_d{}^{\dagger}\left(K^{\text{in}}_{\mathcal{M}}-K^{\text{rev}}_{\mathcal{M}}\right)J_d\right]. \end{split} \tag{2.74}$$

This is the standard structure of quadratic influence phase for an open system in contact

[27] We employ the notational shorthand $\int\frac{d\omega}{2\pi}\,\frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}}\equiv\int_k$ to keep the expressions compact.

with thermal bath [63, 65]. The explicit expression for the Green's functions has already been recorded in (2.36).

2.6.2 Non-Markovian probes

Let us now turn to the non-Markovian probes. All of the above steps can be repeated in parallel with some minor modifications as elucidated in §2.3 and §2.5. We analytically continue $\mathcal{M}$ to $-\mathcal{M}$ and in this process the CFT sources J of the Markovian probe morph into the long-distance open EFT fields $\breve{\Phi}$ of the non-Markovian probe. The analog of (2.64) now reads:

$$\begin{aligned}\varphi^{\text{SK}}_{-\mathcal{M}}(\omega,\zeta,\mathbf{k}) &= G^{\text{in}}_{-\mathcal{M}}\,\left[(n_{_B}+1)\,\breve{\Phi}_{\text{R}} - n_{_B}\,\breve{\Phi}_{\text{L}}\right] - G^{\text{rev}}_{-\mathcal{M}}\,n_{_B}\,\left[\breve{\Phi}_{R} - \breve{\Phi}_{\text{L}}\right]\;e^{\beta\omega(1-\zeta)} \\ &= G^{\text{in}}_{-\mathcal{M}}\,\breve{\Phi}_{a} + \left[\left(n_{_B}+\frac{1}{2}\right)G^{\text{in}}_{-\mathcal{M}} - n_{_B}\,e^{\beta\omega(1-\zeta)}\,G^{\text{rev}}_{-\mathcal{M}}\right]\breve{\Phi}_{d}\,.\end{aligned} \tag{2.75}$$

where

$$\breve{\Phi}_{a} \equiv \frac{1}{2}\left(\breve{\Phi}_{R} + \breve{\Phi}_{\text{L}}\right)\;, \qquad \breve{\Phi}_{d} \equiv \breve{\Phi}_{R} - \breve{\Phi}_{\text{L}}\,. \tag{2.76}$$

One point functions can be computed using the right or left normalizable modes with the appropriate counterterms determined before in §2.5.4. We have

$$\left\langle \mathcal{O}_{\text{L/R}}\right\rangle = \lim_{r\to\infty\pm i0}\left\{\varphi_{-\mathcal{M}} - \frac{1}{\sqrt{f}}\left[-\frac{c^{(0)}_{\pi}}{r} + \frac{c^{(2)}_{\pi}}{r^{3}}\left(k^{2}-\frac{1}{f}\omega^{2}\right)+\cdots\right]\mathbb{D}_{+}\varphi_{-\mathcal{M}}\right\}. \tag{2.77}$$

The divergence and counterterm structures again involve only even powers of ω, so the counterterms that cancel the divergences of $G^{\text{in}}_{-\mathcal{M}}$ also cancels the divergences in $G^{\text{rev}}_{-\mathcal{M}}$. By explicit computation we verify again the analog of (2.56) in the grSK geometry,

$$\left\langle \mathcal{O}_{\text{L/R}}\right\rangle = \breve{\Phi}_{\text{L/R}}\;, \tag{2.78}$$

viz., the long-distance open EFT fields $\breve{\Phi}$ in the CFT can be identified with the dual single-trace primary. A useful identity in deriving the above relation is

$$\lim_{r\to\infty}\;e^{-\beta\omega(1-\zeta)}\left\{1 - \frac{1}{\sqrt{f}}\left[-\frac{c^{(0)}_{\pi}}{r} + \frac{c^{(2)}_{\pi}}{r^{3}}\left(k^{2}-\frac{1}{f}\omega^{2}\right)+\cdots\right]\mathbb{D}_{+}\right\}\left[G^{\text{rev}}_{-\mathcal{M}}\;e^{\beta\omega(1-\zeta)}\right] = 1\,. \tag{2.79}$$

Let us now consider the non-normalizable modes of the non-Markovian designer scalar. These can be computed from the definitions

$$\begin{aligned}\breve{J}_{\text{L}} &= -\lim_{r\to\infty+i0}\;\pi_{-\mathcal{M}} = \lim_{r\to\infty+i0}\;r^{-\mathcal{M}}\mathbb{D}_{+}\varphi_{-\mathcal{M}}\;, \\ \breve{J}_{\text{R}} &= -\lim_{r\to\infty-i0}\;\pi_{-\mathcal{M}} = \lim_{r\to\infty-i0}\;r^{-\mathcal{M}}\mathbb{D}_{+}\varphi_{-\mathcal{M}}\;,\end{aligned} \tag{2.80}$$

leading to the expressions on the grSK geometry:

$$\begin{aligned} \breve{J}_{\text{R}} &= K^{\text{in}}_{-\mathcal{M}} \left[(n_{_B}+1)\,\breve{\Phi}_{\text{R}} - n_{_B}\,\breve{\Phi}_{\text{L}}\right] - n_{_B} K^{\text{rev}}_{-\mathcal{M}} \left[\breve{\Phi}_R - \breve{\Phi}_{\text{L}}\right] \,, \\ \breve{J}_{\text{L}} &= K^{\text{in}}_{-\mathcal{M}} \left[(n_{_B}+1)\,\breve{\Phi}_{\text{R}} - n_{_B}\,\breve{\Phi}_{\text{L}}\right] - (n_{_B}+1) K^{\text{rev}}_{-\mathcal{M}} \left[\breve{\Phi}_R - \breve{\Phi}_{\text{L}}\right] \,, \end{aligned} \tag{2.81}$$

where $K^{\text{in}}_{-\mathcal{M}}$ is the dispersion function defined in (2.42). For completeness we quote here the third order formula derived in (A.20):

$$\begin{split} K^{\text{in}}_{-\mathcal{M}}(\omega,\mathbf{k}) = b^{\mathcal{M}-1} \Bigg[-i\,\mathfrak{w} + \frac{\mathfrak{q}^2}{\mathcal{M}+1} + \mathfrak{w}^2\,\Delta(\mathcal{M},1) + i\,\mathfrak{w}^3 \left(\Delta(\mathcal{M},1)^2 - 2\,H_\omega(\mathcal{M},1)\right) \\ + 2i\,\frac{\mathfrak{w}\,\mathfrak{q}^2}{\mathcal{M}+1}\left(\Delta(\mathcal{M},1) - (\mathcal{M}-1)\,H_k(\mathcal{M},1)\right) + \cdots \Bigg] \end{split} \tag{2.82}$$

It is straightforward to verify, using the values of the gradient expansion functions (A.25) and (A.26), that the above reduces to the retarded Green's function of the Markovian probe with the replacement $\mathcal{M} \to -\mathcal{M}$.

Taking the average and the difference of the equations above, we obtain the Keldysh basis sources

$$\begin{aligned} \breve{J}_a &= K^{\text{in}}_{-\mathcal{M}}\,\breve{\Phi}_a + \left(n_{_B} + \frac{1}{2}\right)\left[K^{\text{in}}_{-\mathcal{M}} - K^{\text{rev}}_{-\mathcal{M}}\right]\breve{\Phi}_d \,, \\ \breve{J}_d &= K^{\text{rev}}_{-\mathcal{M}}\,\breve{\Phi}_d \,. \end{aligned} \tag{2.83}$$

The combination multiplying $\breve{\Phi}_d$ in the expression for $\breve{J}_a$ can be simplified to:

$$\begin{split} \left(n_{_B}+\frac{1}{2}\right)\left[K^{\text{in}}_{-\mathcal{M}} - K^{\text{rev}}_{-\mathcal{M}}\right] &= \frac{d}{2\pi i}\left(\frac{d\,\beta}{4\pi}\right)^{\mathcal{M}-1}\left(1+\left(\frac{4\pi}{d}\right)^2\frac{\mathfrak{w}^2}{12}\right) \\ &\qquad \times\left(1 - 2\left[\mathfrak{q}^2\,H_k(-\mathcal{M},1) + \mathfrak{w}^2\,H_\omega(-\mathcal{M},1)\right] + \cdots\right), \\ &= \frac{d}{2\pi i}\left(\frac{d\,\beta}{4\pi}\right)^{\mathcal{M}-1}\left(1+\left(\frac{4\pi}{d}\right)^2\frac{\mathfrak{w}^2}{12}\right) \\ &\qquad \times\Bigg(1 + 2\Bigg[\mathfrak{q}^2\frac{(\mathcal{M}-1)\,H_k(\mathcal{M},1) - \Delta(\mathcal{M},1)}{\mathcal{M}+1} \\ &\qquad\qquad + \mathfrak{w}^2\left(H_\omega(\mathcal{M},1) - \frac{1}{2}\Delta(\mathcal{M},1)^2\right)\Bigg] + \cdots\Bigg)\,. \end{split} \tag{2.84}$$

The equations that we have derived above should really be thought of as the local equation of motion (or Schwinger-Dyson equations) for the open EFT of the fields $\breve{\Phi}_{\text{R,L}}$. These equations can be solved for the effective fields $\breve{\Phi}_{\text{R,L}}$ in terms of the SK sources $\breve{J}_{\text{R,L}}$ to yield Schwinger-Keldysh Green's functions.

To wrap up the discussion let us also demonstrate that we might have equivalently evaluated the on-shell bulk action which gives the Wilsonian Influence Functional, the effective action corresponding to the above equations of motion. The designer non-Markovian scalar action with the boundary counterterms (2.53) evaluates on-shell to a pure boundary term, in analogy with (2.72) which in the boundary Fourier domain takes the form

$$S[\varphi_{-\mathcal{M}}]\Big|_{\text{on-shell}} = -\frac{1}{2}\lim_{r_c\to\infty}\mathfrak{I}$$
$$\mathfrak{I} = \int_k \left[\left(\varphi_{-\mathcal{M}} + \frac{1}{\sqrt{f}}\left[\frac{c_\pi^{(0)}}{r} - \frac{c_\pi^{(2)}}{r^3}\left(k^2 - \frac{1}{f}\omega^2\right) + \cdots\right]\mathbb{D}_+\varphi_{-\mathcal{M}}\right)^\dagger r^{-\mathcal{M}}\mathbb{D}_+\varphi_{-\mathcal{M}}\right]_{r=r_c+i0}^{r=r_c-i0}, \tag{2.85}$$

Using the explicit solution for the non-Markovian field on the grSK contour we obtain:

$$\begin{aligned} S[\varphi_{-\mathcal{M}}]\Big|_{\text{on-shell}} &= -\frac{1}{2}\int_k \left[\breve{\Phi}_R - \breve{\Phi}_{\text{L}}\right]^\dagger K^{\text{in}}_{-\mathcal{M}}\left[(n_{\!_B}+1)\,\breve{\Phi}_{\text{R}} - n_{\!_B}\,\breve{\Phi}_{\text{L}}\right] \\ &\qquad + \frac{1}{2}\int_k \left[n_{\!_B}\breve{\Phi}_{\text{R}} - (n_{\!_B}+1)\breve{\Phi}_{\text{L}}\right]^\dagger K^{\text{rev}}_{\mathcal{M}}\left[\breve{\Phi}_R - \breve{\Phi}_{\text{L}}\right] \\ &= -\int_k \left[\breve{\Phi}_R - \breve{\Phi}_{\text{L}}\right]^\dagger K^{\text{in}}_{-\mathcal{M}}\left[(n_{\!_B}+1)\,\breve{\Phi}_{\text{R}} - n_{\!_B}\,\breve{\Phi}_{\text{L}}\right]. \end{aligned} \tag{2.86}$$

In the last line, we have redefined $\omega \to -\omega$ in the second integral and have used the identity $1 + n_{\!_B}(\omega) + n_{\!_B}(-\omega) = 0$. This is the expression for the Wilsonian influence functional in the retarded-advanced (RA) basis. In the average-difference or Keldysh basis, we get

$$\begin{aligned} S[\varphi_{-\mathcal{M}}]\Big|_{\text{on-shell}} &= -\int_k \breve{\Phi}_d^\dagger\, K^{\text{in}}_{-\mathcal{M}}\left[\breve{\Phi}_a + \left(n_{\!_B} + \frac{1}{2}\right)\,\breve{\Phi}_d\right] \\ &= -\int_k \left[\breve{\Phi}_d^\dagger K^{\text{in}}_{-\mathcal{M}}\breve{\Phi}_a + \frac{1}{2}\left(n_{\!_B} + \frac{1}{2}\right)\,\breve{\Phi}_d^\dagger\left(K^{\text{in}}_{-\mathcal{M}} - K^{\text{rev}}_{-\mathcal{M}}\right)\breve{\Phi}_d\right]. \end{aligned} \tag{2.87}$$

By further turning on external sources for the hydrodynamic moduli of the form:

$$\int_k \left[\breve{J}_{\text{R}}\breve{\Phi}_{\text{R}}^\dagger - \breve{J}_{\text{L}}\breve{\Phi}_{\text{L}}^\dagger\right] = \int_k \left[\breve{\Phi}_d^\dagger\,\breve{J}_a + \breve{\Phi}_a\,\breve{J}_d^\dagger\right], \tag{2.88}$$

and varying the effective action, we get the hydrodynamic equations of motion quoted above in (2.83). This shows that the bulk on-shell action does indeed generate the correct hydrodynamic equations of motion.

2.6.3 The Gaussian Wilsonian influence functional

To wrap up the discussion let us consider an example of a probe system which comprises of a non-interacting pair of Markovian and non-Markovian fields with Markovianity indices $\mathcal{M}_1$ and $-\mathcal{M}_2$, respectively. Using the results of the preceding sections we can now write down the final answer for the quadratic approximation to the influence phase. We have

$$\mathcal{S}_{\text{WIF}}[J_a, J_d, \breve{\Phi}_a, \breve{\Phi}_d] = -\int_k \left\{ J_d^\dagger \, K^{\text{in}}_{\mathcal{M}_1} \left[J_a + \left(n_{_B} + \frac{1}{2} \right) \, J_d \right] + \breve{\Phi}_d^\dagger \, K^{\text{in}}_{-\mathcal{M}_2} \left[\breve{\Phi}_a + \left(n_{_B} + \frac{1}{2} \right) \, \breve{\Phi}_d \right] \right\} \tag{2.89}$$

where the retarded Green's function to the third order in the gradient expansion are given in (2.36) and (2.82), respectively. As we deduced earlier these functions for the Markovian and non-Markovian fields have the same functional form. We record here for completeness the function $K^{\text{in}}_{\mathcal{M}}$ with all the factors fixed

$$K^{\text{in}}_{\mathcal{M}}(\omega, k) = \left(\frac{4\pi}{d\,\beta} \right)^{\mathcal{M}+1} \left[-i\,\mathfrak{w} - \frac{\mathfrak{q}^2}{\mathcal{M}-1} - \mathfrak{w}^2 \, \Delta(\mathcal{M},1) + 2i\,\mathfrak{w}\mathfrak{q}^2 \, H_k(\mathcal{M},1) + 2i\,\mathfrak{w}^3 \, H_\omega(\mathcal{M},1) + \cdots \right] . \tag{2.90}$$

with the parameters

$$\begin{aligned}
\Delta(\mathcal{M},1) &= \frac{1}{d} \left[\psi\left(\frac{\mathcal{M}+1}{d} \right) - \psi\left(\frac{1-\mathcal{M}}{d} \right) \right] \\
H_k(\mathcal{M},1) &= \frac{1}{d\,(\mathcal{M}-1)} \left[\psi\left(\frac{\mathcal{M}+1}{d} \right) - \psi\left(\frac{2}{d} \right) \right] \\
H_\omega(\mathcal{M},1) &= \frac{\Delta(\mathcal{M},1)}{2d} \, \text{Har}\left(\frac{\mathcal{M}+1}{d} - 1 \right) + H^{(2)}_\omega(\mathcal{M},1) \\
H^{(2)}_\omega(\mathcal{M},1) &= -\frac{\mathcal{M}}{d} \sum_{n=0}^{\infty} \frac{\text{Har}\left(n - 1 + \frac{2}{d} \right)}{(nd + \mathcal{M} - 1)(nd + 1 - \mathcal{M})}
\end{aligned} \tag{2.91}$$

We have written the result in terms of the digamma function $\psi(x)$ and the related Harmonic number function $\text{Har}\,(x)$ which has been used before in the fluid/gravity literature.[28]

2.7 Time-reversal invariant gauge system

We now turn to the study of the Maxwell analogue of the designer scalar and explore the dynamics of an Abelian gauge field coupled to the background geometry and an auxiliary dilaton introduced in §2.2. We will demonstrate below the following claim: the dynamics of this designer gauge system can be completely encoded, in a gauge invariant manner,

[28] For fourth order derivative corrections to $K_{\mathcal{M}}$, check [134] .

in two designer scalars. This statement then generalizes the observation made earlier in (2.9) for a pure Maxwell theory (i.e., $\mathcal{M} = d - 3$). We will sketch the essence of the argument below, supplementing our discussion with further details in Appendix B. In §2.8 we will see that similar statements apply to linearized gravitational perturbations.

Several authors have attempted in the past to give a Schwinger-Keldysh description for the bulk gauge theory and gravity [88, 112–115]. Our treatment here is substantially different with a focus on the physics of outgoing Hawking modes (the dynamics of infalling quasinormal modes has been well understood since the early works of [19–21] and in the fluid/gravity context [22, 23]). As in the scalar problem, we will explicitly keep track of the origin and the effects of the Bose-Einstein distribution in the effective dynamics of the Markovian and non-Markovian modes.

Let us first highlight the key aspects where our discussion differs from earlier literature for systems with bulk gauge symmetry (either abelian gauge symmetry or diffeomorphism).

1. We eschew the use of the radial gauge which is commonly employed in AdS/CFT discussions. In this gauge outgoing Hawking modes are tricky to explore as the gauge explicitly breaks the $\mathbb{Z}_2$ time reversal isometry. The solution to this issue is straightforward: we adopt a gauge invariant scheme for solving the field equations.

2. Systems with gauge invariance contain both Markovian and non-Markovian modes. Ideally one would like to disentangle them, so that one can integrate out the Markovian modes, whilst keeping the non-Markovian modes off-shell. Fortuitously for us, a plane wave decomposition of the perturbation naturally serves to decouple the modes. It furthermore maps them onto the scalar problem we explored earlier.

3. We will solve the radial Gauss constraints arising from the gauge invariance which are dual to boundary conservation equations. In the non-Markovian sector we turn on appropriate sources to keep these modes off-shell.

2.7.1 Decomposition of gauge field modes

We start with the designer gauge system introduced in §2.2. The equations of motion arising from the action (2.4) are simply

$$\partial_A \left(\sqrt{-g}\, r^{\mathcal{M}+3-d}\, \mathcal{C}^{AB}\right) = 0 \tag{2.92}$$

We can examine these directly for the gauge field strengths, or pass as usual to the conventional parameterization in terms of the gauge potential $\mathcal{V}_A$. We will find it convenient to expand the potential $\mathcal{V}$ in terms of plane wave harmonics on $\mathbb{R}^{d-1,1}$. This will have the advantage that we will be easily able to decouple the Markovian and non-Markovian degrees of freedom contained in these equations. We let[29]

$$
\begin{aligned}
\mathcal{V}_r(v,r,\mathbf{x}) &= \int_k \bar{\Psi}_r(r,\omega,\mathbf{k})\,\mathbb{S}(\omega,\mathbf{k}|v,\mathbf{x})\,,\\
\mathcal{V}_v(v,r,\mathbf{x}) &= \int_k \bar{\Psi}_v(r,\omega,k)\,\mathbb{S}(\omega,\mathbf{k}|v,\mathbf{x})\;,\\
\mathcal{V}_i(v,r,\mathbf{x}) &= \int_k \left[\sum_{\alpha=1}^{N_V} \bar{\Phi}_\alpha(r,\omega,\mathbf{k})\,\mathbb{V}_i^\alpha(\omega,\mathbf{k}|v,\mathbf{x}) + i\,\bar{\Psi}_x(r,\omega,\mathbf{k})\,\mathbb{S}_i(\omega,\mathbf{k}|v,\mathbf{x})\right]\\
&= \int_k \left[\sum_{\alpha=1}^{N_V} \bar{\Phi}_\alpha(r,\omega,\mathbf{k})\,\mathbb{V}_i^\alpha(\omega,\mathbf{k}|v,\mathbf{x}) - \bar{\Psi}_x(r,\omega,\mathbf{k})\,\frac{k_i}{k}\,\mathbb{S}(\omega,\mathbf{k}|v,\mathbf{x})\right].
\end{aligned}
\tag{2.93}
$$

Here $\alpha = 1, 2, \ldots, N_V = d-2$ labels the different vector polarizations of the gauge field and r, v, i denote spacetime indices. Our conventions for the harmonics are summarized in Appendix F. The main point to note is that the decomposition is in an orthonormal basis which allows us to decouple the modes of the gauge field.

Plugging in the harmonic decomposition into (2.92) and exploiting the decoupling of the transverse vector sector from the transverse scalar sector, we find the following:

- There is a single equation in the transverse vector sector which can be identified with that of a time-reversal invariant designer scalar with exponent $\mathcal{M}$. Specifically, all the fields $\bar{\Phi}_\alpha$ satisfy:

$$
\frac{1}{r^{\mathcal{M}}}\mathbb{D}_+\left(r^{\mathcal{M}}\,\mathbb{D}_+\bar{\Phi}_\alpha\right) + \left(\omega^2 - k^2 f\right)\bar{\Phi}_\alpha = 0\,. \tag{2.94}
$$

  Comparing with (2.3) we obtain $\bar{\Phi}_\alpha = \varphi_{\mathcal{M}}$ for $\alpha = 1, 2, \ldots, N_V$.

- In the scalar sector, we find a set of three coupled differential equations for the fields $\bar{\Psi}_v$, $\bar{\Psi}_r$, and $\bar{\Psi}_x$. We introduce the following gauge invariant combinations of these fields:

$$
\begin{aligned}
&\bar{\Pi}_v \equiv \frac{d\bar{\Psi}_v}{dr} + i\omega\,\bar{\Psi}_r\,, \qquad \bar{P}_r \equiv \frac{d\bar{\Psi}_x}{dr} + ik\,\bar{\Psi}_r\,,\\
&\bar{\Pi}_x \equiv \mathbb{D}_+\bar{\Psi}_x + ik\,\bar{\Psi}_v + ik\,r^2 f\,\bar{\Psi}_r \equiv r^2 f\,\bar{P}_r + \bar{P}_v\,,\\
&\bar{P}_v \equiv i\,k\,\bar{\Psi}_v - i\omega\,\bar{\Psi}_x\,,
\end{aligned}
\tag{2.95}
$$

[29] The notation for the planar harmonic components of the gauge fields being barred is intentional. Later when we discuss gravity dynamics in §2.8 we will employ closely related notation without the decoration to denote the gravitational degrees of freedom.

in terms of which we find compact expressions for the equations of motion:

$$\begin{aligned}
\mathcal{E}_v^{\text{Max}} &\equiv \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\mathcal{M}+2}\,\bar{\Pi}_v\right) - ik\, r^{\mathcal{M}}\,\bar{P}_r = 0\,,\\
\mathcal{E}_x^{\text{Max}} &\equiv \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\mathcal{M}}\,\bar{\Pi}_x\right) - i\omega\, r^{\mathcal{M}}\,\bar{P}_r = 0\,,\\
\mathcal{E}_r^{\text{Max}} &\equiv -i\omega\, r^{\mathcal{M}+2}\,\bar{\Pi}_v + ik\, r^{\mathcal{M}}\,\bar{\Pi}_x = 0\,.
\end{aligned} \tag{2.96}$$

It is easy to check that the combinations above are closely related to the field strengths in Fourier domain

$$\mathcal{C}_{rv} = \bar{\Pi}_v\,, \qquad \mathcal{C}_{ri} = -\frac{k_i}{k}\,\bar{P}_r\,, \qquad \mathcal{C}_{vi} = -\frac{k_i}{k}\,\bar{P}_v\,. \tag{2.97}$$

One can understand our gauge invariant combinations by realizing that the gauge parameter also admits a decomposition in the plane wave harmonics, i.e.,

$$\Lambda_g(v,r,\mathbf{x}) = \int_k \Lambda(r,\omega,\mathbf{k})\,\mathbb{S}(\omega,\mathbf{k}|v,\mathbf{x})\,. \tag{2.98}$$

It then follows that gauge transformations leave $\bar{\Phi}_\alpha$ invariant whereas the scalar sector variables get shifted as

$$\bar{\Psi}_r \mapsto \bar{\Psi}_r + \frac{\mathrm{d}\Lambda}{\mathrm{d}r}\,, \quad \bar{\Psi}_v \mapsto \bar{\Psi}_v - i\omega\,\Lambda\,, \quad \bar{\Psi}_x \mapsto \bar{\Psi}_x - ik\,\Lambda\,, \tag{2.99}$$

which confirms that the combinations defined in (2.95) are indeed invariant.

The transverse vector modes, which as we noted above, map simply to a collection of $N_V = d-2$ designer scalars. Recalling that Markovianity demands $\mathcal{M} > -1$, and that in the pure Maxwell case with no dilatonic coupling, $\mathcal{M} = d-3$, we see that transverse modes are Markovian when $d > 2$ and marginally non-Markovian for $d = 2$.[30] Physically, this sector describes the real electromagnetic waves or photons (which exist when $d > 2$) that fall into the black brane and are, in turn, Hawking radiated out. The Markovian property tells us that this happens at a short time scale of order the inverse Hawking temperature β of the black brane. At time scales much larger than β, one can integrate out the effects of this physics to get a local description for the remaining degrees of freedom.[31]

[30]Note that this is in keeping with the usual expectation. For a Maxwell field in AdS_{d+1} with $d \leq 2$, the standard boundary conditions are inadmissible [135]. The only physically sensible boundary conditions are to freeze the charge on the boundary and leave the currents unconstrained.

[31]We remind that this statement does not directly translate to the question – how long does it take to Hawking radiate a Markovian mode after a diary has been thrown in? Rather we only say that Markovian fluctuations witnessed by the CFT dissipate within order inverse-temperature time scales, which is a priori uninteresting from the perspective of black hole information paradox. However, it suggests that the Markovianity of the radiation plays an important role in the information recovery problem once we account for the gravitational interactions.

The equations in the transverse scalar sector themselves comprise a gauge system as they retain all the characteristics of the underlying gauge field. We will refer to this sector as the *diffusive gauge system* and it has the following additional properties. From (2.96) we can see the combination $i\omega\, \mathcal{E}_v^{\text{Max}} - ik\, \mathcal{E}_x^{\text{Max}} + \frac{\mathrm{d}}{\mathrm{d}r}\mathcal{E}_r^{\text{Max}} = 0$. In other words, the third equation, $\mathcal{E}_r^{\text{Max}}$ can be considered as the Gauss constraint which is preserved under radial evolution described by the first two equations. It implies that if the first two equations hold everywhere, then it is sufficient to impose the third equation only at the radial slice at infinity.

These statements are quite familiar in AdS/CFT. Recall that a bulk gauge symmetry implies a boundary global symmetry by the standard rules of AdS/CFT. Up to an overall normalization and counterterms, the Noether charge density and Noether current density of the conserved current are given by

$$J_v^{\text{CFT}} = -\lim_{r\to r_c} r^{\mathcal{M}+2}\, \bar{\Pi}_v\,, \qquad J_i^{\text{CFT}} = \frac{k_i}{k} \lim_{r\to r_c} r^{\mathcal{M}}\, \bar{\Pi}_x\,. \tag{2.100}$$

With this understanding the final equation in (2.96) is simply that of current conservation $\nabla_\mu (J^{\text{CFT}})^\mu = 0$ on the boundary. This statement in itself implies that we should expect long-lived charge diffusion modes to be present in the system in the scalar sector. This is indeed the case for standard Maxwell dynamics ($\mathcal{M} = d-3$) in the black hole background [20, 136], which encodes the physics of ohmic conductivity.

Before we proceed to establish the connection, we note that the designer gauge system is time reversal invariant. In particular, one can rewrite (2.96) in a time-reversal invariant form

$$\begin{aligned} \mathbb{D}_+ \left(r^{\mathcal{M}+2}\, \bar{\Pi}_v\right) + ik\; r^{\mathcal{M}}\, \bar{P}_v &= 0\,, \\ \mathbb{D}_+ \left(r^{\mathcal{M}}\, \bar{\Pi}_x\right) + i\omega\, r^{\mathcal{M}}\, \bar{P}_v &= 0\,, \\ -i\omega\, r^{\mathcal{M}+2}\, \bar{\Pi}_v + ik\, r^{\mathcal{M}}\, \bar{\Pi}_x &= 0\,. \end{aligned} \tag{2.101}$$

In writing the first equation in the covariant form above, we have used the radial Gauss law constraint. It is helpful to note that $\{r^2 f\, \bar{\Psi}_r + \bar{\Psi}_v, \bar{\Psi}_x, \bar{\Pi}_x\}$ have positive intrinsic time reversal parity while $\{\bar{\Psi}_v, \bar{P}_v, \bar{\Pi}_v\}$ all have negative intrinsic time reversal parity. Details of these facts are explained in Appendix B.

2.7.2 The non-Markovian charge diffusion scalar

Now that we have explored the basic equations for the diffusive gauge system, we will argue for a solution of (2.96) in terms of a single non-Markovian scalar degree of freedom. Introduce a field $\bar{\Phi}_{\text{D}}$, and fix the variables $\bar{\Pi}_v, \bar{P}_r, \bar{\Pi}_x$ as follows:

$$r^{\mathcal{M}}\, \bar{P}_r = -\frac{\mathrm{d}}{\mathrm{d}r}\left(ik\, \bar{\Phi}_{\text{D}}\right), \quad r^{\mathcal{M}}\, \bar{\Pi}_x = -i\omega(ik\, \bar{\Phi}_{\text{D}})\,, \quad r^{\mathcal{M}+2}\, \bar{\Pi}_v = -ik(ik\, \bar{\Phi}_{\text{D}})\,. \tag{2.102}$$

This clearly solves (2.96). To understand the constraints on $\bar{\Phi}_{\text{D}}$ we proceed as follows: the first and the second equations in (2.102) can be combined to solve for $\bar{\Pi}_v$, obtaining

$$r^{\mathcal{M}}\,\bar{P}_v = ik\,\mathbb{D}_+\bar{\Phi}_{\text{D}} \quad\Longrightarrow\quad \bar{\Psi}_v = r^{-\mathcal{M}}\mathbb{D}_+\bar{\Phi}_{\text{D}} + \frac{\omega}{k}\bar{\Psi}_x\,. \tag{2.103}$$

We have used here the definition of $\bar{P}_v$ given in (2.95). Substituting this back into the third equation in (2.102), we find:

$$\begin{aligned} r^{\mathcal{M}+2}\left(\frac{\mathrm{d}\bar{\Psi}_v}{\mathrm{d}r} + i\omega\,\bar{\Psi}_r\right) &= r^{\mathcal{M}+2}\left(\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{-\mathcal{M}}\mathbb{D}_+\bar{\Phi}_{\text{D}}\Big) + \frac{\omega}{k}\left(\frac{\mathrm{d}\bar{\Psi}_x}{\mathrm{d}r} + ik\,\bar{\Psi}_r\right)\right) \\ &= r^{\mathcal{M}+2}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{-\mathcal{M}}\mathbb{D}_+\bar{\Phi}_{\text{D}}\Big) - i\omega\,r^2\,\frac{\mathrm{d}\bar{\Phi}_{\text{D}}}{\mathrm{d}r} \\ &= \frac{1}{f}\left(r^{\mathcal{M}}\mathbb{D}_+\left(r^{-\mathcal{M}}\mathbb{D}_+\bar{\Phi}_{\text{D}}\right) + \omega^2\bar{\Phi}_{\text{D}}\right). \end{aligned} \tag{2.104}$$

Comparing back again with the third equation, we conclude that our definitions are mutually consistent only if $\bar{\Phi}_{\text{D}}$ satisfies a homogeneous second order ODE, viz.,

$$\mathcal{E}^{\text{D}}[\bar{\Phi}_{\text{D}}] \equiv r^{\mathcal{M}}\,\mathbb{D}_+\left(r^{-\mathcal{M}}\,\mathbb{D}_+\bar{\Phi}_{\text{D}}\right) + (\omega^2 - k^2 f)\bar{\Phi}_{\text{D}} = 0\,. \tag{2.105}$$

We can give a gauge invariant derivation of the same result by first writing the field strengths in terms of $\bar{\Phi}_{\text{D}}$ using (2.97) and (2.102) and then demanding that they satisfy the Bianchi identity. The above result proves our assertion that the scalar sector of the designer gauge field, is indeed encoded in the dynamics of a single non-Markovian scalar degree of freedom, viz., $\bar{\Phi}_{\text{D}} \equiv \varphi_{-\mathcal{M}}$. The statement is not new, the idea of using the underlying gauge invariance to isolate the gauge invariant data has, as we indicated earlier in §2.2.2, a long history. A discussion of gauge fields in gravitational backgrounds can be found for example in [107].

A physical interpretation of this scalar degree of freedom can be easily understood by noting that the Noether charge current is simply:

$$J^{\text{CFT}}_\mu\,dx^\mu = -k^2\,\bar{\Phi}_{\text{D}}\,dv + \omega\,k_i\,\bar{\Phi}_{\text{D}}\,dx^i\,. \tag{2.106}$$

This shows that the constant mode of $\bar{\Phi}_{\text{D}}$, which is a normalizable mode for $\mathcal{M} > 1$, is actually the expectation value of Noether charge density (up to a factor of k^2) in the dual field theory. The time derivative of its gradient is the Noether current density.

To get the exact form of the original vector potential, we need to choose a gauge and invert the gauge invariant equations in (2.102). For example one can solve for $\bar{\Psi}_r, \bar{\Psi}_v, \bar{\Psi}_x$ as:

$$\bar{\Psi}_r = -\frac{1}{r^{\mathcal{M}}}\frac{\mathrm{d}\bar{\Phi}_{\text{D}}}{\mathrm{d}r} + \frac{\mathrm{d}\Lambda}{\mathrm{d}r}\,, \qquad \bar{\Psi}_v = \frac{1}{r^{\mathcal{M}}}\mathbb{D}_+\bar{\Phi}_{\text{D}} - i\omega\,\Lambda\,, \qquad \bar{\Psi}_x = -ik\,\Lambda\,. \tag{2.107}$$

One simple gauge choice is to set $\Lambda = 0$ in the equation above which sets $\bar{\Psi}_x = 0$. In this case, the solution can be written in the manifestly time reversal invariant form,

$$\bar{\Psi}_r\, dr + \bar{\Psi}_v\, dv - \bar{\Psi}_x \frac{k_i}{k}\, dx^i = \frac{1}{r^{\mathcal{M}}} \left(dv\ \mathbb{D}_+ - dr\, \frac{\mathrm{d}}{\mathrm{d}r} \right) \bar{\Phi}_{\text{D}}\,, \tag{2.108}$$

which is the form appearing in (2.9). This is analogous to the Debye gauge used in electromagnetism.

2.7.3 Maxwell action and Wilsonian influence phase

We now have a complete description of the gauge fields in terms of a collection of N_V Markovian scalars and a single non-Markovian scalar. Let us rewrite this in terms of an action including all the boundary terms and counterterms. We first isolate the diffusive gauge system from the Markovian

$$\mathcal{V}^{\text{D}} = \mathcal{V}\Big|_{\bar{\Phi}_\alpha = 0}\,, \qquad \mathcal{C}^{\text{D}} = \mathcal{C}\Big|_{\bar{\Phi}_\alpha=0} \tag{2.109}$$

The action for the designer gauge field takes the form:

$$\begin{aligned} S_{dv} &= -\frac{1}{4}\int d^{d+1}x\,\sqrt{-g}\; e^{\chi_v}\, \mathcal{C}^{AB}\mathcal{C}_{AB} + S_{\text{ct}} \\ S_{\text{ct}} &= -\frac{c^{(2)}_{\mathcal{V}}}{4}\int d^dx\sqrt{-\gamma}\, e^{\chi_v}\, \mathcal{C}^{\mu\nu}\mathcal{C}_{\mu\nu}\,, \qquad c^{(2)}_{\mathcal{V}} = -\frac{1}{\mathcal{M}-1}\,. \end{aligned} \tag{2.110}$$

We draw attention to the fact that we are imposing standard Dirichlet boundary conditions for the gauge field and thus require no variational counterterm.[32] The quadratic counterterm is required to cancel subleading divergences at $\mathcal{O}(\omega^2)$ and $\mathcal{O}(k^2)$ and is naturally built out of the covariant gauge strengths projected onto the boundary.

Using the parameterization in terms of the vector polarization scalars $\bar{\Phi}_\alpha$ and the diffusive gauge field $\mathcal{V}^{\text{D}}$ we decouple the Markovian and non-Markovian parts, viz.,

$$\begin{aligned} S_{dv} &= S^{\text{M}}_{dv} + S^{\text{D}}_{dv} \\ S^{\text{M}}_{dv} &= -\frac{1}{2}\sum_{\alpha=1}^{N_V}\left[\int d^{d+1}x\,\sqrt{-g}\, e^{\chi_s}\, g^{AB}\,\nabla_A\,\bar{\Phi}_\alpha \nabla_B \bar{\Phi}_\alpha + c^{(2)}_{\mathcal{V}}\int d^dx\,\sqrt{-\gamma}\, e^{\chi_s}\,\gamma^{\mu\nu}\,\nabla_\mu\bar{\Phi}_\alpha\,\nabla_\nu\bar{\Phi}_\alpha\right] \\ S^{\text{D}}_{dv} &= -\frac{1}{4}\left[\int d^{d+1}x\,\sqrt{-g}\, e^{\chi_v}\,(\mathcal{C}^{\text{D}})^{AB}(\mathcal{C}^{\text{D}})_{AB} + c^{(2)}_{\mathcal{V}}\int d^dx\sqrt{-\gamma}\, e^{\chi_v}\,(\mathcal{C}^{\text{D}})^{\mu\nu}(\mathcal{C}^{\text{D}})_{\mu\nu}\right] \end{aligned} \tag{2.111}$$

The Markovian dynamics encoded in N_V fields $\bar{\Phi}_\alpha$ matches our earlier scalar discussion, not just at the level of the equation of motion, but also at the level of the variational

[32] This can be checked against the AdS analysis of [127, 135] since the asymptotic boundary conditions are dictated purely by the near-boundary behaviour (and thus state independent).

principle, and the counterterms. Indeed, the quadratic counterterm $c^{(2)}_{\varphi}$, given in (2.34), is inherited in the reduction from the gauge field counterterm $c^{(2)}_{\mathcal{V}}$. This provides a useful cross-check of our analysis in §2.4.3.

The diffusive gauge field $\mathcal{V}^{\text{D}}_A$ and its field strength $\mathcal{C}^{\text{D}}_{AB}$ are parameterized in terms of a single scalar degree of freedom $\bar{\Phi}_{\text{D}}$. Substituting the Debye gauge vector potential (2.108) into S^{D}_{dv} we can write the action in terms of $\bar{\Phi}_{\text{D}}$. The bulk term in S^{D}_{dv} results in a fourth order action that can be massaged into contributions involving the equation of motion, $\mathcal{E}^{\text{D}}[\bar{\Phi}_{\text{D}}]$, given in (2.105):

$$\begin{aligned} S^{\text{D}}_{dv}\Big|_{\text{bulk}} &= -\frac{1}{4}\int d^{d+1}x\,\sqrt{-g}\,e^{\chi_v}\,(\mathcal{C}^{\text{D}})^{AB}(\mathcal{C}^{\text{D}})_{AB} \\ &= \frac{1}{2}\int_k dr\,r^{-\mathcal{M}-2}\,\frac{\mathcal{E}^{\text{D}}[\bar{\Phi}_{\text{D}}]}{f}\left(\frac{\mathcal{E}^{\text{D}}[\bar{\Phi}_{\text{D}}]}{f}+k^2\,\bar{\Phi}_{\text{D}}\right)+\frac{1}{2}\int_k k^2\,r^{-\mathcal{M}}\,\bar{\Phi}_{\text{D}}\mathbb{D}_+\bar{\Phi}_{\text{D}}\,. \end{aligned} \tag{2.112}$$

Integrating by parts we can convert the term $\mathcal{E}^{\text{D}}[\bar{\Phi}_{\text{D}}]\bar{\Phi}_{\text{D}}$ into the canonical form involving $\nabla_A\bar{\Phi}_{\text{D}}$. We may also absorb the factors of k^2 by writing the expression in terms of spatial derivatives ∂_i on $\mathbb{R}^{d-1,1}$ and write the action in position space, instead of the Fourier domain expression above. These steps lead us to[33]

$$\begin{aligned} S^{\text{D}}_{dv}\Big|_{\text{bulk}} = -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\,r^{-\mathcal{M}+1-d}\,\partial_i\nabla^A\bar{\Phi}_{\text{D}}\,\partial_i\nabla_A\bar{\Phi}_{\text{D}} + \int d^dx\,r^{-\mathcal{M}}\,\partial_i\bar{\Phi}_{\text{D}}\,\partial_i\mathbb{D}_+\bar{\Phi}_{\text{D}} \\ + \frac{1}{2}\int_k dr\,r^{-\mathcal{M}-2}\left(\frac{\mathcal{E}^{\text{D}}[\bar{\Phi}_{\text{D}}]}{f}\right)^2 \end{aligned} \tag{2.113}$$

The contribution proportional to $(\mathcal{E}^{\text{D}}[\bar{\Phi}_{\text{D}}])^2$ can be ignored for the variational principle, since its variation vanishes on-shell. The first two terms in the last equality above, we recognize, modulo a factor of k^2 (from ∂_i), to be precisely the action for the non-Markovian scalar (2.48). In particular, we emphasize that the reduction does give directly the variational boundary term $-\pi_{_{-\mathcal{M}}}\varphi_{_{-\mathcal{M}}}$, required to impose Neumann boundary conditions on the non-Markovian scalar.

This is quite satisfying. While the boundary conditions on the non-Markovian scalar were inferred from the standard asymptotic analysis earlier, it is useful to recognize that one is not imposing an artificial boundary condition for the gauge field in AdS. While some of these statements are implicit in earlier discussions [127, 135], especially in terms of what modes to freeze and which to allow to fluctuate (in global AdS), it is also useful to have a clear derivation at the level of the variational principle.

Once we recognize this fact we can also check that the zeroth order counterterm for

[33] We ignore all total derivative terms along the boundary directions, i.e., contributions of the form $\partial_\mu(\cdot)$ are dropped in our action.

the non-Markovian scalar $c_{\pi}^{(0)}$ given in (2.54) follows from the Maxwell counterterm $c_{\mathcal{V}}^{(2)}$ at the quadratic order. The counterterm $c_{\pi}^{(2)}$ corresponds to the contribution from two four-derivative counterterms proportional to $e^{\chi_v}\,\gamma^{\alpha\beta}\,\mathcal{C}^{\mu\nu}\nabla_\alpha\nabla_\beta\mathcal{C}_{\mu\nu}$ and $e^{\chi_v}\,\gamma^{\alpha\beta}\,\nabla_\alpha\mathcal{C}^{\mu\nu}\nabla_\beta\mathcal{C}_{\mu\nu}$, respectively.

All told we can write down the dynamics of the designer Maxwell field as follows:

$$S_{dv} = \sum_{\alpha=1}^{N_V} S_{ds}[\varphi^\alpha_{\mathcal{M}}] + S_{ds}[\partial_i \varphi_{-\mathcal{M}}] \tag{2.114}$$

where the actions for the Markovian and non-Markovian scalars are given in (2.33) and (2.48) (see also (2.53) for the relevant counterterms), respectively.

We are then in a position to write down the Wilsonian influence phase for the standard Maxwell field which has Markovianity index, $\mathcal{M} = d-3$. We let the boundary sources for the Markovian vector modes to be $\mathcal{A}^\alpha_a$ and $\mathcal{A}^\alpha_d$ in the average-difference basis. These correspond to the magnetic components of the boundary gauge field, written out in our plane wave decomposition. For the non-Markovian sector, we recognize that the hydrodynamic mode is the boundary charge mode $\breve{\mathcal{Q}}$ and parameterize the Wilsonian influence phase using this field. The sources and the moduli are written in our plane wave basis, so we are using the notation above to just talk about the mode coefficients. We compute the Wilsonian influence phase via a Legendre transform dropping the variational boundary term required to implement the Neumann boundary condition.

We have from (2.89) the result expressed in terms of the retarded Green's function:

$$\begin{split}\mathcal{S}_{\text{WIF}}[\mathcal{A}^\alpha_a, \mathcal{A}^\alpha_d, \breve{\mathcal{Q}}_a, \breve{\mathcal{Q}}_d] = -\int_k \Bigg\{ \sum_{\alpha=1}^{N_V} (\mathcal{A}^\alpha_d)^\dagger\, K^{\text{in}}_{d-3} \left[\mathcal{A}^\alpha_a + \left(n_{_B} + \frac{1}{2}\right)\, \mathcal{A}^\alpha_d\right] \\ + k^2\, \breve{\mathcal{Q}}^\dagger_d\, K^{\text{in}}_{-d+3} \left[\breve{\mathcal{Q}}_a + \left(n_{_B} + \frac{1}{2}\right)\, \breve{\mathcal{Q}}_d\right]\Bigg\}\end{split} \tag{2.115}$$

Specializing (2.36) and (2.82) to the case $\mathcal{M} = d-3$, we have the explicit expression for the probe Maxwell field in a planar Schwarzschild-AdS$_{d+1}$ background:

$$\begin{split}K_{d-3}(\omega, k) = \left(\frac{4\pi}{d\,\beta}\right)^{d-2} \Bigg\{ -i\,\mathfrak{w} - \frac{\mathfrak{q}^2}{d-4} - \mathfrak{l}_2(d)\,\mathfrak{w}^2 + \frac{2\pi i}{d(d-4)} \cot\left(\frac{2\pi}{d}\right) \mathfrak{w}\mathfrak{q}^2 \\ + i\left[\frac{1}{d^2}\,\text{Har}\left(-\frac{2}{d}\right)\mathfrak{l}_2(d) + \mathfrak{l}_3(d)\right]\mathfrak{w}^3 + \cdots \Bigg\}\,.\end{split} \tag{2.116}$$

and

$$K_{_{3-d}}(\omega,k)=\left(\frac{4\pi}{d\,\beta}\right)^{2-d}\left\{-i\,\mathfrak{w}+\frac{\mathfrak{q}^2}{d-2}+\mathfrak{l}_2(d)\,\mathfrak{w}^2+\frac{2i}{d(d-2)}\left[\psi\left(\frac{2}{d}\right)-\psi\left(\frac{4-d}{d}\right)\right]\mathfrak{w}\mathfrak{q}^2\right.$$
$$\left.+i\left[-\frac{1}{d^2}\,\mathrm{Har}\left(\frac{4-2d}{d}\right)\mathfrak{l}_2(d)-\mathfrak{l}_3(d)\right]\mathfrak{w}^3+\cdots\right\}. \tag{2.117}$$

The two parameters $\mathfrak{l}_2(d)$ and $\mathfrak{l}_3(d)$ introduced above are given by

$$\begin{aligned}\mathfrak{l}_2(d)&=\Delta(d-3,1)=\frac{1}{d}\left[\mathrm{Har}\left(-\frac{2}{d}\right)-\mathrm{Har}\left(\frac{4-2d}{d}\right)\right]\\ \mathfrak{l}_3(d)&=-\frac{2(d-3)}{d}\sum_{n=0}^{\infty}\frac{\mathrm{Har}\left(n-1+\frac{2}{d}\right)}{(nd+4-d)(nd+d-2)}\end{aligned} \tag{2.118}$$

We note that while we have eschewed the study of the marginal case $\mathcal{M}=-1$ which is relevant for the R-charge diffusion of $\mathcal{N}=4$ SYM, using the dynamics of Maxwell fields in Schwarzschild-AdS$_5$. Nevertheless, we can extract by a pole prescription in a $4-\epsilon$ expansion, the diffusion constant from (2.115). One finds that it agrees with the prediction of [20], viz., $\mathcal{D}=\frac{1}{2\pi T}$ after an appropriate translation of the variables.

2.8 Gravitational perturbations

Let us finally turn to linearized gravitational perturbations about the Schwarzschild-AdS$_{d+1}$ black hole geometry. We will show that a subset of gravitational modes, viz., the tensor and vector sectors, in the plane wave harmonic decomposition, can be mapped as presaged in §2.2.2 onto the designer scalar and gauge field respectively. Since much of the analysis reduces to that of the previous sections, we will be brief in our presentation. Details on some of the statements here can be found in Appendix C.

2.8.1 Dynamics of transverse tensor and vector gravitations

We consider linearized metric perturbations about a planar Schwarzschild-AdS$_{d+1}$ black hole, where only transverse tensor and vector type perturbations are turned on. Explicitly, we have:

$$
\begin{aligned}
ds^2 &= \Big(g_{AB} + (h_{AB})^{\text{Tens}} + (h_{AB})^{\text{Vec}} + \cancel{(h_{AB})}^{\text{Scal}}\Big) dx^A dx^B \\
(h_{AB})^{\text{Tens}}\, dx^A\, dx^B &= r^2 \int_k \sum_{\sigma=1}^{N_T} \Phi_\sigma(r,\omega,\mathbf{k})\, \mathbb{T}^\sigma_{ij}(\omega,\mathbf{k}|v,\mathbf{x})\; dx^i dx^j\ , \\
(h_{AB})^{\text{Vec}}\, dx^A\, dx^B &= r^2 \int_k \sum_{\alpha=1}^{N_V} \Big(2\ (\Psi^\alpha_r(r,\omega,\mathbf{k})\, dr + \Psi^\alpha_v(r,\omega,\mathbf{k}) dv)\, \mathbb{V}^\alpha_i(\omega,\mathbf{k}|v,\mathbf{x}) dx^i \\
&\qquad + i\, \Psi^\alpha_x(r,\omega,\mathbf{k})\ \mathbb{V}^\alpha_{ij}\, dx^i dx^j\Big),
\end{aligned}
\tag{2.119}
$$

with $N_V = d-2$ transverse vector and $N_T = \frac{d(d-3)}{2}$ transverse tensor polarizations of the gravitons (see Appendix F). We defer the analysis of the scalar sector perturbations, $(h_{AB})^{\text{Scal}}$, to chapter 3.

What we are after is the dynamics of the modes Φ_σ and Ψ^α. These can be succinctly described with a slight repacking of data. Given a set of N_T scalar fields Φ_σ, and N_V vectors Ψ^α, introduce a collection of auxiliary fields by repackaging the Fourier modes in the harmonic decomposition above as (see footnote 29)

$$
\begin{aligned}
\Phi_\sigma(v,r,\mathbf{x}) &\equiv \int_k \Phi_\sigma(r,\omega,\mathbf{k})\, \mathbb{S}(\omega,\mathbf{k}|v,\mathbf{x})\,, \\
\mathcal{A}^\alpha_B(v,r,\mathbf{x})\, dx^B &\equiv \int_k \Big((\Psi^\alpha_r(r,\omega,\mathbf{k})\, dr + \Psi^\alpha_v(r,\omega,\mathbf{k}) dv)\, \mathbb{S}(\omega,\mathbf{k}|v,\mathbf{x}) \\
&\qquad - i\, \Psi^\alpha_x(r,\omega,\mathbf{k})\ \mathbb{S}_i\, dx^i\Big),
\end{aligned}
\tag{2.120}
$$

The set of 1-forms $\mathcal{A}^\alpha$ are diffusive Abelian gauge fields with corresponding field strengths $\mathcal{F}^\alpha_{BC} = \partial_B \mathcal{A}^\alpha_C - \partial_C \mathcal{A}^\alpha_B$. By construction, these auxiliary gauge fields only contain scalar type perturbations (compare with (2.93)) and reduce thus to the diffusive gauge field studied in §2.7.2. As a result its photons are all radially polarized and travel tangentially to the black brane. We will identify these as the non-Markovian momentum diffusion modes which survive to late time and long distances. The radially infalling auxiliary photons (polarized along $\mathbf{x}$), which would have had fast Markovian dynamics, are absent. The origin of the gauge symmetry is of course the underlying diffeomorphisms. Equivalently, one can view the N_V modes as the diffusive momentum modes of the boundary energy-momentum tensor $(T^{\text{CFT}})^{\mu\nu}$, as we elaborate in Appendix C.

With this repackaging, it is actually possible to write down the equations of motion for the linearized gravitational perturbations in a compact form. The linearized Einstein's equations for the parameterization read

$$R_{AB} + d\, g_{AB} = 0 \quad \Longrightarrow \quad \nabla_A \nabla^A \Phi^\sigma = 0 \quad \text{and} \quad \nabla_A \left(r^2\, \mathcal{F}_\alpha^{AB} \right) = 0\,. \tag{2.121}$$

We recognize these to be the massless, minimally coupled Klein Gordon equation for the N_T transverse tensor modes, along with a collection of N_V diffusive gauge fields with $\mathcal{M} = d-1$ as indicated in §2.2.2, see (2.10). Both these systems have already been studied in the preceding sections. Hence all results derived heretofore directly apply.

We can further use the results of §2.7 to rewrite the dynamics of the diffusive auxiliary gauge field $\mathcal{A}$ in terms of a non-Markovian scalar $\bar{\Phi}_{\text{D}}$. This proves the assertion made in §2.2.2 and justifies (2.10).

2.8.2 The gravitational action

One can demonstrate explicitly by a straightforward computation that the Einstein-Hilbert action, together with the Gibbons-Hawking boundary terms, and additional boundary counterterms can be completely mapped to the auxiliary system, up to a time-independent DC contribution (which originates from the equilibrium free energy of the black hole). The gravitational dynamics is prescribed by[34]

$$\begin{aligned} S_{\text{grav}} &= \int d^{d+1}x\, \sqrt{-g}\, \Big(R + d(d-1) \Big) + 2 \int d^d x\, \sqrt{-\gamma}\, K + S_{\text{ct}} \\ S_{\text{ct}} &= -\int d^d x\, \sqrt{-\gamma}\, \left[2(d-1) + \frac{1}{d-2}\, {}^\gamma R \right] \end{aligned} \tag{2.122}$$

Here $\gamma_{\mu\nu}$ is the timelike induced metric on the boundary.[35]

We find upon substituting the parameterization (2.119) that the dynamics can be repackaged as

$$\begin{aligned} S_{\text{grav}} &= \sum_{\sigma=1}^{N_T} S[\Phi_\sigma] + \sum_{\alpha=1}^{N_V} S[\Psi_\alpha] + \int d^d x\, \sqrt{-\gamma}\, \left[\sqrt{-\gamma_{\mu\nu} \mathbf{b}^\mu\, \mathbf{b}^\nu}\, \right]^{-d} \\ S[\Phi_\sigma] &= -\frac{1}{2} \left[\int d^{d+1}x\, \sqrt{-g}\, \nabla_A \Phi_\sigma\, \nabla^B \Phi_\sigma + c_\Phi \int d^d x\, \sqrt{-\gamma}\, \nabla_\mu \Phi_\sigma\, \nabla^\nu \Phi_\sigma \right] \\ S[\Psi_\alpha] &= -\frac{1}{4} \left[\int d^{d+1}x\, \sqrt{-g}\, r^2\, (\mathcal{F}^\alpha)_{AB}\, (\mathcal{F}^\alpha)^{AB} + c_{\mathcal{A}} \int d^d x\, \sqrt{-\gamma}\, r^2\, (\mathcal{F}_\alpha)_{\mu\nu} (\mathcal{F}_\alpha)^{\mu\nu} \right] \end{aligned} \tag{2.123}$$

We explain how this works in Appendix C.2 for completeness.

We can understand (2.123) as follows. We recognize here the bulk action for the

[34]To keep the expressions compact we will scale out the usual normalization by $\frac{1}{16\pi G_N}$ in the gravitational action. Boundary CFT results can be obtained by multiplication by $c_{\text{eff}} = \frac{\ell_{\text{AdS}}^{d-1}}{16\pi G_N}$.

[35]We continue to employ the notation g_{AB} and $\gamma_{\mu\nu}$ for the bulk and the boundary metrics respectively. In the gravitational action these include the perturbative corrections, but they are restricted to being just the background values when we write out the auxiliary system of scalars and vectors.

auxiliary scalar and gauge system introduced in (2.120) whose dynamics is given by the equations of motion in (2.121). The counterterm coefficients are fixed by our previous analysis (with $\mathcal{M} = d-1$) and can also be checked to descend from the gravitational counterterm (the boundary Einstein-Hilbert term) and are given by

$$c_\Phi = c_\mathcal{A} = -\frac{1}{d-2}\,. \tag{2.124}$$

What remains is the final term. To write this we have introduced the rescaled thermal vector $\mathbf{b}^\mu$ which is related to the hydrodynamic thermal vector $\boldsymbol{\beta}^\mu$ [62]. For the thermal state on $\mathbb{R}^{d-1,1}$ obtained from the planar Schwarzschild-AdS$_{d+1}$ geometry it is given by $\mathbf{b}^\mu = b\,(\partial_v)^\mu$ and characterizes the dual boundary fluid configuration in local equilibrium. If we switch off the gravitational perturbations this term is the equilibrium free energy $\sim b^{-d}$ that the standard Gibbons-Hawking computation would give us from the on-shell evaluation of the Einstein-Hilbert action. However, once we turn on the perturbation we obtain additional adiabatic contributions (both hydrostatic and Class L terms in the classification of [62]). Since we are working with linearized gravitational perturbations, it is simple to re-express the result in terms of the local temperature T^d_{local} defined in the local inertial frame set by the timelike vector $\mathbf{b}^\mu$. We will have more to say about this below.

As noted above the gravitational action can be further repackaged in terms of the non-Markovian diffusive scalar, by rewriting the auxiliary gauge system parameterized by $\mathcal{A}$ in terms of $\bar{\Phi}_{\text{D}} = \varphi_{_{1-d}}$. In Appendix D we give the explicit expression for the boundary currents in terms of these scalars, see (D.7). We will use this expression to compute the Wilsonian influence phase for the energy-momentum tensor components below.

2.8.3 The Wilsonian influence phase for momentum diffusion

We now have all the pieces in place to write down the Wilsonian influence phase for the transverse tensor and vector graviton modes. We will write the expression in terms of the sources $\boldsymbol{\gamma}^\sigma_a$ and $\boldsymbol{\gamma}^\sigma_d$ that couple to the transverse tensor polarizations, and the diffusive hydrodynamic moduli, $\breve{\mathcal{P}}^\alpha_a$ and $\breve{\mathcal{P}}^\alpha_d$. The former are the transverse tensor components (i.e., the magnetic components) of the boundary metric, though we will assume a harmonic decomposition and not write out the index structure to avoid notational clutter. On the other hand $\breve{\mathcal{P}}^\alpha$ are the momentum flux vectors and capture the shear modes corresponding to momentum diffusion.

We use the result for the designer scalar system (2.89) and express the influence phase in terms of the retarded Green's function of the corresponding modes. The final expression we seek, reads at the quadratic order in amplitudes as, in field theory conventions (see footnote 34)

$$\mathcal{S}_{\text{WIF}}[\boldsymbol{\gamma}_a^\alpha, \boldsymbol{\gamma}_d^\alpha, \breve{\mathcal{P}}_a, \breve{\mathcal{P}}_d] = \mathcal{S}_{\text{ideal}} - c_{\text{eff}} \int_k \Bigg\{ \sum_{\sigma=1}^{N_T} (\boldsymbol{\gamma}_d^\sigma)^\dagger \, K^{\text{in}}_{d-1} \left[\boldsymbol{\gamma}_a^\sigma + \left(n_{_B} + \frac{1}{2}\right) \boldsymbol{\gamma}_d^\sigma\right] + k^2 \sum_{\alpha=1}^{N_V} (\breve{\mathcal{P}}_d^\alpha)^\dagger \, K^{\text{in}}_{-d+1} \left[\breve{\mathcal{P}}_a^\alpha + \left(n_{_B} + \frac{1}{2}\right) \breve{\mathcal{P}}_d^\alpha\right] \Bigg\} \tag{2.125}$$

where $\mathcal{S}_{\text{ideal}}$ is the background thermal contribution to the Wilsonian influence phase arising from the local free energy derived above. We write this more naturally in the LR basis as

$$\mathcal{S}_{\text{ideal}} = c_{\text{eff}} \int d^dx \, \sqrt{-\gamma} \, \left[\sqrt{-(\gamma_{\text{R}})_{\mu\nu} \mathbf{b}^\mu_{\text{R}} \, \mathbf{b}^\nu_{\text{R}}}\,\right]^{-d} - c_{\text{eff}} \int d^dx \, \sqrt{-\gamma} \, \left[\sqrt{-(\gamma_{\text{L}})_{\mu\nu} \mathbf{b}^\mu_{\text{L}} \, \mathbf{b}^\nu_{\text{L}}}\,\right]^{-d} \tag{2.126}$$

The retarded Green's function data entering the expression above can be obtained from the designer scalar analysis. Specializing (2.36) and (2.82) to the case $\mathcal{M} = \pm(d-1)$, one finds surprisingly compact formulae for the parameters as several of the coefficients simplify significantly. To wit,

$$K_{_{d-1}}(\omega, k) = \left(\frac{4\pi}{d\,\beta}\right)^d \left\{ -i\,\mathfrak{w} - \frac{\mathfrak{q}^2}{d-2} + \frac{1}{d} \, \text{Har}\left(\frac{2-2d}{d}\right) \mathfrak{w}^2 + i\,\mathfrak{h}_3(d)\,\mathfrak{w}^3 - \frac{2i}{d(d-2)} \, \text{Har}\left(\frac{2-d}{d}\right) \mathfrak{w}\mathfrak{q}^2 + \cdots \right\} . \tag{2.127}$$

and

$$K_{_{1-d}}(\omega, k) = \left(\frac{4\pi}{d\,\beta}\right)^{2-d} \left\{ -i\,\mathfrak{w} + \frac{\mathfrak{q}^2}{d} - \frac{1}{d} \, \text{Har}\left(\frac{2-2d}{d}\right) \mathfrak{w}^2 - \frac{2i}{d(d-2)} \mathfrak{w}\mathfrak{q}^2 + i\left[\frac{1}{d^2}\left[\text{Har}\left(\frac{2-2d}{d}\right)\right]^2 - \mathfrak{h}_3(d)\right] \mathfrak{w}^3 + \cdots \right\} . \tag{2.128}$$

The parameters $\mathfrak{h}_3$ introduced above is given by the infinite sum:

$$\mathfrak{h}_3(d) = -\frac{2(d-1)}{d^3} \sum_{n=0}^{\infty} \frac{\text{Har}\left(n-1+\frac{2}{d}\right)}{(n-1+\frac{2}{d})} \, \frac{1}{n+1} \, . \tag{2.129}$$

It should be possible to resum this expression in terms of polylogs (see example, [126] for results in $d = 3$), but we will settle for quoting the numeric values in special cases.

2.8.4 Comparison with fluid/gravity

The Wilsonian influence phase for the stress tensor components can be compared directly with the predictions of hydrodynamics. We will focus first on the dispersion relations which have been discussed extensively in the literature, and then turn to the Green's functions of the energy-momentum tensor components.

Shear dispersion: From (2.128) we see that the dispersion relation we derive by setting $K_{1-d}(\omega, k) = 0$ gives

$$\begin{aligned} 0 = -i\omega + \frac{1}{4\pi T}\, k^2 - \frac{1}{4\pi T}\, \text{Har}\left(\frac{2-2d}{d}\right) \omega^2 - \frac{2i}{(d-2)}\, \frac{d}{(4\pi T)^2}\, \omega\, k^2 \\ + i\, \frac{d^2}{(4\pi T)^2} \left[\frac{1}{d^2}\left[\text{Har}\left(\frac{2-2d}{d}\right)\right]^2 - \mathfrak{h}_3(d)\right] \omega^3 + \frac{d^3}{(4\pi T)^3}\, \mathfrak{h}_{0,4}(d)\, k^4 + \cdots \end{aligned} \tag{2.130}$$

Apart from the terms computed before, we have included in the above a quartic contribution proportional to k^4 with a coefficient $\mathfrak{h}_{0,4}(d)$ which is necessary for obtaining $\omega(k)$ accurate to quartic order (see footnote 28). This expression recovers the familiar expression for the shear diffusion constant:

$$\mathcal{D} = \frac{1}{4\pi T} \quad\Longrightarrow\quad \frac{\eta}{s} = \frac{1}{4\pi} \tag{2.131}$$

Per se, this is not a surprise, since the computation one is doing to derive $K^{\text{in}}_{\mathcal{M}}$ is the standard quasinormal mode analysis that was first carried out for gravitons in [19]. Specializing to $\mathcal{N} = 4$ SYM we can write the dispersion relation as

$$\begin{aligned} 0 = & -i\,\omega + \frac{1}{4\pi T}\, k^2 - \frac{1-\ln 2}{2\pi T}\, \omega^2 \\ & - i\, \frac{1}{4(\pi T)^2}\, \omega\, \left[k^2 - \Big((1-\ln 2)^2 - 4\, \mathfrak{h}_3(4)\Big)\, \omega^2\right] + \frac{d^3}{(4\pi T)^3}\, \mathfrak{h}_{0,4}(4)\, k^4 \end{aligned} \tag{2.132}$$

with $\mathfrak{h}_3(4) = -0.432$ when evaluated numerically. Solving the shear dispersion relation to quartic order one finds for $\omega(k)$ the expression

$$\omega(k) = -i\frac{1}{4\pi T}\, k^2 - i\, \frac{1}{(4\pi T)^3}\left[\text{Har}\left(\frac{2}{d} - 2\right) - \frac{2d}{d-2} + d^3\, \mathfrak{h}_{0,4}(d)\right] k^4 + \cdots \tag{2.133}$$

Dispersion relations in this form were the first signal of hydrodynamic behaviour from AdS/CFT [19] who used the quadratic piece in the dispersion to obtain the shear viscosity. The quartic term was also computed in [19] for $\mathcal{N} = 4$ SYM, while [126] obtained the analogous expression for $d = 3$ (ABJM plasma). The dispersions accurate to quartic order obtained in these references are

$$
\begin{aligned}
d = 4: &\quad \omega(k) = -i\frac{1}{4\pi T}\,k^2 - i\frac{1-\ln 2}{32\pi^3\,T^3}\,k^4\,,\\
d = 3: &\quad \omega(k) = -i\frac{1}{4\pi T}\,k^2 - i\frac{9+\sqrt{3}\pi - 9\,\ln 3}{384\pi^3\,T^3}\,k^4\,.
\end{aligned}
\tag{2.134}
$$

Stress tensor correlators: Let us use the Wilsonian influence phase to write out the stress tensor correlation functions for the polarizations we have studied in this chapter. We will focus on the correlation functions for $\mathcal{N}=4$ SYM, and to keep expressions simple pick the momentum to point along the z-direction, $\mathbf{k} = k\,\hat{e}_z$. The overall normalization for $SU(N)$ gauge group is simply $c_{\text{eff}} = \frac{N^2}{8\pi^2}$.

Consider first the tensor polarization of gravitons, which map to minimally coupled massless scalars in the Schwarzschild-AdS$_5$ geometry and is the Markovian sector of the stress tensor. The retarded Green's function for the corresponding component of the boundary stress tensor T_{xy} can be read off directly from (2.127). Specializing to $\mathcal{N}=4$ SYM one finds the retarded correlator

$$
\begin{aligned}
&\left\langle T^{\text{CFT}}_{xy}(-\omega,-\mathbf{k})\,T^{\text{CFT}}_{xy}(\omega,\mathbf{k})\right\rangle^{\text{Ret}}\\
&\quad = i\,c_{\text{eff}}\,(\pi T)^4\left[1 - i\,\mathfrak{w} - \frac{\mathfrak{q}^2}{2} + \frac{1-\ln 2}{2}\,\mathfrak{w}^2 + i\,\frac{\ln 2}{2}\mathfrak{w}\,\mathfrak{q}^2 + i\mathfrak{h}_3(4)\,\mathfrak{w}^3 + \cdots\right]
\end{aligned}
\tag{2.135}
$$

This expression includes the pressure term, which is the spatio-temporally constant, background term, which is required by Kubo formulae analysis [109]. The expression above should be compared with Eq. (4.8) of [109] – we see perfect agreement at quadratic order.[36] We do also include here the cubic order corrections which is a new result.

The Keldysh correlator corresponding to the fluctuations using (2.71). Here we find

$$
\left\langle T^{\text{CFT}}_{xy}(-\omega,-\mathbf{k})\,T^{\text{CFT}}_{xy}(\omega,\mathbf{k})\right\rangle^{\text{Kel}} = \frac{c_{\text{eff}}}{\pi}\,(\pi T)^4\left(1+\frac{\pi^2}{12}\,\mathfrak{w}^2\right)\left(1-\frac{\ln 2}{2}\,\mathfrak{q}^2 - \mathfrak{h}_3(4)\,\mathfrak{w}^2+\cdots\right).
\tag{2.136}
$$

This shows that the correlators obey the KMS relation to the quadratic order in gradients,

$$
\left\langle T^{\text{CFT}}_{xy}(-\omega,-\mathbf{k})\,T^{\text{CFT}}_{xy}(\omega,\mathbf{k})\right\rangle^{\text{Kel}} = \frac{1}{2}\,\coth\left(\frac{\beta\omega}{2}\right)\,\text{Re}\left[\left\langle T^{\text{CFT}}_{xy}(-\omega,-\mathbf{k})\,T^{\text{CFT}}_{xy}(\omega,\mathbf{k})\right\rangle^{\text{Ret}}\right].
\tag{2.137}
$$

The first principles derivation of this expression has not appeared in the literature hitherto, though given the retarded Green's function and the KMS condition one could have easily written it down. Note that we computed the third order gradient terms in the infalling Green's function mainly to get the first non-trivial terms which are not fixed by

[36] The dimensionless frequencies and momenta used in [109] differs from the one we use by a factor of two.

the Bose-Einstein statistics. The above expressions may be written more directly in the average-difference basis from the effective action.

For the transverse vector polarization of gravitons with momentum along $\mathbf{k} = k\,\hat{e}_z$ we will focus on the momentum density T^{CFT}_{vx} and momentum current T^{CFT}_{zx}. These currents are in turn expressed in terms of the non-Markovian scalar $\varphi_{_{1-d}}$ in d-dimensions. To obtain the shear sector momentum flux correlators one needs to invert the non-Markovian inverse Green's functions $K_{_{1-d}}(\omega, k)$ which is straightforward. Accounting for the translation we have the following result for the non-contact or shear part of the stress tensor correlators:

$$\begin{aligned}
\langle T^{\text{CFT}}_{vx}(-\omega,-\mathbf{k})\, T^{\text{CFT}}_{vx}(\omega,\mathbf{k})\rangle_{\text{shear}} &= c_{\text{eff}}^2\, k^4 \left\langle \varphi_{_{-3}}(-\omega,-\mathbf{k})\ \varphi_{_{-3}}(\omega,\mathbf{k})\right\rangle ,\\
\langle T^{\text{CFT}}_{vx}(-\omega,-\mathbf{k})\, T^{\text{CFT}}_{zx}(\omega,\mathbf{k})\rangle_{\text{shear}} &= -c_{\text{eff}}^2\, k^3\, \omega \left\langle \varphi_{_{-3}}(-\omega,-\mathbf{k})\ \varphi_{_{-3}}(\omega,\mathbf{k})\right\rangle ,\\
\langle T^{\text{CFT}}_{zx}(-\omega,-\mathbf{k})\, T^{\text{CFT}}_{zx}(\omega,\mathbf{k})\rangle_{\text{shear}} &= c_{\text{eff}}^2\, k^2\, \omega^2 \left\langle \varphi_{_{-3}}(-\omega,-\mathbf{k})\ \varphi_{_{-3}}(\omega,\mathbf{k})\right\rangle .
\end{aligned} \tag{2.138}$$

In addition we also have a background contact, or ideal contribution which can be obtained directly from $\mathcal{S}_{\text{ideal}}$, though this only contributes to the retarded Green's function. One has

$$\begin{aligned}
\langle T^{\text{CFT}}_{vx}(-\omega,-\mathbf{k})\, T^{\text{CFT}}_{vx}(\omega,\mathbf{k})\rangle^{\text{Ret}}_{\text{ideal}} &= 3\, c_{\text{eff}}\, (\pi T)^4\\
\langle T^{\text{CFT}}_{vx}(-\omega,-\mathbf{k})\, T^{\text{CFT}}_{zx}(\omega,\mathbf{k})\rangle^{\text{Ret}}_{\text{ideal}} &= 0\\
\langle T^{\text{CFT}}_{zx}(-\omega,-\mathbf{k})\, T^{\text{CFT}}_{zx}(\omega,\mathbf{k})\rangle^{\text{Ret}}_{\text{ideal}} &= c_{\text{eff}}\, (\pi T)^4
\end{aligned} \tag{2.139}$$

The shear part of retarded Green's function can be obtained by inverting the dispersion function, modulo a factor of k^2. We thus find:

$$\begin{aligned}
\left\langle \varphi_{_{-3}}(-\omega,-\mathbf{k})\varphi_{_{-3}}(\omega,\mathbf{k})\right\rangle_{\text{Ret}} &= -\frac{i}{c_{\text{eff}}}\,\frac{1}{\mathfrak{q}^2}\\
&\times \left(-i\mathfrak{w} + \frac{1}{4}\,\mathfrak{q}^2 - \frac{1-\ln 2}{2}\,\mathfrak{w}^2 - i\frac{\mathfrak{w}}{4}\left[\mathfrak{q}^2 - \Big((1-\ln 2)^2 - 4\,\mathfrak{h}_3(4)\Big)\,\mathfrak{w}^2\right]\right)^{-1}
\end{aligned} \tag{2.140}$$

These expression should be compared with the shear sector correlators derived in [110] extending the early work of [20]. In that work they choose to expand the correlator about the shear pole, which leads to a non-local expression. However, as we have argued the presence of hydrodynamic moduli can be handled more effectively by invoking a suitable Legendre transform to write the inverse Green's function given in (2.128).

The Keldysh correlator for the shear modes can similarly be obtained from (2.84).

Specializing again to $\mathcal{M} = d-1$ and $d=4$ we find the following

$$\begin{aligned}
&\Big\langle \varphi_{_{-3}}(-\omega,-\mathbf{k})\varphi_{_{-3}}(\omega,\mathbf{k})\Big\rangle_{\text{Kel}} \\
&= \frac{1}{\pi c_{\text{eff}}}\,\frac{1}{\mathfrak{q}^2}\left(1+\frac{\pi^2}{12}\mathfrak{w}^2\right)\left(1+\frac{1}{4}\,\mathfrak{q}^2+\left(\mathfrak{h}_3(4)-\frac{(1-\ln 2)^2}{4}\right)\mathfrak{w}^2+\cdots\right) \\
&\qquad\times\left|-i\mathfrak{w}+\frac{1}{4}\,\mathfrak{q}^2-\frac{1-\ln 2}{2}\,\mathfrak{w}^2-i\frac{\mathfrak{w}}{4}\left[\mathfrak{q}^2-\Big((1-\ln 2)^2-4\,\mathfrak{h}_3(4)\Big)\,\mathfrak{w}^2\right]\right|^{-2}
\end{aligned} \tag{2.141}$$

Once again we see the KMS relations satisfied to the order we have computed, viz.,

$$\Big\langle \varphi_{_{-3}}(-\omega,-\mathbf{k})\varphi_{_{-3}}(\omega,\mathbf{k})\Big\rangle_{\text{Kel}} = \frac{1}{2}\coth\left(\frac{\beta\omega}{2}\right)\text{Re}\left[\Big\langle \varphi_{_{-3}}(-\omega,-\mathbf{k})\varphi_{_{-3}}(\omega,\mathbf{k})\Big\rangle_{\text{Ret}}\right] \tag{2.142}$$

Hydrodynamic effective actions: As a final application to the fluid/gravity correspondence, we consider the non-dissipative contributions to the stress-tensor correlations. The authors of [62] have proposed a conjecture for the hydrodynamic effective action for this sector which they call the Class L Lagrangian. This conjecture relied on the structure of non-dissipative contributions to hydrodynamic stress tensor and the transport data for holographic fluids. In particular, it had been argued that the non-dissipative contribution should be captured by the following boundary Lagrangian density (cf., Eq. (14.37) of [62]):

$$\mathcal{L}^{\mathcal{W}} = c_{\text{eff}}\left(\frac{4\pi T}{d}\right)^d - c_{\text{eff}}\left(\frac{4\pi T}{d}\right)^{d-2}\left[\frac{{}^{\mathcal{W}}R}{d-2}+\frac{1}{2}\omega_{\text{vor}}^2+\frac{1}{d}\,\text{Har}\left(\frac{2}{d}-1\right)\,\sigma_{\text{sh}}^2\right] \tag{2.143}$$

This action is written in Weyl covariant hydrodynamic variables ${}^{\mathcal{W}}R$ is the Weyl covariant curvature scalar on the boundary, ω_{vor} is the fluid vorticity, and σ_{sh} is the shear tensor of the fluid. This action was written down by reverse engineering known fluid transport and was conjectured to hold non-linearly in amplitudes, but in a boundary gradient expansion.

It is simple to check that our results reproduce this structure at quadratic order in amplitudes and gradients. All we need is the defining property of harmonic number function, viz., $\text{Har}\,(x+1) = \text{Har}\,(x) + \frac{1}{x+1}$, to rewrite

$$\Delta(d-1,1) = -\frac{1}{d-2} - \frac{1}{d}\,\text{Har}\left(\frac{2}{d}-1\right). \tag{2.144}$$

This basically says that the contribution from the non-dissipative terms at $\mathcal{O}(\omega^2)$ arises from a combination of the Weyl curvature and shear squared terms in the Class L action. At a numerological level this is not a strong check and one might argue that it was guaranteed by the fact that we get the correct Green's functions.

However, the reason for our optimism here is that unlike in fluid/gravity, we have put the gravitons in the bulk on-shell, evaluated the on-shell action and recovered directly the Class L action. In fact, we get a clear prediction thanks to the grSK geometry. The non-dissipative part of the hydrodynamic effective action obtained from Einstein-Hilbert dynamics directly gives us two copies of Class L, for the R/L degrees of freedom, consistent with the discussion in [62]. This is a promising start and hints at where in the gravitational dynamics one can find the appropriate data to prove this conjecture. The generalization to the dissipative sector also works as expected and is consistent with field theory analyses. However, as noted in footnote 7 the structure at quadratic order in amplitudes is not a strong test, but provides a useful sanity check that one is on the right track.

Chapter 3

Effective theory of energy transport

This chapter is based on [43] written by the author in collaboration with Temple He, R. Loganayagam, Mukund Rangamani and Julio Virrueta.

3.1 The timbre of Hawking gravitons

In the previous chapter, the focus was on constructing an open effective field theory of diffusive modes focusing on momentum diffusion in a neutral holographic plasma.[1] This occurs through shear modes, which carry transverse momentum and shear the fluid elements (hence their name) without compressing the plasma as they propagate. Hence, the shear hydrodynamic mode is present in compressible as well as incompressible fluids alike. The shearing causes transverse viscous drag, resulting in the diffusion of the transverse momentum. These shear modes are long-lived and diffuse slowly through the medium.

On the other hand, compressible fluids have an additional degree of freedom: the sound mode. Sound travels by carrying a longitudinal momentum, applying pressure on the fluid elements, which in turn results in a wave of compressions and rarefactions. As the fluid gets compressed, there is a local change in pressure and energy density, unlike in the case of shear modes (where the local pressure and energy density remain unperturbed). We remind the reader that the relativistic fluids are always compressible since incompressibility requires an instantaneous transmission of pressure which is forbidden within special relativity. Thus, relativistic fluids always have sound modes. The physics of shear and sound are qualitatively different. While shear modes are diffusive and obey parabolic PDEs, sound modes are oscillatory and obey hyperbolic PDEs. The difference owes to the fact that a fluid at rest already has an energy density and pressure. A perturbation over this background results in the sound mode.[2] In contrast, a fluid at rest has no momentum density, so the transverse momentum diffuses, resulting in the shear

[1] See [134] for the discussion of momentum and charge diffusion in a charged holographic plasma.

[2] We will only discuss perturbations which do not change the flat spacetime energy density. A static homogeneous change of temperature or energy density is IR divergent and not included in our analysis.

mode.

We have described until now the sound and the shear mode as being governed by second order PDEs. In a realistic system, higher derivative corrections appear and the effects of the thermal fluctuations need to be considered, resulting in a higher derivative stochastic PDE. In the case of the sound mode, higher derivative corrections describe the sound relaxation whereas fluctuations describe the noise background. More precisely, the sound and the shear modes should be thought in terms of an *open effective theory* incorporating both fluctuation and dissipation. As discussed previously, this is more easily said than done: one first needs to systematically separate out the fast modes from the slow modes and integrate out the former. In weakly coupled theories, we need to further deal with the fact that sound relaxes over non-perturbatively long time scales. A salient result of chapter 2 is that, these issues can be sidestepped if we consider a strongly coupled CFT plasma and study it using holography.

In order to motivate our presentation of the dynamics of sound modes in a strongly coupled plasma using gravitational dynamics, it is useful to recap some important features of the holographic set-up highlighted in chapter 2. Firstly, it makes clear that the dynamics of short-lived and long-lived modes, dubbed Markovian and non-Markovian, respectively, are qualitatively different. At the linearized level, each such mode is described by a scalar field propagating in the AdS black brane background. These fields were constructed as designer scalars non-minimally coupled to the gravitational background, with the coupling modelled as a (auxiliary) background dilaton. Heuristically, the short-lived modes are repelled from the boundary of the spacetime, while the long-lived modes are floppy and have a large wavefunction support near the boundary, which prevents them from decaying away rapidly. Technically, the Markovian nature is captured by the asymptotic behaviour of the dilaton and can be encoded in a single number, the *Markovianity index*, $\mathcal{M}$.

The Markovian nature of a bulk field is characterized by the boundary conditions imposed on it in order to compute correlation functions via the GKPW dictionary.

- Markovian fields are quantized with Dirichlet boundary conditions and have index $\mathcal{M} > -1$.
- Non-Markovian fields are quantized with Neumann boundary conditions and have index $\mathcal{M} < 1$.

Note that this definition mildly updates the definition used in chapter 2. The main distinction is that fields in the window $\mathcal{M} \in (-1, 1)$ can either be Markovian or non-Markovian, depending on the boundary conditions imposed. This window is similar to the usual discussion of relevant operators close to the unitarity bound in AdS/CFT [128], when near-boundary fall-offs are slow enough that one can switch to general multi-trace boundary conditions [129].

Secondly, the linearized metric perturbations which contain both short and long-lived modes can be decoupled by working with diffeomorphism/gauge invariant degrees of freedom. This provides a clean separation of fast and slow modes, allowing one to integrate out the former. It also makes clear what the natural gauge choices are for analyzing perturbations on the grSK geometry.[3] The gauge invariant variables chosen in [42, 134] leads to a smooth solution on the grSK geometry with appropriate boundary conditions on the two asymptotic boundaries (corresponding to the bra and ket pieces of the Schwinger-Keldysh evolution).

Let us now recap the results of chapter 2 pertaining to the specific case of interest, the dynamics of the plasma stress tensor. Given a direction of propagation we could identify polarizations labelled by the little group in the transverse spatial geometry. The stress tensor operator has traceless spin-2 polarizations which are short-lived, and hence were characterised as Markovian with index $\mathcal{M} = d - 1$. The transverse vector spin-1 polarizations were determined to be non-Markovian with index $\mathcal{M} = 1 - d$. They represented the shear modes of the plasma. This left out the single longitudinal mode which is the focus of this chapter – it describes the sound mode resulting from energy transport. These statements translate directly in the dual gravity picture, for the selfsame decomposition can be applied to the linearized gravitons [106, 123].

In chapter 2 we studied the effective dynamics from the dual gravitational perspective for the spin-2 transverse traceless tensor and spin-1 transverse vector modes (see [134] for discussion of the same in a charged plasma). The non-Markovian shear mode was captured by a designer scalar with index $1 - d$. This scalar, which is weakly coupled near the AdS boundary due to the dilatonic modulation, was quantized with Neumann boundary conditions for purposes of computing the generating function of dual stress tensor correlators. This was not an ad-hoc choice, but rather one enforced by the underlying Einstein-Hilbert dynamics which, when distilled through the field redefinitions necessary to arrive at the decoupled designer scalar dynamics, ensured that the variational principle worked out with suitable boundary terms for this choice.

Since the shear mode is long-lived, constructing a local effective field theory requires that we treat it in a Wilsonian fashion. Rather than compute the generating function of Schwinger-Keldysh correlators we therefore chose to compute a Wilsonian influence functional (WIF) parameterized by the long-lived boundary modulus field – the momentum flux operator for shear modes. One could as well obtain this WIF by Legendre transforming the generating function of correlators. Happily, the Legendre transformation was simple and obtained by quantizing the designer non-Markovian field with (renormalized) Dirichlet boundary conditions.

[3]The use of standard gauge fixing in the AdS black hole background leads to solutions which have spurious singularities at the horizon, necessitating artificial boundary conditions in the interior of the spacetime, in tension with the rules of the AdS/CFT correspondence.This is an issue for instance in the solutions discussed in [88] and [108].

Based on the description of sound propagation in relativistic plasma, one expects that the longitudinal modes of the gravitons are similarly captured by an effective designer non-Markovian scalar, which is dual to the boundary energy flux operator. While this observation is morally correct, the technical details are significantly more involved. For one, the field dual to the energy flux operator by itself (which we denote as Θ below) does not have a nice dual geometric description. Rather, a linear combination of this field and the conformal mode of gravitational perturbations has simple autonomous non-Markovian dynamics. Even more curiously, this designer field, denoted $\mathcal{Z}$, has its coupling to the background geometry modulated non-trivially as a function of momentum.

While the technical reason for these statements can be traced through the derivation, we do not understand the physical reason for why this should happen. On the contrary, the field Θ, while exhibiting no such pathologies, does not have simple autonomous dynamics. All of this is in stark contrast with the physics of diffusive modes; even for charged plasmas where the shear mode mixes with the Markovian charge current mode, the dynamical system allows for clear decoupling and relatively simple dynamics for the resulting non-Markovian degrees of freedom [134].

We will primarily focus on the spatially inhomogeneous modes of the plasma. Physically, we imagine cutting-off spatial momenta as $k \geq k_{\text{IR}}$ and examining the dynamics of propagating sound modes above this cut-off. This is sensible for the plasma on $\mathbb{R}^{d-1,1}$ to mitigate IR effects.[4] This perspective will be important for us, since the dynamical system we analyze exhibits a somewhat discontinuous behaviour as a function of momentum. Spatially homogeneous modes (zero spatial momentum) have a qualitatively different behaviour. For one, their dynamics appears to be Markovian, and further there are zero modes that do not merge into $\mathcal{Z}$. One such is the mode which corresponds to a homogeneous static heating of the plasma, which as explained in footnote 2 is unphysical (it changes the background solution). Since these modes do not directly affect the dynamics of sound, we will mostly not discuss them in the main text. Nevertheless, for the sake of completeness, we include an analysis of the homogeneous solution space. We demonstrate that it can be understood as the space of large diffeomorphisms, and moreover demarcate the part which is captured by our designer field in Appendix K (see [137] for an earlier analysis).

Part of the complication in the sound mode sector has to do with the fact that there are many degrees of freedom in the dual gravitational description. One has seven metric functions which can be combined into diffeomorphism invariant combinations. Four of these can be eliminated a-priori, three by gauge fixing and one by using an algebraic constraint arising from the dynamical equations of motion (latter for $k \neq 0$).

[4]The Goldstone mode for sound in Minkowski spacetime has soft modes which may be tamed by considering the plasma on a large compact spatial sphere, i.e., using global AdS to provide a regulator. While this is an interesting problem to study, working with a momentum cut-off will suffice to extract the physics of fluctuating phonons.

The remaining three fields satisfy three linearly independent equations of motion, of which two can be solved by introducing the field Θ and enabling elimination of two functions. The final step is to show that this field combines with the conformal mode to produce the designer field $\mathcal{Z}$ for spatially inhomogeneous modes. Somewhat amazingly, the off-shell Einstein-Hilbert action simplifies considerably when we parameterize the linearized gravitons using $\mathcal{Z}$.

The complexities are all pushed to pure boundary terms (a consequence of the large degree of redundancy in the classical system). These also simplify significantly when we consider asymptotically locally AdS boundary conditions, allowing one to show that for purposes of computing correlation functions, $\mathcal{Z}$ should be quantized with Neumann boundary conditions. Earlier analyses of scalar sector quasinormal modes in AdS black hole geometries have not carefully analyzed the variational principle, leading to some inaccurate statements in the literature.

The reduction to the single scalar field $\mathcal{Z}$ was first ascertained by [106] at the level of equations of motion. Using their results and examining the asymptotics in global Schwarzschild-AdS$_4$, [124] argued that one should impose a Robin boundary condition for the global analog of our field $\mathcal{Z}$. During their study of planar black hole quasinormal modes, [98] argued that the field $\mathcal{Z}$ should have Robin boundary conditions in $d = 3, 4$ but should have Dirichlet boundary conditions in $d \geq 5$.[5] Since we follow the field redefinitions and examine the variational principle, we have a clear prescription to obtain an unambiguous answer: $\mathcal{Z}$ should be quantized with Neumann boundary conditions to obtain boundary stress tensor correlation functions.[6] One way to see the issue is to note that the field $\mathcal{Z}$, and not its conjugate momentum, gets renormalized by higher order counterterms. The renormalization of $\mathcal{Z}$ starts with the counterterm at quartic order in boundary gradients (the leading cosmological constant and boundary Einstein-Hilbert counterterms do not renormalize $\mathcal{Z}$). Usual AdS/CFT dictionary with Dirichlet boundary conditions renormalizes the field momentum and not the field, which is held frozen as the source.

Once we have isolated the dynamics in terms of a single designer scalar degree of freedom, the rest of the analysis follows along the lines described in chapter 2. We first obtain the ingoing boundary to bulk (inverse) Green's function by solving the dynamical equation of motion, order by order in boundary gradient expansion. We discover in this process another surprise: the solution for $\mathcal{Z}$ can be written in terms of the solution to the minimally coupled scalar wave equation (equivalently the tensor mode solution) with

[5] A different combination of the linearized metric components with decoupled dynamics was constructed in [123] in radial gauge. This combination should be quantized with Dirichlet boundary conditions as noted by the authors. We have not made direct contact with their variables, but note that the relation to the field $\mathcal{Z}$ can be recovered from Eq. (4.7) of [98].

[6] This is true even in low dimensions $d = 3, 4$ where the index for $\mathcal{Z}$ lies in the window $\mathcal{M}_{\mathcal{Z}} \in (-1, 1)$ mentioned earlier, allowing for both sets of boundary conditions. So while general boundary conditions are technically allowed, they don't compute stress tensor correlation functions.

some simple replacement rules up to cubic order in gradients. At quartic order there are new functions which reflect the change in the nature of the dynamics.

Armed with the boundary to bulk (inverse) ingoing Green's function we can construct the full boundary to bulk solution on the grSK geometry, parameterized by the expectation value of the corresponding boundary plasma operator. This information then suffices to obtain the WIF at the Gaussian order by evaluating the on-shell action (which gives the saddle point semiclassical answer). This expression can be equivalently written in terms of the energy flux operator by a suitable field redefinition.

This is the primary result of this chapter: we reproduce in the process the expected correlation function of the energy-momentum tensor isolating the locus of the sound pole and obtaining thus the sound dispersion relation of the holographic plasma. Our results are consistent with the earlier computations of [109] and [126], who obtained the non-linear sound dispersion for $\mathcal{N} = 4$ SYM ($d = 4$) and ABJM plasma ($d = 3$), respectively, and generalize them to arbitrary dimensions.

3.1.1 Outline of the chapter

The outline of the chapter is as follows: In §3.2 we argue that the dynamics of Einstein's equations can be distilled into one designer field. We primarily present the final result of the analysis and give a bird's-eye view of the arguments leading up to capturing the dynamics into a single field. We describe in §3.3 the solutions of the designer field and its on-shell action on the grSK geometry. We then use this data in §3.4 to write down the open effective action for the energy flux operator, completing thereby the task initiated in chapter 2 of understanding fluctuating hydrodynamics in a neutral plasma.

To avoid cluttering the text with technical details, we have tried to summarize the essential points as much as possible. Readers interested in understanding the steps leading to our statements in detail are invited to consult the appendices. Appendix G gives a detailed argument of how to assemble the gauge invariant data and use the dynamics implied by Einstein's equations to deduce the field $\mathcal{Z}$ for spatially inhomogeneous modes. While the final result is not original, with a close relative of $\mathcal{Z}$ having already been motivated in [106], we have tried to make more transparent the origins of the designer field. On the other hand, Appendix H contains a careful examination of the Einstein-Hilbert dynamics at the level of the off-shell action, which has hitherto not appeared in the literature. For completeness, we give two presentations, one in terms of metric functions after suitable gauge fixing to make connection with the dynamical equations, and another in terms of the designer field $\mathcal{Z}$. The latter is important for our analysis, since we wish to establish the boundary conditions for $\mathcal{Z}$ – we prove that it satisfies Neumann boundary conditions asymptotically for computing the generating function of correlations. In Appendix I we give the expressions for the boundary observables which we use in §3.4. Finally, Appendix J compiles details of the solution we obtain in gradient expansion. In particular, we give the near-boundary asymptotics of the functions which enter the computation of the boundary observables.

For completeness, we also include Appendix K where we characterize the dynamics of the spatially homogeneous modes and relate them to the large diffeomorphisms of the background geometry. We demonstrate that there are additional zero modes in the problem, which do not smoothly connect to the solution space parameterized by $\mathcal{Z}$.

3.2 Dynamics and the designer sound field

As discussed in §3.1, to understand energy transport and the sound modes in the plasma, it suffices to focus on metric perturbations involving scalar plane waves and their derivatives. These scalar polarized gravitons will be the only set of modes we analyze in this chapter. For this decomposition we pick a direction for the spatial momentum $\mathbf{k}$ and define harmonics on $\mathbb{R}^{d-1,1}$ to be the $SO(d-2)$ harmonics of the corresponding little group.

Concretely, we consider metric perturbations of the form

$$\begin{aligned} ds^2_{(1)} &= \left(h_{AB}\, dx^A dx^B\right)^{\text{Scal}} \\ &= \int_k \Bigg\{ \left(2\,\Psi_{\text{S}}\, ds^2_{(0)} + \Psi_{vv}\, dv^2 + 2\,\Psi_{vr}\, dvdr + \Psi_{rr}\, dr^2\right)\mathbb{S} \\ &\qquad - \left[2\,r\,(\Psi_{vx}\, dv + \Psi_{rx}\, dr)\,\mathbb{S}_i\, dx^i - 2\,r^2\,\Psi_{\text{T}}\,\mathbb{S}^{\text{T}}_{ij}\, dx^i dx^j\right]\Bigg\}. \end{aligned} \tag{3.1}$$

Here $\mathbb{S} = e^{i\mathbf{k}\cdot\mathbf{x}-i\omega v}$ is the scalar plane wave on $\mathbb{R}^{d-1,1}$ and $\mathbb{S}_i$ and $\mathbb{S}^{\text{T}}_{ij}$ are derived harmonics, defined as

$$\mathbb{S}_i = \frac{1}{k}\,\partial_i\,\mathbb{S}\,, \qquad \mathbb{S}^{\text{T}}_{ij} = \frac{1}{k^2}\left(\partial_i\,\partial_j - \frac{\delta_{ij}}{d-1}\,\partial^2\right)\mathbb{S}\,. \tag{3.2}$$

$\mathbb{S}^{\text{T}}_{ij}$ is traceless but not transverse, $\partial_i \mathbb{S}^{\text{T}}_{ij} = \frac{d-2}{d-1}\,\partial^2\,\partial_j$. It is thus distinguished from the derived harmonic $\mathbb{S}_{ij}$ used in chapter 2 (see Appendix F), which is neither transverse nor traceless.[7]

There are seven metric components in the perturbation above. A-priori we expect that these seven functions obey seven coupled radial ODEs arising from the linearized Einstein equations. Owing to diffeomorphism invariance not all of these dynamical equations are independent. One must first identify the pure gauge modes from the physical

[7] In the conventions of Appendix F, $\mathbb{S}^{\text{T}}_{ij}$ satisfies the normalisation

$$\langle \mathbb{S}^{\text{T}}_{ij}(\omega_1,\mathbf{k}_1|v,\mathbf{x}),\, \mathbb{S}^{\text{T}}_{ij}(\omega_2,\mathbf{k}_2|v,\mathbf{x})\rangle = \frac{d-2}{d-1}\times(2\pi)^d\,\delta^d(k_1+k_2)\,. \tag{3.3}$$

perturbations and focus on their dynamics. This problem has been analyzed in detail in [106], whose discussion we can adapt for our purposes. Following their analysis we will refer to the two-dimensional $\{v, r\}$ spacetime as the *orbit space*. The physical sound mode ends up being described as a non-Markovian scalar field in this orbit space.

We now explain how to efficiently distill the dynamics into a single gauge invariant degree of freedom inspired by the previous analysis of gauge dynamics in [42, 134].[8] The key point is that there are seven metric functions while the classical phase space, parameterized by gauge invariant data, has only one physical degree of freedom in the scalar sector of gravitational perturbations. To arrive at this result we make use of the following observations which are elaborated upon in Appendix G:

- There are seven diffeomorphism invariant combinations of the metric perturbations Ψ_{AB} which can be organized as an orbit space traceless tensor, an orbit space vector, and two orbit space scalars. One first deduces that the orbit space vector (Ψ_{vx} and Ψ_{rx}) and one scalar (Ψ_{T}) can be gauge fixed to vanish, leaving four functions, which are essentially $\Psi_{\text{S}}, \Psi_{vv}, \Psi_{vr}, \Psi_{rr}$.

- Time-reversal involution is an orbit space diffeomorphism: $\Psi_{\text{S}}, \Psi_{vv}, \Psi_{vr} + \frac{1}{2} r^2 f \, \Psi_{rr}$ are time-reversal even, while $\Psi_{vv} + r^2 f \, \Psi_{vr}$ is time-reversal odd. This information is useful to constrain the structure of the equations of motion and the action.

- Among the equations of motion we find an algebraic constraint, which allows elimination of Ψ_{rr} for non-zero spatial momentum. Two other equations are the momentum constraint equation (which is the boundary energy-momentum tensor conservation) and a first order radial equation. These two equations can be used to solve for Ψ_{vv} and Ψ_{vr} in terms of a function Θ, which is related to the boundary stress tensor component $(T_{\text{CFT}})_v{}^i$.

One therefore finds that the metric can be parameterized by two functions: an overall Weyl rescaling of the background geometry and a function which encodes the physical data of the boundary stress tensor. We will label these fields as Φ_{W} and Θ, respectively, and judiciously define them with suitable factors of r to simplify the dynamical equations;

$$\Psi_{\text{S}} \equiv \frac{1}{2r^{d-2}} \Phi_{\text{W}} , \qquad \Psi_{vv} + r^2 f \, \Psi_{vr} \equiv -\frac{i\omega}{r^{d-3}} \Theta \, . \tag{3.4}$$

The metric parameterized by these fields, subject to the gauge fixing where all the metric components involving derived scalar spherical harmonics are set to zero, is said to

[8] We will exclusively describe the dynamics of modes carrying non-vanishing spatial momentum in the text, relegating the analysis of spatially homogeneous modes to Appendix K.

be in the *Debye gauge*. Our scalar perturbations are then captured by[9]

$$ds^2_{(1)} = \frac{\Phi_{\rm W}}{r^{d-2}}\, ds^2_{(0)} + 2\,\frac{f}{r^{d-5}}\,\frac{\mathrm{d}\Theta}{\mathrm{d}r}\,\frac{dr}{r^2 f}\left(\frac{dr}{r^2 f} - dv\right) + \frac{\mathbb{D}_+\Theta}{r^{d-3}} dv^2 - (d-1)\,\frac{f\,\Phi_{\rm W}}{r^{d-4}}\left(\frac{dr}{r^2 f}\right)^2 . \tag{3.5}$$

We have written the metric in the basis of the cotangent space that is adapted to the time-reversal involution of the background.

The last step involves analyzing the remaining dynamical equations of motion and discerning that they can be solved if one further introduces a field $\mathcal{Z}$ to parameterize $\Phi_{\rm W}$ and Θ as

$$\Theta = \frac{r}{\Lambda_k}\left(\mathbb{D}_+ - \frac{1}{2}\, r^2 f'\right)\mathcal{Z}\,, \qquad \Phi_{\rm W} = \frac{1}{\Lambda_k}\left(r\,\mathbb{D}_+ + \frac{k^2}{d-1}\right)\mathcal{Z}\,. \tag{3.6}$$

The function Λ_k is curious. It is a non-trivial function of spatial momentum (indicated by the subscript). It will turn out to be a designer dilaton for the field $\mathcal{Z}$ and is given by

$$\Lambda_k(r) \equiv k^2 + \frac{d-1}{2}\, r^3 f' = k^2 + \frac{d(d-1)}{2\, b^d\, r^{d-2}}\,. \tag{3.7}$$

Because of the momentum dependence, the field $\mathcal{Z}$ should be seen as residing in the orbit space and not in the entire background geometry.[10] Note that there is a linear relation between Θ and $\Phi_{\rm W}$ from (3.6)

$$\Theta = \Phi_{\rm W} - \frac{1}{(d-1)}\,\mathcal{Z}\,. \tag{3.8}$$

Our main claim is the following: linearized Einstein equations for scalar perturbations of the Schwarzschild-AdS$_{d+1}$ geometry are satisfied provided the field $\mathcal{Z}$ obeys a second order linear differential equation

$$r^{d-3}\,\Lambda_k(r)^2\,\mathbb{D}_+\left(\frac{1}{r^{d-3}\,\Lambda_k(r)^2}\,\mathbb{D}_+\mathcal{Z}\right) + \left(\omega^2 - k^2 f\left[1 - \frac{d\,(d-2)}{b^d\, r^{d-2}\,\Lambda_k(r)}\right]\right)\mathcal{Z} = 0\,. \tag{3.9}$$

This equation is the 'master field equation' obtained in [106], which we have obtained in a somewhat different parameterization.[11]

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[9] We have written the metric directly in position space as the fields $\Phi_{\rm W}$ and Θ are simply Fourier transformed with the scalar harmonic $\mathbb{S}$.

[10] The origin of Λ_k is analogous to the Ohmic function $h(r)$ which appears in the analysis of vector perturbations of the Reissner-Nordström-AdS$_{d+1}$ black hole [134]. The modulation there was due to the background charge whereas here it directly relates to the momentum carried by the perturbation.

[11] The authors of [106] prefer to write the equations in Schwarzschild coordinates and express it as a Schrödinger equation in orbit space for a field $\mathcal{S}_{\rm KI}$. To bring our equation to their form, one first transforms from our ingoing coordinates to Schwarzschild coordinates and implements a field redefinition: $\mathcal{Z} = r^{\frac{d-5}{2}}\,\Lambda_k\,\mathcal{S}_{\rm KI}$.

We have written the effective dynamics in a manifest time-reversal invariant form. This implies that for analyzing the solution on the grSK geometry (1.33) we only need to solve (3.9) in a single copy background (1.36) with ingoing boundary conditions. One can then use the time-reversal properties to construct the outgoing solution and thence the full grSK solution with boundary conditions specified on the two asymptotic boundaries at $r \to \infty \pm i0$ as described in [61]. Arriving here was a key step in the analysis: had one worked with other gauge fixing methods traditionally employed in AdS/CFT such as radial gauge, while one would have been able to solve the ingoing problem, the corresponding outgoing solution would be singular. The issue is similar to the problems encountered with gauge fields and momentum diffusion analyzed in chapter 2.

Not only does one end up with a single second order differential equation to solve in the scalar sector of gravity, but there is also a remarkable simplification of the Einstein-Hilbert action. Plugging in the parameterization, we find after a series of algebraic simplifications the dynamics of $\mathcal{Z}$ to be governed by a simple dynamical system[12]

$$
\begin{aligned}
\frac{1}{c_{\text{eff}}}\, S[\mathcal{Z}] &= -\int_k \frac{d}{8}\,\nu_s\, k^4 \int dr\, \sqrt{-g}\; e^{\chi_z} \left[\frac{1}{r^2 f}\, \mathbb{D}_+\mathcal{Z}^\dagger\, \mathbb{D}_+\mathcal{Z} + V_{\mathcal{Z}}(r)\mathcal{Z}^\dagger\, \mathcal{Z}\right] + S_{\text{bdy}}[\mathcal{Z}]\,, \\
V_{\mathcal{Z}}(r) &= -\frac{\omega^2}{r^2 f} + \frac{k^2}{r^2}\left(1 - \frac{(d-2)\, r^3\, f'}{\Lambda_k}\right).
\end{aligned}
\tag{3.10}
$$

The dilaton χ_z which modulates the gravitational interaction is[13]

$$
e^{\chi_z} \equiv \frac{1}{r^{2(d-2)}\, \Lambda_k^2}\,, \tag{3.11}
$$

while the normalization is fixed by a parameter ν_s, which in turn is

$$
\nu_s \equiv \frac{2\,(d-2)}{d\,(d-1)}\,. \tag{3.12}
$$

This is a remarkable simplification given the complexities inherent in the scalar perturbation; our field redefinitions in (3.6) imply that the metric in Debye gauge is a function of $\{\mathcal{Z}, \mathbb{D}_+\mathcal{Z}, \mathbb{D}_+^2\mathcal{Z}\}$, so the truncation of two derivative dynamics is indeed a welcome surprise. As one might expect, much of the complication is hidden in the boundary term $S_{\text{bdy}}[\mathcal{Z}]$ in (3.10). It captures all the contributions from the Gibbons-Hawking term, additional boundary terms encountered while writing the action in terms of $\mathcal{Z}$, and counterterms. In Appendix H, we explain how to obtain the action and the variational principle for the field $\mathcal{Z}$ starting from the Einstein-Hilbert dynamics.

This action (3.10) is of the general form of a non-Markovian designer scalar introduced

---

[12] The effective central charge is defined as $c_{\text{eff}} = \frac{\ell_{\text{AdS}}^{d-1}}{16\pi G_N}$.

[13] χ_z was denoted as χ_s in [43], which we have re-expressed in order to avoid confusion with the dilaton associated with designer scalar probes of chapter 2.

in chapter 2, albeit with some additional novelties. Firstly, the dilaton χ_z modulates the gravitational interaction non-trivially as a function of spatial momentum $\mathbf{k}$. The Markovianity index,[14] which depends on the asymptotic behaviour of the dilaton, is determined to be

$$\lim_{r\to\infty} e^{\chi_z} = \frac{1}{k^4\, r^{2(d-2)}} \implies \mathcal{M}_{k\geq k_{\text{IR}}} = 3-d\,. \tag{3.13}$$

We have made explicit our spatial momentum cut-off for clarity. Thus, we see that for non-zero momentum the field $\mathcal{Z}$ is a non-Markovian designer scalar with index $\mathcal{M} = 3-d$. Note that while the field $\mathcal{Z}$ has a non-trivial potential which we will need to take into account, the potential does not, as in the analysis of [134], modify the Markovianity properties. The latter is purely governed by the radial kinetic operator.

While our distillation of the dynamics into the designer field $\mathcal{Z}$ is only valid for spatially inhomogeneous modes, for purposes of finding the solution, we can examine the behaviour of the dilaton and the wave equation (3.9) at zero spatial momentum. On this locus Markovianity index changes to $\mathcal{M} = d-1$, which suggests that the zero mode sector comprises of short-lived modes (sound of course doesn't propagate without momentum). The full dynamics of $k=0$ modes is a bit more involved, though a part of the solution space is indeed captured by $\mathcal{Z}$ with an effective Markovian dynamics as (3.9) suggests.

From a pragmatic standpoint, this switch between Markovian and non-Markovian behaviour in §3.3.2 will prove very useful when we solve (3.9) in a gradient expansion. Thanks to this observation, we will be able to write down the solution directly in terms of known solutions of the Markovian wave equation with $\mathcal{M} = d-1$ at low orders in the gradient expansion. As mentioned earlier, an analysis of zero modes can be found in Appendix K, where we describe how $\mathcal{Z}(r,\omega,\mathbf{0})$ connects onto the set of large diffeomorphisms.

3.3 Sounding out the grSK geometry

For the rest of the discussion we will focus on the dynamics of the gravitational system encoded in $\mathcal{Z}$ and recover the physics of sound propagation with attenuation. We begin with the solution of equation (3.9) with ingoing boundary conditions. Subsequently, using the time-reversal involution of the grSK geometry we will construct the full linearized solution parameterized by the expectation value of a boundary stress tensor component at the L and R boundaries.

[14] The Markovianity index was defined in chapter 2 with a minimally coupled massless scalar having index $d-1$.

3.3.1 Solving the designer equation

To better understand the nature of the designer scalar $\mathcal{Z}(r, \omega, \mathbf{k})$, we first analyze the asymptotics of the wave equation (3.9). As described above, while the derivation of the $\mathcal{Z}$ equation is valid for $\mathbf{k} \neq 0$, we can consider $k \geq 0$, since part of the homogeneous mode solutions merge into $\mathcal{Z}$.

Let us first focus on zero frequency solutions, but examine both the zero momentum and non-zero momentum behaviour separately, owing to (3.13). At zero spatial momentum, we have the asymptotic behaviour determined by

$$\mathcal{Z}(r, 0, \mathbf{0}) = \tilde{c}_a + \tilde{c}_m \,\frac{1}{r^d}\,, \tag{3.14}$$

where the constant mode c_a is analytic, but the monodromy mode c_m typically has a logarithmic branch cut emanating from the horizon once the finite frequency corrections are included. On the other hand at non-zero spatial momentum one finds the expected non-Markovian behaviour:

$$\mathcal{Z}(r, 0, \mathbf{k}) = c_a + c_m \, r^{d-4}\,. \tag{3.15}$$

Per se, this is not unexpected given the general analysis of Markovian and non-Markovian degrees of freedom as described in chapter 2. This is an explicit realization of the observation above that naively there is a change in character as a function of momentum.

Let us pause here to comment on two special cases: $d = 3$ and $d = 4$. The situation in $d = 4$ is marginal; the monodromy mode behaves logarithmically. When we present solutions we will be careful to normalize them as $\frac{r^{d-4}}{d-4}$ so the limit $d \to 4$ can be easily taken by replacing this function by $\log r$. On the other hand, in $d = 3$, it appears from the fall-offs that the field $\mathcal{Z}$ is Markovian since the index vanishes. This is misleading since, as noted in §3.1, the field $\mathcal{Z}$ is actually non-Markovian for all $d \geq 3$. This is not manifest from just analyzing fall-offs, for we also need to take into consideration the variational principle and the boundary conditions for $\mathcal{Z}$, which we discuss in Appendix H.3. We demonstrate there that $\mathcal{Z}$ has Neumann boundary conditions imposed on it for purposes of computing the generating function of correlators and hence it is non-Markovian for all $d \geq 3$.

Returning to the dynamical problem we can solve (3.9) by disentangling the physical non-Markovian behaviour from the auxiliary Markovian one. We first rewrite the differential equation for $\mathcal{Z}$ in terms of the designer scalar wave operator:

$$\mathfrak{D}_{\mathcal{M}} = r^{-\mathcal{M}}\, \mathbb{D}_+ \left(r^{\mathcal{M}}\, \mathbb{D}_+\right) + \left(\omega^2 - k^2 f\right)\,. \tag{3.16}$$

When expanded in powers of momentum we find a remarkably simple form for (3.9):

$$\mathfrak{D}_{d-1}\,\mathcal{Z} - k^2\nu_s\, f\left(1 - \frac{\nu_s\, k^2}{(d-2)\, r^2\,(1-f)} + \mathcal{O}\big(k^4\big)\right)\left(\frac{2}{r(1-f)}\,\mathbb{D}_+ - d\right)\mathcal{Z} = 0\,, \tag{3.17}$$

where the parameter ν_s was defined earlier in (3.12).

Since the operator $\mathfrak{D}_{d-1}$ annihilates a Markovian scalar of index $d-1$, which we write as the field $\varphi_{_{d-1}}$ in the notation of [42], one can subtract out this piece from $\mathcal{Z}$ and write a general solution as

$$\mathcal{Z}(r,\omega,\mathbf{k}) = \varphi_{_{d-1}}(r,\omega,\mathbf{k}) - \nu_s\,\mathfrak{q}^2\,\widetilde{\mathcal{Z}}(r,\omega,\mathbf{k})\,. \tag{3.18}$$

We are almost done: $\widetilde{\mathcal{Z}}$ satisfies an inhomogeneous linear differential equation of the form $\mathfrak{D}_{d-1}\widetilde{\mathcal{Z}}$ = source. The source is simply determined in terms of a Markovian field $\varphi_{_{d-1}}$ in the background Schwarzschild-AdS$_{d+1}$ geometry, which has already been solved for in chapter 2. That analysis has already inverted the operator $\mathfrak{D}_{d-1}$, so one simply needs to add in a particular solution to determine the full behaviour.

Not only do we have access to the solution space, but the nature of the change in asymptotics is now transparent. At zero momentum the second term in (3.17) vanishes and simply has $\mathfrak{D}_{d-1}$ annihilating $\mathcal{Z}(r,\omega,\mathbf{0})$. At non-zero momentum however, the source provided by the Markovian solution is scaled up because of the $\frac{2}{r(1-f)}\,\mathbb{D}_+$ term. This accounts for the change in behaviour. We will parameterize the solutions in a particularly convenient manner so that the non-Markovian behaviour in fact only sets in at quartic order.[15] This will turn out to be another artifact of the field $\mathcal{Z}$; translating back to the metric function Φ_{w}, or even Θ, we will see a change at quadratic order in momenta.

We now present the ingoing solution for the field $\mathcal{Z}$ and use it to determine the full solution on the grSK geometry. Our analysis will be accurate to quartic order in the boundary gradient expansion.

[15] In fact, this behaviour is quite similar conceptually to that observed for transverse vector polarizations of photons and gravitons in [134]. There it was found that one had a Markovian mode which mixes with a non-Markovian mode and picked up additional divergent terms. In that discussion, there were two independent degrees of freedom which were decoupled to isolate the long-lived and short-lived modes. Here there is only a single mode whose character changes owing to the underlying gauge invariance.

3.3.2 Ingoing solution in gradient expansion

Working with the dimensionless variables introduced in (2.19), we obtain the ingoing inverse Green's function for the field $\mathcal{Z}$, normalized to satisfy $\lim_{\xi\to\infty} G^{\text{in}}_{\mathcal{Z}}(\xi,\omega,\mathbf{0}) = 1$:

$$
\begin{aligned}
G^{\text{in}}_{\mathcal{Z}}(\xi,\omega,\mathbf{k}) = e^{-i\,\mathfrak{w}\,F(\xi)}\Big\{ & 1 - \mathfrak{w}^2\, H_\omega(\xi) - \mathfrak{p}_s^2\, H_k(\xi) + i\,\mathfrak{w}\,\mathfrak{p}_s^2\, I_k(\xi) + i\,\mathfrak{w}^3 I_\omega(\xi) \\
& + \mathfrak{p}_s^4\, J_k(\xi) + \mathfrak{w}^2\,\mathfrak{p}_s^2\, J_{\omega k}(\xi) + \mathfrak{w}^4\, J_\omega(\xi) \\
& - \frac{\mathfrak{q}^2}{d(d-1)}\left[4\,(d-2)^2\mathfrak{w}^2\, J_k(\xi) + 2\,\frac{K_s(\omega,\mathbf{k})}{d-2}\, V_k(\xi)\right] + \cdots \Big\}.
\end{aligned}
\tag{3.19}
$$

We have introduced two new functions of frequencies and momenta in parameterizing (3.19). The first, $K_s(\omega,\mathbf{k})$, which we will later confirm to be the sound dispersion function, is defined as

$$
\begin{aligned}
K_s(\omega,\mathbf{k}) &\equiv -\mathfrak{w}^2 + \frac{\mathfrak{q}^2}{d-1} + \nu_s\,\mathfrak{q}^2\,\Gamma_s(\omega,\mathbf{k})\,, \\
\Gamma_s(\omega,\mathbf{k}) &= -i\,\mathfrak{w} - \mathfrak{w}^2\left[(d-2)\, H_k(1) - \frac{1}{d-2}\right] + \frac{d-3}{(d-1)\,(d-2)}\,\mathfrak{q}^2 + \cdots .
\end{aligned}
\tag{3.20}
$$

Up to quadratic order K_s captures the propagation of sound while Γ_s encodes its attenuation. The second parameter, $\mathfrak{p}_s$, may be viewed as a 'deformed momentum' parameter arising from the spatial modulation of the dilaton and is

$$
\mathfrak{p}_s^2 \equiv \mathfrak{q}^2\left(-\frac{d-3}{d-1} + 2\,\nu_s\,\Gamma_s\right). \tag{3.21}
$$

We have judiciously combined terms from the solution for $\varphi_{_{d-1}}$ and $\widetilde{\mathcal{Z}}$ to write the result in this compact form (introducing the parameter $\mathfrak{p}_s$ in the process).[16]

All of the functions that appear above, except for $V_k(\xi)$, are defined using the ingoing solution for massless Klein-Gordon scalar $\varphi_{_{d-1}}$ in the Schwarzschild-AdS$_{d+1}$ background. More precisely,

$$
\begin{aligned}
G^{\text{in}}_{_{d-1}}(\xi,\omega,\mathbf{k}) \equiv e^{-i\,\mathfrak{w}\,F(\xi)}\Big\{ & 1 - \mathfrak{w}^2\, H_\omega(\xi) - \mathfrak{q}^2\, H_k(\xi) + i\,\mathfrak{w}\,\mathfrak{q}^2\, I_k(\xi) + i\,\mathfrak{w}^3\, I_\omega(\xi) \\
& + \mathfrak{q}^4\, J_k(\xi) + \mathfrak{w}^2\,\mathfrak{q}^2\, J_{\omega k}(\xi) + \mathfrak{w}^4\, J_\omega(\xi) + \cdots \Big\}
\end{aligned}
\tag{3.22}
$$

[16] One way to observe that $\mathfrak{p}_s^2$ starts off as $-\frac{d-3}{d-1}\,\mathfrak{q}^2$ is to note that setting $\Lambda_k = \frac{d-1}{2}\, r^3 f'$ in (3.9) reduces it to

$$
\frac{1}{r^{d-1}}\,\mathbb{D}_+\left(r^{d-1}\,\mathbb{D}_+\mathcal{Z}\right) + \left(\omega^2 + \frac{d-3}{d-1}\,k^2\right)\mathcal{Z} = 0\,,
$$

which has the form of a Markovian wave operator with analytically continued momenta.

Function	Source	Function	Source
F	$(d-1)\,\xi^{d-1}\,(\xi^d-1)$	I_ω	$\frac{2}{d-2}\,(\xi^{d-2}-1)-2\,(d-2)\,\widehat{H}_k(\xi)$
H_ω	$1-\xi^{2(d-1)}$	$J_{\omega k}$	$\left(1-\xi^{2(d-1)}\right)H_k(\xi)-4\,\widehat{H}_k(\xi)$ $+\xi^{d-2}\,(\xi^d-1)\,H_\omega(\xi)$
H_k	$\xi^{d-2}(\xi^d-1)$	J_ω	$\left(1-\xi^{2(d-1)}\right)H_\omega(\xi)-4\,\widehat{H}_\omega(\xi)$
I_k	$-\frac{2}{d-2}\,(\xi^{d-2}-1)$	J_k	$\xi^{d-2}\,(\xi^d-1)\,H_k(\xi)$

Table 3.1: The sources for the functions parameterizing the ingoing solution for $\mathcal{Z}$ in (3.19) which enter their integral representation (3.24).

solves $\mathfrak{D}_{d-1}\varphi_{_{d-1}} = 0$ with ingoing boundary conditions, $\lim_{r\to\infty}\varphi_{_{d-1}} = 1$, and thus is the ingoing bulk to boundary Green function for the massless scalar described in chapter 2. Therefore, the eight functions $\{F, H_\omega, H_k, I_\omega, I_k, J_\omega, J_{\omega k}, J_k\}$ are obtained by setting $\mathcal{M} = d-1$ in the corresponding functions defined for general values of $\mathcal{M}$ in chapter 2.[17] The asymptotic boundary condition implies that they all vanish as $r\to\infty$. Note that conversion from the massless scalar to the field $\mathcal{Z}$ involves replacing factors $\mathfrak{q}^2$ with $\mathfrak{p}_s^2$ in various places up to the quartic order: this in particular means that $G^{\text{in}}_{\mathcal{Z}}$ remains Markovian up to that order (for reasons explained in footnote 16).

The key function that controls the asymptotic growth is $V_k(\xi)$, which is defined by the integral

$$V_k(\xi) \equiv -\frac{\xi^{d-4}}{d-4} + \int_\xi^\infty \frac{y^{d-4}-1}{y(y^d-1)}dy\ . \tag{3.23}$$

For $d>4$ this function grows as $r\to\infty$, unlike the other functions. We have extracted the leading divergence as r^{d-4}, which is precisely what one expects for a non-Markovian field of index $\mathcal{M} = 3-d$ as described around (3.15).

Since we are effectively inverting the Markovian operator $\mathfrak{D}_{d-1}$ to find the functions above, they can all be given formal integral representations, assuming that the sources are regular on the horizon and do not grow too fast at infinity. Just as in chapter 2, we can write in general

$$\mathfrak{F}(\xi) = \int_\xi^\infty \frac{dy}{y\,(y^d-1)}\int_1^y \frac{dy'}{y'\,(y'^d-1)}\,\widehat{\mathfrak{J}}(y')\,, \tag{3.24}$$

with $\widehat{\mathfrak{F}}(\xi) = \mathfrak{F}(\xi)-\mathfrak{F}(1)$ defined to measure function values relative to that on the horizon. We tabulate the data for the functions appearing in the gradient expansion originating from the minimally coupled scalar in Table 3.1. Further details of the solution, including asymptotic expansions and expressions for the gravitational data from our solution for $\mathcal{Z}$, are compiled in Appendix J.

[17] Note that the general $\mathcal{M}$ solutions of chapter 2 are explicitly constructed only up to cubic order in gradients.

There is one important subtlety which the reader should be aware of in the way we have presented the solution for $\mathcal{Z}$. The solution to (3.9) up to the quartic order in gradients does not determine Γ_s owing to the explicit factor of $\mathfrak{q}^2$ multiplying $V_k(\xi)$ in (3.19). This would be unfortunate since Γ_s will turn out to be the sound attenuation function. However, if we compute the functions Φ_{W} and Θ from $\mathcal{Z}$ (or any other metric function) then this overall factor of $\mathfrak{q}^2$ disappears and we obtain Γ_s accurate to quadratic order by examining the coefficient of the non-normalizable mode in the solution. A physical way to say this is to note that the boundary source is the conjugate momentum with a factor of k^2 stripped off, see (3.32). This behaviour is manifest in the ingoing Green's function of the field Φ_{W} which is given in (J.10) (the information in Θ is similar, but not independent owing to the linear relation (3.8)). To obtain Γ_s directly from $\mathcal{Z}$ as written above, we would have to compute the solution accurately to sextic order, another peculiarity of the fact that it is related to radial derivatives of the metric functions.

In fact, we would like to conjecture that one can actually get Γ_s to quartic order in gradients. From the leading divergent mode in Φ_{W} (or Θ), which scales as r^{d-2}, we get an expression for K_s to quartic order (see (J.10)). However, we can also examine the constant mode, scaling as r^0, near the boundary. It has a coefficient which is a non-trivial function of ω and $\mathbf{k}$. By judiciously parameterizing the solution for Φ_{W}, one finds that this function starts off as Γ_s and gets corrected at cubic and quartic corrections (the same quantity determines the constant mode of Θ). If one uses a parameterization of the solution in terms of Γ_s, simply noting that it starts at linear order, then it turns out that the on-shell action is finite up to sextic order.[18] Based on this observation and the nature of the explicit solution, we predict an expression for Γ_s up to the fourth order in boundary gradients, which we record for completeness in (J.9) (where for clarity it is denoted as $\widetilde{\Gamma}_s(\omega, \mathbf{k})$). See (J.13) for the prediction for the sound dispersion function valid up to quintic order using the above conjecture.

3.3.3 The grSK solution for the designer field

The ingoing Green's function for $\mathcal{Z}$ suffices for us to determine the full solution on the grSK geometry thanks to the time-reversal invariance of (3.9). We want to impose suitable boundary conditions at the two boundaries of the grSK geometry at $r \to \infty \pm i0$. It was argued in chapter 2 that a non-Markovian fields should be quantized with Neumann boundary conditions if we wish to compute their correlation functions. Equivalently, the asymptotic field value does not correspond to the boundary source, but rather gives the dual boundary operator (akin to the alternate quantization of low lying operators). This was cleanly formulated for probe non-Markovian fields and established to be the case for

[18]To clarify, we mean that in addition to the assumption that Γ_s starts at linear order in derivatives, all we need to input is its relation to K_s via the first line of (3.20).

diffusive modes in the aforementioned reference and [134].

The question we should first ask is what the Einstein-Hilbert dynamics with its usual Dirichlet boundary conditions for computing correlation functions of dual energy-momentum tensor correlation functions implies for the field $\mathcal{Z}$. It turns out that while the dynamics of $\mathcal{Z}$ is pretty simple in the bulk, as evidenced from (3.10), it has a pretty involved set of boundary terms owing to the redefinitions in (G.27). The boundary terms are a general quadratic form in $\mathcal{Z}$, $\mathbb{D}_+\mathcal{Z}$, and $\mathbb{D}_+^2\mathcal{Z}$, implying that generically we need to fix a combination of these three quantities for stationarity of the action. However, one can do better: armed with the asymptotic fall-offs of the field, one learns that the leading set of boundary terms in a near-boundary expansion are simpler, and one indeed finds that $\mathcal{Z}$ ought to be quantized with Neumann boundary conditions to obtain the dual stress tensor correlators. The detailed argument analyzing the variational principle for the designer field $\mathcal{Z}$ is given in Appendix H.3.

While the field $\mathcal{Z}$ is quantized with Neumann boundary conditions in order to compute the generating function of boundary correlators, the resulting correlators have a sound pole, reflecting the non-Markovian nature of the field. For this reason, it was proposed in chapter 2 that one should, for purposes of computing a real-time Wilsonian influence functional, parameterize the solution for $\mathcal{Z}$ in terms of the normalizable mode, which for a non-Markovian field corresponds to the conjugate momentum.

With this understanding we will now parameterize the general solution on the grSK geometry in terms of the sound modulus, which is the expectation value of the dual boundary operator $\breve{\mathcal{O}}_{\scriptscriptstyle\mathcal{Z}}$ (thinking of $\mathcal{Z}$ probe field in the fixed Schwarzschild-AdS$_{d+1}$ background). We will denote the modulus associated with the auxiliary field $\mathcal{Z}$ as $\breve{\mathcal{Z}}$ and write

$$\Big\langle(\breve{\mathcal{O}}_{\scriptscriptstyle\mathcal{Z}})_{\text{L}}\Big\rangle = \breve{\mathcal{Z}}_{\text{L}}\,, \qquad \Big\langle(\breve{\mathcal{O}}_{\scriptscriptstyle\mathcal{Z}})_{\text{R}}\Big\rangle = \breve{\mathcal{Z}}_{\text{R}}\,, \tag{3.25}$$

with

$$\breve{\mathcal{Z}}_{\text{L/R}} = \lim_{r\to\infty\pm i0}\left[\mathcal{Z} + \text{counterterms}\right]. \tag{3.26}$$

In terms this boundary modulus field $\breve{\mathcal{Z}}$ we can write the full grSK solution in the average-difference basis as

$$\mathcal{Z}^{\text{SK}}(\zeta,\omega,\mathbf{k}) = G^{\text{in}}_{\scriptscriptstyle\mathcal{Z}}\,\breve{\mathcal{Z}}_a + \left[\left(n_{\scriptscriptstyle B} + \frac{1}{2}\right) G^{\text{in}}_{\scriptscriptstyle\mathcal{Z}} - n_{\scriptscriptstyle B}\, e^{\beta\omega(1-\zeta)}\, G^{\text{rev}}_{\scriptscriptstyle\mathcal{Z}}\right]\breve{\mathcal{Z}}_d\,, \tag{3.27}$$

with $G^{\text{rev}}_{\scriptscriptstyle\mathcal{Z}}(\zeta,\omega,\mathbf{k}) = G^{\text{in}}_{\scriptscriptstyle\mathcal{Z}}(\zeta,-\omega,\mathbf{k})$ being the time-reversed propagator. It is useful to note that $\mathcal{Z}$ has mass dimension $d-2$ and hence $\breve{\mathcal{Z}}_{a,d}$ are likewise boundary fields with this dimension. We will use this information in the next section to present the effective action for the phonon modes in the relativistic plasma.

3.4 Effective dynamics of sound and energy transport

With the grSK solution for the designer sound mode, we can evaluate the on-shell action parameterized by the fields $\breve{\mathcal{Z}}_{a,d}$, which then gives us the open effective field theory of sound propagation in the holographic plasma at the Gaussian order in amplitudes. We first outline the details of this effective action and then turn to the boundary stress tensor, which we will express in terms of $\breve{\mathcal{Z}}$ and background polarization terms involving the sources $\breve{\boldsymbol{\zeta}}$.

3.4.1 The sound Wilsonian influence functional

Since the background Schwarzschild-AdS$_{d+1}$ geometry has a non-vanishing free energy, we expect to see two contributions to the on-shell action. One is a piece from the background, which based on fluid/gravity intuition should correspond to the ideal fluid free energy. In addition, there will be the true dynamical data corresponding to the Wilsonian influence functional (WIF) of the field $\mathcal{Z}$. These two contributions are cleanly separated in the shear and tensor sectors of a neutral fluid because, in those sectors, we do not have a propagating mode.

A relativistic ideal fluid has a propagating phonon mode, so the split in this sound sector is not a-priori manifest. We will therefore not identify the ideal fluid contribution at this stage, but simply separate the on-shell action into the WIF and contact terms. Subsequently, working out the stress tensor will enable us to understand which pieces should be regarded as part of the ideal fluid contribution. With this preamble, let us write the full action as a sum of two pieces, viz.,

$$S[\mathcal{Z}] = S_{\text{contact}}[\mathcal{Z}] + S_{\text{WIF}}[\mathcal{Z}]\,. \tag{3.28}$$

We now summarize the final result for the Schwinger-Keldysh effective action obtained by computing the on-shell action for $\mathcal{Z}$ with the gradient expansion solution given in §3.3.2. The reader can find details of the evaluation in Appendix I.3.

Let us start with the contribution to the WIF, which is ascertained to be

$$\frac{1}{c_{\text{eff}}}\, S_{\text{WIF}}[\mathcal{Z}] = -\int_k k^2 \left(\breve{\mathcal{Z}}_d^\dagger\, K^{\text{in}}_{\mathcal{Z}} \left[\breve{\mathcal{Z}}_a + \left(n_{_B} + \frac{1}{2}\right)\breve{\mathcal{Z}}_d\right] + \text{cc}\right), \tag{3.29}$$

where

$$K^{\text{in}}_{\mathcal{Z}}(\omega,\mathbf{k}) = \frac{b^{d-2}}{2\,d\,(d-1)^2}\, K_s(\omega,\mathbf{k})\,. \tag{3.30}$$

From this expression we solve for the boundary source of the field $\mathcal{Z}$ in terms of the moduli field $\breve{\mathcal{Z}}$ and obtain

$$\begin{aligned}
\breve{\zeta}_a &= K^{\text{in}}_{z}\, \breve{\mathcal{Z}}_a + \left(n_{B} + \frac{1}{2}\right) \left[K^{\text{in}}_{z} - K^{\text{rev}}_{z}\right] \breve{\mathcal{Z}}_d\,, \\
\breve{\zeta}_d &= K^{\text{rev}}_{z}\, \breve{\mathcal{Z}}_d\,.
\end{aligned} \tag{3.31}$$

One can give an alternate expression for the source directly from the conjugate momentum of the field $\mathcal{Z}$, after stripping off a factor of k^2, viz.,

$$\breve{\zeta}_{\text{L/R}} = -\lim_{r\to\infty\pm i0} \frac{\Pi_{z}}{k^2}\,. \tag{3.32}$$

It can be checked that our identification of the source in (3.32) agrees with the expectation from the WIF (3.31) (which we expect on general grounds from chapter 2) and is verified in Appendix H. Isolating a factor of k^2 in the WIF results in stripping off a similar factor from the conjugate momentum (3.32).[19] The rescaling by a factor of k^2 is only allowed since we are focusing on spatially inhomogeneous modes and are implicitly working with $k > k_{\text{IR}}$.

The source for the designer field can be given several equivalent expressions in terms of the metric functions. Using the asymptotics of the solutions obtained in Appendix J.2 (see in particular (J.10) and (J.12)) one can show

$$\breve{\zeta}_{\text{L/R}} = \frac{1}{4\,(d-1)} \lim_{r\to\infty\pm i0} \frac{\mathbb{D}_+\Theta}{r^{d-1}} = \frac{d\,\nu_s}{8} \lim_{r\to\infty\pm i0} \frac{\Phi_{\text{W}}}{r^{d-2}}\,. \tag{3.33}$$

These source terms are basically capturing the deformation of the boundary metric order by order in the gradient expansion. Indeed, upon examining the induced metric $\gamma_{\mu\nu}$ on the boundary

$$\begin{aligned}
(\gamma_{\text{L/R}})_{\mu\nu} &= \lim_{r\to\infty\pm i0} \left[\frac{\mathbb{D}_+\Theta}{r^{d-1}}\, dv^2 + \left(1 + \frac{\Phi_{\text{W}}}{r^{d-2}}\right) \eta_{\mu\nu}\, dx^\mu\, dx^\nu\right] \\
&= -\left(1 - \frac{4\,(d-1)\,(d-3)}{d-2}\, \breve{\zeta}_{\text{L/R}}\right) dv^2 + \left(1 + \frac{4\,(d-1)}{d-2}\, \breve{\zeta}_{\text{L/R}}\right) d\mathbf{x}^2\,,
\end{aligned} \tag{3.34}$$

we see that spatial and temporal components of the perturbed boundary metric can be viewed as sources for $\mathcal{Z}$. Since we have only one physical degree of freedom in the longitudinal sector, we do not have independent metric perturbations, but rather see that the red-shift factor captured by the temporal term is related up to a dimension dependent constant to Weyl rescaling of the background.

With the identification of the sources we can now present the contact term part of the action, which is a functional of these sources. Ignoring the background free energy term

[19] In the vector sector, the passage to Debye gauge already factors out a piece proportional to k^2, even in the off-shell action for the designer fields [42, 134]. This does not happen in the off-shell action (3.10) for $\mathcal{Z}$ due to the momentum dependent dilaton, but is reinstated in the on-shell action.

for simplicity, we find the following result at linear and quadratic order in amplitudes:

$$\frac{1}{c_{\text{eff}}}\, S_{\text{contact}}[\mathcal{Z}] = \frac{2\,(d-1)^2}{b^d} \int_k \left[\breve{\zeta}_{\text{R}} - \breve{\zeta}_{\text{L}} + \frac{(d-1)\,(d-6)}{(d-2)} \left(\breve{\zeta}^{\dagger}_{\text{R}}\, \breve{\zeta}_{\text{R}} - \breve{\zeta}^{\dagger}_{\text{L}}\, \breve{\zeta}_{\text{L}}\right)\right]. \tag{3.35}$$

In §3.4.4 we will argue that this contact term can be understood as arising from the on-shell action of an ideal fluid propagating on (3.34). This includes the somewhat counter-intuitive numerical factor in the quadratic term, which vanishes in $d = 6$.

We will explain the individual contribution to the action, in particular the split between ideal and non-ideal parts, after we discuss the stress tensor. With that understanding we will be able to cleanly identify the ideal fluid contribution. Along the way, we will also argue that the non-dissipative part of the hydrodynamic action, the Class L terms in the terminology of [62], can be extracted from the WIF. For the scalar sector we will for example see the curvature coupling of the fluid at quadratic order in gradients. These statements will be elaborated in §3.4.4.

Let us take stock of some physical implications from the grSK solution for $\mathcal{Z}$, and the results for the Wilsonian influence functional. We see that the inverse Green's function computed from the WIF is proportional to $K_s(\omega, \mathbf{k})$. In other words, the retarded Green's function of $\breve{\mathcal{O}}_{\mathcal{Z}}$ has a pole at the vanishing locus of K_s. This function also appears as the coefficient of the divergent (non-normalizable) mode in $\mathcal{Z}$. As described in chapter 2, non-Markovian fields have a completely normalizable solution on a codimension-1 locus in the boundary Fourier domain. For $\mathcal{Z}$ this is the vanishing locus of K_s; it will end up defining the dispersion function for sound. This also implies that Γ_s defined in (3.20) is the rate of attenuation of sound speed due to viscosity. We will elaborate on this further below when we discuss the physical sound degree of freedom and compute boundary energy-momentum tensor correlators.

3.4.2 The boundary stress tensor

With the Wilsonian influence functional parameterized by the boundary value of the field $\mathcal{Z}$ at hand, we can now turn to the physical boundary observables, which are the scalar polarizations of the boundary energy-momentum tensor density. The stress tensor has both a background contact piece and a contribution given in terms of the fluctuating field $\breve{\mathcal{O}}_{\mathcal{Z}}$. In the scalar sector, the presence of a non-trivial boundary metric (3.34) means that the result we quote for the contact terms depends on the index positions and whether or not we work with stress tensor densities. We will work with tensor densities, quoting the mixed components for the stress tensor operator, but compute the correlator for the operator with both indices raised.[20] For the sake of notational simplicity, we define

$$\widehat{T}^{\mu\nu} \equiv \frac{1}{c_{\text{eff}}} \, T^{\mu\nu}_{\text{CFT}} \, . \tag{3.36}$$

Here $T^{\mu\nu}_{\text{CFT}}$ is the counterterm corrected Brown-York stress tensor density given in (I.4).

We will take the viewpoint that the standard rules of the extrapolate dictionary in holography (which can be extended to grSK geometry [61]) as applied to non-Markovian operators dictates a canonical split between operator (or vev) and source contributions. As explained in Appendix I.2, this can be seen by examining the Brown-York stress tensor (corrected by the counterterms) directly in terms of the metric fields Φ_{w} and Θ. This tensor must satisfy conservation and be traceless (up to the conformal anomaly, which we do not access in $d > 4$). This statement follows naturally from the momentum constraint equation in the bulk geometry and underlies the original identification in [138].

With this choice the Schwinger-Keldysh stress tensor operator has the following representation:

$$\begin{split}
\left(\widehat{T}_v{}^v\right)_{\text{L/R}} &= -\frac{d-1}{b^d} + \int_k \mathbb{S} \left[\frac{2\,(d-1)^2}{b^d}\, \breve{\zeta}_{\text{L/R}} - \frac{k^2}{d-1}\; (\breve{\mathcal{O}}_{\mathbb{z}})_{\text{L/R}}\right],\\
\left(\widehat{T}_v{}^i\right)_{\text{L/R}} &= i \int_k \frac{k\,\omega}{d-1}\, \mathbb{S}_i \; (\breve{\mathcal{O}}_{\mathbb{z}})_{\text{L/R}} \,,\\
\left(\widehat{T}_i{}^j\right)_{\text{L/R}} &= \frac{\delta_i^{\ j}}{b^d} + \int_k \mathbb{S} \frac{2\,(d-1)^2}{b^d}\, \delta_i^{\ j}\, \breve{\zeta}_{\text{L/R}} \\
&\quad + \int_k \left[\frac{1}{d-1}\left(\omega^2 - \nu_s\, \Gamma_s\, k^2\right) \delta_i^{\ j}\, \mathbb{S} - \frac{1}{d-2}\, \nu_s\, \Gamma_s\, k^2\, (\mathbb{S}^{\text{T}})_i^{\ j}\right] (\breve{\mathcal{O}}_{\mathbb{z}})_{\text{L/R}} \, .
\end{split} \tag{3.37}$$

We recognize the background contribution which says that the unperturbed planar Schwarzschild-AdS$_{d+1}$ black hole is a conformal plasma. The terms linear in $\breve{\zeta}$ and $\breve{\mathcal{O}}_{\mathbb{z}}$ are the terms we should understand.

As it is written, the stress tensor is not manifestly traceless. Nor is the conservation Ward identify obvious on the induced boundary geometry (3.34). The two do hold, and are, in fact, equivalent to the relation between sources and vevs (3.31), which picks out the sound dispersion locus:[21]

$$\left\langle \nabla_\mu \widehat{T}^{\mu\nu} \right\rangle = 0 = \left\langle \widehat{T}_\mu{}^\mu \right\rangle \implies \left(\omega^2 - \frac{k^2}{d-1} - k^2\, \nu_s\, \Gamma_s\right) \breve{\mathcal{Z}}_{\text{L/R}} + \frac{2\,d\,(d-1)^2}{b^d}\, \breve{\zeta}_{\text{L/R}} = 0 \, . \tag{3.38}$$

We should view (3.38) as giving us the on-shell condition for the sound mode which occurs when the solution is purely normalizable, i.e., when the source contribution is set

[20] In the AdS/CFT context, it is natural, as from any effective action, to extract the boundary stress tensor density, since it only requires variation with respect to the boundary metric and no removal of metric determinants (recall that the stress tensor operator is $T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta \gamma_{\mu\nu}}$).

[21] The covariant derivative in the conservation equation is the one appropriate for the stress tensor density, cf., footnote 20.

to vanish. This confirms that the operator K_s gives us the dispersion relation for sound in the plasma and identifies $\Gamma_s(\omega, \mathbf{k})$ as the attenuation function.

With this identification we can now give an alternate presentation of the stress tensor. Let us use the dispersion relation (3.38) to shift the source and vev contributions in the spatial part of the stress tensor, i.e., use the replacement rule

$$\omega^2 - \nu_s \, \Gamma_s \, k^2 \quad \mapsto \quad \frac{k^2}{d-1} - \frac{1}{b^2} \, K_s \, . \tag{3.39}$$

Re-expressing $K_s \, \breve{\mathcal{Z}}$ in terms of $\breve{\boldsymbol{\zeta}}$ we find that the stress tensor in (3.37) can be equivalently presented as

$$\begin{aligned}
\left\langle \widehat{T}_v{}^v \right\rangle_{\text{L/R}} &= -\frac{d-1}{b^d} + \int_k \mathbb{S} \left[\frac{2\,(d-1)^2}{b^d} \, \breve{\boldsymbol{\zeta}}_{\text{L/R}} - \frac{k^2}{d-1} \, \breve{\mathcal{Z}}_{\text{L/R}} \right] , \\
\left\langle \widehat{T}_v{}^i \right\rangle_{\text{L/R}} &= i \int_k \frac{k\,\omega}{d-1} \, \mathbb{S}_i \, \breve{\mathcal{Z}}_{\text{L/R}} \, , \\
\left\langle \widehat{T}_i{}^j \right\rangle_{\text{L/R}} &= -\frac{1}{d-1} \left\langle \widehat{T}_v{}^v \right\rangle_{\text{L/R}} \delta_i^{\,j} - \int_k \frac{k^2}{d-2} \, \nu_s \, \Gamma_s \, (\mathbb{S}^{\text{T}})_i^{\ j} \, \breve{\mathcal{Z}}_{\text{L/R}} \, .
\end{aligned} \tag{3.40}$$

The representation of the CFT stress tensor (3.40) renders the trace Ward identity manifest. Additionally, it also isolates the sound attenuation contribution captured by Γ_s solely into the spatial part of the stress tensor. In fact, the contribution is governed by the longitudinal trace-free tensor structure $\mathbb{S}^{\text{T}}_{ij}$. In this presentation, the tracelessness Ward identity is manifest, but conservation now implies the on-shell condition (3.38).

One could go a step further and replace the Γ_s term in the spatial part of the stress tensor once again using (3.39), i.e., express it as

$$\begin{aligned}
\left\langle \widehat{T}_i{}^j \right\rangle_{\text{L/R}} = -\frac{1}{d-1} \left\langle \widehat{T}_v{}^v \right\rangle_{\text{L/R}} \delta_i^{\,j} &+ \frac{1}{d-2} \int_k \left(\frac{k^2}{d-1} - \omega^2 \right) (\mathbb{S}^{\text{T}})_i^{\ j} \, \breve{\mathcal{Z}}_{\text{L/R}} \\
&- \frac{2\,d\,(d-1)^2}{(d-2)\,b^d} \int_k (\mathbb{S}^{\text{T}})_i^{\ j} \, \breve{\boldsymbol{\zeta}}_{\text{L/R}} \, .
\end{aligned} \tag{3.41}$$

In this manner of presentation, both the conservation and tracelessness Ward identities of the stress tensor are identically satisfied. The operator contribution encoded in $\breve{\mathcal{Z}}$ has the right momentum and frequency dependent factors for the conservation to be rendered trivial, while the terms involving $\breve{\boldsymbol{\zeta}}$, one can check, respect ${}^{\gamma}\nabla_\mu \, \widehat{T}^{\mu\nu} = 0$ by themselves.

While there appear to be three distinct parameterizations, the form given in (3.40) is the one that separates the ideal fluid contribution from the dissipative part. We will demonstrate this in §3.4.4 after writing down the stress tensor correlators.

3.4.3 Correlation functions

From the Wilsonian influence functional (3.29) we can read off the correlation functions of the $\breve{\mathcal{O}}_{\mathcal{Z}}$ field operator on the boundary. The retarded Green's function is given by

$$\left\langle \breve{\mathcal{O}}_{\mathcal{Z}}(-\omega,-\mathbf{k})\, \breve{\mathcal{O}}_{\mathcal{Z}}(\omega,\mathbf{k}) \right\rangle^{\text{Ret}} = \frac{1}{i\, c_{\text{eff}}\, k^2\, K^{\text{in}}_{\mathcal{M}} \mathcal{Z}(\omega,\mathbf{k})} = -i\, \frac{2\, d\, (d-1)^2}{c_{\text{eff}}\, b^d}\, \frac{1}{\mathfrak{q}^2\, K_s(\omega,\mathbf{k})}\,. \tag{3.42}$$

The structure of the WIF respects the KMS condition, as explained in the earlier works, implying that the Keldysh correlator satisfies the fluctuation dissipation condition, viz.,

$$\left\langle \breve{\mathcal{O}}_{\mathcal{Z}}(-\omega,-\mathbf{k})\, \breve{\mathcal{O}}_{\mathcal{Z}}(\omega,\mathbf{k}) \right\rangle^{\text{Kel}} = -\frac{1}{2\, c_{\text{eff}}} \coth\left(\frac{\beta\omega}{2}\right) \frac{\text{Im}\left[K^{\text{in}}_{\mathcal{M}} \mathcal{Z}(\omega,\mathbf{k})\right]}{k^2 \left| K^{\text{in}}_{\mathcal{M}} \mathcal{Z}(\omega,\mathbf{k})\right|^2}\,. \tag{3.43}$$

We can use this information to write down the stress tensor correlators given the explicit expressions for the components in (3.37). From this expression it is clear that the result is given by the two-point functions of $\breve{\mathcal{O}}_{\mathcal{Z}}$, suitably dressed to account for the derivative operators present (the functions of ω, k in frequency/momentum domain).

Before doing so however, we should note that we have the boundary metric determined by a single source function (3.34), which implies relation between sources of various components (and an absence of source for the spatial-temporal component). We will view the boundary metric as defining the source for the energy density. Thus, we can obtain the energy density two-point function naturally. Once we have this piece of data, we will use flat spacetime Ward identities, as explained in [21], to fix the remaining correlation functions. In particular, we demand that the contact terms in the retarded correlation functions are fixed so that flat spacetime momentum conservation holds – this implies that correlation functions with at least one temporal index vanish at zero momentum. To fix the purely spatial components we utilize full energy-momentum tensor conservation.

With this understanding, we now quote the result for the physical retarded correlator for the energy-momentum tensor density operator $T^{\mu\nu}_{\text{CFT}}$. We parameterize these as

$$\left\langle T^{\mu\nu}_{\text{CFT}}(-\omega,-\mathbf{k})\, T^{\mu\nu}_{\text{CFT}}(\omega,\mathbf{k}) \right\rangle^{\text{Ret}} = \frac{2d}{i\, c_{\text{eff}}\, b^d}\, \frac{\mathfrak{G}^{\mu\nu,\rho\sigma}(\omega,\mathbf{k})}{K_s(\omega,\mathbf{k})} + \text{analytic}\,. \tag{3.44}$$

To avoid writing involved expressions we pick a spatial direction for sound propagation, setting $\mathbf{k} = k\,\hat{x}$, decomposing $\mathbb{R}^{d-1}$ coordinates into $\{x, x_s\}$ with $s = 2, \cdots, d-1$. One can check that with this choice $\mathbb{S}^{\text{T}}_{ij} = \frac{1}{d-1}\,\text{diag}\{-(d-2), 1, \cdots, 1\}$. We choose a representation where $\mathfrak{G}^{\mu\nu,\rho\sigma}(\omega, \mathbf{k})$ are polynomials in ω, k and take the form:

$$\begin{aligned}
\mathfrak{G}^{vv,vv} &= \mathfrak{q}^2\,, & \mathfrak{G}^{vv,vx} &= \mathfrak{q}\,\mathfrak{w}\,,\\
\mathfrak{G}^{vv,xx} &= \mathfrak{q}^2\left(\frac{1}{d-1}+\nu_s\,\Gamma_s\right), & \mathfrak{G}^{vv,ss} &= \mathfrak{q}^2\left(\frac{1}{d-1}-\frac{1}{d-2}\,\nu_s\,\Gamma_s\right),\\
\mathfrak{G}^{vx,vx} &= \mathfrak{q}^2\left(\frac{1}{d-1}+\nu_s\,\Gamma_s\right), & \mathfrak{G}^{vx,xx} &= \mathfrak{w}\,\mathfrak{q}\left(\frac{1}{d-1}+\nu_s\,\Gamma_s\right),\\
\mathfrak{G}^{vx,ss} &= \mathfrak{w}\,\mathfrak{q}\left(\frac{1}{d-1}-\frac{1}{d-2}\,\nu_s\,\Gamma_s\right), & \mathfrak{G}^{xx,xx} &= \mathfrak{w}^2\left(\frac{1}{d-1}+\nu_s\,\Gamma_s\right),\\
\mathfrak{G}^{xx,ss} &= \mathfrak{w}^2\left(\frac{1}{d-1}-\frac{1}{d-2}\,\nu_s\,\Gamma_s\right), & &\\
\mathfrak{G}^{ss,ss} &= \mathfrak{w}^2\left(\frac{1}{d-1}-\frac{1}{d-2}\,\nu_s\,\Gamma_s\right)-\frac{d-1}{d-2}\,\nu_s\,\Gamma_s^*\left(\mathfrak{w}^2-\frac{d-1}{d-2}\,\mathfrak{q}^2\,\nu_s\,\Gamma_s\right). & &
\end{aligned} \tag{3.45}$$

Any other representation of $\mathfrak{G}^{\mu\nu,\rho\sigma}$ would differ from the above by factors of K_s. The Keldysh propagator follows naturally from the fluctuation dissipation theorem. Our choice for non-analytic part of the retarded Green's function recovers the result for the analytically continued Wightman function given in [21] for $d = 4$.

The physical aspect of the result which is interesting is the fact that the stress tensor correlators have a sound pole at the dispersion locus characterized by the vanishing of K_s. As noted around (3.38) this function gives us the on-shell condition for the phonon mode. At the leading order in gradients it enforces the expected equation of state condition, which fixes the speed of sound in a conformal plasma to be $\frac{1}{\sqrt{d-1}}$. At higher orders the Γ_s pieces serve to attenuate the propagation and predicts its half-life.

Solving (3.38), and using the expression for Γ_s given in (3.20), we find[22]

$$\omega = \frac{k}{\sqrt{d-1}} - i\,\frac{d-2}{d\,(d-1)}\,b\,k^2 + \frac{d-2}{2\,d^2\,(d-1)^{\frac{3}{2}}}\left[d+2+2\ \mathrm{Har}\left(\frac{2-d}{d}\right)\right] b^2\,k^3 + \cdots\,. \tag{3.46}$$

We have indicated only one branch of the solution in the above expression.

As particular cases, note that with $d = 4$ we recover the results for the $\mathcal{N} = 4$ SYM plasmas,[23]

$$\mathfrak{w} = \frac{\mathfrak{q}}{\sqrt{3}} - \frac{i}{6}\,\mathfrak{q}^2 + \frac{3-2\log 2}{24\sqrt{3}}\,\mathfrak{q}^3 + \cdots\,, \tag{3.47}$$

which was first obtained to cubic order in [109] (extending the original result of [21]). Setting $d = 3$ we obtain the corresponding result for the ABJM plasma obtained in

[22] We used $H_k(1) = -\frac{1}{d\,(d-2)}\ \mathrm{Har}\left(\frac{2-d}{d}\right)$ where $\mathrm{Har}\,(x)$ is the Harmonic number function.

[23] We work with dimensionless frequencies and momenta defined in (2.19). This definition differs from normalizations used in earlier references. Our normalization is twice that used in [109], while [126] uses a normalization set by temperature and not (inverse) horizon size, which in $d = 3$ differs by a factor of $\frac{3}{4\pi}$.

[98, 126]:

$$\mathfrak{w} = \frac{\mathfrak{q}}{\sqrt{2}} - \frac{i}{6}\,\mathfrak{q}^2 + \frac{15+\sqrt{3}\,\pi - 9\,\log 3}{108\,\sqrt{2}}\,\mathfrak{q}^3 + \cdots . \tag{3.48}$$

As we describe in Appendix J.2, from our solution we can actually extract higher order corrections to the dispersion, and quote the result accurate to quintic order in (J.13).

3.4.4 A fluid dynamical perspective

We have now reproduced the physical results expected for sound propagation as encoded in the stress tensor correlators. Let us therefore try to analyze features of the solution from a hydrodynamic perspective and, in particular, attempt to understand the contributions to the action (3.35). The gravitational calculation gives us the answer all at once, but we can attempt to decompose it using hydrodynamic intuition.

The physical question of interest is how one delineates the ideal and non-ideal parts of the effective action. Addressing this question will make clear that one should think of the action as comprising of a Schwinger-Keldysh factorized part corresponding to sound propagation and the physical influence phase which is the part that governs sound attenuation, as we indicate in (3.58) below.

The non-ideal part, by definition, includes all the gradient contributions in the stress tensor parameterized in hydrodynamic variables, whether or not they lead to dissipation. We will use the stress tensor in the form parameterized in (3.40), which judiciously isolates the non-ideal contributions into Γ_s. As has been argued earlier [62], not all higher order transport is dissipative. While dissipative transport leads to entropy production, in general, there exists non-dissipative transport which is adiabatic and leads to no entropy production. While at leading order $\Gamma_s(\omega, \mathbf{k})$ captures sound attenuation, which originates from the dissipative shear viscosity term, it also includes contributions from higher order non-dissipative gradient terms as we demonstrate below.

Parameterization of the ideal fluid: To keep the discussion transparent, we will first identify the ideal fluid contribution, which by definition only captures zeroth order terms in the gradient expansion of the stress tensor. Having understood this part, we will then attempt to address higher order (spatial) gradient terms, which capture non-dissipative transport.

Consider an ideal fluid with energy density and pressure related by the conformal equation of state $\epsilon = (d-1)\,p$, viz., the stress tensor density (nb: $\widetilde{T}$ is dimensionless for simplicity)[24]

$$T^{\mu\nu}_{\text{ideal}} = \sqrt{-\gamma}\left(\frac{\widetilde{T}}{b}\right)^{d} \left(\gamma^{\mu\nu} + d\,u^\mu\,u^\nu\right). \tag{3.49}$$

[24] We are not keeping track of the normalization factor translating between the horizon size parameter b and the physical temperature T, cf., (1.34).

We claim that the part of our solution which is insensitive to sound attenuation, i.e., with $\Gamma_s \to 0$, describes the dynamics of such an ideal fluid on the boundary geometry (on either L or R boundary). Specifically, we assert

$$T^{\mu\nu}_{\text{ideal}} = \left\langle \widehat{T}^{\mu\nu} \right\rangle \Big|_{\Gamma_s \to 0} \tag{3.50}$$

where $\left\langle \widehat{T}^{\mu\nu} \right\rangle$ is the stress tensor density as parameterized in (3.40). We can confirm this by solving the conservation equations arising from (3.49) on the induced boundary geometry (3.34).

We can parameterize the temperature and velocity by a field $\breve{\mathcal{Z}}$, which is a-priori unrelated to the stress tensor expectation value

$$\begin{aligned} \widetilde{T}_{\text{L/R}} &= 1 - \int_k \left[\frac{2\,(d-1)}{d-2}\, \mathbb{S}\, \breve{\boldsymbol{\zeta}}_{\text{L/R}} - \frac{b^{d-2}\, \mathfrak{q}^2}{d\,(d-1)^2}\, \mathbb{S}\, \breve{\mathcal{Z}}_{\text{L/R}} \right], \\ (u_\mu\, dx^\mu)_{\text{L/R}} &= \left(-1 + \int_k \frac{2\,(d-1)\,(d-3)}{d-2}\, \mathbb{S}\, \breve{\boldsymbol{\zeta}}_{\text{L/R}} \right) dv - \int_k \frac{i\, b^{d-2}\, \mathfrak{q}\, \mathfrak{w}}{d\,(d-1)}\, \mathbb{S}_i\, \breve{\mathcal{Z}}_{\text{L/R}}\, dx^i\,. \end{aligned} \tag{3.51}$$

Imposing the conservation equation one finds that $\breve{\mathcal{Z}}$ and $\breve{\boldsymbol{\zeta}}$ must satisfy the relation (3.38) with $\Gamma_s = 0$.[25] With this constraint recovered, we may identify $\breve{\mathcal{Z}}$ as the physical phonon mode, i.e., as the boundary value of the non-Markovian field. We have effectively isolated the dynamical sound mode, which importantly does exist even in the absence of dissipation, on the inhomogeneous dynamical boundary spacetime (3.34). One can, furthermore, use the on-shell relation to write the temperature more suggestively as

$$\widetilde{T}_{\text{L/R}} = 1 + \int_k \left[\frac{2\,(d-1)\,(d-3)}{d-2}\, \mathbb{S}\, \breve{\boldsymbol{\zeta}}_{\text{L/R}} + \frac{b^{d-2}\, \mathfrak{w}^2}{d\,(d-1)}\, \mathbb{S}\, \breve{\mathcal{Z}}_{\text{L/R}} \right]. \tag{3.52}$$

In this presentation, we see that the source $\breve{\boldsymbol{\zeta}}$ contribution to the temperature is just the red-shift effect for a fluid propagating on (3.34). The contribution from $\breve{\mathcal{Z}}$ parameterizes the response of the fluid.

Curvature corrections to stress tensor: Beyond the ideal fluid, the first correction comes from the dissipative shear viscosity term, which physically leads to the damping of the sound in the medium. This is the leading $i\omega$ contribution to Γ_s. To isolate any

[25] This parameterization can be motivated by considering a phonon mode for a relativistic plasma in flat spacetime. All one needs is the statement that the dynamics is captured by conservation of the stress energy tensor. It is not hard to check that for a linearized perturbation about an equilibrium plasma in flat spacetime,

$$T = T_0 + \int_k \frac{k^2}{d-1}\, \mathbb{S}\, \breve{\mathcal{Z}}\,, \qquad u_\mu\, dx^\mu = -dv - \int_k i\, \omega\, k\, \mathbb{S}_i\, \breve{\mathcal{Z}}\, dx^i\,,$$

satisfies the conservation law at linear order in amplitudes provided $\left(-\omega^2 + \frac{k^2}{d-1}\right) \breve{\mathcal{Z}} = 0$. The latter equation picks out the sound dispersion locus in the absence of a source.

non-dissipative contributions we must therefore switch off time-dependence and focus on equilibrium data. In stationary equilibrium, $\Gamma_s(0,\mathbf{k}) = \frac{d-3}{(d-1)\,(d-2)}\, b^2\, k^2 + \cdots$. The $\mathbb{S}^{\mathrm{T}}_{ij}$ part of the spatial stress tensor can be identified as the coupling of the fluid to background curvature. For a conformal plasma it takes the form

$$T^{\mu\nu}_{\text{CFT}} \supset \kappa \left({}^{\gamma}C^{\mu\alpha\nu\beta}\, u_\alpha\, u_\beta + \sigma_{\text{sh}}^{<\mu\alpha}\, (\sigma_{\text{sh}})^{\nu>}{}_{\alpha} + \omega_{\text{vor}}^{<\mu\alpha}\, (\omega_{\text{vor}})_{\alpha}{}^{\nu>} \right). \tag{3.53}$$

Here ${}^{\gamma}C^{\mu\alpha\nu\beta}$ is the Weyl tensor of the boundary geometry (3.34), ω_{vor} is the fluid vorticity, σ_{sh} is the shear tensor of the fluid, and the angle brackets indicate transverse projection.[26] The value of the transport coefficient $\kappa = 2\, c_{\text{eff}}\, T^{d-2}$ is known for Schwarzschild-AdS$_{d+1}$ black holes [132] (it was initially derived in $d=4$ in [109]). Using the temperature and velocity profiles identified above, one can directly check that our result captures this contribution to the stress tensor.

Having understood the contributions at the level of the stress tensor we can now explain how to interpret the contact terms in the action. Furthermore, we can also isolate terms corresponding to the Class L adiabatic action (equivalent to the Lagrangian density (2.143)):

$$S^{\mathcal{W}} = b^{-d} \int d^dx\, \sqrt{-\gamma} \left(\widetilde{T}^d - b^2\, \widetilde{T}^{d-2} \left[\frac{{}^{\mathcal{W}}R}{d-2} + \frac{1}{2}\omega_{\text{vor}}^2 + \frac{1}{d}\,\text{Harmonic}\left(\frac{2}{d}-1\right)\, \sigma_{\text{sh}}^2 \right] \right). \tag{3.54}$$

Recall that ${}^{\mathcal{W}}R$ is the Weyl covariant curvature scalar on the boundary.

Contact terms from ideal fluid: Let us begin with the ideal fluid part which is the leading contribution in (3.54). To understand this we first note that the contribution to $S[\mathcal{Z}]$ can be understood directly from the variational definition of the stress energy tensor. Specifically, the on-shell action with $\Gamma_s \to 0$, which prior to our Legendre transformation is the usual generating function of correlators, is given by contracting the ideal stress tensor (3.49) with the change in the background metric from flat spacetime, viz.,

$$S_{\text{ideal}}[\mathcal{Z}] = \frac{1}{2} \int d^dx\, T^{\mu\nu}_{\text{ideal}}\, (\gamma_{\mu\nu} - \eta_{\mu\nu}) \quad \xrightarrow[\text{transform}]{\text{Legendre}} \quad S[\mathcal{Z}]\Big|_{\Gamma_s \to 0}. \tag{3.55}$$

Here we have dropped the background constant free energy part and focused on the pieces arising from the solution to the linearized equations of motion.

Having understood the connection between the gravitational on-shell action and the stress tensor, we can connect to the adiabatic effective action (3.54). Prior to the Legendre transformation, the ideal part is simply the free energy evaluated on the sound mode solution. On the grsK solution we can represent it using the rescaled thermal vector

[26] We have written this term in the second order stress tensor in a form inspired by the classification of hydrodynamic transport introduced in [62].

$\mathbf{b}^\mu_{\text{L/R}} = \left(\frac{u^\mu}{T}\right)_{\text{L/R}}$, which is

$$\mathbf{b}^\mu_{\text{L/R}} = b\left[\left(1 - \int_k \mathbb{S}\, \frac{b^{d-2}\,\mathfrak{w}^2}{d\,(d-1)}\,\breve{\mathcal{Z}}_{\text{L/R}}\right)\frac{\partial}{\partial v} - \int_k i\,\frac{b^{d-2}\,\mathfrak{q}\,\mathfrak{w}}{d\,(d-1)}\,\mathbb{S}_i\,\breve{\mathcal{Z}}_{\text{L/R}}\,\frac{\partial}{\partial x^i}\right]. \tag{3.56}$$

In order to compute the quadratic part of the action it will suffice to know rescaled thermal vector accurate to linear order in amplitude, consistent with our identification using the stress tensor. The reason is that the amplitude expansion of the ideal fluid action

$$S_{\text{ideal}}[\mathfrak{Z}] = \int d^dx\,\sqrt{-\gamma_{\text{R}}}\,\left[-(\gamma_{\text{R}})_{\mu\nu}\,\mathbf{b}^\mu_{\text{R}}\,\mathbf{b}^\nu_{\text{R}}\right]^{-\frac{d}{2}} - \int d^dx\,\sqrt{-\gamma_{\text{R}}}\,\left[-(\gamma_{\text{R}})_{\mu\nu}\,\mathbf{b}^\mu_{\text{R}}\,\mathbf{b}^\nu_{\text{R}}\right]^{-\frac{d}{2}}\,, \tag{3.57}$$

results in two terms. One is the contribution which leads to the ideal fluid stress tensor in (3.49) (from the variation of the background metric $\gamma_{\mu\nu}$), and the other the 'heat current' term, which originates from the change of the rescaled thermal vector (the variation $\delta\mathbf{b}^\mu = \mathbf{b}^\mu - \frac{1}{b}\partial_v$), cf., [62]. Using the on-shell relation for the ideal fluid one can check that these two contributions nicely sum up (up to the aforementioned Legendre transform) into the terms arising from (3.57).

To summarize, we find that the contact term contribution (3.35) in its entirety originates from the propagation of an ideal fluid on (3.34). It should now be clear that the curious factor of $(d-6)$ in (3.35) is just a numerical accident; it arises due to the relation between the metric components in $\gamma_{\mu\nu}$.[27]

This is structurally similar to the Wilsonian influence functional in the vector sector, which captures the shear modes driving momentum diffusion in chapter 2. The main difference in that case was that since there was no propagating mode; the ideal piece was purely expressible in terms of a contact term, and moreover could be isolated directly from the boundary terms of the Einstein-Hilbert action.

For the sound mode we can re-express the on-shell action as an ideal piece and a term that captures sound attenuation. To wit,

$$\frac{1}{c_{\text{eff}}}\,S[\mathfrak{Z}] = S_{\text{ideal,LT}}[\mathfrak{Z}] - \frac{d-2}{d^2\,(d-1)^3}\int_k b^d\,k^4\left(\breve{\mathcal{Z}}_d^\dagger\,\Gamma_s(\omega,\mathbf{k})\left[\breve{\mathcal{Z}}_a + \left(n_{\!B} + \frac{1}{2}\right)\breve{\mathcal{Z}}_d\right] + \text{cc}\right). \tag{3.58}$$

Here the 'LT' term in the subscript is present to remind us that one should Legendre transform the ideal fluid part to account for the fact that we have a sound pole.

[27] There is nothing special about relativistic conformal fluids in six spacetime dimensions, nor are Schwarzschild-AdS$_7$ black holes (and the dual $(0,2)$ SCFT plasma) in any way singled out.

Curvature corrections within Γ_s: From the Γ_s term we can also reproduce the Weyl curvature contribution in (3.54) above, as promised. One finds

$$S[\mathcal{Z}] \supset -\int_k \frac{8\,(d-1)^2\,(d-3)}{b^d}\,\mathfrak{q}^2 \left(\breve{\boldsymbol{\zeta}}^\dagger_{\rm R}\,\breve{\boldsymbol{\zeta}}_{\rm R} - \breve{\boldsymbol{\zeta}}^\dagger_{\rm L}\,\breve{\boldsymbol{\zeta}}_{\rm L}\right). \tag{3.59}$$

We have used the on-shell relation (3.31) between $\breve{\mathcal{Z}}$ and $\breve{\boldsymbol{\zeta}}$ to make clear that this term arises from the spatial curvature of the boundary geometry. Note that the numerical coefficient in (3.59) is determined by the stationary limit of the attenuation function, $\Gamma_s(0, \mathbf{k})$, identified before.

To summarize, the non-dissipative content within $S[\mathcal{Z}]$ agrees well with the class L action (3.54). The contact terms and the two derivative kinetic terms of the phonon are packaged cleanly within the ideal fluid part. The remaining terms from the influence functional describe the attenuation of the phonon as it propagates along the plasma and also higher derivative non-dissipative corrections predicted by the class L action.

Chapter 4

Discussion and directions

In this thesis we have presented the construction of a model of fluctuating hydrodynamics and therefore open quantum systems with memory. We imagine coupling some probe degrees of freedom to conserved currents of a neutral plasma modelled as a holographic conformal field theory. While a subset of modes of the conserved currents relax back to equilibrium quickly (typically in a time-scale of order the inverse temperature), conservation ordains the presence of long-lived modes. Any coupling to such modes will retain long-term memory leading to challenges in constructing an open effective description. While one can eschew the standard low energy gradient expansion and come up with alternate ways to tackle the problem, we propose a Wilsonian effective field theory approach and instead construct a local effective action for the long-lived hydrodynamic fields.[1]

The key idea we employed is to exploit holography and use the bulk gravity as a guidepost for constructing such an effective description. Taking inspiration from studies of systems with gauge invariance, we establish a formalism in terms of designer scalar systems that captures the essential physical content. Our primary thesis is that for systems with both fast and slow modes, one should decouple their dynamics at leading quadratic order and understand each in their own terms. In particular, for the slow modes which pertain to memory, one should isolate the subspace of low energy Goldstone-like degrees of freedom, the hydrodynamic moduli space, and use them instead of the background sources to parameterize the effective action.

Motivated by the above reasoning we argued that the contribution from a particular point in hydrodynamic moduli space is characterized by the Wilsonian influence

[1] Note that alternatively one could restrict themselves to evaluating on-shell observables and take up an approach prioritizing only the correlation functions, or one could work with non-local descriptions which manifestly keep track of the long-term memory. As long as they reproduce the intricate long-time behaviour of hydrodynamic correlators they are admissible alternatives. However, the criterion for a good physical theory is that it allows to progressively and systematically take into account corrections from quantum and statistical fluctuations within the fluid. This makes the Wilsonian EFT approach highly attractive.

functional $\mathcal{S}_{\text{WIF}}$, which we have chosen to parameterize in terms of the sources for the Markovian (fast) modes, and the fields (or vevs) for the non-Markovian (slow) modes. We established that this object can be effectively computed using the holographic dual description and used it to describe a universal framework to understand charge and momentum diffusion in chapter 2. The corresponding bulk description involved a simple characterization of the designer scalars fields in terms of a single Markovianity index parameter. Our main result is the Wilsonian influence functional of the diffusive modes expressed as a functional of the expectation values of charge and momentum densities of the boundary plasma. In chapter 3 we extended this analysis to include Goldstone modes with a decay width. Specifically, we analyzed the dynamics of energy transport and the physics of associated sound modes in a relativistic thermal plasma. The key result is the Wilsonian influence functional parameterized directly in terms of the boundary expectation value of the energy flux operator.

While the field theory result for energy transport shares many characteristics with the corresponding effective action for diffusive modes, there are interesting technical peculiarities in the gravitational description. For the diffusive modes, one was able to repackage the bulk dynamics into non-minimally coupled designer scalar fields (one per polarization), where the non-minimal coupling was captured by an auxiliary dilaton, whose primary characterization was its asymptotic fall-off rate (the Markovianity index). This auxiliary dilaton however modulated only the interactions in the radial direction, i.e., as a function of energy scale in the field theory, but was spatially homogeneous. This no longer holds for the bulk dual of the sound mode; the auxiliary dilaton has a non-trivial modulation along the spatial directions of the boundary. It nevertheless remains true that the dual field has a non-Markovianity index $3-d$ for spatially inhomogeneous modes.

Owing to the spatial modulation of the designer field dynamics in the gravitational description, one finds there to be an interesting discontinuity in the dynamics between vanishing and non-vanishing spatial momentum. As explained at the outset, in order to isolate the physics of sound, it suffices to imagine there being an infra-red cut-off in momenta and to study modes which are long-wavelength above this cut-off scale. To understand the physics of the soft zero modes however, needs a bit more work. These modes can be understood as large diffeomorphisms of the background, but we have not attempted to quantize this system. It would be interesting to do so.[2]

The gravitational analysis of chapter 3 gives a beautiful picture for the dynamics of energy transport. The physical phonon degree of freedom is part of the ideal fluid, and thus should be captured by the hydrodynamic sigma model (Class L) actions of [62]. Owing to the presence of a gapless mode, one should not construct the sigma model

[2]Alternately, one could work with a physical cut-off, say by placing the plasma on a compact spatial volume, e.g., on $\mathbf{S}^{d-1}$.

action directly, but rather write the Wilsonian analog, which effectively captures the two derivative kinetic term of the Goldstone mode. Since this part is conservative, the resulting Schwinger-Keldysh effective action is factorized into L and R pieces, as indicated in (3.58). The dynamical information, viz., the dispersion relation (3.31), is obtained from this action by the constrained variational principle outlined in the aforementioned reference. The Class L action also captures higher order non-dissipative contributions, like the background curvature coupling (3.59).[3]

Once we have separated out the propagating mode, what is left is the physics of sound attenuation. Since this is driven by the leading order dissipative terms, the shear viscosity of a conformal plasma, the structure is isomorphic to that found for momentum diffusion in chapter 2. In other words, the physical influence phase of the sound mode is the non-factorized part of the Wilsonian influence functional, with a physical kernel $\Gamma_s(\omega, \mathbf{k})$. The dissipative part of this kernel is not captured by the Class L sigma model actions, as it should be; it is these frictional effects which drive the plasma to behave as an open quantum system. So in a sense, $\Gamma_s(\omega, \mathbf{k})$ is the physical influence functional for phonons, though their complete dynamics also requires the kinetic operator arising from the conservative part of the action.

At a conceptual level, it is satisfying to see elements of the fluid/gravity correspondence emerge directly from the gravitational analysis. We emphasize that while earlier discussions either focused solely on the retarded response to get hydrodynamic correlators from the study of quasinormal modes, or left the conservation equations off-shell as in the fluid/gravity correspondence, we are explicitly solving all of the dynamical equations and evaluating the boundary observables. In our results we see clear glimmers of the structure of hydrodynamic effective field theories. Whether the localization to the hydrodynamic moduli space can be understood broadly in terms of the vision outlined in [139] remains however to be checked.

From a certain perspective, it is quite amazing that the analysis of the bulk theory instructs us to directly focus on computing the Wilsonian influence functional for the non-Markovian fields. Recall that, one usually wants to freeze the classical sources (non-normalizable modes) while functionally integrating over the normalizable modes to obtain a generating functional for the correlators. As we saw in our discussion, freezing the non-normalizable modes results in localizing the normalizable modes to the hydrodynamic moduli space. However, if we focus on solving the non-Markovian dynamics with alternate (Neumann) boundary conditions one directly lands on the Wilsonian influence functional. It is rather remarkable that, despite many years of familiarity with the AdS/CFT correspondence, the gravitational description naturally implements these considerations by allowing for a simple change of boundary conditions to implement these

[3] There is a specific prediction for fourth order (in gradients) transport data contained in the $\mathfrak{q}^4$ terms of (J.9). We have not attempted to classify the terms in the Class L action that are responsible for it.

Legendre transforms.

Having the Wilsonian influence functional at hand, we can go back and understand how to construct the generating functional of conserved current correlators. This can be achieved following the usual construction of the quantum effective action. One can obtain the (non-local) Schwinger-Keldysh influence functional in terms of the sources of both Markovian and non-Markovian fields, by first solving for the non-Markovian sources in terms of the hydrodynamic moduli and then performing a constrained functional integral over these light fields.

While our analysis was focused on fluctuating hydrodynamics of a neutral plasma, it can be readily extended to include fluids with non-vanishing amount of background conserved charges. This gives rise to interesting transport behaviour where the dynamics of energy transport and momentum diffusion couple to that of the charge current. In the case of vector perturbations, it leads to the mixing of the modes describing Markovian conserved currents and the non-Markovian momentum diffusion. For scalar perturbations the analogous effect is the mixing of two non-Markovian modes corresponding to charge diffusion and energy transport. However, in agreement with the paradigm we put forth of obtaining decoupled dynamics of slow and fast modes at the quadratic order, it is possible to decouple the dynamics as shown in [134, 140]. It will be interesting to test our ideas on more complicated models such as holographic superfluids and holographic metals as envisioned in [93].

Through out this work we have preferred to characterize the Markovianity of a probe using its Markovianity index identified from the corresponding asymptotic fall-offs. It is useful to ask if one could come up with alternate criteria to characterize these probes. For example, from the fluid picture it is tempting to think that compared to non-Markovian modes which dissipate at a slower rate, Markovian excitations which quickly dissolve down should back-react weakly on the microscopic environment. From the perspective of the bulk gravitational theory this suggests that an alternate characterization of the Markovianity of a probe could be possible from the strength of its back reaction on the near horizon geometry. A near horizon diagnostic for Markovianity is also compelling from the perspective of the old membrane paradigm.

As a closely related point, recall that AdS/CFT dictionary accommodates alternate quantization choices for specific windows of operator dimensions in the CFT. This phenomenon is known to interplay beautifully with the physics of RG flows to deep IR phases in the CFT [111]. In the context of holographic Wilsonian open effective theories this foreshadows the following interesting situation. Notice that the Markovianity index of a probe reverses between alternate quantization schemes. Therefore a probe which would have been identified as Markovian in the UV theory could undergo a severe metamorphosis under RG flow to become a non-Markovian mode in the IR and vice versa. It would be interesting to explore this phenomenon from the perspective of open effective theories.

We wish to point out a peculiar feature of how the grSK geometry encodes the thermality of the dynamics around the black brane. Recall that the standard derivation of the Hawking effect begins by arguing that natural quantization appropriate for a future observer is distinguished from that for an early time observer. The thermality of the late time correlations follows from correctly identifying the corresponding field modes, which are related to the early time modes via a Bogoliubov transformation. In contrast, the Hawking effect on the grSK saddle directly follows from analytical structure of the geometry. Thus we reach the unusual conclusion that Hawking radiation which is inherently thought of as a quantum phenomenon, is a classical deformation about the Schwinger-Keldysh saddle. However, from the perspective of real-time path integrals this is not a completely surprising result. Many inherently quantum effects like tunneling across potential barriers and decay of meta-stable vacua are dominated by classical field configurations along the Schwinger-Keldysh time contour [25]. The reader should compare such results with computations of instanton effects in field theory, where fundamentally quantum processes get repackaged as classical dynamics along the imaginary time direction. In fact, similar interpretations of Hawking radiation have been proposed by various authors where the radiation originates from the tunneling of an interior particle to the outside [141, 142]. However, the relation of these computations to our treatment based on the grSK geometry is not clear.

In this thesis we study the Wilsonian influence functional only up to quadratic order in field amplitudes. Correspondingly, in the bulk we analyse the Einstein-Hilbert action expanded to the same order in metric perturbations considering them as small deformations above the black brane. One may take the amplitude expansion to be controlled by either the bulk Newton's constant or the background energy density of the boundary plasma. While it is promising that we get consistent results for the quadratic theory, a convincing evaluation of our proposal is possible only after constructing an interacting theory of fluctuating hydrodynamics. In order to do this, we need to derive the corrections to the WIF due to gravitational interactions by computing appropriate Feynman-Witten diagrams [17]. One could further, systematically add in corrections due to bulk loop diagrams, which are subleading compared to tree diagrams at any given order in amplitude expansion. An interesting import from the real-time approach facilitated by the grSK saddle is that the thermality in the fluid fluctuations is qualitatively different from the quantum corrections in the bulk theory; though gravitational loop effects and Hawking fluctuations are both considered to be suppressed in the large N limit, their contributions are cleanly distinguished on the grSK saddle.[4] It would be interesting to check if the boundary conditions we proposed continue to yield a local Wilsonian influence functional in the presence of interactions.

It is worth contrasting the analysis of real-time fluctuating hydrodynamics with the

[4]See [28, 143] for early discussions of gravitational loop contributions to hydrodynamics.

earlier work on the fluid/gravity correspondence [22, 23]. The focus in that work and extensions thereof was to construct the gravity dual of a fluid flow of the boundary CFT. In particular, given a holographic system whose stress tensor one-point function can be parameterized in terms of hydrodynamic variables, viz., temperatures and velocities, obeying the conservation equations, the fluid/gravity paradigm constructs a spacetime geometry characterized by this hydrodynamic data. By virtue of focusing on thermal one-point functions, that analysis had a technical advantage of being able to work with $SO(d-1)$ tensor decomposition, but more importantly was fully non-linear in amplitudes of departures from thermal equilibrium.

The open effective field theory paradigm however addresses a slightly different question: What is the gravitational dual of a fluctuating plasma? More precisely, realizing that the plasma consists of both short-lived and long-lived modes, we seek to parameterize its dynamics in terms of the sources for the former and the operators (or fields) corresponding to the latter. This was the philosophy originally outlined in [42] for the study of the Wilsonian influence functional of the plasma. One has to not only keep track of the dissipative pieces which relate to infalling quanta in the dual gravity, but also the Hawking quanta that correspond to stochastic fluctuations. But this is precisely what has been achieved in terms of the designer fields, which now parameterize the fluctuating bulk metric. While they are not manifestly $SO(d-1)$ covariant, and our analysis thus far has been restricted to linear order in amplitudes, the close resemblance of the ingoing part of our solutions to those obtained in the fluid/gravity literature makes it highly suggestive that it should be possible to bootstrap onto a non-linear solution.

Finally, we remind that a crucial ingredient of our analysis, the grSK saddle, is only a conjectured prescription which has survived many consistency checks so far. The efficacy of this prescription beyond the probe approximation (after including back-reaction on the geometry) is a question of high importance. A clear derivation of the grSK saddle could inform us how to define it away from equilibrium, thereby facilitating the derivation of a more complete description of fluctuating hydrodynamics in the boundary. Another related direction is to identify gravitational saddles which are dual to generalised Schwinger-Keldysh time contours in the boundary field theory. A pertinent application of such a geometry would be to construct an effective theory of out-of-time ordered hydrodynamics.[5] It would be fascinating if this goal can be realized.

[5]Such a theory would perhaps make precise the relation between hydrodynamics and quantum chaos suggested in [144, 145].

Appendices

A Designer scalar analysis

The designer scalar wave equation (2.3) can be written in several equivalent forms that are useful not only for comparison with similar expressions that exist in literature, but also to intuit some of the general features by quick inspection. We record a few of these for the benefit of the reader:

- A trivial rewriting that follows from (2.3) is

$$(\mathbb{D}_+)^2\varphi_{\mathfrak{M}} + \mathfrak{M}\, r\, f\, \mathbb{D}_+\varphi_{\mathfrak{M}} + \left(\omega^2 - k^2 f\right)\varphi_{\mathfrak{M}} = 0\,. \tag{A.1}$$

  which when converted to the explicit radial derivatives using (1.40) leads to:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\mathfrak{M}+2} f\,\frac{\mathrm{d}\varphi_{\mathfrak{M}}}{\mathrm{d}r}\right) - i\,\omega\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\mathfrak{M}}\,\varphi_{\mathfrak{M}}\right) + r^{\mathfrak{M}}\,\frac{\mathrm{d}\varphi_{\mathfrak{M}}}{\mathrm{d}r}\right] - k^2\, r^{\mathfrak{M}-2}\,\varphi_{\mathfrak{M}} = 0\ . \tag{A.2}$$

- Usual study of linearized fields in a black hole background is carried out in the Schwarzschild time coordinate rather than the ingoing time v. We can recover this form by a simple field redefinition, $\varphi^{\text{Schw}}_{\mathfrak{M}} \equiv e^{\frac{\beta\omega}{2}\zeta}\,\varphi_{\mathfrak{M}}$, which leads to

$$\left[\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)^2 + \left(\mathfrak{M} + \frac{d(1-f)}{f}\right) r^3\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\omega^2 - k^2 f}{f^2}\right]\varphi^{\text{Schw}}_{\mathfrak{M}} = 0\ , \tag{A.3}$$

  Time reversal for these modes are simply achieved by $\omega \mapsto -\omega$: this is evident from the fact that their equations have no linear terms in ω.

- A Schrödinger-like form in the regular tortoise coordinate can be obtained by defining $\varphi^{\text{Schr}}_{\mathfrak{M}} \equiv r^{\frac{\mathfrak{M}}{2}}\,\varphi^{\text{Schw}}_{\mathfrak{M}} = r^{\frac{\mathfrak{M}}{2}}\,e^{\frac{\beta\omega}{2}\zeta}\,\varphi_{\mathfrak{M}}$. One obtains a standard eigenvalue equation:

$$\left[\left(r^2 f\frac{\mathrm{d}}{\mathrm{d}r}\right)^2 - V_{\mathfrak{M}} + \omega^2\right]\varphi^{\text{Schr}}_{\mathfrak{M}} = 0\ , \tag{A.4}$$

with

$$V_{\mathcal{M}} \equiv f \left[k^2 + \frac{d}{2}\, \mathcal{M}\, r^2 (1-f) + \frac{1}{4} \mathcal{M}(\mathcal{M}+2)\, r^2 f \right] . \tag{A.5}$$

This form is often useful to ascertain stability of the solution to linearized perturbations and argue for the self-adjointness of the linearized fluctuation operator. In the tensor and vector sector of Einstein equations, these coincide with the Regge-Wheeler-Zerelli master fields (a characterization we avoid in the text) studied in the literature as can be inferred from Eq. (3.10) and Eq. (3.13) of [98]. The precise identifications are: $\varphi_T^{\rm RWZ} = \varphi_{d-1}^{\rm Schr}\,, \quad \varphi_V^{\rm RWZ} = \varphi_{1-d}^{\rm Schr}$. We furthermore note that the differential operator

$$\nabla_{\rm o}^2 = \left(r^2 f \frac{\partial}{\partial r} \right)^2 - \frac{\partial^2}{\partial t^2} \tag{A.6}$$

is the so-called orbit-space Laplacian which is used in the parameterization of the gravitational perturbations [106]

A.1 Gradient expansion of the Green's function

In the main text we have considered the gradient expansion of the designer scalar to quadratic order in the frequency and momenta. We now give a general analysis in particular highlighting some of the central features and explicit results accurate to third order, viz., $\mathcal{O}(\omega^3, k^2\,\omega)$. Firstly, we note that parameterizing the ingoing solution as

$$G^{\rm in}_{\mathcal{M}}(\omega, r, \mathbf{k}) \equiv e^{-i\,\mathfrak{w}\, F(\mathcal{M},\xi)}\, \Xi(\mathcal{M},\xi)\,, \tag{A.7}$$

we can then rewrite (2.11) in terms of the following recursion relation for $\Xi(\mathcal{M},\xi)$:

$$\begin{split} \xi^{\mathcal{M}+2-d}(\xi^d-1)\, \frac{\rm d}{{\rm d}\xi} \Xi(\mathcal{M},\xi) &= 2i\,\mathfrak{w}\, \left[\Xi(\mathcal{M},\xi) - \Xi(\mathcal{M},1)\right] \\ &\quad + \int_1^\xi dy\; \Xi(\mathcal{M},y) \left[\mathfrak{q}^2\, y^{\mathcal{M}-2} + \mathfrak{w}^2\, \frac{{\rm d}\Delta(\mathcal{M},y)}{{\rm d}y} \right] . \end{split} \tag{A.8}$$

As before, we have performed one integral and fixed the constant of integration to remove the pole at the horizon. This recursion relation can then be used to readily give the appropriate first order ODEs at any order in derivative expansion. Till the third order in the derivative expansion, we can parametrize the ingoing solution as in (2.18).

Solution up to quadratic order: Upon expanding out we find the differential equations for the functions $\{F, H_k\}$ to be given quite simply since the leading order contribution comes from $G^{\rm in}_{\mathcal{M}}(\omega = 0, r, \mathbf{k} = 0) = 1$. This then implies that these functions satisfy simple differential equations that can be integrated up once immediately. One finds them to satisfy.

$$\begin{aligned}\frac{\mathrm{d}F(\mathcal{M},\xi)}{\mathrm{d}\xi} + \xi^{d-2}\,\frac{\xi^{\mathcal{M}}-1}{\xi^{\mathcal{M}}(\xi^d-1)} &= 0\,,\\ \frac{\mathrm{d}H_k(\mathcal{M},\xi)}{\mathrm{d}\xi} + \frac{\xi^{d-2}}{\mathcal{M}-1}\,\frac{\xi^{\mathcal{M}-1}-1}{\xi^{\mathcal{M}}(\xi^d-1)} &= 0\end{aligned} \tag{A.9}$$

As advertised, we have performed the subtraction needed to make the derivatives of the functions analytic near $\xi = 1$. Comparing these equations with the defining equation for the incomplete beta function (E.2) we see that we can immediately write down the solution as advertised in (2.25).

The function $H_\omega(\mathcal{M},\xi)$ obeys a second order ODE sourced by $F(\mathcal{M},\xi)$ which can also be written down:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{\mathcal{M}+2-d}(\xi^d-1)\,\frac{\mathrm{d}H_\omega(\mathcal{M},\xi)}{\mathrm{d}\xi}\right) + (\xi^{\mathcal{M}}+1)\frac{\mathrm{d}F(\mathcal{M},\xi)}{\mathrm{d}\xi} = 0\,. \tag{A.10}$$

Before solving this equation, it is useful to attend to the function $\Delta(\mathcal{M},\xi)$ introduced in (2.26). This function satisfies a first order ODE

$$\frac{\mathrm{d}\Delta(\mathcal{M},\xi)}{\mathrm{d}\xi} - (\xi^{\mathcal{M}}+1)\,\frac{\mathrm{d}F(\mathcal{M},\xi)}{\mathrm{d}\xi} = \frac{\mathrm{d}\Delta(\mathcal{M},\xi)}{\mathrm{d}\xi} + \xi^{d-2-\mathcal{M}}\,\frac{\xi^{2\mathcal{M}}-1}{\xi^d-1} = 0\,. \tag{A.11}$$

A simple calculation then reveals that we can massage (A.10) into a more tractable form of a first order ODE using our definition of $\Delta(\mathcal{M},\xi)$ using (A.11). We have

$$\frac{dH_\omega(\mathcal{M},\xi)}{d\xi} + \xi^{d-2}\frac{\widehat{\Delta}(\mathcal{M},\xi)}{\xi^{\mathcal{M}}(\xi^d-1)} = 0\ . \tag{A.12}$$

where as mentioned in the main text around (2.23) we see an explicit occurrence of a hatted function in the source term for H_ω. Using the explicit parameterization of $\hat{\Delta}(\mathcal{M},\xi)$ in terms of the incomplete beta functions written in their defining series form (E.1) we can again trivially integrate (A.12) to arrive at (2.27).

In addition to the differential equations and the explicit solutions in terms of incomplete beta functions, it is also helpful to have at hand the asymptotic form of these functions which will play an important role in our analysis of normalizability and boundary conditions. We can use the defining series of the incomplete beta functions (E.1) and write down immediately,

$$\begin{aligned} F(\mathcal{M},\xi) &= -\sum_{n=0}^{\infty}\frac{1}{(nd+1+\mathcal{M})\xi^{nd+1+\mathcal{M}}}+\sum_{n=0}^{\infty}\frac{1}{(nd+1)\xi^{nd+1}}\,, \\ (\mathcal{M}-1)H_k(\mathcal{M},\xi) &= -\sum_{n=0}^{\infty}\frac{1}{(nd+1+\mathcal{M})\xi^{nd+1+\mathcal{M}}}+\sum_{n=0}^{\infty}\frac{1}{(nd+2)\xi^{nd+2}}\,, \\ \Delta(\mathcal{M},\xi) &= -\sum_{n=0}^{\infty}\frac{1}{(nd+1+\mathcal{M})\xi^{nd+1+\mathcal{M}}}+\sum_{n=0}^{\infty}\frac{1}{(nd+1-\mathcal{M})\xi^{nd+1-\mathcal{M}}}\,. \end{aligned} \tag{A.13}$$

Similarly, a double series expansion can be written down for H_ω.

The functions at third order in gradients: At the next order we obtain for our functions I_k and I_ω, the differential equations

$$\begin{aligned} \frac{\mathrm{d}I_k(\mathcal{M},\xi)}{\mathrm{d}\xi}+2\,\xi^{d-2}\,\frac{\widehat{H}_k(\mathcal{M},\xi)}{\xi^{\mathcal{M}}(\xi^d-1)} &= 0\,, \\ \frac{\mathrm{d}I_\omega(\mathcal{M},\xi)}{\mathrm{d}\xi}+2\,\xi^{d-2}\,\frac{\widehat{H}_\omega(\mathcal{M},\xi)}{\xi^{\mathcal{M}}(\xi^d-1)} &= 0\,. \end{aligned} \tag{A.14}$$

We have employed the hat decoration to simplify the presentation of the source terms. The solution to the functions $I_a(\mathcal{M},\xi)$ can be obtained using the lower order functions we have derived. At each order the trick is to use series representation of the incomplete beta function (E.1) to convert the final integral into the ODE for the incomplete beta function (E.2). Since $H_k(\mathcal{M},\xi)$ was a simple combination of incomplete beta functions, it follows that I_k will be given a single series representation (like H_ω in (2.27)). On the other hand I_ω will be written in terms of a double series representation. We will outline the structure of these solutions elsewhere, but for now will focus on obtaining the results for the non-Markovian sector $\mathcal{M} < -1$ assuming that we have solved the equations for the Markovian sector $\mathcal{M} > -1$.

To obtain the solutions in the non-Markovian case, especially with the view towards determining their asymptotic behaviour, we will exploit a strategy similar to the one used to determine the functions $F(-\mathcal{M},\xi)$, $H_\omega(-\mathcal{M},\xi)$ and $H_k(-\mathcal{M},\xi)$. The idea is to define judicious combinations that simplify the analytic continuation from $\mathcal{M}\to-\mathcal{M}$. Consider the following functions:

$$
\begin{aligned}
\Delta_k(\mathcal{M},\xi) &= (\mathcal{M}+1)\Big[I_k(-\mathcal{M},\xi) + \widehat{\Delta}(-\mathcal{M},\xi)\,\widehat{H}_k(-\mathcal{M},\xi) - \Delta(-\mathcal{M},1)\,H_k(-\mathcal{M},1)\Big] \\
&\quad + (\mathcal{M}-1)\Big[I_k(\mathcal{M},\xi) + \widehat{\Delta}(\mathcal{M},\xi)\,\widehat{H}_k(\mathcal{M},\xi) - \Delta(\mathcal{M},1)\,H_k(\mathcal{M},1)\Big] \\
\Delta_\omega(\mathcal{M},\xi) &= I_\omega(-\mathcal{M},\xi) + \widehat{\Delta}(-\mathcal{M},\xi)\,\widehat{H}_\omega(-\mathcal{M},\xi) - \Delta(-\mathcal{M},1)\,H_\omega(-\mathcal{M},1) \\
&\quad + I_\omega(\mathcal{M},\xi) + \widehat{\Delta}(\mathcal{M},\xi)\,\widehat{H}_\omega(\mathcal{M},\xi) - \Delta(\mathcal{M},1)\,H_\omega(\mathcal{M},1) \\
&\quad + \frac{1}{2(\mathcal{M}-1)}\,\Delta_k(\mathcal{M},\xi) - \frac{1}{6}\Big[\widehat{\Delta}(\mathcal{M},\xi)^3 + \Delta(\mathcal{M},1)^3\Big]
\end{aligned}
\tag{A.15}
$$

By explicit computation we can check that these functions satisfy the following ODEs:

$$
\begin{aligned}
\frac{\mathrm{d}\Delta_k(\mathcal{M},\xi)}{\mathrm{d}\xi} + 2\,\frac{\xi^{d-3}}{(\xi^d-1)}\,\widehat{\Delta}(\mathcal{M},\xi) &= 0\,, \\
\frac{\mathrm{d}\Delta_\omega(\mathcal{M},\xi)}{\mathrm{d}\xi} + \frac{\xi^{d-2}}{\xi^{\mathcal{M}}\,(\xi^d-1)}\,\widehat{\Delta}(\mathcal{M},\xi)\left[\widehat{\Delta}(\mathcal{M},\xi) + \frac{\xi^{\mathcal{M}-1}}{\mathcal{M}-1}\right] &= 0
\end{aligned}
\tag{A.16}
$$

We can solve these equations at large ξ in a Taylor expansion and use the freedom of picking the integration constant to demand the asymptotic expansion

$$
\begin{aligned}
\lim_{\xi\to\infty}\left\{\Delta_k(\mathcal{M},\xi) - 2\frac{\xi^{\mathcal{M}-3}}{(\mathcal{M}-1)(\mathcal{M}-3)}\right\} &= 0\,, \\
\lim_{\xi\to\infty}\Delta_\omega(\mathcal{M},\xi) &= 0\,.
\end{aligned}
\tag{A.17}
$$

The asymptotic growth of the functions $I_k(-\mathcal{M},\xi)$ and $I_\omega(-\mathcal{M},\xi)$ can then be found by using the asymptotic solution for the Markovian sector, and inverting (A.15).

At fourth order: Continuing thus, if we parameterize the fourth order contribution to the function $\Xi(\mathcal{M},\xi)$ as $\Xi(\mathcal{M},\xi) = \mathfrak{q}^4\,J_k(\mathcal{M},\xi) + \mathfrak{w}^4\,J_\omega(\mathcal{M},\xi) + \mathfrak{w}^2\mathfrak{q}^2\,J_{\omega k}(\mathcal{M},\xi)$ we obtain the equations for the functions J_a to be

$$
\begin{aligned}
&\frac{\mathrm{d}J_k(\mathcal{M},\xi)}{\mathrm{d}\xi} + \frac{\xi^{d-2}}{\xi^{\mathcal{M}}(\xi^d-1)}\int_1^\xi dy\,H_k(\mathcal{M},y)\,y^{\mathcal{M}-2} = 0\;, \\
&\frac{\mathrm{d}J_\omega(\mathcal{M},\xi)}{\mathrm{d}\xi} + 2\,\frac{\xi^{d-2}}{\xi^{\mathcal{M}}(\xi^d-1)}\left[\widehat{I}_\omega(\mathcal{M},\xi) + \int_1^\xi dy\,H_\omega(\mathcal{M},y)\frac{\mathrm{d}\Delta(\mathcal{M},y)}{\mathrm{d}y}\right] = 0\;, \\
&\frac{\mathrm{d}J_{\omega k}(\mathcal{M},\xi)}{\mathrm{d}\xi} + 2\,\frac{\xi^{d-2}}{\xi^{\mathcal{M}}\,(\xi^d-1)}\left[\widehat{I}_k(\mathcal{M},\xi) + \int_1^\xi dy\,\left(H_\omega(\mathcal{M},y)\,y^{\mathcal{M}-2} + H_k(\mathcal{M},y)\,\frac{\mathrm{d}\Delta(\mathcal{M},y)}{\mathrm{d}y}\right)\right] = 0\;.
\end{aligned}
\tag{A.18}
$$

It is clear that these can again be tackled as detailed above. We hope to report on useful explicit parameterizations of the functions at a later date.

A.2 Non-Markovian ingoing Green's function at third order

We can now put together all the functions up to third derivative order and write the ingoing solution for the non-Markovian case in a manner similar to (2.39) and obtain a conveniently factorized form:

$$\begin{aligned} G^{\text{in}}_{-\mathcal{M}} = e^{-i\,\mathfrak{w} F(\mathcal{M},\xi)} \left[1 - \frac{K^{\text{in}}_{-\mathcal{M}}}{b^{\mathcal{M}-1}}\, \Xi_{\text{nn}}(\mathcal{M},\xi)\right] \Bigg[1 + \frac{\mathcal{M}-1}{\mathcal{M}+1}\,\mathfrak{q}^2\, H_k(\mathcal{M},\xi) + \mathfrak{w}^2\, H_\omega(\mathcal{M},\xi) \\ - i\mathfrak{w}\,\mathfrak{q}^2\,\frac{\mathcal{M}-1}{\mathcal{M}+1}\, I_k(\mathcal{M},\xi) - i\,\mathfrak{w}^3\, I_\omega(\mathcal{M},\xi) + \cdots\Bigg]. \end{aligned} \tag{A.19}$$

We have written this expression in terms of the Markovian data using (2.26) and (A.16) and parameterized it further in terms of a particular mode function $\Xi_{\text{nn}}(\mathcal{M},\xi)$ which is a non-normalizable mode function and a dispersion function $K^{\text{in}}_{-\mathcal{M}}$. These two pieces of data are given by:

$$\begin{aligned} K^{\text{in}}_{-\mathcal{M}}(\omega,\mathbf{k}) &\equiv b^{\mathcal{M}-1}\Bigg[-i\,\mathfrak{w} + \frac{\mathfrak{q}^2}{\mathcal{M}+1} - \mathfrak{w}^2\,\Delta(-\mathcal{M},1) \\ &\qquad + 2i\,\mathfrak{w}\left[\mathfrak{q}^2\, H_k(-\mathcal{M},1) + \mathfrak{w}^2\, H_\omega(-\mathcal{M},1)\right] + \cdots\Bigg] \\ &= b^{\mathcal{M}-1}\Bigg[-i\,\mathfrak{w} + \frac{\mathfrak{q}^2}{\mathcal{M}+1} + \mathfrak{w}^2\,\Delta(\mathcal{M},1) + i\mathfrak{w}^3\left(\Delta(\mathcal{M},1)^2 - 2\,H_\omega(\mathcal{M},1)\right) \\ &\qquad + 2i\,\frac{\mathfrak{w}\,\mathfrak{q}^2}{\mathcal{M}+1}\left(\Delta(\mathcal{M},1) - (\mathcal{M}-1)\,H_k(\mathcal{M},1)\right) + \cdots\Bigg], \end{aligned} \tag{A.20}$$

and

$$\begin{aligned} \Xi_{\text{nn}}(\mathcal{M},\xi) \equiv \Delta(\mathcal{M},\xi) - 2\left[\frac{\mathcal{M}-1}{\mathcal{M}+1}\,\mathfrak{q}^2\, H_k(\mathcal{M},\xi) + \mathfrak{w}^2\, H_\omega(\mathcal{M},\xi)\right]\widehat{\Delta}(\mathcal{M},\xi) \\ + \left(\frac{\mathfrak{q}^2}{\mathcal{M}+1} - \frac{\mathfrak{w}^2}{2(\mathcal{M}-1)}\right)\Delta_k(\mathcal{M},\xi) + \mathfrak{w}^2\,\Delta_\omega(\mathcal{M},\xi) + \cdots\,, \end{aligned} \tag{A.21}$$

respectively.

We note that, unlike the Markovian case (where there was no ingoing normalizable mode), we now have ingoing normalizable modes for $(\omega,\mathbf{k})$ satisfying the dispersion relation $K^{\text{in}}_{-\mathcal{M}}(\omega,\mathbf{k}) = 0$. In other words, there exists a codimension-1 locus, a hypersurface, in $(\omega, \mathbf{k})$ space where ingoing normalizable modes exist and are given by writing

$$\begin{aligned}\widetilde{G}^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k}) &= G^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k})\Big|_{K^{\text{in}}_{-\mathcal{M}}(\omega,\mathbf{k})=0} \\ &= e^{-ib\omega F(\mathcal{M},\xi)}\left[1 + \frac{\mathcal{M}-1}{\mathcal{M}+1}\,\mathfrak{q}^2\, H_k(\mathcal{M},\xi) + \mathfrak{w}^2\, H_\omega(\mathcal{M},\xi)\right. \\ &\qquad\qquad \left. -i\,\frac{\mathcal{M}-1}{\mathcal{M}+1}\,\mathfrak{w}\,\mathfrak{q}^2\, I_k(\mathcal{M},\xi) - i\,\mathfrak{w}^3\, I_\omega(\mathcal{M},\xi) + \cdots\right].\end{aligned} \tag{A.22}$$

This ingoing normalizable mode (which only exists on the locus $K^{\text{in}}_{-\mathcal{M}}(\omega, \mathbf{k}) = 0$) bears a close resemblance to the ingoing non-normalizable mode in the Markovian case, cf., (2.18). As mentioned in the text there are some differences in the normalization of the various functions appearing at each order in the gradient expansion, but the overall structure is closely related. One can indeed given the Markovian Green's function guess at the non-Markovian normalizable mode.

When $K^{\text{in}}_{-\mathcal{M}} \neq 0$, we get a non-normalizable mode $\Xi_{\text{nn}}(\mathcal{M}, \xi)$ whose dominant growth at large ξ is given by $\Delta(\mathcal{M}, \xi)$. Using (A.13) we see that this function grows as $\xi^{\mathcal{M}-1}$. While the functions $H_\omega(-\mathcal{M}, \xi)$, $I_k(-\mathcal{M}, \xi)$ and $I_\omega(-\mathcal{M}, \xi)$ have higher powers of $\Delta(\mathcal{M}, \xi)$ in their parameterization, the powers higher than unity all cancel against the contributions of higher powers of $F(-\mathcal{M}, \xi)$. This is a good sanity check and consistent with the requirement familiar in other holographic examples that the power of r giving the dominant growth of non-normalizable mode is independent of $(\omega, \mathbf{k})$ and hence cannot change as we go to higher orders in derivative expansion.

For the computation of the conjugate momentum and counterterms we record that asymptotically

$$\lim_{r\to\infty} \Xi_{\text{nn}}(\omega, r, \mathbf{k}) \sim -\frac{r^{\mathcal{M}-1}}{\mathcal{M}-1} + \mathcal{O}\Big(r^{\mathcal{M}-2}\Big) \tag{A.23}$$

which follows from the asymptotics of the function $\Delta(\mathcal{M}, \xi)$, cf., (A.13). Likewise

$$\lim_{r\to\infty} \widetilde{G}^{\text{in}}_{-\mathcal{M}}(\omega, r, \mathbf{k}) \sim 1 + \mathcal{O}\Big(r^{-\mathcal{M}-1}\Big) \tag{A.24}$$

A.3 Horizon values and transport data

In addition to the specific form of the functions appearing in the gradient expansion it will also be useful to record the values of these functions at the horizon $\xi = 1$. These determine the coefficients in our Green's functions and in fact directly parameterize the transport data for the hydrodynamic moduli.

To obtain them we can use the fact that the difference of two incomplete beta functions we encountered, has a finite limit as we take $\xi \to 1$, cf., (E.6). This allows to extract the following values for the functions appearing in the gradient expansion

$$\Delta(\mathcal{M},1) = \mathsf{s}\Bigg[\psi\left(\mathsf{s}(\mathcal{M}+1)\right) - \psi\left(\mathsf{s}(1-\mathcal{M})\right)\Bigg] \tag{A.25}$$

while

$$\begin{aligned} F(\mathcal{M},1) &= \mathsf{s}\left[\psi\left(\mathsf{s}(\mathcal{M}+1)\right) - \psi(\mathsf{s})\right] \\ H_k(\mathcal{M},1) &= \frac{\mathsf{s}}{\mathcal{M}-1}\left[\psi\left(\mathsf{s}(\mathcal{M}+1)\right) - \psi(2\mathsf{s})\right] \\ H_\omega(\mathcal{M},1) &= \frac{\mathsf{s}^2}{2}\,\psi\left(\mathsf{s}(\mathcal{M}+1)\right)\left[\psi\left(\mathsf{s}(\mathcal{M}+1)\right) - \psi\left(\mathsf{s}(1-\mathcal{M})\right)\right] \\ &\qquad + \sum_{n=0}^{\infty}\left[\frac{1}{n+\mathsf{s}(1+\mathcal{M})} - \frac{1}{n+\mathsf{s}(1-\mathcal{M})}\right]\psi\left(n+2\mathsf{s}\right) \end{aligned} \tag{A.26}$$

Function	$\mathcal{M} = d-3$	$\mathcal{M} = d-1$
$\Delta(\mathcal{M},1)$	$\frac{1}{d}\left[\text{Har}\left(-\frac{2}{d}\right) - \text{Har}\left(\frac{4-2d}{d}\right)\right]$	$-\frac{1}{d}\,\text{Har}\left(\frac{2-2d}{d}\right)$
$H_k(\mathcal{M},1)$	$\frac{\pi}{d(d-4)}\,\cot\left(\frac{2\pi}{d}\right)$	$-\frac{1}{d(d-2)}\,\text{Har}\left(\frac{2-d}{d}\right)$
$H_\omega^{(1)}(\mathcal{M},1)$	$\frac{1}{2d^2}\,\text{Har}\left(-\frac{2}{d}\right)\left[\text{Har}\left(-\frac{2}{d}\right) - \text{Har}\left(\frac{4}{d}-2\right)\right]$	0

Function	$\mathcal{M} = 3-d$	$\mathcal{M} = 1-d$
$\Delta(\mathcal{M},1)$	$-\frac{1}{d}\left[\text{Har}\left(-\frac{2}{d}\right) - \text{Har}\left(\frac{4-2d}{d}\right)\right]$	$\frac{1}{d}\,\text{Har}\left(\frac{2-2d}{d}\right)$
$H_k(\mathcal{M},1)$	$\frac{1}{d(d-2)}\left[\text{Har}\left(\frac{2}{d}-1\right) - \text{Har}\left(\frac{4-2d}{d}\right)\right]$	$-\frac{1}{d(d-2)}$
$H_\omega^{(1)}(\mathcal{M},1)$	$-\frac{1}{2d^2}\,\text{Har}\left(\frac{4-2d}{d}\right)\left[\text{Har}\left(-\frac{2}{d}\right) - \text{Har}\left(\frac{4}{d}-2\right)\right]$	$\frac{1}{2d^2}\left[\text{Har}\left(\frac{2-2d}{d}\right)\right]^2$

Table 1: Horizon values of the special functions appearing in the Wilsonian influence phase and determining transport data for Maxwell fields and gravitons.

We can simplify $H_\omega(\mathcal{M},1)$ using the harmonic sum representation of the digamma function

$$\psi(z) = \sum_{m=0}^{\infty}\left(\frac{1}{m+1} - \frac{1}{m+z}\right) - \gamma_E\,. \tag{A.27}$$

Using this rewriting we may reassemble the contributions to $H_\omega(\mathcal{M},1)$ as

$$\begin{aligned} H_\omega(\mathcal{M},1) &= H_\omega^{(1)}(\mathcal{M},1) + H_\omega^{(2)}(\mathcal{M},1) \\ H_\omega^{(1)}(\mathcal{M},1) &= \frac{s}{2}\,\text{Har}\left(-1+\mathsf{s}(1+\mathcal{M})\right)\,\Delta(\mathcal{M},1) \\ H_\omega^{(2)}(\mathcal{M},1) &= -\sum_{n=0}^{\infty}\frac{\mathcal{M}\,\mathsf{s}^3\,\text{Har}\left(n+2\mathsf{s}-1\right)}{(n+\mathsf{s}(1+\mathcal{M}))(n+\mathsf{s}(1-\mathcal{M}))} \end{aligned} \tag{A.28}$$

Note that $H_\omega^{(2)}(\mathcal{M},1) = -H_\omega^{(2)}(-\mathcal{M},1)$. One should be able to find an expression for this quantity in terms of polygamma values, but we will not attempt to do so here.

Of interest to us are the special cases $\mathcal{M} = \pm(d-3)$ relevant for probe Maxwell fields, and $\mathcal{M} = \pm(d-1)$ relevant for probe gravitons. In those cases we record the horizon values of the functions that enter the Wilsonian influence phase in Table 1. In writing the expressions above we have employed the Harmonic number function that has appeared before in the transport data of holographic plasmas [132] which is defined by

$$\text{Har}\,(x-1) = \gamma_E + \psi(x)\,, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}\,. \tag{A.29}$$

B Further details for the gauge system

The equations of motion for the designer Maxwell probe (2.4) given in (2.92) can be explicitly written out in terms of the covariant field strengths as:

$$\begin{aligned} \partial_r(r^{\mathcal{M}+2}\mathcal{C}_{rv}) + \partial_i(r^{\mathcal{M}}\mathcal{C}_{ri}) &= 0\,, \\ \partial_r\left(r^{\mathcal{M}}(r^2 f\,\mathcal{C}_{ri} + \mathcal{C}_{vi})\right) + r^{\mathcal{M}}\,\partial_v\mathcal{C}_{ri} - r^{\mathcal{M}-2}\,\partial_j\mathcal{C}_{ij} &= 0\,, \\ r^{\mathcal{M}+2}\,\partial_v\mathcal{C}_{rv} - r^{\mathcal{M}}\,\partial_i\left(r^2 f\,\mathcal{C}_{ri} + \mathcal{C}_{vi}\right) &= 0\,. \end{aligned} \tag{B.1}$$

We have used here $\sqrt{-g} = r^{d-1}$ as well as the relations

$$\mathcal{C}^{vr} = \mathcal{C}_{rv}\,, \quad \mathcal{C}^{ir} = -r^{-2}(\,\mathcal{C}_{vi} + r^2 f\,\mathcal{C}_{ri})\,, \quad \mathcal{C}^{iv} = -r^{-2}\,\mathcal{C}_{ri}\,, \quad \mathcal{C}^{ij} = r^{-4}\,\mathcal{C}_{ij}\;. \tag{B.2}$$

In terms of the potentials, the equations of motion take the form

$$\begin{aligned} \frac{\partial}{\partial r}\left(r^{\mathcal{M}+2}\left(\frac{\partial\mathcal{V}_v}{\partial r} - \frac{\partial\mathcal{V}_r}{\partial v}\right)\right) + r^{\mathcal{M}}\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{V}_i}{\partial r} - \frac{\partial\mathcal{V}_r}{\partial x^i}\right) &= 0\,, \\ \frac{\partial}{\partial r}\left(r^{\mathcal{M}}\left(\mathbb{D}_+\mathcal{V}_i - \partial_i(r^2 f\,\mathcal{V}_r + \mathcal{V}_v)\right)\right) + r^{\mathcal{M}}\,\frac{\partial}{\partial v}\left(\frac{\partial\mathcal{V}_i}{\partial r} - \frac{\partial\mathcal{V}_r}{\partial x^i}\right) &= 0\,, \\ -r^{\mathcal{M}-2}(\partial_i\partial_j - \partial_k\partial_k\delta_{ij})\mathcal{V}_j &= 0\,, \\ r^{\mathcal{M}+2}\frac{\partial}{\partial v}\left(\frac{\partial\mathcal{V}_v}{\partial r} - \frac{\partial\mathcal{V}_r}{\partial v}\right) + r^{\mathcal{M}}\,\frac{\partial}{\partial x^i}\left(\mathbb{D}_+\mathcal{V}_i - \frac{\partial}{\partial x^i}\left(r^2 f\,\mathcal{V}_r + \mathcal{V}_v\right)\right) &= 0\,. \end{aligned} \tag{B.3}$$

These equations are invariant under the gauge redundancy $\mathcal{V}_A \mapsto \mathcal{V}_A + \partial_A\overline{\Lambda}$ as can be verified directly. Using the harmonic plane wave decomposition (2.93) one can infer the equations (2.94) and (2.96) quoted in the main text.

B.1 Action of $\mathbb{Z}_2$ time reversal

The time reversal $\mathbb{Z}_2$ isometry $v \mapsto i\beta\zeta - v, \omega \mapsto -\omega$ leaves the background metric invariant. The transverse vector equation of motion (2.94) has been explicitly demonstrated to be invariant under this involution. We now claim that the designer gauge system in (2.96) is also invariant under this $\mathbb{Z}_2$. The action of time reversal on the fields may be determined by the transformation of the 1-form

$$\mathcal{V}_s = \bar{\Psi}_r \, dr + \bar{\Psi}_v \, dv - \bar{\Psi}_x \frac{k_i}{k} dx^i \,. \tag{B.4}$$

One finds:

$$\begin{aligned}
&\bar{\Psi}_r(\omega, r, \mathbf{k}) \mapsto e^{-\beta\omega\zeta} \left(\bar{\Psi}_r(-\omega, r, \mathbf{k}) + \frac{2}{r^2 f} \, \bar{\Psi}_v(-\omega, r, \mathbf{k})\right), \\
&\bar{\Psi}_v(\omega, r, \mathbf{k}) \mapsto -e^{-\beta\omega\zeta} \, \bar{\Psi}_v(-\omega, r, \mathbf{k}), \\
&\bar{\Psi}_x(\omega, r, \mathbf{k}) \mapsto e^{-\beta\omega\zeta} \, \bar{\Psi}_x(-\omega, r, \mathbf{k}) \,.
\end{aligned} \tag{B.5}$$

Furthermore, using the relation $-\beta\omega \frac{d\zeta}{dr} = 2\frac{i\omega}{r^2 f}$ one can check that the following transformations hold:

$$\begin{aligned}
&\bar{P}_r(\omega, r, \mathbf{k}) \mapsto e^{-\beta\omega\zeta} \left(\bar{P}_r(-\omega, r, \mathbf{k}) + \frac{2}{r^2 f} \, \bar{P}_v(-\omega, r, \mathbf{k})\right), \\
&\bar{P}_v(\omega, r, \mathbf{k}) \mapsto -e^{-\beta\omega\zeta} \, \bar{P}_v(-\omega, r, \mathbf{k}) \,, \\
&\bar{\Pi}_v(\omega, r, \mathbf{k}) \mapsto -e^{-\beta\omega\zeta} \, \bar{\Pi}_v(-\omega, r, \mathbf{k}) \,.
\end{aligned} \tag{B.6}$$

The last two combinations, $\bar{P}_v$ and $\bar{\Pi}_v$, transform covariantly under $\mathbb{Z}_2$ with an odd time reversal parity. A third $\mathbb{Z}_2$ covariant, even time reversal parity combination can be formed from the first two and implies that

$$\bar{\Pi}_x(\omega, r, \mathbf{k}) \mapsto e^{-\beta\omega\zeta} \, \bar{\Pi}_x(-\omega, r, \mathbf{k}) \,. \tag{B.7}$$

B.2 Radial gauge analysis of the gauge system

In the standard discussion of gauge systems in AdS spacetime, one often tends to a-priori pick a gauge. In this context the gauge choice that is most natural for the ingoing mode analysis is the radial gauge. We have discussed the potential advantages of our gauge invariant formalism in the main text, but for completeness let us examine now the solution of the ingoing modes in the radial gauge and recover the standard story of diffusion therefrom.

We set $\bar{\Psi}_r = 0$ in the gauge system (2.96) to get the dynamical equations for the remaining components $\bar{\Psi}_v$ and $\bar{\Psi}_x$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r^{\mathcal{M}}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\mathcal{M}+1}\,\bar{\Psi}_v\right)\right] - \frac{ik}{r}\frac{\mathrm{d}\bar{\Psi}_x}{\mathrm{d}r} = 0 \;, \\
\frac{1}{r^{\mathcal{M}}}\mathbb{D}_+\left[r^{\mathcal{M}}\mathbb{D}_+\bar{\Psi}_x\right] + \omega^2\bar{\Psi}_x + ik\,r^2 f \;\times\; \frac{1}{r^{\mathcal{M}}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\mathcal{M}}\,\bar{\Psi}_v\right) = 0 \;, \\
-i\omega\; r^{\mathcal{M}+2}\,\frac{\mathrm{d}\bar{\Psi}_v}{\mathrm{d}r} + ik\,r^{\mathcal{M}}\left(\mathbb{D}_+\bar{\Psi}_x + ik\,\bar{\Psi}_v\right) = 0 \,.
\end{aligned} \tag{B.8}$$

where we have used the identity $\frac{\mathrm{d}}{\mathrm{d}r}[\frac{1}{r^{\mathcal{M}}}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{\mathcal{M}+1}\,\bar{\Psi}_v\Big)] = \frac{1}{r^{\mathcal{M}+1}}\frac{\mathrm{d}}{\mathrm{d}r}[r^{\mathcal{M}+2}\frac{\mathrm{d}\bar{\Psi}_v}{\mathrm{d}r}]$ to simplify the equations. For now, we will ignore the third equation, which is the radial Gauss constraint, and solve the first two equations in derivative expansion.

To second order in derivative expansion, the most general ingoing or analytic solution in this radial gauge is given by:

$$\begin{aligned}
\bar{\Psi}_v &= C_v - \mu_{\mathcal{M}}\left(\frac{1}{\xi^{\mathcal{M}+1}} - (\mathcal{M}+1)\,\mathfrak{q}^2\int_\xi^\infty \frac{dy}{y^{\mathcal{M}+2}}\,H_k(-\mathcal{M},y)\right) \\
&\qquad\qquad + \mathfrak{q}\,[\mathfrak{q}\,(\mu_{\mathcal{M}} - C_v) + \mathfrak{w}\,C_x]\int_\xi^\infty \frac{dy}{y^{\mathcal{M}+2}}\,F(-\mathcal{M},y) + \cdots\ , \\
\bar{\Psi}_x &= C_x + i\,\mathfrak{q}\,\mu_{\mathcal{M}}\left[F(\mathcal{M}+1,\xi) - i\mathfrak{w}\,\widetilde{H}_k(\mathcal{M},\xi)\right] \\
&\qquad - i\,[\mathfrak{q}\,(\mu_{\mathcal{M}} - C_v) + \mathfrak{w}\,C_x]\left[F(\mathcal{M},\xi) - i\mathfrak{w}\left(H_\omega(\mathcal{M},\xi) + \frac{1}{2}\,F(\mathcal{M},\xi)^2\right)\right] + \cdots\,.
\end{aligned} \tag{B.9}$$

where we introduced the combination

$$\widetilde{H}_k(\mathcal{M},\xi) \equiv \int_\xi^\infty \frac{y^{d-2}\,dy}{y^{\mathcal{M}}(y^d-1)}\left[y^{\mathcal{M}}F(\mathcal{M}+1,y) - F(\mathcal{M}+1,1) + (\mathcal{M}+1)\widehat{H}_k(-\mathcal{M},y)\right]. \tag{B.10}$$

The remaining functions $\{F, H_k, H_\omega\}$ are the familiar ones which we encountered in the scalar field gradient expansion analysis in §2.4.

The coefficients C_v and C_x are the two non-normalizable modes (both of which are analytic) whereas $\mu_{\mathcal{M}}$ is the unique analytic normalizable mode. The coefficients C_v, C_x are fixed by the Dirichlet conditions

$$\lim_{r\to\infty}\bar{\Psi}_v = C_v\ , \qquad \lim_{r\to\infty}\bar{\Psi}_x = C_x\ , \tag{B.11}$$

and they correspond to the boundary source perturbations. One can verify all these statements using the asymptotic expansions in Appendix A.1.

The normalizable mode can be related to the Noether charge density via

$$(J^{\text{CFT}})^v \equiv \lim_{r\to\infty}\left[r^{\mathcal{M}+2}\left(\frac{d\bar{\Psi}_v}{dr} + i\omega\bar{\Psi}_r\right) - k\frac{r^{\mathcal{M}-1}}{\mathcal{M}-1}(k\,C_v - \omega\,C_x)\right] = (\mathcal{M}+1)\frac{\mu_{\mathcal{M}}}{b^{\mathcal{M}+1}} \tag{B.12}$$

We subtracted a temperature-independent, gauge invariant counterterm to get a finite result to this order in derivative expansion above. The corresponding Noether current density, after an analogous counterterm subtraction, is given by

$$\begin{aligned}(J^{\text{CFT}})^i &\equiv \frac{k_i}{k} \lim_{r\to\infty}\left[r^{\mathcal{M}}\left(\mathbb{D}_+\bar{\Psi}_x + ik\,\bar{\Psi}_v + ik\,r^2 f\,\bar{\Psi}_r\right) - \omega\,\frac{r^{\mathcal{M}-1}}{\mathcal{M}-1}(k\,C_v - \omega\,C_x)\right]\\ &= -i(bk_i)\left[1 + i\,\mathfrak{w}\left(F(\mathcal{M}+1,1) - (\mathcal{M}-1)\,H_k(\mathcal{M},1)\right) + \cdots\right]\frac{\mu_{\mathcal{M}}}{b^{\mathcal{M}+1}}\\ &\qquad + \frac{1}{b^{\mathcal{M}}}\left[1 - i\,\mathfrak{w}\left(F(\mathcal{M},1) - F(-\mathcal{M},1)\right) + \cdots\right]\frac{ik_i}{k}\,(k\,C_v - \omega\,C_x) + \cdots\,.\end{aligned} \tag{B.13}$$

Here we have used (2.38) to write the functions corresponding to the exponent $-\mathcal{M}$ in terms of the functions corresponding to the exponent $\mathcal{M}$. In the final expression for J^i, the penultimate line in (B.13) gives the diffusion current while the final line denotes the drift current due to the external applied field. The coefficients here are the (frequency dependent) diffusion constant and conductivity respectively.

The qualitative structure of the ingoing solution in the generalized gauge system is now clear. Unlike the Markovian sectors, we have here a long-lived generalized charge mode in the CFT indicated by the presence of an analytic normalizable mode. The physics here is the generalized diffusion of this Noether charge density with generalized chemical potential $\mu_{\mathcal{M}}$ (Fick's law) along with a drift in the charge due to external forcing by potentials C_v and C_x (Ohm's law).

The normalizable mode $\mu_{\mathcal{M}}$ however is not arbitrary. Substituting our solution into the radial Gauss constraint, which we recall we have left off-shell, we get a radius independent relation between the normalizable and the non-normalizable modes (as we should). We obtain the following constraint:

$$\begin{aligned}&\Big\{-i\omega\,(\mathcal{M}+1) + bk^2\left[1 + i\,\mathfrak{w}\;\left(F(\mathcal{M}+1,1) - (\mathcal{M}-1)\,H_k(\mathcal{M},1)\right) + \cdots\right]\Big\}\,\mu_{\mathcal{M}}\\ &\qquad - \mathfrak{q}\left\{1 - i\,\mathfrak{w}\left[F(\mathcal{M},1) - F(-\mathcal{M},1)\right] + \cdots\right\}(k\,C_v - \omega\,C_x)\ = 0\,.\end{aligned} \tag{B.14}$$

The above equation is the generalized diffusion equation in the Fourier domain describing the diffusion of CFT charge density corresponding to $\mu_{\mathcal{M}}$ along with an ohmic drift due to external field produced by C_v and C_x. If we set the external forcing to zero (i.e., fix $C_v = C_x - 0$) we get the inverse of the diffusion Green function

$$G^{-1}_{\mathcal{M},\text{diff}} \equiv -i\,\omega\,(\mathcal{M}+1) + bk^2\left(1 + i\,\mathfrak{w}\left[F(\mathcal{M}+1,1) - (\mathcal{M}-1)\,H_k(\mathcal{M},1)\right] + \cdots\right)\,. \tag{B.15}$$

This is a diffusion equation with a frequency dependent diffusion constant

$$\mathcal{D} = \frac{1}{\mathcal{M}+1}\,\frac{d\,\beta}{4\pi}\left(1 + \frac{\beta\omega}{2\pi}\,d\left[F(\mathcal{M}+1,1) - (\mathcal{M}-1)\,H_k(\mathcal{M},1)\right] + \cdots\right) \tag{B.16}$$

Thus, in the absence of forcing we get a charge density that slowly diffuses and equilibrates over the planar Schwarzschild-AdS$_{d+1}$ black hole horizon.

When we do having external sources forcing the system, we get a superposition of the free normalizable diffusion mode with the normalizable drift mode due to external driving (parametrized as heretofore mentioned by C_v and C_x):

$$\mu_{\mathcal{M}} = \mu_{\mathcal{M}}^{\text{diff}} + \mathfrak{q}\, G_{\mathcal{M},\text{diff}} \left[1 - i\,\mathfrak{w}\left(F(\mathcal{M},1) - F(-\mathcal{M},1)\right) + \cdots\right] (k\, C_v - \omega\, C_x)\,. \tag{B.17}$$

The analysis in the radial gauge has some useful pointers for our general solution. As we see above, the moment we impose the Gauss constraint, the third equation of (B.8), we see that the gradient expansion breaks down. However, the physics of this breakdown is simple it is associated with the presence of long-lived, long wavelength modes that we are integrating out.

Thus, once the radial Gauss constraint is fully solved for, we get a breakdown of derivative expansion and the Markovian approximation, but only via the diffusion pole in $G_{\mathcal{M},\text{Diff}}$.

C On the gravitational perturbations

In this appendix we give some of the details regarding the reduction of gravitational dynamics encoded in the Einstein-Hilbert action onto the designer scalar and gauge fields that we have discussed. The main aim is to show that the transverse tensor polarizations of gravitons are a Markovian scalar (a minimally coupled massless scalar with $\mathcal{M} = d-1$) while the transverse vector polarizations leads to a diffusive gauge field with $\mathcal{M} = 1-d$. We do not discuss the scalar polarization, which are expected to be non-Markovian, since they comprise the low energy sound mode.

C.1 Linearized diffeomorphisms and abelian gauge symmetries

Let us begin by quickly motivating the abelian gauge symmetry encountered in the auxiliary gauge system in (2.120). This symmetry is inherited from vector diffeomorphisms of the form

$$x^i \mapsto x^i + \int_k \sum_{\alpha=1}^{N_V} \Lambda_\alpha(r,\omega,\mathbf{k})\, \mathbb{V}_i^\alpha(\omega,\mathbf{k}|v,\mathbf{x})\ . \tag{C.1}$$

This is the only allowed diffeomorphism once we ignore terms with scalar plane waves in the harmonic decomposition. To see how this works, we feed the shift

$$dx^i \mapsto dx^i + \int_k \sum_{\alpha=1}^{N_V} \left[\frac{\mathrm{d}\Lambda_\alpha}{\mathrm{d}r}\, dr + \Lambda_\alpha(r,\omega,\mathbf{k})\ (dv\, \partial_v + dx^j\ \partial_j)\right] \mathbb{V}_\alpha^i(\omega,\mathbf{k}|v,\mathbf{x})\,, \tag{C.2}$$

into the background black brane metric (1.36). Retaining terms linear in Λ_α and ignoring further the modifications coming from the perturbations to the geometry parameterized in (2.119) (which are non-linear effects), we can read off the effect of the diffeomorphism on the fields parameterizing h_{AB}. One finds,

$$\Psi_r^\alpha \mapsto \Psi_r + \frac{\mathrm{d}\Lambda_\alpha}{\mathrm{d}r} \;, \quad \Psi_v^\alpha \mapsto \Psi_v - i\omega\; \Lambda_\alpha \;, \quad \Psi_x^\alpha \mapsto \Psi_x - ik\, \Lambda_\alpha \,, \tag{C.3}$$

which is of course equivalent to (cf., (2.99))

$$\mathcal{A}_B^\alpha\, dx^B \mapsto \mathcal{A}_B^\alpha\, dx^B + \partial_B \int_k \Lambda_\alpha\, \mathbb{S}(\omega, \mathbf{k}|v, \mathbf{x})\, dx^B \,, \tag{C.4}$$

Thus to this leading order we are justified in thinking of the transverse vector gravitons as an auxiliary gauge field.

An equivalent way to think about the emergence of this auxiliary gauge field is from the dual CFT: we know that the shear sector of the CFT sector carries a divergence free momentum density. We can then expand such a momentum density (the vi components of the energy-momentum tensor $(T^{\text{CFT}})^{\mu\nu}$) in the basis of vector plane waves

$$(T^{\text{CFT}})^{vi}(v, \mathbf{x}) \equiv \int_k \sum_{\alpha=1}^{N_V} P^\alpha(\omega, \mathbf{k})\, \mathbb{V}_\alpha^i(\omega, \mathbf{k}|v, \mathbf{x}) \;. \tag{C.5}$$

The diffusion of this momentum density is then equivalent to the diffusion of $N_V = d-2$ charges defined via

$$\int_k P^\alpha(\omega, k)\, \mathbb{S}(\omega, \mathbf{k}|v, \mathbf{x})\,. \tag{C.6}$$

In holography, as we saw in §2.7 the diffusion of a charge density is dual to the physics of radially polarized photons travelling tangentially to the boundary. This then naturally leads to a construction of $N_V = d-2$ radially polarized gauge fields in the bulk, as we have described above.

C.2 Graviton dynamics repackaged

The Einstein's equations (2.121) are of course derived from the Einstein-Hilbert action, along with the Gibbons-Hawking boundary term. In addition, one has to specific boundary covariant counterterms that regulate the UV divergences and give finite boundary observables, viz., the CFT stress tensor $(T^{\text{CFT}})^{\mu\nu}$ and its correlation functions. This is given in (2.122).

The auxiliary system of scalars and gauge fields introduced in (2.120) and their resulting dynamics in (2.121) can be derived from the following action:

$$
\begin{aligned}
S_{\text{Aux}} &= -\int d^{d+1}x\,\sqrt{-g}\left[\frac{1}{2}\sum_{\sigma=1}^{N_T}\nabla_A\Phi_\sigma\,\nabla^A\Phi_\sigma+\frac{r^2}{4}\sum_{\alpha=1}^{N_V} g^{AC}\,g^{BD}\,\mathfrak{F}^\alpha_{AB}\,\mathfrak{F}^\alpha_{CD}\right]+S_{\text{Aux, ct}}\\
S_{\text{Aux, ct}} &= \frac{1}{d-2}\int d^dx\,\sqrt{-\gamma}\left[\frac{1}{2}\sum_{\sigma=1}^{N_T}\gamma^{\mu\nu}\,\partial_\mu\Phi_\sigma\,\partial_\nu\Phi_\sigma+\frac{r^2}{4}\sum_{\alpha=1}^{N_V}\gamma^{\mu\sigma}\,\gamma^{\nu\lambda}\mathfrak{F}^\alpha_{\mu\nu}\mathfrak{F}^\alpha_{\sigma\lambda}\right]
\end{aligned}
\tag{C.7}
$$

The boundary counterterms in $S_{\text{Aux, ct}}$ are fixed by the following requirements: they should be diagonal in the auxiliary fields since the equations of motion do not mix the fields. They also ought to obey the original symmetries of S_{Aux}: the shift invariance in Φ_σ and gauge invariance of Ψ^α_B. The terms written above respect these conditions and furthermore only include terms that are at most quadratic in the auxiliary fields. The coefficients are fixed by demanding that the counterterms cancel the divergences of S_{Aux} when we evaluate the boundary observables.

The induced metric can be obtained by taking the boundary limit of (2.119), leading to[6]

$$
\begin{aligned}
ds^2_{\text{bdy}} &= \left(\gamma_{\mu\nu}+\left({}^h\gamma_{\mu\nu}\right)_{\text{Tens}}+\left({}^h\gamma_{\mu\nu}\right)_{\text{Vec}}\right)\,dx^\mu\,dx^\nu\\
\gamma_{\mu\nu}\,dx^\mu\,dx^\nu &= -r^2\,f(r)\,dv^2+r^2\,d\mathbf{x}^2\,,\\
\left({}^h\gamma_{\mu\nu}\right)_{\text{Tens}}\,dx^\mu\,dx^\nu &= r^2\int_k\sum_{\sigma=1}^{N_T}\Phi_\sigma\,\mathbb{T}^\sigma_{ij}(\omega,\mathbf{k}|v,\mathbf{x})\,dx^i dx^j\,,\\
\left({}^h\gamma_{\mu\nu}\right)_{\text{Vec}}\,dx^\mu\,dx^\nu &= r^2\int_k\sum_{\alpha=1}^{N_V}\left[2\,\Psi^\alpha_v(r,\omega,\mathbf{k})\,\mathbb{V}^\alpha_i(\omega,\mathbf{k}|v,\mathbf{x})\,dvdx^i+i\,\Psi^\alpha_x(r,\omega,\mathbf{k})\,\mathbb{V}_{ij}(\omega,\mathbf{k}|v,\mathbf{x})\,dx^i\,dx^j\right].
\end{aligned}
\tag{C.8}
$$

We further note that the auxiliary action (C.7) is defined with only the background metric, while the Einstein-Hilbert action (2.122) includes the tensor and vector perturbations as indicated in (2.119).

For the purposes of computing the two-point function of the energy-momentum tensor we need an expression for the on-shell action of the gravitational system. One quick way to obtain this is to use the auxiliary fields and evaluate their on-shell action. We find:

$$
S_{\text{Aux}}\big|_{\text{On-shell}} = -\frac{1}{2}\int d^dx\left[r^{d-1}\sum_{\sigma=1}^{N_T}\Phi_\sigma\mathbb{D}_+\Phi_\sigma+r^{d+1}\sum_{\alpha=1}^{N_V}g^{BC}\,\mathcal{A}^\alpha_B(\mathfrak{F}^\alpha_{Cv}+r^2 f\,\mathfrak{F}^\alpha_{Cr})\right]. \tag{C.9}
$$

We now explain how one obtains the dynamics captured by the auxiliary system directly from the Einstein-Hilbert action along with the Gibbons-Hawking and boundary

[6] We will write the induced metric with explicit coordinate r for simplicity. It is to be understood that we are considering the metric induced at a fixed radial cut-off $r = r_c$ and interested in the limit $r_c \to \infty$. This also applies below when we discuss the boundary counterterms.

counterterms in (2.122). Plugging in the perturbation ansatz (2.119) one finds by direct evaluation:

$$S_{\text{EH}} \equiv \int d^{d+1}x \sqrt{-g} \left(R + d(d-1)\right) = S_{\text{Aux}} - \int d^dx \, L_{\text{EH},bdy} + \mathcal{O}\left(h^3\right). \tag{C.10}$$

The fact that the equations of motion agree in fact guarantee that the above equality should hold, up to the boundary terms in $L_{\text{EH},bdy}$. In obtaining this result we have dropped all the total derivatives along the boundary – terms of the form $\partial_\mu F$ are not included in the above. The boundary term $L_{\text{EH},bdy}$ comes from total derivatives along the radial direction and can be computed to be

$$\begin{aligned} L_{\text{EH},bdy} = 2\, r^d \Bigg\{ & f - \frac{f}{2}\frac{\partial}{\partial r}\left(r \left[\sum_{\sigma=1}^{N_T} \Phi_\sigma^2 + r^2 \sum_{\alpha=1}^{N_V} \gamma^{\mu\nu} \mathcal{A}_\mu^\alpha \mathcal{A}_\nu^\alpha \right]\right) \\ & + \sum_{\alpha=1}^{N_V} \left[\frac{d}{4f}(1+f)\,(\mathcal{A}_v^\alpha + r^2 f\, \mathcal{A}_r^\alpha)^2 + \frac{d}{4f}\,(1-f)\,(\mathcal{A}_v^\alpha)^2 \right] \Bigg\}. \end{aligned} \tag{C.11}$$

A good consistency check is that this total radial derivative term should be cancelled by the standard Gibbons-Hawking term. We indeed find plugging in the ansatz (2.119) that

$$\begin{aligned} S_{\text{GH}} &\equiv 2 \int d^dx \sqrt{-\gamma}\, K = \int d^dx \, \left(L_{\text{EH},bdy} + L_{\text{ideal}}\right), \\ L_{\text{ideal}} &= r^d \left\{ (d + (d-2) f) \left(1 - \frac{1}{2}\sum_{\sigma=1}^{N_T} \Phi_\sigma^2 - \frac{1}{2}\sum_{\alpha=1}^{N_V} \mathcal{A}_i^\alpha \mathcal{A}_i^\alpha \right) + (d-1) \sum_{\alpha=1}^{N_V} (\mathcal{A}_v^\alpha)^2 \right\}. \end{aligned} \tag{C.12}$$

L_{ideal} is by itself still divergent, but these UV divergences are cancelled by the gravitational counterterms encoded in S_{ct} given in (2.122). Once again computing this quantity by plugging in (C.8) we find we can express the answer in terms of the counterterms evaluated for the auxiliary system of scalars and gauge fields along with an additional piece. The final answer is given as

$$\begin{aligned} S_{\text{ct}} &= S_{\text{Aux, ct}} + \int d^dx \, L_{\text{ideal, ct}} \\ L_{\text{ideal,ct}} &= 2(d-1)\, r^d \sqrt{f} \left[-1 + \frac{1}{2}\sum_{\sigma=1}^{N_T} \Phi_\sigma^2 + \frac{1}{2}\sum_{\alpha=1}^{N_V} \mathcal{A}_i^\alpha \mathcal{A}_i^\alpha - \frac{1}{2f}\sum_{\alpha=1}^{N_V} (\mathcal{A}_v^\alpha)^2 \right]. \end{aligned} \tag{C.13}$$

With this parameterization we find the finite combination for the piece we have characterized by the adjective 'ideal':

$$\lim_{r\to\infty}\left(L_{\text{ideal}}+L_{\text{ideal, ct}}\right)=-\lim_{r\to\infty}\frac{1}{b^d}\left[-1+\frac{1}{2}\sum_{\sigma=1}^{N_T}\Phi_\sigma^2+\frac{1}{2}\sum_{\alpha=1}^{N_V}\left(\mathcal{A}_i^\alpha\,\mathcal{A}_i^\alpha+(d-1)\,(\mathcal{A}_v^\alpha)^2\right)\right]$$
$$=\int d^dx\,\sqrt{-\gamma}\,\left[\sqrt{-\gamma_{\mu\nu}\,\mathbf{b}^\mu\,\mathbf{b}^\nu}\,\right]^{-d} \tag{C.14}$$

where we introduce the thermal vector $\mathbf{b}^\mu$, which in the limit where we probe the equilibrium state of the static Schwarzschild-AdS$_{d+1}$ black hole is given by $\mathbf{b}^\mu\partial_\mu = b\,\partial_v$. The adjective 'ideal' is now easily explained: the finite contribution appearing in (C.14) is the ideal fluid free energy in the thermal state on the boundary. This action is explicitly invariant under the Weyl transformations of the CFT metric.

D Conserved currents: gauge theory and gravity

In this appendix, we would like to describe the normalizable modes in the gauge theory and gravity and relate it to normalizable modes of the auxiliary scalar fields. By standard AdS/CFT dictionary, these correspond to the CFT expectation values of the global current and the energy-momentum tensor respectively. We will show that the normalizable modes in the original gauge or gravity description matches with the normalizable modes in the designer scalar description. This is perhaps expected given the reduction to the designer scalars described in the main text.

We remind the reader that the definition of normalizable modes needs counterterms built out of non-normalizable modes. The corresponding statement about counterterms would be that the known counterterms for gauge/gravity variables reproduce the required counterterms for the scalar fields.

We begin with the time-reversal invariant gauge system. Including the boundary Maxwell counterterm corrections (see (2.110)) and the Markovian contributions into (2.100), we obtain

$$\begin{aligned}
J_v^{\text{CFT}} &= -\lim_{r\to\infty} r^{\mathcal{M}}\left\{r^2\mathcal{C}_{rv}+c_v^{(2)}\partial_i\frac{\mathcal{C}_{vi}}{r\sqrt{f}}\right\}\\
&= -\int_k k^2\lim_{r\to\infty}\left\{\bar{\Phi}_{\text{D}}+\frac{C_\pi^{(0)}}{r\sqrt{f}}\mathbb{D}_+\bar{\Phi}_{\text{D}}\right\}\,,\\
J_i^{\text{CFT}} &= -\lim_{r\to\infty} r^{\mathcal{M}}\left\{r^2 f\mathcal{C}_{ri}+\mathcal{C}_{vi}+c_v^{(2)}\frac{\partial_v\mathcal{C}_{vi}-f\,\partial_j\mathcal{C}_{ji}}{r\sqrt{f}}\right\}\\
&= -\lim_{r\to\infty}\sum_{\alpha=1}^{N_V}\int_k \mathbb{V}_i^\alpha\left\{r^{\mathcal{M}}\mathbb{D}_+\bar{\Phi}_\alpha+c_\varphi^{(2)}\sqrt{f}\,r^{\mathcal{M}-1}\left(k^2-\frac{1}{f}\omega^2\right)\bar{\Phi}_\alpha\right\}\\
&\quad+\int_k \omega k_i\lim_{r\to\infty}\left\{\bar{\Phi}_{\text{D}}+\frac{C_\pi^{(0)}}{r\sqrt{f}}\mathbb{D}_+\bar{\Phi}_{\text{D}}\right\}\,,
\end{aligned} \tag{D.1}$$

where we have used the parametrization given in (2.102). We have also used the fact that the counterterm coefficient $c^{(2)}_{\mathcal{V}}$ is equal to the leading counterterms $c^{(2)}_{\varphi}$ and $c^{(0)}_{\pi}$ for the corresponding scalars. The derivation above shows that the normalizable modes of the gauge theory map directly to the normalizable mode of the corresponding designer scalars. Further, note that the CFT current appears in a form where the current conservation is automatic, even after the couterterm is added. We can go further and add in the four-derivative counterterms proportional to $e^{\chi_v}\,\gamma^{\alpha\beta}\,\mathfrak{C}^{\mu\nu}\nabla_\alpha\nabla_\beta\mathfrak{C}_{\mu\nu}$ and $e^{\chi_v}\,\gamma^{\alpha\beta}\,\nabla_\alpha\mathfrak{C}^{\mu\nu}\nabla_\beta\mathfrak{C}_{\mu\nu}$ in order to reproduce $c^{(2)}_{\pi}$ counterterm for the non-Markovian contribution.

Once the map between the normalizable modes is established, we can directly import the results from the subsections §2.6.1 and §2.6.2. The expectation value of the CFT current is given by adopting (2.67) and (2.78) to the gauge problem. This then gives the CFT current in terms of Markovian sources and the non-Markovian effective fields parameterizing the hydrodynamic moduli space to the SK one-point functions :

$$
\begin{split}
\left\langle J^{\text{CFT}}_{v,\text{R}}\right\rangle &= -\int_k k^2\,\breve{\mathcal{Q}}_{\text{R}}\,, \qquad \left\langle J^{\text{CFT}}_{v,\text{L}}\right\rangle = -\int_k k^2\,\breve{\mathcal{Q}}_{\text{L}}\,,\\
\left\langle J^{\text{CFT}}_{i,\text{R}}\right\rangle &= -\sum_{\alpha=1}^{N_V}\int_k \mathbb{V}^\alpha_i\left(K^{\text{in}}_{\mathcal{M}}\left[(n_{_B}+1)\,\mathcal{A}^\alpha_{\text{R}} - n_{_B}\,\mathcal{A}^\alpha_{\text{L}}\right] + n_{_B}\,K^{\text{rev}}_{\mathcal{M}}\left[\mathcal{A}^\alpha_{\text{R}}-\mathcal{A}^\alpha_{\text{L}}\right]\right)\\
&\qquad + \int_k \omega\,k_i\,\breve{\mathcal{Q}}_{\text{R}}\,,\\
\left\langle J^{\text{CFT}}_{i,\text{L}}\right\rangle &= -\sum_{\alpha=1}^{N_V}\int_k \mathbb{V}^\alpha_i\left(K^{\text{in}}_{\mathcal{M}}\left[(n_{_B}+1)\,\mathcal{A}^\alpha_{\text{R}} - n_{_B}\,\mathcal{A}^\alpha_{\text{L}}\right] + (n_{_B}+1)K^{\text{rev}}_{\mathcal{M}}\left[\mathcal{A}^\alpha_{\text{R}}-\mathcal{A}^\alpha_{\text{L}}\right]\right)\\
&\qquad + \int_k \omega\,k_i\,\breve{\mathcal{Q}}_{\text{L}}\,.
\end{split}
\tag{D.2}
$$

Here $\mathcal{A}$ denotes the source for the Markovian sector (magnetic part of the CFT current source) whereas $\breve{\mathcal{Q}}$ denotes the effective fields in the non-Markovian sector (the charge diffusion mode). It can be readily checked that these expressions coincide with expectation values obtained by varying the influence phase computed in the main text.

The above discussion can be extended to gravity. The analysis goes through with minor modifications. The energy momentum tensor is given by the boundary limit of the Brown-York tensor with appropriate counterterms:

$$
\sqrt{-g^{\text{CFT}}}\,T^{\text{CFT}}_{\mu\nu} = c_{\text{eff}}\lim_{r\to\infty} r^{-2}\sqrt{-\gamma}\,\left(2K\gamma_{\mu\nu} - 2K_{\mu\nu} - 2(d-1)\gamma_{\mu\nu} + \frac{2}{d-2}\,{}^{\gamma}G_{\mu\nu}\right), \tag{D.3}
$$

where ${}^{\gamma}G_{\mu\nu}$ is the Einstein tensor of the induced boundary metric and $c_{\text{eff}} = \frac{\ell^{d-1}_{\text{AdS}}}{16\pi G_N}$. This expression follows from the on-shell variation of the gravity action with its counterterms.

As a warm-up, let us compute the CFT stress tensor for the background black-brane. This gives

$$
T^{\text{CFT}}_{\mu\nu}\,dx^\mu dx^\nu = \frac{c_{\text{eff}}}{b^d}\left[(d-1)dv^2 + dx_i dx_i\right] = T^{\text{Ideal}}_{\mu\nu}\,dx^\mu dx^\nu\,. \tag{D.4}
$$

We recognize here the energy-momentum tensor of an ideal conformal fluid at rest with a pressure $p = c_{\text{eff}}\, b^{-d}$ and an energy density $\varepsilon = (d-1)p = (d-1)\, c_{\text{eff}}\, b^{-d}$. We remind the reader that the ideal fluid stress tensor for a fluid with an energy density ε, a pressure p and a spacetime velocity field u^μ is given by

$$T^{\mu\nu}_{\text{Ideal}} = \varepsilon\ u^\mu u^\nu + p\,(\eta^{\mu\nu} + u^\mu u^\nu) = p\,(\eta^{\mu\nu} + d\, u^\mu u^\nu)\ . \tag{D.5}$$

In the last step we have used the relation $\varepsilon = (d-1)p$ arising from the fact that $T^{\mu\nu}$ should be trace free in a CFT.

Next, we turn on the tensor/vector perturbations to ask how the CFT stress tensor changes under such a deformation. Since we are not turning on any scalar perturbations, the contribution to the energy density T^{CFT}_{vv} due to these perturbations is zero. The components T^{CFT}_{vi} and T^{CFT}_{ij} take the form

$$\begin{aligned}
T^{\text{CFT}}_{vi,\text{Non-Ideal}} &= -c_{\text{eff}} \sum_{\alpha=1}^{N_V} \int_k k^2 \mathbb{V}_i^\alpha \lim_{r\to\infty} \left\{ \bar{\Phi}_{\text{D}} + \frac{c^{(0)}_{\pi}}{r\sqrt{f}} \mathbb{D}_+ \bar{\Phi}_{\text{D}} \right\}\ , \\
T^{\text{CFT}}_{ij,\text{Non-Ideal}} &= -c_{\text{eff}} \sum_{\sigma=1}^{N_T} \int_k \mathbb{T}_{ij}^\sigma \lim_{r\to\infty} \left\{ r^{d-1}\mathbb{D}_+\Phi_\sigma + c^{(2)}_{\varphi} \sqrt{f}\ r^{d-2} \left(k^2 - \frac{1}{f}\omega^2\right) \Phi_\sigma \right\} \\
&\quad + c_{\text{eff}} \sum_{\alpha=1}^{N_V} \int_k \omega \left(k_i \mathbb{V}_j^\alpha + k_j \mathbb{V}_i^\alpha\right) \lim_{r\to\infty} \left\{ \bar{\Phi}^\alpha_{\text{D}} + \frac{c^{(0)}_{\pi}}{r\sqrt{f}} \mathbb{D}_+ \bar{\Phi}^\alpha_{\text{D}} \right\}\ .
\end{aligned} \tag{D.6}$$

These expressions are the gravitational analogues of the gauge theory expressions that were derived above. The counterterm coefficients in the above equation are evaluated by setting $\mathcal{M} = d-1$. The normalizable modes in the non-ideal part again map to the normalizable mode of the corresponding designer scalars and the energy-momentum conservation is automatic. To get the four-derivative counterterms in the non-Markovian sector $c^{(2)}_{\pi}$ with $\mathcal{M} = d-1$, we need to add in the corresponding Riemann square counterterms in gravity.

Evaluating the above expression on our solution, we can derive the SK one point functions of the CFT energy momentum tensor as

$$
\begin{aligned}
\left\langle T^{\rm CFT}_{vi,{\rm R}}\right\rangle &= -\sum_{\alpha=1}^{N_V}\int_k k^2\,\breve{\mathcal{P}}^\alpha_{\rm R}\;, \qquad \left\langle T^{\rm CFT}_{vi,{\rm L}}\right\rangle = -\sum_{\alpha=1}^{N_V}\int_k k^2\,\breve{\mathcal{P}}^\alpha_{\rm L}\;,\\
\left\langle T^{\rm CFT}_{ij,{\rm R}}\right\rangle &= -\sum_{\sigma=1}^{N_T}\int_k \mathbb{T}^\sigma_{ij}\left(K^{\rm in}_{d-1}\left[(n_{_B}+1)\,\boldsymbol{\gamma}^\sigma_{\rm R} - n_{_B}\,\boldsymbol{\gamma}^\sigma_{\rm L}\right] + n_{_B}\,K^{\rm rev}_{d-1}\left[\boldsymbol{\gamma}^\sigma_{\rm R}-\boldsymbol{\gamma}^\sigma_{\rm L}\right]\right)\\
&\quad+\sum_{\alpha=1}^{N_V}\int_k \omega\,\left(k_i\mathbb{V}^\alpha_j + k_j\mathbb{V}^\alpha_i\right)\breve{\mathcal{P}}^\alpha_{\rm R}\;,\\
\left\langle T^{\rm CFT}_{ij,{\rm L}}\right\rangle &= -\sum_{\sigma=1}^{N_T}\int_k \mathbb{T}^\sigma_{ij}\left(K^{\rm in}_{d-1}\left[(n_{_B}+1)\,\boldsymbol{\gamma}^\alpha_{\rm R} - n_{_B}\,\boldsymbol{\gamma}^\alpha_{\rm L}\right] + (n_{_B}+1)K^{\rm rev}_{d-1}\left[\boldsymbol{\gamma}^\alpha_{\rm R}-\boldsymbol{\gamma}^\alpha_{\rm L}\right]\right)\\
&\quad+\sum_{\alpha=1}^{N_V}\int_k \omega\left(k_i\mathbb{V}^\alpha_j + k_j\mathbb{V}^\alpha_i\right)\breve{\mathcal{P}}^\alpha_{\rm L}\;.
\end{aligned}
\tag{D.7}
$$

Here $\boldsymbol{\gamma}$ denotes the source for the Markovian sector (the magnetic part of the CFT metric) whereas $\breve{\mathcal{P}}$ denotes the effective fields in the non-Markovian sector (the momentum diffusion or shear mode). It can be readily checked that these expressions coincide with expectation values obtained by varying the influence phase computed in the main text. We will leave for future a detailed comparison of the above expressions to the results from fluid/gravity correspondence.

E On some incomplete Beta functions

We briefly review the properties of the subclass of incomplete Beta functions that have been used to parameterize the solutions in the gradient expansion. We start by noting their definition in terms of the hypergeometric series expansion and outline a few identities that are helpful in verifying some of our statements. We recall, [92, Eq. 8.17.E7]

$$
{\rm B}\left(\alpha,0;z\right) \equiv \frac{z^\alpha}{\alpha}\;\; {}_2F_1\left(1,\alpha,1+\alpha;z\right) = \sum_{m=0}^{\infty}\frac{z^{m+\alpha}}{m+\alpha}\;. \tag{E.1}
$$

This series can be written down for any α except when α is a negative integer. It is absolutely convergent for $|z|<1$, as can be easily shown by ratio test. We conclude that this series gives a well-defined function for any $\alpha\notin\mathbb{Z}_-$ and for all $|z|<1$. Further absolute convergence legitimizes term by term differentiation, integration etc., of this series representation.

The central fact relevant for our purpose is that via term by term differentiation, it can be shown that the incomplete Beta function solves the following inhomogeneous first order ODE:

$$
\xi^2\left(1-\xi^{-d}\right)\frac{d}{d\xi}\left[\frac{1}{d}\,{\rm B}\left(\frac{k+1}{d},0;\frac{1}{\xi^d}\right)\right]+\frac{1}{\xi^k}=0\;. \tag{E.2}
$$

This can also be equivalently shown via the following integral representation valid for $\alpha > 0$ and $|z| < 1$:

$$\mathrm{B}\left(\alpha,0;z\right) = \int_0^z \frac{t^{\alpha-1}dt}{1-t} \,. \tag{E.3}$$

One can check that a binomial expansion of the denominator in the integrand gives us back the original series. The differential equation above uniquely determines the incomplete Beta function up to an additive constant. For $\alpha > 0$, the additive constant can be fixed by demanding that the function vanishes as $z \to 0$. The $\alpha < 0$ case can then be thought of as an analytic continuation from $\alpha > 0$ case.

The incomplete Beta function has a logarithmic branch cut at $z = 1$: the series expansion as we take $z \to 1^-$ is given by

$$-\,\mathrm{B}\left(\alpha,0;z\right) = \gamma_E + \psi(\alpha) + \ln(1-z) + \sum_{n=1}^{\infty} \frac{(-)^n}{n} \frac{\Gamma(\alpha)}{n!\;\Gamma(\alpha-n)}(1-z)^n \,. \tag{E.4}$$

Here γ_E is the Euler constant and $\psi(\alpha)$ is the digamma function. The presence of a branch cut is easily seen when α is a positive integer. In this case, the defining series can be recognized as the Madhava-Taylor series of the logarithm with the first few terms dropped:

$$\mathrm{B}\left(\alpha,0;z\right) = \ln(1-z) + \sum_{n=1}^{\alpha-1} \frac{z^k}{k} \quad \text{for} \quad \alpha \in \mathbb{Z}_+ \,. \tag{E.5}$$

The logarithmic branch cut is evident in this case. Based on the above note in particular that the difference of two incomplete Beta functions is analytic near $z = 1$, and is given in terms of the digamma function $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log\Gamma(x)$

$$\lim_{z\to 1^-} \left[\mathrm{B}\left(\alpha_1,0;z\right) - \mathrm{B}\left(\alpha_2,0;z\right)\right] = \psi(\alpha_2) - \psi(\alpha_1) = \sum_{n=0}^{\infty}\left[\frac{1}{n+\alpha_2} - \frac{1}{n+\alpha_1}\right] . \tag{E.6}$$

F Plane wave harmonics

We give a quick summary of our conventions for harmonic decomposition on $\mathbb{R}^{d-1,1}$. We will classify the harmonics to be scalar, vector and tensor, based on their transformation of the spatial $SO(d-2)$ rotation group transverse to a fixed momentum vector $\mathbf{k}$. This is standard in most of the literature, see for example [106] for a general discussion. We note that a more compact presentation would have used $SO(d-1)$ representation theory as is employed in the fluid/gravity literature [22, 132]. Our discussion differs in a minor way from the bulk of the literature in that we directly work in $\mathbb{R}^{d-1,1}$ including the time-frequency dependence in our definition.

- **Scalar plane waves**: These are simply scalar plane waves

$$\mathbb{S}(\omega, \mathbf{k}|v, \mathbf{x}) \equiv e^{i\,\mathbf{k}\cdot\mathbf{x} - i\,\omega v} \tag{F.1}$$

  We will also need its spatial derivatives in what follows, which we denote by $\mathbb{S}_i$ and $\mathbb{S}_{ij}$.

- **Vector plane waves**: These are transverse vector plane waves, denoted $\mathbb{V}_i^\alpha$. They are transverse to the momentum vector $\mathbf{k}$, satisfying $k_i\,\mathbb{V}_i^\alpha = 0$ and transform in a spin-one (vector) representation of $SO(d-2)$, which is the rotation group in spatial directions normal to $\mathbf{k}$. They will be taken to furnish an orthonormal basis for the transverse vectors and are $N_V = d-2$ in number.

- **Tensor plane waves**: These are transverse, symmetric, trace-free, tensor plane waves, obeying $k_i\,\mathbb{T}_{ij}^\sigma = 0$ and $\mathbb{T}_{ii}^\sigma = 0$. There are clearly $N_T = \frac{d(d-3)}{2}$ such modes which furnish a spin-2 representation of $SO(d-2)$.

All these functions $\{\mathbb{S}, \mathbb{V}_i^\alpha, \mathbb{T}_{ij}^\sigma\}$ appearing above are, as noted, plane waves, i.e., eigenfunctions of $\{i\,\frac{\partial}{\partial v}, -i\frac{\partial}{\partial x^i}\}$ with eigenvalue $\{\omega, k_i\}$. On the plane waves we define orthonormality with respect to the flat measure on $\mathbb{R}^{d-1,1}$. Let

$$\langle \mathcal{P}(\omega_1, \mathbf{k}_1|v, \mathbf{x}), \mathcal{Q}(\omega_2, \mathbf{k}_2|v, \mathbf{x})\rangle = \int d^dx\, \mathcal{P}(\omega_1, \mathbf{k}_1|v, \mathbf{x})\, \mathcal{Q}(\omega_2, \mathbf{k}_2|v, \mathbf{x})\,, \tag{F.2}$$

where we adopt the notational shorthand

$$d^dx = dv\, d^{d-1}\mathbf{x}\,, \qquad \delta^d(k_1 + k_2) = \delta(\omega_1 + \omega_2) \times \delta^{d-1}(\mathbf{k}_1 + \mathbf{k}_2)\,. \tag{F.3}$$

The statement of orthonormality is then simply the standard plane wave normalization in flat spacetime dressed with Kronecker deltas for the discrete labels, viz.,

$$\begin{aligned}
\langle \mathbb{S}(\omega_1, \mathbf{k}_1|v, \mathbf{x}), \mathbb{S}(\omega_2, \mathbf{k}_2|v, \mathbf{x})\rangle &= (2\pi)^d\, \delta^d(k_1 + k_2) \\
\langle \mathbb{V}_i^{\alpha_1}(\omega_1, \mathbf{k}_1|v, \mathbf{x}), \mathbb{V}_i^{\alpha_2}(\omega_2, \mathbf{k}_2|v, \mathbf{x})\rangle &= \delta_{\alpha_1\alpha_2} \times (2\pi)^d\, \delta^d(k_1 + k_2)\,, \\
\frac{1}{2}\, \Big\langle \mathbb{T}_{ij}^{\sigma_1}(\omega_1, \mathbf{k}_1|v, \mathbf{x}), \mathbb{T}_{ij}^{\sigma_2}(\omega_2, \mathbf{k}_2|v, \mathbf{x})\Big\rangle &= \delta_{\sigma_1\sigma_2} \times (2\pi)^d\, \delta^d(k_1 + k_2)\ .
\end{aligned} \tag{F.4}$$

The basis of vector and tensor plane waves are also orthonormal with respect to derived objects obtained by differentiating scalar or vector plane waves. Define:

$$\begin{aligned}
\mathbb{S}_i(\omega, \mathbf{k}|v, \mathbf{x}) &\equiv \frac{1}{k}\partial_i \mathbb{S}(\omega, \mathbf{k}|v, \mathbf{x})\ , \\
\mathbb{S}_{ij}(\omega, \mathbf{k}|v, \mathbf{x}) &\equiv \frac{1}{k^2}\partial_i\partial_j \mathbb{S}(\omega, \mathbf{k}|v, \mathbf{x})\ , \\
\mathbb{V}_{ij}^\alpha(\omega, \mathbf{k}|v, \mathbf{x}) &\equiv \frac{1}{k}\left(\partial_i \mathbb{V}_j^\alpha(\omega, \mathbf{k}|v, \mathbf{x}) + \partial_j \mathbb{V}_i^\alpha(\omega, \mathbf{k}|v, \mathbf{x})\right)\ ,
\end{aligned} \tag{F.5}$$

These derived wave harmonics are also normalized as

$$\begin{aligned}\langle \mathbb{S}_i(\omega_1,\mathbf{k}_1|v,\mathbf{x}),\mathbb{S}_i(\omega_2,\mathbf{k}_2|v,\mathbf{x})\rangle &= (2\pi)^d\,\delta^d(k_1+k_2)\\ \frac{1}{2}\left\langle \mathbb{V}^{\alpha_1}_{ij}(\omega_1,\mathbf{k}_1|v,\mathbf{x}),\mathbb{V}^{\alpha_2}_{ij}(\omega_2,\mathbf{k}_2|v,\mathbf{x})\right\rangle &= \delta_{\alpha_1\alpha_2}\times(2\pi)^d\,\delta^d(k_1+k_2)\ ,\\ \frac{1}{2}\,\langle \mathbb{S}_{ij}(\omega_1,\mathbf{k}_1|v,\mathbf{x}),\mathbb{S}_{ij}(\omega_2,\mathbf{k}_2|v,\mathbf{x})\rangle &= \frac{1}{2}\times(2\pi)^d\,\delta^d(k_1+k_2)\ .\end{aligned} \tag{F.6}$$

The relative orthogonality between the derived plane waves and the vector/tensor plane waves is simply the statement that

$$\begin{aligned}\langle \mathbb{V}^{\alpha}_{i}(\omega_1,\mathbf{k}_1|v,\mathbf{x}),\mathbb{S}_i(\omega_2,\mathbf{k}_2|v,\mathbf{x})\rangle &= 0\,,\\ \left\langle \mathbb{T}^{\sigma}_{ij}(\omega_1,\mathbf{k}_1|v,\mathbf{x}),\mathbb{V}^{\alpha}_{ij}(\omega_2,\mathbf{k}_2|v,\mathbf{x})\right\rangle &= 0\,,\\ \left\langle \mathbb{T}^{\sigma}_{ij}(\omega_1,\mathbf{k}_1|v,\mathbf{x}),\mathbb{S}_{ij}(\omega_2,\mathbf{k}_2|v,\mathbf{x})\right\rangle &= 0\,,\\ \left\langle \mathbb{V}^{\alpha}_{ij}(\omega_1,\mathbf{k}_1|v,\mathbf{x}),\mathbb{S}_{ij}(\omega_2,\mathbf{k}_2|v,\mathbf{x})\right\rangle &= 0\,.\end{aligned} \tag{F.7}$$

These vector plane waves may be familiar to the reader as the plane wave solutions for massless vector fields in $\mathbb{R}^{d-1,1}$ and they fill out a $(d-2)$ dimensional representation of the little group $SO(d-2)$. From the point of AdS/CFT, long distance (i.e., small $\{\omega, k_i\}$) physics of these modes on the gravity side is dual to the physics of the shear modes in the dual CFT fluid. Shear modes are a diffusive branch of solutions of Navier-Stokes equations where the fluid velocity or momentum density is divergence-free. They describe the shear viscosity driven diffusion of momentum density transverse to the wave vector direction. Because of the divergence free property, these modes occur both in compressible as well as incompressible fluids.

These tensor plane waves may be familiar to the reader as the plane wave solutions for massless spin 2 (graviton) fields in $\mathbb{R}^{d-1,1}$. There are $\frac{d(d-3)}{2}$ such graviton polarizations in d spacetime dimensions. In AdS/CFT, these modes do not survive the long distance (i.e., small $\{\omega, k_i\}$) limit. They are Markovian and hence do not have a dual CFT fluid analogue.

G Dynamics of scalar gravitons

Our starting point for analyzing the action is simply the Einstein-Hilbert action with its Gibbons-Hawking variational boundary term and appropriate counterterms. We are going to be working to quartic order in gradients. A-priori we expect that we would need counterterms accurate to that order. However, as we shall see there are some additional subtleties in this system which will allow us to obtain certain finite results from the quadratic counterterms alone. Irrespective of this we will quote here the full counterterm action accurate to fourth order in boundary derivatives.[7]

The gravitational dynamics we consider is governed by[8]

$$S_{\text{grav}} = \int d^{d+1}x\, \sqrt{-g}\, \left(R + d(d-1)\right) + 2\int d^d x\, \sqrt{-\gamma}\, K + S_{\text{ct}}$$
$$S_{\text{ct}} = \sqrt{-\gamma}\left(-2(d-1) - \frac{1}{d-2}\, {}^\gamma\! R - \frac{1}{(d-4)\,(d-2)^2}\left({}^\gamma\! R_{\mu\nu}\, {}^\gamma\! R^{\mu\nu} - \frac{d}{4(d-1)}\, {}^\gamma\! R^2\right)\right). \tag{G.1}$$

We will first examine the equations of motion which we write as

$$\mathbb{E}_{AB} = R_{AB} - \frac{1}{2}\, R\, g_{AB} - \frac{1}{2}\, d(d-1)\, g_{AB} = 0\,, \tag{G.2}$$

and then proceed to analyze the variational principle.

G.1 Gauge invariant data and time-reversal

To understand the dynamics of the scalar gravitational perturbations and deduce that the dynamics can be captured by a single field $\mathcal{Z}$ we will proceed in a series of steps. Our first task will be to identify the diffeomorphism invariant combinations of the metric perturbations for the ansatz (3.1). A natural way to capture this information is to look at the curvature tensors which we write in terms of orbit space tensors. It will be convenient to define a connection on this part of the geometry:[9]

$$\Upsilon \equiv \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 f\right) = d\, r - (d-2)\, r f\ . \tag{G.3}$$

Some useful identities which we have used to simplify the expressions are:

$$\left(\mathbb{D}_+ - \frac{1}{2}\Upsilon\right)\mathfrak{F} = r\sqrt{f}\ \mathbb{D}_+\left(\frac{\mathfrak{F}}{r\sqrt{f}}\right), \qquad \left(\mathbb{D}_+ - \frac{1}{2}\,\Upsilon\right)(r\,\mathfrak{F}) = r\left(\mathbb{D}_+ - \frac{1}{2}\, r^2 f'\right)\mathfrak{F}. \tag{G.4}$$

We start with the metric parameterized in terms of Ψ_{AB} as presented in (3.1). For this geometry the gauge invariant combinations organized into the orbit space tensors are [106]:

- An orbit space vector $\mathbb{X}^a$ whose dual one-form has components

$$\mathbb{X}_v \equiv k\, r\, \Psi_{vx} - i\omega\, r^2\, \Psi_{\text{T}}\ , \quad \mathbb{X}_r \equiv k\, r\, \Psi_{rx} + r^2\, \frac{\mathrm{d}\Psi_{\text{T}}}{\mathrm{d}r}\,. \tag{G.5}$$

---

[7] The fourth order counterterms will turn out to be the leading regularization for $\mathcal{Z}$ which receives no corrections at lower orders.

[8] We eschew the overall normalization by $\frac{1}{16\pi G_N}$ to keep the expressions simple. Boundary quantities will be obtained by multiplying by c_{eff} at the end.

[9] We will use lowercase early alphabet Latin characters to indicate orbit space tensors in addition to the conventions specified in §1.3.

- An orbit space symmetric traceless rank 2 tensor, $\mathbb{Y}_{ab}$, with components

$$\begin{aligned}
\mathbb{Y}_{vv} &\equiv k^2\left(\Psi_{vv} - 2\,r^2 f\,\Psi_{\text{S}}\right) - 2\,i\omega\,\mathbb{X}_v - \Upsilon\left(\mathbb{X}_v + r^2 f\,\mathbb{X}_r\right)\,,\\
\mathbb{Y}_{vr} \equiv \mathbb{Y}_{rv} &\equiv k^2\left(\Psi_{vr} + 2\,\Psi_{\text{S}}\right) + \left(\Upsilon - i\omega\right)\,\mathbb{X}_r + \frac{\mathrm{d}\mathbb{X}_v}{\mathrm{d}r}\,,\\
\mathbb{Y}_{rr} &\equiv k^2\,\Psi_{rr} + 2\,\frac{\mathrm{d}\mathbb{X}_r}{\mathrm{d}r}\,.
\end{aligned} \tag{G.6}$$

- And finally, we have orbit space scalars

$$\begin{aligned}
\mathbb{Y}_{\text{S}} &\equiv k^2\left(\Psi_{\text{S}} + \frac{\Psi_{\text{T}}}{d-1}\right) + \frac{1}{r}\left(\mathbb{X}_v + r^2 f\,\mathbb{X}_r\right)\,,\\
\frac{1}{2}\,\mathbb{Y}^a_a &= \mathbb{Y}_{vr} + \frac{1}{2}r^2 f\;\;\mathbb{Y}_{rr} = k^2\left(\Psi_{vr} + 2\,\Psi_{\text{S}} + \frac{1}{2}\,r^2 f\;\;\Psi_{rr}\right) + \frac{d\mathbb{X}_v}{dr} + (\mathbb{D}_+ + \Upsilon)\mathbb{X}_r\,.
\end{aligned} \tag{G.7}$$

To understand the time-reversal properties of these combinations we use the observation that on the orbit space time-reversal is just a diffeomorphism. Hence we conclude that $\mathbb{Y}^a_a$ and $\mathbb{Y}_{\text{S}}$ have even time-reversal parity. For the reminder we use the fact that the orbit space vectors can be decomposed into the basis adapted to time-reversal introduced above (1.40),

$$\mathbb{X}_a\,dx^a = \mathbb{X}_v\left(dv - \frac{dr}{r^2 f}\right) + \left(\mathbb{X}_v + r^2 f\,\mathbb{X}_r\right)\frac{dr}{r^2 f}\,, \tag{G.8}$$

and use the fact that $dv - \frac{dr}{r^2 f}$ is odd under time-reversal and $\frac{dr}{r^2 f}$ is even. Similar decomposition for the tensors leads to

$$\begin{aligned}
\mathbb{Y}_{ab}\,dx^a\,dx^b = \mathbb{Y}_{vv}&\left\{\left(dv - \frac{dr}{r^2 f}\right)^2 + \left(\frac{dr}{r^2 f}\right)^2\right\} + 2\,r^2 f\left(\mathbb{Y}_{vr} + \frac{1}{2}r^2 f\;\;\mathbb{Y}_{rr}\right)\left(\frac{dr}{r^2 f}\right)^2\\
&+\;2\left(r^2 f\mathbb{Y}_{vr} + \mathbb{Y}_{vv}\right)\frac{dr}{r^2 f}\left(dv - \frac{dr}{r^2 f}\right)\,.
\end{aligned} \tag{G.9}$$

For the purposes of analyzing the equations of motion it is helpful to define some rescaled combinations of fields which have definite time-reversal parity. We introduce:

$$\Phi_{\text{E}} = r^{d-3}\,\Psi_{vv}\,,\quad \Phi_{\text{O}} = r^{d-3}\left(\Psi_{vv} + r^2 f\,\Psi_{vr}\right)\,,\quad \Phi_{\text{W}} = 2\,r^{d-2}\,\Psi_{\text{S}}\,. \tag{G.10}$$

We summarize the essential information from this analysis in Table 2.

G.2 Dynamics in the Debye gauge

In Appendix G.1 we introduced the gauge invariant combinations of metric perturbations. One can however fix some of these metric functions by using the diffeomorphism freedom.

TR Parity	Gauge invariants	Metric components	Debye gauge data
Even	$\mathbb{Y}_{\text{S}},\ \mathbb{Y}_{vv}$ $\mathbb{Y}_{vr}+\frac{1}{2}r^2f\,\mathbb{Y}_{rr}$ $\mathbb{X}_v+r^2f\,\mathbb{X}_r$	$\Psi_{\text{S}},\ \Psi_{vv},\ \Psi_{\text{T}}$ $\Psi_{vr}+\frac{1}{2}r^2f\,\Psi_{rr}$ $\Psi_{vx}+r^2f\,\Psi_{rx}$	$\Phi_{\text{E}},\ \Phi_{\text{W}}$ $\Theta,\ \mathcal{Z}$
Odd	$r^2f\,\mathbb{Y}_{vr}+\mathbb{Y}_{vv}$ $\mathbb{X}_v$	$\Psi_{vv}+r^2f\,\Psi_{vr}$ Ψ_{vx}	Φ_{O}

Table 2: Time-reversal parity of the scalar perturbation of Schwarzschild-AdS$_{d+1}$.

A-priori we can gauge fix three functions, leaving behind four of the seven perturbation functions appearing in (3.1). We will implement this by working with a set of 4 functions $\{\Psi_{\text{S}},\Psi_{vv},\Psi_{vr},\Psi_{rr}\}$ by first rescaling out a factor of k^2 from the gauge invariant scalar and tensor data, $\mathbb{Y}_{\text{S}}$ and $\mathbb{Y}_{ab}$, i.e., setting $\mathbb{X}_a=\Psi_{\text{T}}=0$. Equivalently, we have the gauge conditions

$$\text{Debye Gauge}:\quad \Psi_{vx}=\Psi_{rx}=\Psi_{\text{T}}=0\,. \tag{G.11}$$

We can interpret $\mathbb{Y}_{\text{S}}$ and $\mathbb{Y}_{ab}$ in terms of metric components in the scalar sector in a Debye gauge.[10] This is a coordinate chart such that metric has no derivatives of scalar plane waves under plane wave decomposition, viz., the perturbation can be recast into the form

$$ds^2_{(1)}=\int_k\Big\{(\Psi_{vv}-2r^2f\,\Psi_{\text{S}})\,dv^2+2\,(\Psi_{vr}+2\,\Psi_{\text{S}})\,dvdr+\Psi_{rr}\,dr^2+2\,r^2\,\Psi_{\text{S}}\,d\mathbf{x}^2\Big\}\,\mathbb{S}\,. \tag{G.12}$$

This was the gauge choice adopted in [106].

We will now present the linearized Einstein equations in terms of these scalars by decomposing (G.2) into plane waves. Employing the definitions in (G.10) and further introducing the combination:

$$\Psi_{rr}=-\frac{1}{r^{d+1}f^2}\left[2(\Phi_{\text{O}}-\Phi_{\text{E}})+r f\,(d-1)\,\Phi_{\text{W}}+\Phi_{\text{B}}\right], \tag{G.13}$$

we end up the metric which at linear order takes the form:

$$\begin{split}ds^2_{(1)}=\frac{\Phi_{\text{E}}-rf\,\Phi_{\text{W}}}{r^{d-3}}\,dv^2+\frac{2}{r^{d-1}f}\,(\Phi_{\text{O}}-\Phi_{\text{E}}+rf\,\Phi_{\text{W}})\,dv\,dr+r^2\,\frac{\Phi_{\text{W}}}{r^{d-2}}\,d\mathbf{x}^2\\ -\frac{1}{r^{d+1}f^2}\left[2(\Phi_{\text{O}}-\Phi_{\text{E}})+rf\,(d-1)\,\Phi_{\text{W}}+\Phi_{\text{B}}\right]dr^2\,.\end{split} \tag{G.14}$$

With this choice of gauge the vector gauge invariants vanish $\mathbb{X}_a=0$, while the re-

[10]This statement is true for spatially inhomogeneous modes. For spatially homogeneous modes all the invariants $\mathbb{Y}_{ab}$ and $\mathbb{Y}_{\text{S}}$ are determined in terms of the vector invariant $\mathbb{X}_a$, which has been set to zero here by our gauge choice. Most of our analysis will be for $k\neq 0$ where this is not an issue. We will highlight this when we study the homogeneous modes in Appendix K.

maining scalar and tensor invariants simplify in the parameterization (G.14) to

$$
\begin{aligned}
\mathbb{Y}_{\text{S}} &\equiv \frac{k^2}{2\,r^{d-2}}\,\Phi_{\text{W}}\,, \\
\mathbb{Y}_{\text{E}} &\equiv \mathbb{Y}_{vv} = \frac{k^2}{r^{d-3}}(\Phi_{\text{E}} - r f\,\Phi_{\text{W}})\,, \\
\mathbb{Y}_{\text{O}} &\equiv \mathbb{Y}_{vv} + r^2 f\,\mathbb{Y}_{vr} = \frac{k^2}{r^{d-3}}\Phi_{\text{O}}\,, \\
\mathbb{Y}_{\text{B}} &\equiv \mathbb{Y}_{vr} + \frac{1}{2}\,r^2 f\,\mathbb{Y}_{rr} = -\frac{k^2}{2\,r^{d-1} f}\,(\Phi_{\text{B}} + (d-3)\,r f\,\Phi_{\text{W}})\,.
\end{aligned}
\tag{G.15}
$$

It will be helpful to assemble the equations of motion (G.2) into time-reversal invariant orbit space tensor combinations as above. We first have the scalar equation, which involves only Φ_{B} and takes a simple form:[11]

$$
\mathbb{E}_{\text{T}} = -\frac{k^2}{2\,r^{d-1}}\Phi_{\text{B}}\,. \tag{G.16}
$$

This equation is actually a simple algebraic constraint on the invariants: $\mathbb{E}_{\text{T}} = \mathbb{Y}_{\text{B}} + (d-3)\,\mathbb{Y}_{\text{S}}$.

The orbit space tensor equations assembled again into time-reversal invariant combinations take the form:

$$
\begin{aligned}
\mathbb{E}_1 &\equiv \frac{2\,r^{d-1}}{d-1}\,\mathbb{E}_{vv} \\
&= \mathbb{D}_+\,(\mathcal{T} - r\,\Phi_{\text{B}}) + \frac{k^2}{d-1}\,(\Phi_{\text{E}} - \Phi_{\text{B}})\,, \\
\mathbb{E}_2 &\equiv \frac{2\,r^{d-1}}{d-1}\,(\mathbb{E}_{vv} + r^2 f\,\mathbb{E}_{vr}) \\
&= -i\omega\,(\mathcal{T} - r\,\Phi_{\text{B}}) + \frac{k^2}{d-1}\Phi_{\text{O}}\,, \\
\mathbb{E}_{\text{B}} &\equiv \frac{2\,r^{d+1} f}{d-1}\left(\mathbb{E}_{vr} + \frac{1}{2}\,r^2 f\,\mathbb{E}_{rr}\right) \\
&= -\mathbb{D}_+\left(\mathcal{T} - \frac{r}{2}\,\Phi_{\text{B}}\right) - i\omega\,r\,\Phi_{\text{O}} - \frac{r}{2}\,(\mathbb{D}_+ - \Upsilon + r f)\,[\mathbb{D}_+\Phi_{\text{W}} - (d-2)\,r f\,\Phi_{\text{W}}] \\
&\quad + \frac{r}{2}\,(\omega^2 - k^2 f)\,\Phi_{\text{W}} + \frac{k^2 + d(d-1)\,r^2}{2\,(d-1)}\,\Phi_{\text{B}}\,.
\end{aligned}
\tag{G.17}
$$

[11] With $\Psi_{\text{T}} \neq 0$, this equation gets modified to

$$
\mathbb{E}_{\text{T}} = -\frac{k^2}{2\,r^{d-1}}\Phi_{\text{B}} + \left[\frac{1}{r^{d-1}}\,\mathbb{D}_+\left(r^{d-1}\,\mathbb{D}_+\right) + \omega^2 + \frac{d-3}{d-1}\,k^2 f\right]\Psi_{\text{T}}\,.
$$

Now Ψ_{T} is a Markovian field of index $\mathcal{M} = d-1$, albeit one with an analytically continued momentum $k^2 \to -\frac{d-3}{d-1}\,k^2$ and sourced by Φ_{B}. The operator acting on Ψ_{T} is the one acting on $\mathcal{Z}$ in (3.9) with the specification $\Lambda_k = \frac{d-1}{2}\,r^3 f'$. The Markovian part of the $\mathcal{Z}$ solution in (3.19) is homogeneous solution of this operator.

We introduced here the quantity $\mathcal{T}$, which is defined to be

$$\mathcal{T} \equiv r\,\Phi_{\text{E}} - \left(\mathbb{D}_+ - \frac{\Upsilon}{2}\right)(r\,\Phi_{\text{W}}) = r\left[\Phi_{\text{E}} - \mathbb{D}_+\Phi_{\text{W}} + \frac{r^2 f'}{2}\Phi_{\text{W}}\right]. \tag{G.18}$$

This leaves us with the vector equations which being coefficients of $\mathbb{S}_i$ have an explicit momentum factor, keeping track of which will be important for understanding spatial zero modes. We find:

$$\begin{aligned}
\mathbb{E}_4 &\equiv 2\,r^{d-1} f\,\mathbb{E}_{vi} = ik_i\,\widetilde{\mathbb{E}}_4 \\
&= ik_i\left[\mathbb{D}_+\Phi_{\text{O}} + i\omega\,(\Phi_{\text{E}} - \Phi_{\text{B}})\right], \\
\mathbb{E}_5 &\equiv 2\,r^{d-1} f\,\left(\mathbb{E}_{vi} + r^2 f\,\mathbb{E}_{ri}\right) = ik_i\,\widetilde{\mathbb{E}}_5 \\
&= ik_i\left[\mathbb{D}_+\Phi_{\text{E}} + i\omega\,\Phi_{\text{O}} - (d-1)\left(\mathbb{D}_+ - \frac{1}{2}\Upsilon\right)(rf\Phi_{\text{W}}) - \frac{r}{2}\,(d+(d-2)f)\,\Phi_{\text{B}}\right].
\end{aligned} \tag{G.19}$$

The tilded equations strip out the momentum factor which is convenient to do. The remaining equations which are orbit space scalars picking out the trace and the $\mathbb{S}_{ij}$ part of the spatial harmonics can be naturally expressed in terms of them as

$$\begin{aligned}
\mathbb{E}_6 &\equiv -\frac{2\,r^{d-1} f}{d-1}\sum_{i=1}^{d-1}\mathbb{E}_{ii} \\
&= \mathbb{D}_+\left(\frac{\widetilde{\mathbb{E}}_5}{f}\right) + i\omega\,\frac{\widetilde{\mathbb{E}}_4}{f} + 2\,\frac{d-2}{d-1}\,r^{d-1}\,\mathbb{E}_{\text{T}}\,, \\
\mathbb{E}_7 &\equiv f\,\mathbb{E}_{ij} = \frac{k_i\,k_j}{k^2}\,\mathbb{E}_{\text{T}}\,.
\end{aligned} \tag{G.20}$$

Finally, a natural way to combine the equations involves taking a particular combination of $\mathbb{E}_{\text{B}}$ and $\mathbb{E}_5$:

$$\begin{aligned}
\mathbb{E}_3 &= \frac{2}{r}\,\mathbb{E}_{\text{B}} + \widetilde{\mathbb{E}}_5 \\
&= (\mathbb{D}_+ + 2rf)\left[\mathbb{D}_+\Phi_{\text{W}} - \Phi_{\text{E}} + \Phi_{\text{B}}\right] - i\omega\,\Phi_{\text{O}} + \frac{\Lambda_k}{(d-1)\,r}\Phi_{\text{B}} \\
&\quad + \left(\omega^2 - k^2 f + \frac{d-3}{2}\,r^3 f f'\right)\Phi_{\text{W}}\,.
\end{aligned} \tag{G.21}$$

G.3 Parameterizing the solution space: $k \neq 0$

Since there are only four physical functions, we should only have to use four of the equations of motion. One can check that not all the equations given above are independent (explicitly visible for example in $\mathbb{E}_6$ and $\mathbb{E}_7$), which suggests that a judicious choice of four equations should suffice to distill the dynamics into a manageable form. For $k \neq 0$ an efficient choice turns out to be the set $\{\mathbb{E}_{\text{T}}, \mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3\}$, satisfying which will ensure that the reminder are also upheld. We will now analyze the equations introducing Θ and $\mathcal{Z}$ to simplify the dynamics in the process.

Let us begin with $\mathbb{E}_{\text{T}} = 0$ which says that $\Phi_{\text{B}}(r,\omega,k) = 0$, as long as we focus on non-zero k, spatially inhomogeneous modes. We will use this to set $\Phi_{\text{B}} = 0$ in this subsection and return to the case where we have a spatially homogeneous function in Appendix K.

In the rest of this section we will give a brief discussion of how one simplifies the dynamics to that of a single scalar field. We have three independent linearized Einstein's equations in the set (G.17), (G.19) for the fields $\{\Phi_{\text{E}}, \Phi_{\text{O}}, \Phi_{\text{W}}\}$. It will be convenient to pick the following combinations as our independent Einstein's equations, setting $\Phi_{\text{B}} = 0$ in the process to simplify our expressions:

$$
\begin{aligned}
\mathbb{E}_1 &= \mathbb{D}_+\mathfrak{T} + \frac{k^2}{d-1}\,\Phi_{\text{E}}\,, \\
\mathbb{E}_2 &= -i\omega\,\mathfrak{T} + \frac{k^2}{d-1}\,\Phi_{\text{O}}\,, \\
\mathbb{E}_3 &= (\mathbb{D}_+ + 2\,r f)\left[\mathbb{D}_+\Phi_{\text{W}} - \Phi_{\text{E}}\right] - i\omega\,\Phi_{\text{O}} + \left(\omega^2 - k^2 f + \frac{(d-3)}{2}\,r^3 f f'\right)\Phi_{\text{W}}\,.
\end{aligned}
\tag{G.22}
$$

We will see shortly that $\mathbb{E}_4$ in (G.19) will be accounted for (actually it can be eliminated algebraically using an algebraic identity). The combination $\mathbb{E}_3$ above will simultaneously take care of $\mathbb{E}_5$ and $\mathbb{E}_{\text{B}}$ by definition.

The Weyl factor and momentum flux fields: To solve these equations we adopt a strategy similar to the one employed in the analysis of gauge field equations in [42, 134]. One notes that $\mathbb{E}_2$ is the energy conservation equation; in fact Φ_{O} is the only time-reversal odd field which is related to the momentum flux. This suggests we should algebraically solve this equation by letting $\Phi_{\text{O}} \propto \omega$. We express $\mathfrak{T}$ in terms of the same variable and then fix Φ_{E} using the first equation. To wit,

$$
\mathfrak{T} = -\frac{k^2}{d-1}\,\Theta\,, \qquad \Phi_{\text{O}} = -i\omega\,\Theta\,, \qquad \Phi_{\text{E}} = \mathbb{D}_+\Theta\,. \tag{G.23}
$$

This choice ensures that the first two equations in (G.22) are satisfied. We are then left with third equation $\mathbb{E}_3$, which can be viewed as a relation between Φ_{W} and Θ. This gives a constraint on the parameterization, isolating the true dynamical equation.

The designer field for sound: At this point, based on the experience with vector polarizations and diffusive mode, one would expect that Θ is the physical variable that should parameterize the designer field dual to the sound mode in the plasma. While this is physically correct (as we will justify) there is however a technical obstacle. The parameterization (G.23) does not immediately give an autonomous equation for Θ but rather leads to a coupled system between Θ and Φ_{W} from $\mathbb{E}_3$.

One can however isolate a new field $\mathcal{Z}$ by realizing that Φ_{W} and Θ are not independent but related to each other through the relation

$$\Phi_{\text{E}} - \frac{1}{r}\mathcal{T} = \left(\mathbb{D}_+ - \frac{1}{2}r^2 f'\right)\Phi_{\text{W}} = \left(\mathbb{D}_+ + \frac{k^2}{d-1}\frac{1}{r}\right)\Theta\,. \tag{G.24}$$

The first equality follows from (G.18) and the second from (G.23). This can be solved by introducing an auxiliary field $\mathcal{Z}$ and solving for Φ_{W} and Θ in terms of it.[12] This results in the expression (3.6) quoted above.

This explains the origin of the designer field $\mathcal{Z}$ and the momentum dependent factor Λ_k, which originates during decoupling the Θ and Φ_{W} dynamics. Once we arrive here, it is straightforward to check that the final constraint equation on the system is the equation of motion for $\mathcal{Z}$ given earlier in (3.9). It is easy to see that the resulting equation is second order once one appreciates that Θ, $\mathcal{Z}$, and Φ_{W} satisfy a linear relation from (3.6)

$$\Theta = \Phi_{\text{W}} - \frac{1}{(d-1)}\mathcal{Z}\,. \tag{G.25}$$

This allows us to write $\mathbb{D}_+\Phi_{\text{W}} - \Phi_{\text{E}} = \mathbb{D}_+\Phi_{\text{W}} - \mathbb{D}_+\Theta = \frac{1}{d-1}\mathbb{D}_+\mathcal{Z}$ which then reduces $\mathbb{E}_3$ to

$$\frac{r^{d-3}\Lambda_k^2}{f}\left(\mathbb{D}_+ - \frac{\Upsilon}{2}\right)\left[\frac{f}{r^{d-3}\Lambda_k^2}\left(\mathbb{D}_+ - \frac{\Upsilon}{2}\right)\right](r\,\mathcal{Z}) + \left(\omega^2 - k^2 f + \frac{d^2}{4}r^2(1-f)^2\right) r\,\mathcal{Z} = 0\,. \tag{G.26}$$

A slight simplification of (G.26) using the explicit expression for Λ_k and (G.4) leads to the equation of motion (3.9) quoted in the main text. We emphasize that the dynamics is governed by a time-reversal invariant equation, which as explained in [61], allows one to construct smooth solutions on the grSK geometry.

The parameterization of the metric functions in terms of $\mathcal{Z}$ is easily obtained to be

$$\begin{aligned}
\Phi_{\text{E}} &= \mathbb{D}_+\left(\frac{r}{\Lambda_k}\left[\mathbb{D}_+ - \frac{r^2 f'}{2}\right]\mathcal{Z}\right), \\
\Phi_{\text{O}} &= -\frac{i\omega\, r}{\Lambda_k}\left[\mathbb{D}_+ - \frac{r^2 f'}{2}\right]\mathcal{Z}, \\
\Phi_{\text{W}} &= \frac{1}{\Lambda_k}\left[r\,\mathbb{D}_+ + \frac{k^2}{d-1}\right]\mathcal{Z}\,.
\end{aligned} \tag{G.27}$$

This suffices to determine the linearized geometry (3.5) once we know the solution for $\mathcal{Z}$. The change of variables involves $\mathcal{Z}$, $\mathbb{D}_+\mathcal{Z}$, and $\mathbb{D}_+^2\mathcal{Z}$ and appears to be necessary to ensure that the classical phase space is only two-dimensional, parameterized by an effective source and a corresponding dual CFT plasma operator. We will focus on parameterizing

[12] To do so we use the observation that equations of the form $(\partial + A)X = (\partial + B)Y$ can be solved by setting $X = \frac{1}{A-B}(\partial + B)Z$ and $Y = \frac{1}{A-B}(\partial + A)Z$.

the phase space for the present by boundary values of $\mathcal{Z}$ and subsequently argue that the physical solution space is best parameterized by the stress tensor component $(T_{\text{CFT}})^i_v$ or equivalently by Θ.

H Variational principle in the scalar sector

We have indicated in (G.1) that L will refer to the Lagrangian including the measure factor. Unless explicitly indicated, we will write the terms in the action in a series of steps below, quoting at each stage this Lagrangian in momentum space. Integrations over momenta and over the bulk radial coordinate can thus be avoided in the expressions, which themselves tend to be pretty long. We also use $\dagger$ to indicate the frequency and momentum reversed field, viz., $\Phi^\dagger(\omega, \mathbf{k}) = \Phi(-\omega, -\mathbf{k})$, and thus use $+$ cc to account for symmetrization. This analysis is restricted to $\mathbf{k} \neq 0$ as we seek to establish the variational principle for $\mathcal{Z}$ at the end of the day.

H.1 Action for time-reversal invariant fields

Since the background Schwarzschild-AdS$_{d+1}$ solution has a non-vanishing on-shell action, when we expand the Einstein-Hilbert action with the perturbation ansatz, we will have terms starting at the zeroth order in the amplitudes of the perturbation. We will separate out the zeroth and first order contributions out ab-initio – they do not contribute to the dynamics of the linearized modes. Rather, these terms correspond to the background free energy and represent the ideal fluid contribution of the boundary action.[13] We therefore will write:

$$S_{\text{grav}} = S_{\text{grav,lin}} + S_{\text{grav,quad}}\,. \tag{H.1}$$

We work with the fields $\{\Phi_{\text{E}}, \Phi_{\text{O}}, \Phi_{\text{W}}\}$ having chosen to eliminate Ψ_{rr} using (G.13) (and use (G.16) to set $\Phi_{\text{B}} = 0$).

To begin, let us look at the contribution from the background and the linear terms in the fluctuations takes the form

$$\begin{aligned} S_{\text{grav,lin}} = \int d^dx \Bigg\{ & r^d \left(d + (d-2)\, f - 2\,(d-1)\sqrt{f}\right) \\ & + (d-1)\left[d\left(\frac{1+f}{2} - \sqrt{f}\right) r^2\, \Phi_{\text{W}} - r\,\Phi_{\text{E}}\left(1 - \frac{1}{\sqrt{f}}\right)\right]\Bigg\}\,. \end{aligned} \tag{H.2}$$

Up to this order, we can express this result as an ideal fluid action on the induced

[13] In the analysis of the tensor and vector modes in [42, 134] we in fact even extracted a part of the quadratic terms which assembled nicely to give ideal fluid contribution at the outset. In the present case, given the relative complexity of the dynamics, we find it useful to keep the quadratic pieces together and only isolate the part which involves terms at most linear in the fluctuation fields.

boundary geometry. This is similar to the earlier discussion in the vector and tensor cases [42]. One can equivalently write

$$S_{\text{grav,lin}} = \int d^d x\, \sqrt{-\gamma}\, \left[-\gamma_{\mu\nu}\, \mathbf{b}^\mu\, \mathbf{b}^\nu\right]^{-\frac{d}{2}} , \qquad \mathbf{b}^\mu\, \partial_\mu = b\, \partial_v. \tag{H.3}$$

Here $\mathbf{b}^\mu$ is the (rescaled) thermal vector that picks out the inertial frame. We will later see that on our solution there are corrections to the thermal vector, which we will need to account for, to get the correct ideal fluid action at quadratic order as noted in footnote 13.

Turning to the quadratic part, we will proceed in a series of steps, outlining independently the contributions of the bulk Einstein-Hilbert and Gibbons-Hawking terms. This will suffice to demonstrate that the dynamics is governed by the familiar Dirichlet boundary conditions for the aforementioned fields (and hence for Ψ_{AB}). Finally, we will outline the contribution from the counterterms that render the on-shell action finite.

The bulk Einstein-Hilbert term can be shown to decompose into a bulk piece, a boundary term, and a total temporal derivative term, viz.,

$$\sqrt{-g}\, \left(R + d(d-1)\right) = L^{\text{bulk}}_{\text{EOW}} - \frac{\partial}{\partial r} L^{\text{bdy}}_{\text{EOW}} + \frac{\partial}{\partial v} L^{\text{dot}}_{\text{EOW}} \,. \tag{H.4}$$

We start with the bulk term which can be simplified to the form:

$$\begin{aligned}
&L^{\text{bulk}}_{\text{EOW}}[\Phi_{\text{E}}, \Phi_{\text{O}}, \Phi_{\text{W}}] \\
&= \frac{d-1}{4\, f\, r^d}\Bigg\{ d\, r\, \mathbb{D}_+\Phi_{\text{W}}\, \mathbb{D}_+\Phi^\dagger_{\text{W}} - \frac{\mathbb{D}_+\Phi^\dagger_{\text{W}}\, \mathbb{D}_+\Phi_{\text{E}} + \text{cc}}{f} + (d-3)\, r \left(\Phi^\dagger_{\text{E}}\mathbb{D}_+\Phi_{\text{W}} + \text{cc}\right) \\
&\quad + (d-2)\, r \left(\Phi^\dagger_{\text{W}}\mathbb{D}_+\Phi_{\text{E}} + \text{cc}\right) + r^2\Big(d - (d-2)\,(d+1)\, f\Big) \left(\Phi^\dagger_{\text{W}}\mathbb{D}_+\Phi_{\text{W}} + \text{cc}\right) \\
&\quad + \frac{2i\omega}{(d-1) f}\left[\Big((d-2)\, \Phi^\dagger_{\text{W}}\, \mathbb{D}_+\Phi_{\text{O}} - \Phi_{\text{O}}\, \mathbb{D}_+\Phi^\dagger_{\text{W}} - \text{cc}\Big) - \frac{1}{r f}\left(\Phi^\dagger_{\text{E}}\, \mathbb{D}_+\Phi_{\text{O}} + \Phi_{\text{O}}\, \mathbb{D}_+\Phi^\dagger_{\text{E}} - \text{cc}\right)\right] \\
&\quad + \frac{i\omega\, r}{(d-1)\, f}\Big(d\,(d-3) + (d^2 - 5\, d + 8)\, f\Big)\left(\Phi_{\text{W}}\Phi^\dagger_{\text{O}} - \text{cc}\right) + \frac{4i\omega\,(d-f)}{(d-1)\, r^2}\left(\Phi^\dagger_{\text{E}}\, \Phi_{\text{O}} - \text{cc})\right) \\
&\quad + \frac{2k^2}{(d-1)\, r f}\left(\Phi^\dagger_{\text{O}}\, \Phi_{\text{O}} - \Phi^\dagger_{\text{E}}\, \Phi_{\text{E}}\right) - \frac{\omega^2 - k^2 f + (d-2)(d-3)\, r^2 f^2}{f}(\Phi^\dagger_{\text{E}}\Phi_{\text{W}} + \text{cc}) \\
&\quad + r\left[(d-2)(\omega^2 - k^2 f) + r^2 f\left(-2\, d(2d-5) + (d+2)\,(d-2)^2\, f\right)\right]\, \Phi^\dagger_{\text{W}}\, \Phi_{\text{W}}\Bigg\}.
\end{aligned} \tag{H.5}$$

This part of the action is obtained by direct evaluation and integrating by parts to isolate the boundary terms. It is interesting to observe that only Φ_{W} has a quadratic kinetic term and Φ_{E} appears in the kinetic part only coupled to Φ_{W}. Since Φ_{O} is the only time-reversal odd field its appearance in the action is highly constrained (and it only shows up with explicit time-derivatives). Note also that the field Φ_{W} has a wrong sign kinetic term (from the last line) reflecting the familiar issue with the conformal mode in gravity.

This already suggests that despite appearances $\Phi_{\rm W}$ is not the physical field.

The temporal boundary term $L^{\rm dot}_{\rm EOW}$ does not enter the analysis and can be dropped ab-initio. The radial boundary term cancels against a similar contribution from the Gibbons-Hawking term. The precise form of these terms will therefore not be necessary for us.

$$2\sqrt{-\gamma}\,K = L^{\rm GH}_{\rm EOW} - L^{\rm bdy}_{\rm EOW}\,. \tag{H.6}$$

This is a good consistency check for our computation ensuring that the Einstein-Hilbert action together with the Gibbons-Hawking term has a good variational principle. The remaining part of the Gibbons-Hawking term turns out to be:[14]

$$\begin{split} L^{\rm GH}_{\rm EOW}[\Phi_{\rm E},\Phi_{\rm O},\Phi_{\rm W}] = \frac{1}{2\,r^d\,f}\Bigg\{ & i\omega\,(d-2)\,r^2\left(\Phi^\dagger_{\rm O}\Phi_{\rm W} - {\rm cc}\right) - i\omega\,\frac{r}{f}\left(\Phi^\dagger_{\rm O}\Phi_{\rm E} - {\rm cc}\right) \\ & -(d-1)\,(d-2)\,r^3 f\,\left(\Phi^\dagger_{\rm E}\,\Phi_{\rm W} + {\rm cc}\right) \\ & + \frac{d-1}{2}\,r^4 f\left[d\,(d-3) + (d+1)\,(d-2)\,f\right]\Phi^\dagger_{\rm W}\,\Phi_{\rm W}\Bigg\}\,. \end{split} \tag{H.7}$$

The variation of (H.5) gives us three independent equations. One of these is the momentum conservation equation $\mathbb{E}_1$ from (G.22) which comes from varying $\Phi_{\rm O}$. The other two equations are linear combinations of the ones we have given above. It is interesting to note that the variation does not directly produce the $\mathbb{E}_3$ equation which was crucial to derive the autonomous second order equation for $\mathcal{Z}$. One aspect that is clear from the variational analysis is that the fields $\{\Phi_{\rm E},\Phi_{\rm O},\Phi_{\rm W}\}$ obey Dirichlet boundary conditions. This is manifest from the structure of the Gibbons-Hawking term which is a quadratic form in these three fields.

Finally, the counterterm action is given as

$$\begin{split} L^{\rm ct}_{\rm EOW}[\Phi_{\rm E},\Phi_{\rm O},\Phi_{\rm W}] = \frac{1}{4\,r^d\,f^{\frac{3}{2}}}\Bigg\{ & \left[(d-1)\,r^2 - \frac{1}{(d-1)\,(d-2)\,(d-4)}\,\frac{k^4}{r^2}\right]\Phi^\dagger_{\rm E}\,\Phi_{\rm E} \\ & + r f\,\left[k^2 + (d-1)\,(d-2)\,r^2\right]\left(\Phi^\dagger_{\rm E}\,\Phi_{\rm W} + {\rm cc}\right) \\ & + (d-1)\,r^2 f\left[\left(\omega^2 - k^2 f\right) - d\,(d-2)\,r^2 f\right]\Phi^\dagger_{\rm W}\,\Phi_{\rm W}\Bigg\}\,. \end{split} \tag{H.8}$$

We will quote results accurate to quartic order in the gradient expansion for which it suffices to include the boundary counterterm that is quadratic in derivatives (i.e., it only includes the boundary Einstein-Hilbert term in (G.1)). We have included here the contribution from the quartic counterterm for completeness.

[14] In [42, 134] this contribution was referred to as the ideal piece, since in those cases, it corresponds to the bare ideal fluid action. We will refrain from employing that notation here; the ideal fluid contribution to the on-shell action is a bit more involved in the scalar sector, owing to the presence of a propagating mode.

H.2 The designer scalar action

We would now like to distill the action in terms of the designer scalar field $\mathcal{Z}$. To do so we can directly compute the terms the bulk Lagrangian and boundary terms from the Einstein-Hilbert dynamics defined in (G.1) with the metric ansatz (3.5) and expand to quadratic order. We could equivalently begin with the action given in terms of the $\{\Phi_{\text{E}}, \Phi_{\text{O}}, \Phi_{\text{W}}\}$ fields and use the substitutions given in (G.27). This is a bit more useful, since we have already removed in the process redundant boundary terms. We can therefore focus on just the three terms computed earlier: $L^{\text{bulk}}_{\text{EOW}}$, $L^{\text{GH}}_{\text{EOW}}$, and $L^{\text{ct}}_{\text{EOW}}$.

Direct substitution of (G.27) into the bulk term $L^{\text{bulk}}_{\text{EOW}}$ leads to formidable expression, denoted as $L_{\text{EH}}[\mathcal{Z}]$. There is however a nice structure beneath this mess. Lets first see why $L_{\text{EH}}[\mathcal{Z}]$ reduces into a two-derivative action in the bulk, with the complications relegated to the boundary terms. Since Φ_{E} appears with a single radial derivative in (H.5) and the change of variables to $\mathcal{Z}$ involves a double radial derivative (G.27), we end up with a action with higher derivative terms. The highest derivative term is $\frac{\mathrm{d}^3}{\mathrm{d}r^3}\mathcal{Z}^\dagger\,\frac{\mathrm{d}^2\mathcal{Z}}{\mathrm{d}r^2}$. If we naively vary this action with respect to the field $\mathcal{Z}$ we expect to get a quintic order equation of motion which should be implied by (G.26), i.e., the resulting equation must be some combination of $\mathbb{E}_{\mathcal{Z}}$ and derivatives thereof. Carrying out the exercise we find however

$$\left(-\frac{\mathrm{d}^3}{\mathrm{d}r^3}\frac{\delta}{\delta\mathcal{Z}'''} + \frac{\mathrm{d}^2}{\mathrm{d}r^2}\frac{\delta}{\delta\mathcal{Z}''} - \frac{\mathrm{d}}{\mathrm{d}r}\frac{\delta}{\delta\mathcal{Z}'} + \frac{\delta}{\delta\mathcal{Z}}\right) L_{\text{EH}}[\mathcal{Z}] = \frac{1}{4}\,\frac{d-2}{d-1}\,\frac{k^4}{r^{d-1}\,f\,\Lambda_k^2}\,\mathbb{E}^\dagger_{\mathcal{Z}}\,. \tag{H.9}$$

The fact that the higher order action leads to a second order equation of motion is a sign of the hidden simplicity. Once one knows this it is a matter of corralling the higher derivative terms and showing they are total derivatives. With some effort one can show that

$$L^{\text{bulk}}_{\text{EOW}}\left[\Phi_{\text{E}}, \Phi_{\text{O}}, \Phi_{\text{W}}\right] \longrightarrow L_{\text{EH}}[\mathcal{Z}] = L[\mathcal{Z}] + \frac{\mathrm{d}}{\mathrm{d}r}L_\partial[\mathcal{Z}]\,. \tag{H.10}$$

The bulk action in momentum space is given in (3.10) which we reproduce here

$$\begin{split} L[\mathcal{Z}] = -\frac{\sqrt{-g}}{4}\left(\frac{d-2}{d-1}\right)\frac{k^4}{r^{2(d-2)}\,\Lambda_k^2} \\ \times\left[\frac{\mathbb{D}_+\mathcal{Z}^\dagger\,\mathbb{D}_+\mathcal{Z}}{r^2 f} - \left(\frac{\omega^2}{r^2 f} - \frac{k^2}{r^2}\left(1 - \frac{(d-2)\,r^3\,f'}{\Lambda_k}\right)\right)\mathcal{Z}^\dagger\,\mathcal{Z}\right]. \end{split} \tag{H.11}$$

The complicated boundary terms can be understood as follows. Firstly, the leading $\frac{\mathrm{d}^3}{\mathrm{d}r^3}\mathcal{Z}^\dagger\,\frac{\mathrm{d}^2\mathcal{Z}}{\mathrm{d}r^2}$ term being absent in (H.11) suggests that $L^\partial_{\text{EH}}[\mathcal{Z}]$ begins with $\frac{\mathrm{d}^2}{\mathrm{d}r^2}\mathcal{Z}^\dagger\,\frac{\mathrm{d}^2\mathcal{Z}}{\mathrm{d}r^2}$. The Gibbons-Hawking term $L^{\text{GH}}_{\text{EOW}}$ does not have a corresponding term with this high derivative order, but the counterterm does (from $\Phi_{\text{E}}\,\Phi^\dagger_{\text{E}}$). The cleanest presentation of the boundary terms turns out to be to combine the contributions from $L_\partial[\mathcal{Z}]$ and $L^{\text{GH}}_{\text{EOW}}$ and express the result as a general quadratic form in the variables $\mathbb{D}_+\Theta \sim \mathbb{D}_+^2\mathcal{Z}$, $\Phi_{\rm w} \sim \mathbb{D}_+\mathcal{Z}$, and $\mathcal{Z}$ itself. We will refer to this total collection of boundary terms as the variational boundary terms of $\mathcal{Z}$ and write:

$$L_{\rm var}[\mathcal{Z}] = L^{\rm GH}_{\rm EOW}\left[\Phi_{\rm E}, \Phi_{\rm O}, \Phi_{\rm W}\right] + L_{\partial}[\mathcal{Z}] \,. \tag{H.12}$$

We find

$$\begin{aligned} L_{\rm var}[\mathcal{Z}] = -\frac{d-1}{4\,r^{d-2} f}\Big\{&\mathbb{D}_+\Theta\,\mathbb{D}_+\Theta^\dagger + c_1\left(\Phi^\dagger_{\rm W}\,\mathbb{D}_+\Theta + {\rm cc}\right) + c_2\left(\mathcal{Z}^\dagger\,\mathbb{D}_+\Theta + {\rm cc}\right) \\ &+ c_3\,\Phi^\dagger_{\rm W}\,\Phi_{\rm W} + c_4\left(\mathcal{Z}^\dagger\,\Theta + {\rm cc}\right) + c_5\,\mathcal{Z}^\dagger\,\mathcal{Z}\Big\}\,, \end{aligned} \tag{H.13}$$

with coefficient functions

$$\begin{aligned} c_1 &= \frac{k^2}{(d-1)\,r} + \frac{1}{2}\,r\,(d+(d-4)\,f)\;, \\ c_2 &= -\frac{k^2}{(d-1)^2\,r}\,, \\ c_3 &= -\frac{d(d-2)}{2}\,r^2 f\;(1+f) + \frac{d}{2\,(d-1)}\,(1-f)\,\Lambda_k + (\omega^2 - k^2\,f)\,, \\ c_4 &= -\frac{d\,k^2}{2\,(d-1)^2}\,(1-f)\,, \\ c_5 &= -\frac{d^3\,(d-2)\,r^6\,f\,(1-f)^3}{8\,\Lambda_k^2} + \frac{d^2\,r^4\,(1-f)^2\,(d+(3\,d-8)\,f)}{8\,(d-1)\,\Lambda_k} - \frac{k^2\,(\omega^2-k^2\,f)}{(d-1)^2\,\Lambda_k} \\ &\quad - \frac{d\,(d-2)}{2\,(d-1)^2}\,r^2 f\,(1-f) - \frac{d}{2\,(d-1)^3}\,(1-f)\,\Lambda_k\,. \end{aligned} \tag{H.14}$$

One can check directly that

$$L^{\rm bulk}_{\rm EOW} + \frac{\rm d}{{\rm d}r} L^{\rm GH}_{\rm EOW} = L[\mathcal{Z}] + \frac{\rm d}{{\rm d}r} L_{\rm var}[\mathcal{Z}]\,. \tag{H.15}$$

Having dealt with the bulk and boundary terms let us turn to the counterterms. These can be evaluated by direct substitution, though we note that the presence of the $\Phi_{\rm E}\,\Phi_{\rm E}^\dagger$ does mean that the counterterms are functionals of $\mathbb{D}_+^2\mathcal{Z}$, which is somewhat unusual. The counterterm action can be corralled into:

$$L_{\rm ct}[\mathcal{Z}] = \frac{d-1}{4\,r^{d-2}\,f^{3/2}}\left\{b_0\,\mathbb{D}_+\Theta^\dagger\,\mathbb{D}_+\Theta + b_1\left(\Phi^\dagger_{\rm W}\mathbb{D}_+\Theta + {\rm cc}\right) + b_2\,\Phi^\dagger_{\rm W}\,\Phi_{\rm W}\right\}, \tag{H.16}$$

with coefficient functions

$$
\begin{aligned}
b_0 &= 1 - \frac{k^4}{(d-1)^2\,(d-2)\,(d-4)\,r^4}\,, \\
b_1 &= \frac{1}{2}\,(d-4+d\,f)\,rf + \frac{f}{(d-1)\,r}\,\Lambda_k\,, \\
b_2 &= -d\,(d-2)\,r^2\,f^2 + (\omega^2 - k^2\,f)\,f\,.
\end{aligned}
\tag{H.17}
$$

H.3 The variational principle for $\mathcal{Z}$

We have all the pieces at our disposal to deduce the variational principle for the field $\mathcal{Z}$. Our first task will be to work out the variational principle that leads to the generating function of boundary correlators, viz., the usual boundary conditions in the AdS/CFT parlance. We will then work out the appropriate Legendre transform that computes the WIF of the boundary theory from the grSK contour.

For the variational analysis we will treat the factor $\frac{k^4}{\Lambda_k^2}$ as an overall pre-factor that we will account for at the end of the day. Equivalently, we work with an auxiliary system for $\mathcal{Z}$ where the action has this factor scaled out.

With this understanding let us first record the momentum conjugate the field $\mathcal{Z}$. From (H.11) we find:

$$
\Pi_{\mathcal{Z}} = -\frac{d\,\nu_s}{8}\,\frac{1}{r^{d-3}}\,\mathbb{D}_+\mathcal{Z}\,. \tag{H.18}
$$

Further, using the Green's function (3.19) we can check that the asymptotically the conjugate momentum is constant and parameterized by the inverse Green's function K_s. With our conventions for $G^{\text{in}}_{\mathcal{Z}}$ we have

$$
\Pi_{\mathcal{Z}} = -\frac{k^2\,K_s(\omega,\mathbf{k})}{2\,d\,(d-1)^2}\,b^{d-2}\,\text{coeff}_{\xi^0}\left[G^{\text{in}}_{\mathcal{Z}}\right] + \cdots\,. \tag{H.19}
$$

We see that the conjugate momentum $\Pi_{\mathcal{Z}}$ is finite; the ellipses is (H.19) denote the subleading terms of $\mathcal{O}\left(\xi^{2-d}\right)$.

This behaviour of the field $\mathcal{Z}$ and its conjugate $\Pi_{\mathcal{Z}}$ is indeed what one expects from a non-Markovian field based on the analysis of [42]. At this point we can even guess that the boundary conditions for $\mathcal{Z}$ are Neumann for the purposes of computing correlation functions. We will however want to compute the Wilsonian Influence Functional (WIF) parameterized by the boundary moduli fields for $\mathcal{Z}$, which will turn out to be computed by quantizing the field with renormalized Dirichlet boundary conditions.

We will now argue for this directly by analyzing the variational principle for the action $S[\mathcal{Z}]$. Recall that we have organized the classical action for the designer field as

$$
S_{\text{grav}}[\mathcal{Z}] = \int_k \int dr\; L[\mathcal{Z}] + \int_k \left(L_{\text{var}}[\mathcal{Z}] + L_{\text{ct}}[\mathcal{Z}]\right). \tag{H.20}
$$

Let us begin with the bulk term whose variation is simply:

$$\delta L[\mathcal{Z}] = \frac{k^4}{\Lambda_k^2}\, \Pi_{\scriptscriptstyle \mathcal{Z}}\, \delta\mathcal{Z}^\dagger\,. \tag{H.21}$$

The variation of the boundary term $L_{\text{var}}[\mathcal{Z}]$ produces various terms which can be expressed as combinations of δZ, $\delta\mathbb{D}_+\mathcal{Z} \propto \delta\Pi_{\scriptscriptstyle \mathcal{Z}}$ and $\delta^2\mathbb{D}_+\mathcal{Z}$. To deduce this we note that Θ and $\Phi_{\scriptscriptstyle \mathrm{W}}$ contain $\mathbb{D}_+\mathcal{Z}$ while $\mathbb{D}_+\Theta$ has a piece that behaves as $\mathbb{D}_+^2\mathcal{Z}$. Putting this together we expect that the stationarity of the action demands

$$\frac{k^4}{\Lambda_k^2}\left[\Pi_{\scriptscriptstyle \mathcal{Z}}^\dagger\, \delta\mathcal{Z} + A_1\, \delta\mathcal{Z} + A_2\, \delta\, \mathbb{D}_+\mathcal{Z} + A_3\, \delta\, \mathbb{D}_+^2\mathcal{Z}\right] = 0\,, \tag{H.22}$$

where A_i depends on $(\mathcal{Z}, \mathbb{D}_+\mathcal{Z}, \mathbb{D}_+^2\mathcal{Z})$ and background metric data through the coefficient functions defined in (H.14). We factored out the $\frac{k^4}{\Lambda_k^2}$ piece as advertised which will be helpful when we use our knowledge of the solution in the gradient expansion (at this stage it was not strictly necessary). This complicated second order boundary condition is what is necessary for ensuring the stationarity of the action at a generic radial hypersurface. We are however interested in an asymptotically locally AdS geometry, so we should understand what the behaviour of the boundary condition is in the $r \to \infty$ limit (see similar discussion in [134]). For this we need not only the asymptotics of the coefficient functions c_i in (H.14) (which is clear from their definition), but also the large r behaviour of the functions Θ, $\Phi_{\scriptscriptstyle \mathrm{W}}$, $\mathbb{D}_+\Theta$. These are of course easy to extract given the solution (3.19) for $\mathcal{Z}$.

Generally, for the purposes of the variational principle it suffices to focus on the bulk and the boundary terms as we have done above. One ignores the counterterms – they are important to ensure that we have a finite norm on the phase space. Crucially, they should be expressed in terms of the intrinsic data on the boundary that are being held fixed by the boundary conditions. Since we inherit the counterterm action from those for the Einstein-Hilbert dynamics (G.1) they are naturally expressed in terms of variables that are held fixed. This is easy to see from our discussion of the dynamics in terms of the triple $\{\Phi_{\scriptscriptstyle \mathrm{E}}, \Phi_{\scriptscriptstyle \mathrm{O}}, \Phi_{\scriptscriptstyle \mathrm{W}}\}$ in Appendix H.1. We have a complicated boundary action as evidenced above, and if we wish to discern the asymptotic behaviour of (H.22) it will be helpful to work with a regularized phase space. Consequently, we will include the counterterms in our variation and discern what they tell us about the variational principle asymptotically.

Before we evaluate the variation of the boundary terms and counterterms it will helpful to corral them into a nicer form. Using (H.13) and (H.16) we find:

$$
\begin{aligned}
L_{\text{var+ct}}[\mathcal{Z}] &\equiv L_{\text{var}}[\mathcal{Z}] + L_{\text{ct}}[\mathcal{Z}] \\
&= -\frac{d-1}{4\,r^{d-2} f}\Bigg[\left(1-\frac{b_0}{\sqrt{f}}\right)\mathbb{D}_+\Theta^\dagger\,\mathbb{D}_+\Theta + \left(c_1-\frac{b_1}{\sqrt{f}}\right)\left[\Phi_{\text{W}}^\dagger\,\mathbb{D}_+\Theta + \text{cc}\right] \\
&\qquad + \left(c_3-\frac{b_2}{\sqrt{f}}\right)\Phi_{\text{W}}^\dagger\,\Phi_{\text{W}} + \left[\mathcal{Z}^\dagger\left(c_2\,\mathbb{D}_+\Theta + c_4\,\Theta\right)+\text{cc}\right] + c_5\,\mathcal{Z}^\dagger\,\mathcal{Z}\Bigg].
\end{aligned}
\tag{H.23}
$$

We now simplify this in the following steps:

- The coefficient functions are explicitly known, so we can use the background data to estimate the leading large r behaviour. To do so we must carefully extract the pieces that scale as $r^{-2(d-1)}$ since the functions are highly divergent. Carrying out this exercise we notice that some terms drop out (eg., the $\Phi_{\text{W}}^\dagger\,\Phi_{\text{W}}$ term).
- Next using the asymptotics of the solution given in (J.10), (J.12), and (H.19) we learn that at large r

$$
\frac{\Phi_{\text{W}}}{r^{d-2}} = \frac{1}{d-2}\,\frac{\mathbb{D}_+\Theta}{r^{d-1}} + \cdots\,, \qquad k^2\,\frac{\mathbb{D}_+\Theta}{r^{d-1}} = -4\,(d-1)\,\Pi_{\mathcal{z}} + \cdots\,. \tag{H.24}
$$

  This implies that we can for purposes of the large r behaviour carry out the replacements above. We will not directly eliminate all occurrences of $\mathbb{D}_+\Theta$ since the relation to $\Pi_{\mathcal{z}}$ involves a factor of k^2.

Implementing this we find the boundary action (H.23) reduces to

$$
L_{\text{var+ct}}[\mathcal{Z}] = \frac{(d-1)\,(d-6)}{8\,(d-2)\,b^d}\left(\frac{\mathbb{D}_+\Theta^\dagger}{r^{d-1}}\,\frac{\mathbb{D}_+\Theta}{r^{d-1}}\right) - \left(\Pi_{\mathcal{z}}^\dagger\,\mathcal{Z} + \text{cc}\right) + \text{subleading counterterms}\,. \tag{H.25}
$$

The two terms that are indicated above are the leading contribution near the boundary while the terms we have dropped have subleading pieces that serve as counterterms. For the purposes of ascertaining the variation principle the terms we have retained suffice.

With this simplified boundary term in hand, we can now understand the full variational principle. Taking a variation of (H.25) and including the bulk contribution (H.21) we find:

$$
\delta S[\mathcal{Z}] = \int_k \frac{k^4}{\Lambda_k^2}\left[\frac{(d-1)\,(d-6)}{8\,(d-2)}\,\frac{\mathbb{D}_+\Theta}{r^{2(d-1)}}\,\delta\mathbb{D}_+\Theta^\dagger - \mathcal{Z}\,\delta\Pi_{\mathcal{z}}^\dagger\right]. \tag{H.26}
$$

Finally, using (H.24) we conclude that the first term is subdominant and the total variation of the action is proportional to $\mathcal{Z}\,\delta\Pi_{\mathcal{z}}^\dagger$.[15] Thus, the conventional boundary conditions

[15] Strictly speaking this is not necessary; from (H.24) we note that $\delta\mathbb{D}_+\Theta \sim \delta\Pi_{\mathcal{z}}$ so the total variation is indeed proportional to $\Delta\Pi_{\mathcal{z}}^\dagger$ implying the Neumann boundary condition deduced above.

used to compute the generating function of stress tensor correlators is a Neumann boundary condition for $\mathcal{Z}$.

We emphasize that this is a highly non-trivial statement relying on the nature of the asymptotic fall-offs for the fields involved. At finite radial position we would have a complicated mixed boundary condition fixing some relation of the schematic form given in (H.22). In fact, this is what would have been suggested if we examined the field redefinitions (G.27) and used the fact that $\{\Phi_{\text{E}}, \Phi_{\text{O}}, \Phi_{\text{W}}\}$ have Dirichlet boundary conditions imposed on them. It is also not something we have imposed by hand, but rather it is completely inherited from the original gravitational dynamics (G.1).[16]

I Boundary observables in the scalar sector

We now have understood the dynamics of the gravitational system encoded in $\mathcal{Z}$. This information can be used to decipher the boundary observables directly. We describe the computation of the on-shell action, the boundary stress tensor, and then turn to a brief discussion of the relative merits of the field $\mathcal{Z}$ versus Θ. We will continue to drop an overall factor of c_{eff} which we have restored in the main text.

I.1 The boundary sources and operators

We have determined that the $\mathcal{Z}$ should be quantized with Neumann boundary conditions for purposes of computing correlation functions. This suffices for us to see that $\mathcal{Z}$ does behave like a regular non-Markovian field introduced in [42]. The field $\mathcal{Z}$ should, by virtue of this boundary condition, limit to the dual boundary operator $\breve{\mathcal{O}}_{\mathcal{Z}}$. Taking the limit on the grsK geometry leads to the statements asserted in §3.3.3, in particular, (3.25) and (3.26).

The field $\mathcal{Z}$ is only divergent starting at $\mathcal{O}\left(r^{d-4}\right)$, which is lower than what one would expect for a field that is supposed to encode the physics of the boundary stress tensor. A consequence of this is that the renormalized field $\mathcal{Z}$ is not modified by the boundary counterterms up to the quartic order (i.e., it is uncorrected by the boundary cosmological constant and Einstein-Hilbert counterterm). It only gets renormalized by the quartic R^2 counterterm in (G.1) which is contained in the coefficient b_0 at $\mathcal{O}(k^4)$ in (H.17). Taking this into account we learn that the correction comes from the k^4 contribution to b_0 in

[16] This is analogous to what was seen earlier in the analysis of vector modes in Reissner-Nordström-AdS$_{d+1}$ background, see [134] for details.

(H.17). The renormalized field $\mathcal{Z}_{\text{ren}}$ can be determined to be

$$\begin{aligned}\mathcal{Z}_{\text{ren}} &= \mathcal{Z} - \frac{k^2}{(d-2)\,(d-4)}\,\frac{\mathbb{D}_+\Theta}{r^{d-1}}\,r^{d-4} + \cdots \\ &= \mathcal{Z} + \frac{4\,(d-1)}{(d-2)\,(d-4)}\,r^{d-4}\,\Pi_{\mathcal{Z}} + \cdots\,.\end{aligned} \tag{I.1}$$

Having identified the boundary conditions for $\mathcal{Z}$ and its renormalized counterpart $\mathcal{Z}_{\text{ren}}$, let us turn to identifying the source. Since $\mathcal{Z}$ is quantized with Neumann boundary condition, the source should be defined in terms of the conjugate momentum $\Pi_{\mathcal{Z}}$. However, the relations in (H.24) suggests that the conjugate momentum can be traded for the fields $\frac{\mathbb{D}_+\Theta}{r^{d-1}}$ and $\frac{\Phi_{\text{W}}}{r^{d-2}}$. This is consistent with the fact that these fields determined the induced boundary geometry. Indeed, including the background piece the induced metric on the boundary evaluates (on either boundary) to

$$\gamma_{\mu\nu}\,dx^\mu\,dx^\nu = \left(1 + \frac{\Phi_{\text{W}}}{r^{d-2}}\right)\eta_{\mu\nu}\,dx^\mu\,dx^\nu + \frac{\mathbb{D}_+\Theta}{r^{d-1}}\,dv^2\,. \tag{I.2}$$

It therefore makes sense to identify the temporal and spatial components of the boundary metric as the sources. These are however not independent from each other or from that conjugate momentum $\Pi_{\mathcal{Z}}$ owing to (H.24). We define therefore the boundary source as in (3.32) and note that

$$\breve{\zeta}_{\text{L/R}} = \lim_{r\to\infty\pm i0}\frac{1}{4\,(d-1)}\,\frac{\mathbb{D}_+\Theta}{r^{d-1}} = \lim_{r\to\infty\pm i0}\frac{d-2}{4\,(d-1)}\,\frac{\Phi_{\text{W}}}{r^{d-2}}\,. \tag{I.3}$$

This definition of the boundary source for $\mathcal{Z}$ in terms of $\mathbb{D}_+\Theta$ has a proper gradient expansion, unlike $\Pi_{\mathcal{Z}}$ which has an additional k^2. As noted above, it is also the physically correct variable; the temporal component of the boundary metric Ψ_{vv} that couples to the energy density is indeed $\Phi_{\text{E}} = \mathbb{D}_+\Theta$ up to a factor of r^{d-1}. Fixing all the normalization factors we find that the induced boundary metric can be expressed as in (3.34).

I.2 The boundary stress tensor

The boundary stress tensor density is given by varying the boundary Gibbons-Hawking term and the counterterms given in (G.1). This leads to the following expression accurate to quartic order in gradients

$$
\begin{aligned}
T^{\text{CFT}}_{\mu\nu} = \lim_{r\to\infty} \frac{2\sqrt{-\gamma}}{r^2} \Bigg[& K\,\gamma_{\mu\nu} - K_{\mu\nu} - (d-1)\,\gamma_{\mu\nu} + \frac{1}{d-2}\,{}^\gamma G_{\mu\nu} \\
& + \frac{1}{(d-2)^2\,(d-4)} \Bigg({}^\gamma\nabla^2\,{}^\gamma R_{\mu\nu} + 2\,{}^\gamma R_{\mu\rho\nu\sigma}\,{}^\gamma R^{\rho\sigma} \\
& \quad + \frac{1}{2\,(d-1)} \left[-(d-2)\,{}^\gamma\nabla_\mu\,{}^\gamma\nabla_\nu\,{}^\gamma R - d\,{}^\gamma R\,{}^\gamma R_{\mu\nu} \right] \\
& \quad - \frac{1}{2}\,\gamma_{\mu\nu} \left({}^\gamma R_{\rho\sigma}\,{}^\gamma R^{\rho\sigma} - \frac{d}{4(d-1)}\,{}^\gamma R^2 + \frac{1}{d-1}\,{}^\gamma\nabla^2\,{}^\gamma R \right) \Bigg) \Bigg] .
\end{aligned}
\tag{I.4}
$$

We will now present the result for $(T^{\text{CFT}})_\mu^{\ \nu}$ which makes it easier to see the traceless condition by inspection. At the first order in amplitudes one evaluates the components of the stress tensor from the Brown-York analysis supplemented with counterterms (I.4). We quote the results for the individual components in turn.

First up, the spatio-temporal pieces are

$$
\begin{aligned}
(T^{\text{CFT}})_v^{\ i} &= -f\,(T^{\text{CFT}})_i^{\ v} = -\lim_{r\to\infty\pm i0} ik\,\mathbb{S}_i\,T_1 \,, \\
T_1 &= i\,\Phi_{\text{O}} - \omega\sqrt{f}\,\Phi_{\text{W}} - \frac{\omega\,k^2}{(d-1)\,(d-2)\,(d-4)}\frac{1}{r^3\sqrt{f}}\,\Phi_{\text{E}}, \\
&= -\frac{\omega}{d-1}\,\mathcal{Z}_{\text{ren}} + \omega\left(1-\sqrt{f}\right)\Phi_{\text{W}} \,,
\end{aligned}
\tag{I.5}
$$

where $\mathcal{Z}_{\text{ren}}$ was defined in (I.1). Notice that the Φ_{W} term limits to zero as we approach the asymptopia. As such, from the original expression it appears that we should regard Θ as the field dual to the energy flux operator, since $\sqrt{f}\,\Phi_{\text{W}}$ is a counterterm contribution. However, as assembled in the last line, it is somewhat transparent that the boundary operator is $\mathcal{Z}$ which is cleanly renormalized by the quartic counterterm. We will return to this in Appendix I.4.

The temporal component is a bit more complicated but can be evaluated straightforwardly. We find:

$$
\begin{aligned}
(T^{\text{CFT}})_v^{\ v} &= \lim_{r\to\infty\pm i0} \mathbb{S}\,T_2 \,, \\
T_2 &= (d-1)\,r\left[\mathbb{D}_+\Phi_{\text{W}} - \left(2-\frac{1}{\sqrt{f}}\right)\Phi_{\text{E}} - d\,(1-\sqrt{f})\,r\sqrt{f}\,\Phi_{\text{W}} + \Phi_{\text{B}}\right] \\
&\quad - k^2\sqrt{f}\,\Phi_{\text{W}} - \frac{k^4}{(d-1)\,(d-2)\,(d-4)}\,\frac{1}{r^3\sqrt{f}}\,\Phi_{\text{E}} \,, \\
&= -(d-1)\left[\left(1-\frac{1}{\sqrt{f}}\right) r\,\Phi_{\text{E}} + \mathcal{T} - r\,\Phi_{\text{B}} + \frac{k^2}{d-1}\,\Theta\right] - \frac{k^2}{d-1}\,\mathcal{Z}_{\text{ren}} \\
&\quad + (1-\sqrt{f})\left[\frac{d\,(d-1)}{2}\,r^2\,(1-\sqrt{f}) + k^2\right]\Phi_{\text{W}} \,.
\end{aligned}
\tag{I.6}
$$

Once again we have combined terms so that the last line with the $\Phi_{\rm W}$ terms is vanishing in the limit $r \to \infty \pm i0$. If we also use (G.23) and set $\Phi_{\rm B} = 0$ as we are allowed by the $\mathbb{E}_{\rm T}$ equation, the first parenthesis also simplifies to $\frac{\Phi_{\rm E}}{r^{d-1}}$ which we recognize as a source contribution.

Finally, the spatial stress tensor has contributions from two tensor structures, δ_{ij} and $\mathbb{S}^{\rm T}_{ij}$. We will write these as the pressure and shear-stress contribution as follows:

$$
\begin{split}
(T^{\rm CFT})_i^{\ j} &= \lim_{r\to\infty\pm i0} \left[\frac{1}{f}\, T_P\, \delta_i^{\ j}\, \mathbb{S} + T_Y\, (\mathbb{S}^{\rm T})_i^{\ j}\right] \\
T_P &= (d-1) r\sqrt{f}(1-\sqrt{f})\,\Phi_{\rm E} - i\omega\,\Phi_{\rm O} + \omega^2\sqrt{f}\,\Phi_{\rm W} + \frac{k^2}{d-1}\frac{\sqrt{f}}{r}\left(\Phi_{\rm E} - (d-2)\, r f\,\Phi_{\rm W}\right) \\
&\quad + \frac{d\,(d-1)}{2}\left(1-\sqrt{f}\right)^2 r^2 f\,\Phi_{\rm W} - \widetilde{\mathbb{E}}_5 \\
T_Y &= -\frac{k^2}{r\sqrt{f}}\left[r f\,\Phi_{\rm W} - \frac{\Phi_{\rm E}}{d-2}\right].
\end{split}
\tag{I.7}
$$

We refrained from writing the quartic order in gradient term which renormalizes $\mathcal{Z}$ and also have exploited the fact that there is an explicit contribution proportional to the $\mathbb{E}_5$ equation to simplify the answer. The first summand in T_P simplifies to $\frac{\Phi_{\rm E}}{2\,r^{d-1}}$, which is a source term, while the designer field $\mathcal{Z}$ assembles from the subsequent pieces involving $\Phi_{\rm O}$ and $\Phi_{\rm W}$ which constitute part of the operator contribution. One gets a term proportional to $\omega^2\mathcal{Z}$ from them. The contribution from $\Phi_{\rm E} - (d-2)\, r\,\Phi_{\rm W}$ is however also an operator contribution, the leading order terms in this difference cancels, as the reader can verify from (J.10) and (J.12).

With this information we can evaluate the stress tensor on our solution parameterized by the designer field. The result of this exercise is what is reported in (3.37) where we have adhered to the identification of source and vev terms. Specifically, contributions of the form $\frac{\Phi_{\rm E}}{r^{d-1}}$ and $\frac{\Phi_{\rm W}}{r^{d-2}}$ are written in terms of $\breve{\boldsymbol{\zeta}}$ using (I.3).

I.3 The on-shell action

Once we have identified the boundary conditions for $\mathcal{Z}$, we can evaluate the on-shell action on the grSK solution. At quadratic order this is just a boundary term and can be easily evaluated on the grSK geometry. One will obtain from this the generating function of boundary correlation functions with sources on the two boundaries of the grSK geometry, viz.,

$$
S_{\rm grav}[\mathcal{Z}] = S^S_{\rm R} - S^S_{\rm L}\,. \tag{I.8}
$$

We would however like to evaluate the Wilsonian influence function, for which we need to perform a Legendre transformation of the generating functional. This requires one to evaluate the on-shell action after including a suitable boundary term to carry out the Legendre transform:

$$S[\mathcal{Z}] = \left[S_{\text{grav}}[\mathcal{Z}] + \int_k \left(\Pi_{_\mathcal{Z}}\, \mathcal{Z}^\dagger + \text{cc}\right)\right]_{\text{on-shell}} . \tag{I.9}$$

As noted in [42] this amounts to quantizing $\mathcal{Z}$ without the additional variational boundary term, i.e., we quantize $\mathcal{Z}$ with (renormalized) Dirichlet boundary conditions as have noted in (I.1).

We will find two distinct contributions to the on-shell action: one arises from terms of the form $\Pi_{_\mathcal{Z}}\, \mathcal{Z}^\dagger$ and originates from a combination of bulk action, various boundary terms, and the Legendre transform. The other contribution will turn out to be purely a functional of the source originating from the $\mathbb{D}_+\Theta^\dagger\, \mathbb{D}_+\Theta$ in (H.25). Using (3.32) this piece can be written as a factorized source contribution on the two boundaries of the grSK geometry. We end up with the result quoted in (3.28) in the form of a contact term (the source contribution) and the genuine influence functional.

The explicit evaluation of the on-shell action is straightforward since we have already deduced the asymptotic behaviour of the field and the conjugate momentum. Let us start with the non-contact term and record the influence functional. The grsK solution for $\mathcal{Z}$ is given in (3.27). Accounting for the contribution from the Legendre transform we find this evaluates to

$$\begin{aligned} S_{\text{WIF}}[\mathcal{Z}] &= \int_k \frac{1}{2}\, \left(\mathcal{Z}^\dagger_{\text{ren}}\, \Pi_{_\mathcal{Z}} + \Pi^\dagger_{_\mathcal{Z}}\, \mathcal{Z}_{\text{ren}}\right)\Bigg|^{r=r_c-i0}_{r=r_c+i0} \\ &= -\int_k k^2 \left(\breve{\mathcal{Z}}^\dagger_d\, \frac{b^{d-2}}{2\,d\,(d-1)^2}\, K_s(\omega,\mathbf{k})\, \left[\breve{\mathcal{Z}}_a + \left(n_{_B} + \frac{1}{2}\right) \breve{\mathcal{Z}}_d\right] + \text{cc}\right), \end{aligned} \tag{I.10}$$

where we introduced a large radius regulator at $r = r_c$. This is the result quoted in (3.29).

The contact term may likewise be evaluated directly. From the definition of the sources (3.32) and the form of the boundary terms in (H.25) we see that this is simply given in terms of the boundary source. The contributions furthermore factorize leading us to the following expression at quadratic order:

$$S_{\text{contact}}[\mathcal{Z}] = \int_k \frac{2\,(d-1)^3\,(d-6)}{(d-2)\,b^d}\, \left[\breve{\zeta}^\dagger_{\text{R}}\, \breve{\zeta}_{\text{R}} - \breve{\zeta}^\dagger_{\text{L}}\, \breve{\zeta}_{\text{L}}\right] . \tag{I.11}$$

This contact term contribution is quite peculiar. However, as we describe in §3.4.4, it is nothing but the contact term part of an ideal fluid on the boundary geometry (I.2). We also explain there how one can isolate various hydrodynamic transport data from our answer and connect to the discussion of Class L adiabatic fluid lagrangians of [62].

I.4 On field redefinitions and boundary operators

The solution for the designer sound field $\mathcal{Z}$ on the grSK contour can be repackaged directly in terms of field theory data. Recall that we expect a single mode of the boundary stress tensor that captures the effective dynamics in the low energy limit. The conservation equation of the stress tensor turns out to be a constraint on the field Θ, cf., the discussion around (G.22) and (G.23). Equivalently, examination of the induced boundary stress reveals that the leading contribution comes directly from Θ,

$$(T_{\text{CFT}})_v{}^i = \int_k \mathbb{S}\,\omega\, k_i\, \left(\Theta + \text{counterterms}\right). \tag{I.12}$$

The full expression for the stress tensor including the counterterms can be found in (I.5), where the $\sqrt{f}\,\Phi_{\text{W}}$ term can be seen to arise from counterterm contributions.

These arguments suggest to us that the holographic dual of the sound mode in the plasma should be identified with Θ. However, Θ does not by itself have simple dynamics.[17] Consequently, we rely on $\mathcal{Z}$ as an intermediate auxiliary field to analyze the problem and translate the physical data back onto Θ therefrom.

The relative choice between $\mathcal{Z}$ and Θ is effectively a field redefinition in the boundary. To appreciate this, let us obtain the grSK solution for Θ by first constructing the inverse Green's function G^{in}_{Θ} for Θ which is reported in (J.11). We find

$$\Theta^{\text{SK}}(\zeta,\omega,\mathbf{k}) = G^{\text{in}}_{\Theta}\, \breve{\mathcal{Z}}_a + \left[\left(n_{_B}+\frac{1}{2}\right) G^{\text{in}}_{\Theta} - n_{_B}\, e^{\beta\omega(1-\zeta)}\, G^{\text{rev}}_{\Theta}\right] \breve{\mathcal{Z}}_d\,. \tag{I.13}$$

The key point to note is that the coefficient of radially homogeneous mode G^{in}_{Θ} from (J.11) is

$$\text{coeff}_{\xi^0}\left[G^{\text{in}}_{\Theta}\right] = -\frac{1}{d-1}\left(1+\frac{2}{d}\,\Gamma_s\right), \tag{I.14}$$

which suggests that

$$\begin{aligned} \breve{\mathcal{E}}_{\text{L,R}} &\equiv \lim_{r\to\infty\pm i0}\left[\Theta + \text{counterterms}\right] \\ &= -\frac{1}{d-1}\left(1+\frac{2}{d}\,\Gamma_s\right)\breve{\mathcal{Z}}_{\text{L,R}}\,. \end{aligned} \tag{I.15}$$

One could, if one wished to do so, convert the expressions in the main text directly to expressions involving $\breve{\mathcal{E}}$, but we have refrained from doing so to avoid complicating the already involved discussion.

[17] We were able to derive a third order radial differential equation for Θ directly. At each order in the gradient expansion this equation turned out to be a second order inhomogeneous equation for $\frac{\mathrm{d}}{\mathrm{d}r}\Theta$ which suggests again that there is further simplification possible by passing onto $\mathcal{Z}$.

J Further details of the gradient expansion solutions

The solution for the designer field $\mathcal{Z}$ which satisfies (3.9) with ingoing boundary conditions was given in §3.3.2. This form was chosen to make direct contact with the functions appearing in the fluid/gravity literature [132] and the earlier analysis in [42]. We compile some useful results about the functions appearing in the expansion in this appendix.

The functions $\{F, H_\omega, H_k, I_\omega, I_k, J_\omega, J_{\omega k}, J_k\}$ that parameterize the solution for $\mathcal{Z}$ in (3.19) have compact integral expressions tabulated in Table 3.1. In Appendix J.1 we collect several useful facts about them and determine their asymptotic behaviour. Using this data we record in Appendix J.2 the asymptotic expansions for the fields $\mathcal{Z}$, Θ, and the metric functions $\Phi_{\text{E}}, \Phi_{\text{W}}$ which will prove useful for the phase space and boundary condition analysis. Subsequently, in Appendix J.3 we present the solution in the alternate form parameterized in [134] for ease of comparison.

J.1 Asymptotics of the solution

As noted in §3.3.2 the functions parameterizing the solution for $\mathcal{Z}$ in gradient expansion are cleanly written in terms of a double-integral transform of a source function $\mathfrak{J}$, cf., (3.24). The sources for the various functions are collated in Table 3. Examining this data we immediately see that there are some useful relations:

- First, the sources for the functions H_ω and H_k determine a useful identity for I_k:

$$H_\omega(\xi) + H_k(\xi) = \frac{d-2}{2}\, I_k(\xi)\,. \tag{J.1}$$

- In some cases, the inner integral in (3.24) can be performed, resulting in a representation involving only one integral, e.g.,

$$\begin{aligned}
F(\xi) &\equiv \int_\xi^\infty \frac{y^{d-1}-1}{y(y^d-1)} dy\,,\\
H_k(\xi) &\equiv \frac{1}{d-2}\int_\xi^\infty \frac{y^{d-2}-1}{y(y^d-1)} dy\,,\\
H_\omega(\xi) &\equiv -H_k(\xi) + (d-2)\int_\xi^\infty \frac{H_k(y)-H_k(1)}{y(y^d-1)} dy\,,\\
I_\omega(\xi) &\equiv 2\int_\xi^\infty \frac{H_\omega(y)-H_\omega(1)}{y(y^d-1)} dy = -I_k(\xi) + (d-2)\int_\xi^\infty \frac{I_k(y)-I_k(1)}{y(y^d-1)} dy\,.
\end{aligned} \tag{J.2}$$

These integral expressions allow us to write down the asymptotic solution for the functions quite efficiently at low orders in gradient expansion. As we proceed to higher orders this structure is lost, and we have to use the nested integral representation (3.24) to deduce the asymptotics. We now record the behaviour of the functions to a sufficiently large order to ensure that we can recover the part of the metric functions which contribute to finite boundary data.

Up to the third order in the gradient expansion, we have the functions $\{F, H_k, H_\omega, I_k, I_\omega\}$, whose asymptotics can be determined from (J.2) to be

$$
\begin{aligned}
F(\xi) &= \xi^{-1} - \frac{\xi^{-d}}{d} + \frac{\xi^{-d-1}}{d+1} - \frac{\xi^{-2d}}{2d} + \frac{\xi^{-2d-1}}{2d+1} + \cdots\,, \\
H_k(\xi) &= \frac{\xi^{-2}}{2(d-2)} - \frac{\xi^{-d}}{d(d-2)} + \frac{\xi^{-d-2}}{d^2-4} - \frac{\xi^{-2d}}{2d(d-2)} + \frac{\xi^{-2d-2}}{2(d+1)(d-2)} + \cdots\,, \\
H_\omega(\xi) &= -\frac{\xi^{-2}}{2(d-2)} - \frac{(d-2)^2 H_k(1) - 1}{d(d-2)}\xi^{-d} + \frac{d-4}{2(d^2-4)}\xi^{-d-2} \\
&\quad - \frac{d(d-2)^2 H_k(1) - 2}{2d^2(d-2)}\xi^{-2d} + \frac{d^2-12}{4(d+1)(d^2-4)}\xi^{-2d-2} + \cdots\,, \\
I_k(\xi) &= -\frac{2H_k(1)}{d}\xi^{-d} + \frac{\xi^{-d-2}}{d^2-4} - \frac{d(d-2)H_k(1)+1}{d^2(d-2)}\xi^{-2d} + \frac{(d+4)\,\xi^{-2d-2}}{2(d+1)(d^2-4)} + \cdots\,, \\
I_\omega(\xi) &= -\frac{2H_\omega(1)}{d}\xi^{-d} - \frac{\xi^{-d-2}}{d^2-4} \\
&\qquad - \frac{d(d-2)H_\omega(1) + (d-2)^2 H_k(1) - 1}{d^2(d-2)}\xi^{-2d} - \frac{3\,\xi^{-2d-2}}{(d+1)(d^2-4)} + \cdots\,.
\end{aligned}
\tag{J.3}
$$

In addition at the quartic order we have four functions. Three of them $\{J_\omega, J_k, J_{\omega k}\}$ are finite and have the following asymptotic behaviour:

$$
\begin{aligned}
J_\omega(\xi) &= \frac{\xi^{-4}}{8(d-2)(d-4)} + \frac{4H_\omega(1) + \lambda_\omega d}{d^2}\xi^{-d} + \frac{1-(d-2)^2 H_k(1)}{2d(d^2-4)}\,\xi^{-d-2} \\
&\quad - \frac{d-16}{4(d^2-4)(d^2-16)}\,\xi^{-d-4} + \frac{\lambda_\omega}{2d}\,\xi^{-2d} - \frac{2(d^2-2d-1) + d(d+5)(d-2)^2 H_k(1)}{4d^2(d^2-4)(d+1)}\xi^{-2d-2} \\
&\quad - \frac{5(d^2-20d-32)}{8(d^2-4)(d^2-16)(d+1)(d+2)}\xi^{-2d-4} + \cdots\,, \\
J_{\omega k}(\xi) &= -\frac{\xi^{-4}}{4(d-2)(d-4)} + \frac{4H_k(1) + \lambda_{\omega k} d}{d^2}\xi^{-d} - \frac{2-(d-2)^2 H_k(1)}{2d(d^2-4)}\xi^{-d-2} \\
&\quad - \frac{6}{(d^2-4)(d^2-16)}\xi^{-d-4} + \frac{\lambda_{\omega k}}{2d}\xi^{-2d} + \frac{d^2-7d-2+d(d+3)(d-2)^2 H_k(1)}{4d^2(d^2-4)(d+1)}\xi^{-2d-2} \\
&\quad + \frac{d^3-16d^2-144d-192}{8(d^2-4)(d^2-16)(d+1)(d+2)}\xi^{-2d-4} + \cdots\,, \\
J_k(\xi) &= \frac{\xi^{-4}}{8(d-2)(d-4)} + \frac{\lambda_k}{d}\xi^{-d} + \frac{1}{2d(d^2-4)}\xi^{-d-2} + \frac{d^2+6d-16}{4(d-2)(d^2-4)(d^2-16)}\xi^{-d-4} \\
&\quad + \frac{\lambda_k}{2d}\xi^{-2d} + \frac{d+3}{4d(d+1)(d^2-4)}\xi^{-2d-2} + \frac{d^3+11d^2+48d+48}{8(d^2-4)(d^2-16)(d+1)(d+2)}\xi^{-2d-4} + \cdots\,,
\end{aligned}
\tag{J.4}
$$

while the fourth function is the one that captures all the divergences in $\mathcal{Z}$ and asymptotes to

$$V_k(\xi) = -\frac{\xi^{d-4}}{d-4} + \frac{\xi^{-4}}{4} - \frac{\xi^{-d}}{d} + \frac{\xi^{-d-4}}{d+4} - \frac{\xi^{-2d}}{2d} + \frac{\xi^{-2d-4}}{2d+4} + \cdots \ . \tag{J.5}$$

In writing these expressions we have introduced three numerical constants $\{\lambda_\omega, \lambda_{\omega k}, \lambda_k\}$, which are defined via the large r limits:

$$\begin{aligned} \lambda_\omega + \frac{4}{d} H_\omega(1) &\equiv \lim_{y\to\infty} \left\{ -\frac{y^{d-4}}{2(d-2)(d-4)} + \int_1^y \frac{\mathfrak{J}_\omega(z)\ dz}{z(z^d-1)} \right\}, \\ \lambda_{\omega k} + \frac{4}{d} H_k(1) &\equiv \lim_{y\to\infty} \left\{ \frac{y^{d-4}}{(d-2)(d-4)} + \int_1^y \frac{\mathfrak{J}_{\omega k}(z)\ dz}{z(z^d-1)} \right\}, \\ \lambda_k &\equiv \lim_{y\to\infty} \left\{ -\frac{y^{d-4}}{2(d-2)(d-4)} + \int_1^y \frac{\mathfrak{J}_k(z)\ dz}{z(z^d-1)} \right\}, \end{aligned} \tag{J.6}$$

where $\{\mathfrak{J}_\omega, \mathfrak{J}_{\omega k}, \mathfrak{J}_k\}$ are the source functions for $\{J_\omega, J_{\omega k}, J_k\}$ given in Table 3.

J.2 The metric functions in gradient expansion

Armed with the expressions for the asymptotics of the functions appearing in the gradient expansion, we can estimate the near-boundary behaviour of $\mathcal{Z}$. Firstly, for $\mathcal{Z}$ we find the asymptotic expansion:

$$G^{\text{in}}_{\mathcal{Z}}(\xi,\omega,\mathbf{k}) = \frac{2\,\mathfrak{q}^2\,K_s(\omega,\mathbf{k})\,\xi^{d-4}}{d\,(d-1)\,(d-2)(d-4)} + 1 - \frac{i\mathfrak{w}}{\xi} + \sum_{i=2}^{4} \frac{\mathfrak{a}_i}{\xi^i} + \frac{1}{2\,d}\frac{\widetilde{\Gamma}_s(\omega,\mathbf{k})}{\xi^d} + \mathcal{O}\Big(\xi^{-d-1}\Big)\,, \tag{J.7}$$

where

$$\begin{aligned} \mathfrak{a}_2 &= \frac{2\,\mathfrak{q}^2\,\widetilde{\Gamma}_s(\omega,\mathbf{k})}{d(d-1)} + \frac{d-3}{2\,(d-2)}\left(\frac{\mathfrak{q}^2}{d-1} - \mathfrak{w}^2\right) \\ \mathfrak{a}_3 &= i\mathfrak{w}\left[\frac{\widetilde{\Gamma}_s(\omega,\mathbf{k})}{d^2\,(d^2-1)}\left(\frac{(d^2-1)\,(d+3)}{3}\,\mathfrak{w}^2 - (d^2+2\,d+5)\,\mathfrak{q}^2\right)\right. \\ &\qquad \left. -\frac{d-3}{2(d-2)}\left(\frac{\mathfrak{q}^2}{d-1} - \frac{d-5}{d-3}\,\mathfrak{w}^2\right) + \frac{i\mathfrak{w}}{d^2}\left(\frac{d^2-5}{d^2-1}\,\mathfrak{q}^2 + (d+3)\frac{\mathfrak{w}^2}{3}\right)\right] \\ \mathfrak{a}_4 &= \frac{1}{8\,(d-2)(d-4)}\left[\frac{(d-5)\,(d-7)\,\mathfrak{w}^4}{3} + \left(\frac{16}{d\,(d-1)} + d - 5\right)\mathfrak{q}^2\left(\frac{\mathfrak{q}^2}{d-1} - 2\,\mathfrak{w}^2\right)\right]. \end{aligned} \tag{J.8}$$

Furthermore, we have introduced a new function $\widetilde{\Gamma}_s(\omega, \mathbf{k})$, which is determined up to the quartic order in gradients to be

$$\begin{aligned}\widetilde{\Gamma}_s(\omega, \mathbf{k}) \equiv i\mathfrak{w} + \frac{1}{d-2}\left(\mathfrak{p}_s - \mathfrak{w}^2 + (d-2)^2 H_k(1)\,\mathfrak{w}^2\right) - 2i\mathfrak{w}\left(H_k(1)\,\mathfrak{p}_s^2 + H_\omega(1)\,\mathfrak{w}^2\right) \\ + \frac{2\,\mathfrak{q}^2\,K_s}{d\,(d-1)\,(d-2)} + \left(\lambda_\omega + \frac{4}{d}H_\omega(1)\right)\mathfrak{w}^4 \\ + \left(\lambda_{\omega k} + \frac{4}{d}H_k(1)\right)\mathfrak{w}^2\,\mathfrak{p}_s^2 + \lambda_k\left(\mathfrak{p}_s^2 + \frac{4(d-2)^2}{d(d-3)}\mathfrak{w}^2\right) + \cdots .\end{aligned} \tag{J.9}$$

We will have more to say about this function below, but note that it agrees with $-\Gamma_s$ up to quadratic order in gradients upon using the definition of $\mathfrak{p}_s$ in (3.21).

The rescaled metric functions $\{\Phi_{\rm E}, \Phi_{\rm O}, \Phi_{\rm W}\}$ and the field Θ can be recovered from the above solution for $\mathcal{Z}$ using (G.27). We can express these functions in terms of the ingoing boundary to bulk Green's function. We normalize this inverse propagator using the solution for $\mathcal{Z}$ so the asymptotic values are obtained in terms of the modulus field $\breve{\mathcal{Z}}_{\rm L,R}$ on the grSK geometry.

Denoting the ingoing Green's function of the Weyl factor $\Phi_{\rm W}$ as $G^{\rm in}_{\rm W}$ we can evaluate directly (we report all the divergent non-normalizable terms, but only the leading normalizable ones)

$$\begin{aligned}G^{\rm in}_{\rm W}(\xi, \omega, \mathbf{k}) &= \frac{1}{\Lambda_k}\left[r\,\mathbb{D}_+ + \frac{k^2}{d-1}\right] G^{\rm in}_{\mathcal{Z}}(\xi, \omega, \mathbf{k}) \\ &= \frac{2\,K_s}{d\,(d-1)\,(d-2)}\,\xi^{d-2}\left[1 - \frac{i\mathfrak{w}}{\xi} - \frac{(d-5)(d-1)\,\mathfrak{w}^2 + (d-3)\,\mathfrak{q}^2}{2\,(d-1)\,(d-4)}\,\frac{1}{\xi^2}\right] \\ &\quad - \frac{2\,\widetilde{\Gamma}_s(\omega, \mathbf{k})}{d\,(d-1)}\left[1 - \frac{i\mathfrak{w}}{\xi}\right] + \mathcal{O}\!\left(\xi^{-2}\right).\end{aligned} \tag{J.10}$$

The data above is sufficient to obtain Θ, which after all is just a linear combination of $\mathcal{Z}$ and $\Phi_{\rm W}$ from (G.25). Scaling Θ with a factor of $i\mathfrak{w}$ gives us the function $\Phi_{\rm O}$. For completeness let us record the leading terms in Θ:

$$\begin{aligned}G^{\rm in}_{\Theta}(\xi, \omega, \mathbf{k}) &= \frac{r}{\Lambda_k}\left(\mathbb{D}_+ - \frac{1}{2}\,r^2\,f'\right) G^{\rm in}_{\mathcal{Z}}(\xi, \omega, \mathbf{k}) \\ &= \frac{2\,K_s}{d\,(d-1)\,(d-2)}\,\xi^{d-2}\left[1 - \frac{i\mathfrak{w}}{\xi} - \frac{(d-5)\,\mathfrak{w}^2 + \mathfrak{q}^2}{2\,(d-4)}\,\frac{1}{\xi^2}\right] \\ &\quad - \frac{1}{d-1}\left(1 + \frac{2}{d}\,\widetilde{\Gamma}_s(\omega, \mathbf{k})\right)\left[1 - \frac{i\mathfrak{w}}{\xi}\right] + \mathcal{O}\!\left(\xi^{-2}\right).\end{aligned} \tag{J.11}$$

The final piece of data is the metric function $\Phi_{\rm E}$ (we don't need to evaluate $\Phi_{\rm O}$ since it is just Θ up to a factor of $i\omega$). The easiest way to obtain its inverse Green's function, denoted G^{in}_{E}, is by using $\Phi_{\text{E}} = \mathbb{D}_+\Theta$. We find that it has the following asymptotic expansion:

$$\begin{aligned} G^{\text{in}}_{\text{E}}(\xi,\omega,\mathbf{k}) &= \mathbb{D}_+\left(\frac{r}{\Lambda_k}\left[\mathbb{D}_+ - \frac{r^2 f'}{2}\right]\right) G^{\text{in}}_{z}(\xi,\omega,\mathbf{k}) = \mathbb{D}_+ G^{\text{in}}_{\Theta}(\xi,\omega,\mathbf{k}) \\ &= \frac{2\,K_s}{d\,(d-1)\,b}\,\xi^{d-1}\left[1 - \frac{i\mathfrak{w}}{\xi} - \frac{(d-3)\,\mathfrak{w}^2 + \mathfrak{q}^2}{2\,\xi^2} + i\,\mathfrak{w}\,\frac{(d-5)\,\mathfrak{w}^2 + \mathfrak{q}^2}{2\,(d-4)\,\xi^3}\right] + \mathcal{O}\big(\xi^0\big)\,. \end{aligned} \tag{J.12}$$

As noted in the main text below (3.21), by examining the leading non-normalizable mode of Φ_{W}, or equivalently Θ, we deduce the coefficient K_s accurate to quartic order in gradients, enabling us to get the sound attenuation function $\Gamma_s(\omega,\mathbf{k})$ to quadratic order. However, if one parameterizes the solution by a function $\Gamma_s(\omega,\mathbf{k})$, which is at least first order in derivatives, then we find that the constant mode in Φ_{W} and Θ, given in terms of $\widetilde{\Gamma}_s(\omega,\mathbf{k})$ above, can equivalently be expressed in terms of this attenuation function (see also footnote 18). Our explicit expression determines this coefficient to be $\widetilde{\Gamma}_s(\omega,\mathbf{k})$, suggesting that Γ_s and $\widetilde{\Gamma}_s(\omega,\mathbf{k})$ agree not just up to quadratic order, but rather that $\widetilde{\Gamma}_s(\omega,\mathbf{k})$ determines $\Gamma_s(\omega,\mathbf{k})$ all the way to quartic order, with the specific relation $\widetilde{\Gamma}_s(\omega,\mathbf{k}) = -\Gamma_s(\omega,\mathbf{k})$. While we have not checked this statement explicitly, we conjecture this to be true to all orders in the gradient expansion. As evidence we offer the observation that expressions for the on-shell action and the stress tensor can be entirely parameterized in terms of Γ_s, suggesting that the solution must likewise be given in terms of it.

Sound dispersion to quintic order: If, as conjectured above, the function $\widetilde{\Gamma}_s(\omega,\mathbf{k})$ is indeed the negative of the attenuation function Γ_s, one can deduce the dispersion locus $\omega(k)$ to quintic order in momenta. This is because with the knowledge of $\widetilde{\Gamma}_s(\omega,\mathbf{k})$ to quartic order, we actually have K_s accurate to sextic order in gradients. Assuming our conjecture $\widetilde{\Gamma}_s(\omega,\mathbf{k}) = -\Gamma_s(\omega,\mathbf{k})$, we find

$$\begin{aligned} \mathfrak{w}(\mathfrak{q}) = \frac{\mathfrak{q}}{\sqrt{d-1}} - i\,\frac{\nu_s}{2}\mathfrak{q}^2 + \frac{\nu_s}{2\sqrt{d-1}}\left(1 - \frac{d-1}{4}\,\nu_s - (d-2)\,H_k(1)\right)\mathfrak{q}^3 \\ + i\nu_s\,\mathfrak{h}_4\,\mathfrak{q}^4 + \frac{\nu_s}{\sqrt{d-1}}\,\mathfrak{h}_5\,\mathfrak{q}^5\,, \end{aligned} \tag{J.13}$$

with

$$\mathfrak{h}_4 = \frac{\nu_s}{2\,(d-2)} + \frac{1}{d-1}\,H_\omega(1) - \frac{d-4}{d(d-1)}\,H_k(1),$$

$$\begin{aligned}\mathfrak{h}_5 = &-\frac{(d+2)}{8\,(d-2)}\,\nu_s + \frac{(d-1)\,(4\,d\,(\,d+4)-4)}{64\,d\,(d-2)}\,\nu_s^2 - \frac{3\,d^3-38\,d^2+60\,d+24}{4\,d^2\,(d-1)}\,H_k(1)\\ &+\frac{3}{8}\,(d-2)^2\,\nu_s\,H_k(1)^2 + \frac{3\,d-8}{d\,(d-1)}\,H_\omega(1) - \frac{1}{2(d-1)}\,\lambda_\omega + \frac{d-2}{2\,(d-1)}\,\lambda_{\omega k}\\ &-\frac{d^4-9\,d^3+31\,d^2-43\,d+16}{2\,d\,(d-1)\,(d-3)}\,\lambda_k\,.\end{aligned} \tag{J.14}$$

The constant $H_k(1)$ is known in terms of the Harmonic number function as noted in footnote 22, while $H_\omega(1)$ has an expression in terms of an infinite sum (cf., Eq. (A.28) of [42]). We have not attempted to derive similar expressions for the constants λ_ω. λ_k, and $\lambda_{\omega k}$ defined in (J.6).

J.3 The designer field solution repackaged

To facilitate comparisons with the analysis of [134] we present first the solution of $\mathcal{Z}$ in a slightly different form, using the exponentiated form of the gradient expansion ansatz. We introduce

$$\mathcal{Z}(r) = \frac{1}{b^{d-2}}\,\exp\left(\sum_{n,m=1}^{\infty}(-i)^m\,\mathfrak{w}^m\,\mathfrak{q}^n\,\varphi_{_\mathcal{Z}}^{m,n}(\xi)\right). \tag{J.15}$$

The functions $\varphi_{_\mathcal{Z}}^{m,n}(\xi)$ can be determined almost entirely in terms of the solution for the $\mathcal{M} = d-1$ Markovian scalar, $\varphi_{_{d-1}}^{m,n}(\xi)$. The deviations from Markovian behaviour only occurs for the momentum dependent pieces, as explained in §3.3.2. Therefore,

$$\varphi_{_\mathcal{Z}}^{m,0}(\xi) = \varphi_{_{d-1}}^{m,0}(\xi)\,. \tag{J.16}$$

These functions $\varphi_{_{d-1}}^{m,n}(\xi)$ are compiled in Table 1 of [134] for general Markovianity index $\mathcal{M}$ and can be specialized to $\mathcal{M} = d-1$. To write compact expressions we introduce an integral transform:[18]

$$\mathfrak{T}\big[\mathfrak{g}\big](\xi) \equiv \int_\xi^\infty \frac{dy}{y^2\,f}\,\mathfrak{g}(y), \qquad \hat{\mathfrak{T}}\big[\mathfrak{g}\big](\xi) \equiv \int_\xi^1 \frac{dy}{y^2\,f}\,\mathfrak{g}(y)\,. \tag{J.17}$$

In terms of these, the auxiliary functions $\Delta_{_{d-1}}^{m,n}(\xi)$ are defined at low orders in the gradient

[18] The data given in [134] is written in terms of the inverse radial variable $\varrho = \frac{1}{\xi}$ which we have translated here to the dimensionless radial variable ξ.

expansion as

$$
\begin{aligned}
\hat{\Delta}^{2,0}_{\mathcal{M}}(\xi) &= \hat{\mathfrak{T}}\left[\xi^{1-d} - \xi^{d-1}\right] , \\
\hat{\Delta}^{1,2}_{\mathcal{M}}(\xi) &= -\hat{\mathfrak{T}}\left[\xi^{-1}\, \hat{\Delta}^{2,0}_{\mathcal{M}}(\xi)\right] , \\
\hat{\Delta}^{3,0}_{\mathcal{M}}(\xi) &= -\hat{\mathfrak{T}}\left[\xi^{1-d}\, \hat{\Delta}^{2,0}_{\mathcal{M}}(\xi)^2\right] .
\end{aligned}
\tag{J.18}
$$

The data entering the solution can then be presented compactly in Table 3.

$\mathfrak{T}\left[\mathfrak{g}\right]$	$\mathfrak{g}$	Asymptotics
$\varphi^{1,0}_{d-1}$	$1-\xi^{1-d}$	$\frac{1}{\xi} - \frac{\xi^{-d}}{d}$
$\varphi^{0,2}_{d-1}$	$\frac{1}{d-2}\frac{1}{\xi}\left(1-\xi^{2-d}\right)$	$\frac{1}{d-2}\left(\frac{1}{2\xi^2} - \frac{\xi^{-d}}{d}\right)$
$\varphi^{2,0}_{d-1}$	$-\xi^{1-d}\hat{\Delta}^{2,0}_{d-1}(\xi)$	$\frac{1}{2(d-2)}\frac{1}{\xi^2} - \frac{\Delta^{2,0}_{d-1}(1)}{d}\xi^{-d}$
$\varphi^{3,0}_{d-1}$	$2\xi^{1-d}\hat{\varphi}^{2,0}_{d-1}(\xi)$	$-\frac{2\varphi^{2,0}_{d-1}(1)}{d}\xi^{-d} - \frac{\xi^{-2-d}}{(d-2)(d+2)}$
$\varphi^{1,2}_{d-1}$	$2\xi^{1-d}\hat{\varphi}^{0,2}_{d-1}(\xi)$	$\frac{2\varphi^{0,2}_{d-1}(1)}{d}\xi^{-d} - \frac{\xi^{-2-d}}{(d-2)(d+2)}$
$\varphi^{4,0}_{d-1}$	$2\xi^{1-d}\left(\hat{\varphi}^{3,0}_{d-1}(\xi) + \frac{1}{2}\hat{\Delta}^{3,0}_{d-1}(\xi)\right)$	$-\frac{1}{4(d-4)(d-2)^2}\frac{1}{\xi^4} + \frac{\Delta^{3,0}_{d-1}(1)+2\varphi^{3,0}_{d-1}(1)}{d}\xi^{-d} + \frac{\Delta^{2,0}_{d-1}(1)}{(d-2)(d+2)}\xi^{-2-d}$
$\varphi^{2,2}_{d-1}$	$2\xi^{1-d}\left(\hat{\varphi}^{1,2}_{d-1}(\xi) - \frac{1}{d-2}\left(\hat{\Delta}^{1,2}_{d-1}(\xi) - \hat{\varphi}^{2,0}_{d-1}(\xi)\right)\right)$	$-\frac{1}{2(d-4)(d-2)^2}\frac{1}{\xi^4} - \frac{1+(d-2)\Delta^{2,0}_{d-1}(1)}{(d+2)(d-2)^2}\xi^{-2-d}$ $-\frac{2\left(\varphi^{2,0}_{d-1}(1)-\Delta^{1,2}_{d-1}(1)+(d-2)\varphi^{1,2}_{d-1}(1)\right)}{d(d-2)}\xi^{-d}$
$\varphi^{0,4}_{d-1}$	$-\frac{1}{d-2}\left(\hat{\varphi}^{0,2}_{d-1}(\xi) - \hat{\varphi}^{0,2}_{3-d}(\xi)\right)$	$-\frac{1}{4(d-4)(d-2)^2}\frac{1}{\xi^4} + \frac{\varphi^{0,2}_{d-1}(1)-\varphi^{0,2}_{2-d-1}(1)}{d(d-2)}\xi^{-d} + \frac{\xi^{-2-d}}{(d+2)(d-2)^2}$

Table 3: The functions appearing in the gradient expansion of the Markovian φ_{d-1} up to the fourth order in gradients, given in the form of an integral transform defined in Eq. (J.17). We also present the leading asymptotic behaviour of the functions which is used for computing boundary observables.

The solutions for the remaining functions with $n \neq 0$ up to quartic order can be determined to be

$$
\begin{aligned}
\varphi^{0,2}_{\mathfrak{z}}(\xi) &= \varphi^{0,2}_{d-1}(\xi) - \frac{2\,(d-2)}{d-1}\,\varphi^{0,2}_{d-1}(\xi)\,, \\
\varphi^{1,2}_{\mathfrak{z}}(\xi) &= \varphi^{1,2}_{d-1}(\xi) - \frac{2\,(d-2)}{d-1}\,\varphi^{1,2}_{d-1}(\xi) - \frac{4\,(d-2)}{d\,(d-1)}\,\varphi^{0,2}_{d-1}(\xi)\,, \\
\varphi^{2,2}_{\mathfrak{z}}(\xi) &= \varphi^{2,2}_{d-1}(\xi) - \frac{2\,(d-2)}{d-1}\,\varphi^{2,2}_{d-1}(\xi) - \frac{4}{d\,(d-1)}\,\varphi^{2,0}_{d-1}(\xi) + \frac{4}{d\,(d-1)}\,\varphi^{0,2}_{d-1}(\xi) \\
&\quad + \frac{2}{d\,(d-1)}\,\Delta^{2,2}_{\mathfrak{z}}(\xi) + \frac{4}{d\,(d-1)}\,\Delta^{1,2}_{d-1}(\xi) - \frac{2}{d\,(d-1)\,(d-2)\,(d-4)}\,\xi^{d-4}\,, \\
\varphi^{0,4}_{\mathfrak{z}}(\xi) &= \varphi^{0,4}_{d-1}(\xi) - \frac{4\,(d-2)}{(d-1)^2}\,\varphi^{0,4}_{d-1}(\xi) + \frac{4\,(d-3)}{d(d-1)^2}\,\varphi^{0,2}_{d-1}(\xi) + \frac{2}{d\,(d-1)^2(d-2)}\,\Delta^{0,4}_{\mathfrak{z}}(\xi)\,.
\end{aligned}
\tag{J.19}
$$

As indicated we could express the solution almost completely in terms of the Markovian data computed earlier, but had to introduce two additional functions:

$$\begin{aligned}
\Delta^{0,4}_{\mathfrak{z}}(\xi) &= \int_{\infty}^{\xi} \frac{dy}{y^2 f}\left(\frac{1}{y^{d-1}} - y^{d-3}\right),\\
\Delta^{2,2}_{\mathfrak{z}}(\xi) &= \int_{\infty}^{\xi} \frac{dy}{y^2 f}\left[\frac{1}{y^{d-1}}\left(2\,\hat{\Delta}^{0,4}_{\mathfrak{z}}(y) - \frac{1}{d-2}\right) + \frac{1}{d-2}\,y^{d-2}\right].
\end{aligned} \tag{J.20}$$

Of these, only $\Delta^{0,4}_{\mathfrak{z}}(\xi)$ has a divergent behaviour at large ξ. It is defined using the earlier functions in (3.16) by analytic continuation. In this parameterization, both $\varphi^{2,2}_{\mathfrak{z}}(\xi)$ and $\varphi^{0,4}_{\mathfrak{z}}(\xi)$ have divergent corrections (to quartic order), while that used in §3.3.2 the divergence was isolated into $V_k(\xi)$.

K Spatially homogeneous scalar perturbations

This appendix is somewhat outside the main line of development of the chapter 3 and is included for completeness. As we saw above in Appendix G.3 the dynamics of spatially inhomogeneous modes can be distilled into that of a single non-Markovian scalar $\mathfrak{Z}$ satisfying a second order non-Markovian differential equation. We wish now to analyze what the equations of motion imply for spatially homogeneous modes.

We will carry out the analysis in two steps. First we examine the large diffeomorphisms of the Schwarzschild-AdS$_{d+1}$ solution that respect the Debye gauge choice. Subsequently, we look at the solution space for spatially homogeneous modes looks like and parameterize it in terms of the most general allowed data compatible with asymptotically locally AdS asymptotics.

We will find that the two sets of analyses lead to the same set of zero modes. The surprise will be that there are more zero modes than those that can be lifted to physical moduli captured by $\mathfrak{Z}$.

K.1 Large diffeomorphisms of the background

The background Schwarzschild-AdS$_{d+1}$ geometry is parameterized by b which is a measure of the black hole temperature or mass. One has additionally chosen a particular Weyl frame, by making a suitable choice of the radial coordinate r. Consider the now the following diffeomorphism and parameter shift on the orbit space, leaving spatial homogeneity intact:

$$\begin{aligned}
r &\mapsto r + \chi_r(v,r)\,, & \mathbf{x} &\mapsto (1 + C_x)\,\mathbf{x}\,,\\
v &\mapsto v + \chi_v(v,r)\,, & b &\mapsto (1 + C_b)\,b\,.
\end{aligned} \tag{K.1}$$

We have two functions on the orbit space and two constants C_x and C_b which the reader will recognize is precisely the freedom to rescale spatial length scales and the boundary temperature homogeneously.

To check this explicitly, we simply implement this change on the background solution,

$$ds^2_{(0)} = -r^2 f\, dv^2 + 2\, dv\, dr + r^2\, d\mathbf{x}^2\,. \tag{K.2}$$

Retaining terms to linear order in the χ's and C's to be consistent with our linearized analysis, we find that the metric remains in the Debye gauge and can be cast in the form of our linearized ansatz. To wit,

$$\begin{aligned} ds^2_{(0)} \mapsto ds^2_{(0)} + \frac{\Phi_{\text{E}} - r f\, \Phi_{\text{W}}}{r^{d-3}}\, dv^2 + \frac{2}{r^{d-1} f}\,(\Phi_{\text{O}} - \Phi_{\text{E}} + r f\, \Phi_{\text{W}})\, dv\, dr + r^2\, \frac{\Phi_{\text{W}}}{r^{d-2}}\, d\mathbf{x}^2 \\ - \frac{1}{r^{d+1} f^2}\left[2(\Phi_{\text{O}} - \Phi_{\text{E}}) + r f\,(d-1)\,\Phi_{\text{W}} + \Phi_{\text{B}}\right] dr^2 \end{aligned} \tag{K.3}$$

with the metric functions taking the form:

$$\begin{aligned} \Phi_{\text{W}} &= 2\, C_x\, r^{d-2} + 2\, r^{d-3}\, \chi_r\,, \\ \Phi_{\text{O}} &= -r^{d-1} f\, \mathbb{D}_+ \left(\chi_v - \frac{\chi_r}{r^2 f}\right) + r^{d-3} \frac{\partial}{\partial v}\chi_r - r^d f'\, C_b\,, \\ \Phi_{\text{E}} &= -2\, r^{d-1} f\, \frac{\partial}{\partial v}\left(\chi_v - \frac{\chi_r}{r^2 f}\right) - r^{d-1} f'\, \chi_r + 2\, r^{d-1} f\, C_x - r^d f'\, C_b \\ \Phi_{\text{B}} &= -2\, r^{d-1} f\, \frac{\partial}{\partial v}\left(\chi_v - \frac{\chi_r}{r^2 f}\right) - 2\, \mathbb{D}_+ \left(r^{d-3}\, \chi_r\right) - 2\,(d-3)\, r^{d-1} f\, C_x\,. \end{aligned} \tag{K.4}$$

To summarize, we have two functions on the orbit space, which along with two constant parameters characterize the space of large diffeomorphisms. As such, demanding that the spacetime be asymptotically locally AdS$_{d+1}$ constrains the two functions. It is not hard to see that $\chi_r(v,r) \to r\, \chi_r^\infty(v)$, while $\chi_v(v,r) \to \chi_v^\infty(v)$. The former corresponds to a choice of (time-dependent) Weyl frame, while the latter is the boundary time reparameterization mode.

Before we turn to the dynamical equations it is useful to examine the quantity $\mathcal{T}$. We find

$$\mathcal{T} - r\, \Phi_{\text{B}} = \frac{d\,(C_x - C_b)}{b^d}\,. \tag{K.5}$$

We recall that in our solutions this parameter is vanishing which suggests that in our solution space for $k \neq 0$ we only have access to the locus $C_x = C_b$. This says that we are only allowed to change the background temperature (which is rescaled by C_b) provided that we concertedly change the spatial length scales/volume (set by C_x). As presaged early on in our discussion, the overall rescaling of temperature in non-compact space requires injecting an infinite amount of energy, which is unphysical.

It is interesting to evaluate the boundary stress tensor for this family of large diffeomorphisms. One finds using the results of Appendix I.2

$$
\begin{aligned}
T_v{}^v &= \lim_{r\to\infty} \frac{d-1}{2}\,\frac{\Phi_{\text{E}}}{r^{d-1}} - \frac{d\,(d-1)}{b^d}\,(C_x - C_b)\,,\\
T_i{}^j &= \lim_{r\to\infty} \frac{d-1}{2}\,\frac{\Phi_{\text{E}}}{r^{d-1}}\,\delta_i{}^j\,.
\end{aligned}
\tag{K.6}
$$

Notice that the contribution from the asymptotic value of the source Φ_{E} is exactly what the spatially inhomogeneous modes pick up. Our large diffeomorphisms however have an additional contribution in the energy density which comes from $\mathfrak{T} - r\,\Phi_{\text{B}}$ using (K.5); it is this contribution that is vanishing in our designer solution.

K.2 Parameterizing the solution space: $k = 0$

We give now a short complementary perspective on the spatially homogeneous modes from the dynamical equations of motion. As one might anticipate the physical solution space is already fully characterized by the large diffeomorphism modes, so we expect to see the same degree of freedom in the solution space, as we shall verify below. The manner in which this happens is that the dynamics of the system is modified at $k = 0$ resulting in additional moduli in the problem. Technically, at $k = 0$ some of the equations degenerate. For one, the scalar equation $\mathbb{E}_T$, the spatial vector equations $\mathbb{E}_4$ and $\mathbb{E}_5$, and the spatial tensor equation $\mathbb{E}_7$ are trivially satisfied, each being explicitly proportional k_i.

We are then left with a simpler set of equations, which we write as[19]

$$
\begin{aligned}
\mathbb{E}_1 &= \mathbb{D}_+\left(\mathfrak{T}^{0_k} - r\,\Phi^{0_k}_{\text{B}}\right),\\
\mathbb{E}_2 &= \frac{\partial}{\partial v}\left(\mathfrak{T}^{0_k} - r\,\Phi^{0_k}_{\text{B}}\right),\\
\mathbb{E}_{\text{B}} &= -\mathbb{D}_+\left(\mathfrak{T}^{0_k} - \frac{r}{2}\,\Phi^{0_k}_{\text{B}}\right) + r\,\frac{\partial}{\partial v}\Phi^{0_k}_{\text{O}} - \frac{r}{2}\,(\mathbb{D}_+ - \Upsilon + r f)\left[\mathbb{D}_+\Phi^{0_k}_{\text{W}} - (d-2)\,r f\,\Phi^{0_k}_{\text{W}}\right]\\
&\qquad + \frac{r}{2}\left(-\frac{\partial^2}{\partial v^2}\,\Phi^{0_k}_{\text{W}} + d\,r\,\Phi^{0_k}_{\text{B}}\right),\\
\mathbb{E}_6 &= \mathbb{D}_+\left(\frac{\widetilde{\mathbb{E}}_5}{f}\right) + i\omega\,\frac{\widetilde{\mathbb{E}}_4}{f}\,,
\end{aligned}
\tag{K.7}
$$

where we have written the last equation succinctly using the parameterization defined in (G.19) for convenience.

The first two equations in (K.7) imply that the combination $\mathfrak{T}^{0_k} - r\,\Phi^{0_k}_{\text{B}}$ must be a

[19] We use the superscript 0_k to remind us that we are looking at spatially homogeneous modes of our fields.

constant,

$$\mathfrak{T}^{0_k} - r\,\Phi^{0_k}_{\rm B} = \mathfrak{C}_T \quad \Longrightarrow \quad \mathbb{D}_+\Phi^{0_k}_{\rm W} = \Phi^{0_k}_{\rm E} - \Phi^{0_k}_{\rm B} + \frac{r^2 f'}{2}\,\Phi^{0_k}_{\rm W} - \frac{\mathfrak{C}_T}{r}\,. \tag{K.8}$$

We are then left with two equations for effectively three variables. However, the remaining two equations are not independent, for

$$\mathbb{E}_6 + \mathbb{D}_+\left(\frac{2\,\mathbb{E}_{\rm B}}{r f}\right) + \frac{r f'}{f}\,\mathbb{E}_{\rm B} = 0\,. \tag{K.9}$$

The one remaining equation can be written as

$$r^{d-1} f\,\mathbb{D}_+\left(\frac{1}{r^{d-1}\,f}\,\left(\mathbb{D}_+\Phi^{0_k}_{\rm W} + \Phi^{0_k}_{\rm B}\right)\right) + \left(\frac{\partial^2}{\partial v^2}\,\Phi^{0_k}_{\rm W} - 2\,\frac{\partial}{\partial v}\Phi^{0_k}_{\rm O}\right) = 0\,. \tag{K.10}$$

The solution space is parameterized by two functions $\Phi^{0_k}_{\rm W}$ and $\Phi^{0_k}_{\rm O}$. The other two functions $\{\Phi^{0_k}_{\rm E}, \Phi^{0_k}_{\rm B}\}$ can be solved in terms of them using (K.8) and (K.10). Inspired by (K.4), we can w.l.o.g parameterize the functions $\Phi^{0_k}_{\rm W}$ and $\Phi^{0_k}_{\rm O}$

$$\begin{aligned} \Phi^{0_k}_{\rm W} &= 2\,\mathfrak{F}_1(v,r) + 2\,C_x\,r^{d-2}\,, \\ \Phi^{0_k}_{\rm O} &= \frac{\partial}{\partial v}\mathfrak{F}_1 + r^{d-1} f\,\mathbb{D}_+\mathfrak{F}_2(v,r) - \frac{d\,C_b}{b^d\,r}\,, \end{aligned} \tag{K.11}$$

where we demanded that the solution be asymptotically AdS. Here C_x is a coefficient that will shortly get related to integration constants. We recover the other two functions as given in (K.4) as[20]

$$\begin{aligned} \Phi^{0_k}_{\rm B} &= -2\,\mathbb{D}_+\left(\mathfrak{F}_1 + C_x\,r^{d-2}\right) + 2\,r^{d-1} f\,\frac{\partial}{\partial v}\mathfrak{F}_2 + r^{d-1} f\,\mathfrak{C}_B\,, \\ \Phi^{0_k}_{\rm E} &= 2\,r^{d-1} f\,\frac{\partial}{\partial v}\mathfrak{F}_2 - r^2 f'\left(\mathfrak{F}_1 + C_x\,r^{d-2}\right) + \frac{\mathfrak{C}_T}{r} + r^{d-1} f\,\mathfrak{C}_B\,. \end{aligned} \tag{K.12}$$

One can readily see that we may identify $\mathfrak{F}_1 = r^{d-3}\,\chi_r$ and $\mathfrak{F}_2 = -\chi_v + \frac{\chi_r}{r^2 f}$ recovering the background diffeomorphisms. This implies that the time-independent pieces in $\Phi^{0_k}_{\rm B}$ and $\Phi^{0_k}_{\rm E}$ are

$$\Phi^{0_k}_{\rm B}(r) \sim \left(\mathfrak{C}_B - 2\,(d-2)\,C_x\right) r^{d-1} f\,, \qquad \Phi^{0_k}_{\rm E}(r) \sim \frac{\mathfrak{C}_T - b^{-d}\,d\,C_x}{r} + r^{d-1} f\,\mathfrak{C}_B\,. \tag{K.13}$$

Comparing with (K.4) we conclude that we can parameterize the integration constants

[20]The operator $\mathbb{D}_+$ annihilates $e^{-\frac{1}{2}\,\mathfrak{w}\,\zeta}$ but this solution does not satisfy ingoing boundary conditions and hence we restrict to allowing an integration constant $\mathfrak{C}_B$.

by defining them in terms of C_x and C_b:

$$\mathfrak{C}_B = 2\, C_x\,, \qquad \mathfrak{C}_T = \frac{d(C_x - C_b)}{b^d}\,. \tag{K.14}$$

We thus recover the full set of large diffeomorphisms from the analysis of the equations of motion.

Zero modes and designer field: One can ask if this solution is related to the zero momentum solution of the designer field $\mathcal{Z}$. The ingoing Green's function given in (3.19) requires $\mathfrak{C}_T = \mathfrak{C}_B = 0$, which can be inferred by noting that the source terms vanish at $\omega = 0$. In addition, we also note that $\Phi_{\rm B}$ vanishes identically in our parameterization by $\mathcal{Z}$, which demands a relation between $\mathfrak{F}_1$ and $\mathfrak{F}_2$, viz.,

$$\mathbb{D}_+ \mathfrak{F}_1 = r^{d-1} f\, \frac{\partial}{\partial v} \mathfrak{F}_2\,. \tag{K.15}$$

This is consistent with the relations (G.27) between $\Phi_{\rm E}, \Phi_{\rm O}, \Phi_{\rm W}$ and $\mathcal{Z}$ at $k = 0$, which in turn require that $\mathcal{Z}$ is given in terms of the diffeomorphism functions $\mathfrak{F}_1$ and $\mathfrak{F}_2$ as

$$\frac{1}{d-1}\, \mathbb{D}_+ \mathcal{Z}^{0_k} = r^2 f'\, \mathfrak{F}_1\,, \qquad \frac{1}{d-1} \frac{\partial}{\partial v} \mathcal{Z}^{0_k} = \frac{\partial}{\partial v} \mathfrak{F}_1 - r^{d-1} f\, \mathbb{D}_+ \mathfrak{F}_2\,. \tag{K.16}$$

Solving for $\mathfrak{F}_1$ and $\mathfrak{F}_2$ using (K.15) and the first equation of (K.16), we can then write an autonomous equation for $\mathcal{Z}^{0_k}$. The resulting equations turns out to be implied by the Markovian wave equation (with $k = 0$)

$$\frac{1}{r^{d-1}}\, \mathbb{D}_+ \left(r^{d-1}\, \mathbb{D}_+ \mathcal{Z}^{0_k} \right) - \frac{\partial^2}{\partial v^2} \mathcal{Z}^{0_k} = 0\,, \tag{K.17}$$

which is of course the zero momentum limit of our designer field equation (3.9). This is also the zero momentum limit of a minimally coupled massless scalar field in Schwarzschild-AdS$_{d+1}$.

We thus see that a part of the large diffeomorphisms is indeed the homogeneous solution for the designer field. In particular, given a solution for $\mathcal{Z}(r, \omega, \mathbf{k})$ satisfying (K.17), we can determine large diffeomorphism functions $\mathfrak{F}_1$ and $\mathfrak{F}_2$. However, one does not recover the full set of large diffeomorphisms. Specifically, the part parameterized by the constants $\{\mathfrak{C}_T, \mathfrak{C}_B\}$ (or equivalently $\{C_x, C_b\}$) is not recovered from the designer field dynamics.

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