Berezin-type quantization of even-dimensional compact manifolds and pullback coherent states

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of **Prof. Rukmini Dey** at the International Center for Theoretical Sciences-Tata Institute of Fundamental Research (ICTS-TIFR), Bangalore.

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In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Prof. Rukmini Dey

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Dedicated to my thesis supervisor Prof. Rukmini Dey

Contents

Introduction

Let M be a Kähler manifold and let there be a Hilbert space on it with a reproducing kernel. In [2] Berezin developed a method of defining a non-commutative star product on the symbol of operators acting on this Hilbert space (with a reproducing kernel). Under certain conditions on M he shows that the star product satisfies the correspondence principle. The literature on subsequent work after [2] on Berezin quantization is vast. We mention that in [8] the conditions have been relaxed considerably. Another direction the theory has developed is Berezin-Toeplitz quantization. Berezin-Toeplitz quantization has been achieved on general compact Kähler manifolds [4], [13] and have been generalized to symplectic manifolds of bounded geometry [11]. Berezin-Toeplitz quantization on certain open sets of complex manifolds have been achieved in [10].

In this thesis we study the problem of Berezin quantization on a compact even dimensional manifold M^{2d}. We start by removing a skeleton of lower dimension such that what remains is diffeomorphic to $\mathbb{R}^{2d}$. This can always be done by [7]. We identify $\mathbb{R}^{2d}$ with $\mathbb{C}^d$ and embed in $\mathbb{C}P^d$. We get an induced Berezin quantization from $\mathbb{C}P^d$. In other words, we obtain a Hilbert space (with a reproducing kernel) on $M \setminus M_0$ induced from $\mathbb{C}P^d$, where M_0 is of lower dimension. We can define a local Poisson structure and a star product on the symbol of bounded linear operators on the Hilbert space which satisfy the correspondence principle, all objects are induced from $\mathbb{C}P^d$.

The Berezin quantization depends on the diffeomorphism of $M \setminus M_0$ to $\mathbb{R}^{2d}$ but if we choose a different diffeomorphism of $M \setminus M_0$ to $\mathbb{R}^{2d}$, then we obtain a quantization with another reproducing kernel with star product on symbols which satisfy the correspondence principle. These two quantizations need not be equivalent in the sense that there is no natural map between the Hilbert spaces which preserve the reproducing kernel.

The situation is better if $X = M \setminus M_0$ has a complex structure. Suppose we have a biholomorphism of X to $\mathbb{C}^d$ or to a polydisc Δ. Then one can show that $X \setminus X_0$ (X_0 is empty or a set of measure zero) is biholomorphic to $\mathbb{C}^d \setminus N_0$ (N_0 being empty or a set of measure zero). As before we can embed $\mathbb{C}^d \setminus N_0$ in $\mathbb{C}^d$ and then into $\mathbb{C}P^d$. Then we can obtain on $X \setminus X_0$, a Hilbert space with a reproducing kernel and a star product on the symbol of operators which satisfy the correspondence principle, all objects induced from $\mathbb{C}P^d$. Thus we have a Berezin-type quantization on X. Next we define equivalence of two such Berezin-type quantizations. Suppose we have two biholomorphisms from $X \setminus X_0$ to $\mathbb{C}^d \setminus N_0$, then we have a correspondence of the two Hilbert spaces under consideration such that the reproducing kernel of one of them maps to the reproducing kernel of the other and the two quantizations are equivalent.

Next we give a local construction for an arbitrary compact complex manifold where we use local coordinates to induce the quantization from $\mathbb{C}P^d$. We also study the possibility of defining it globally using the correspondence between the Hilbert spaces with reproducing kernels.

Next we give a construction where we embed a compact smooth manifold M^n into $\mathbb{R}^{2n}$ by some means (could be the Whitney embedding). We embedd $\mathbb{R}^{2n}$ inside $\mathbb{C}P^n$ as one of the standard local charts U_0 say. For compact integral Kähler manifold M we could use the Kodaira embedding into $\mathbb{C}P^n$ directly. We pull back the Hilbert space of geometric quantization from $\mathbb{C}P^n$ and define a Hermitian inner product on this space. Next we consider pull back bounded linear operators on the pull back Hilbert space, define the $\mathbb{C}P^n$-symbols on them and show that they have a star product which satisfy the correspondence principle. This construction depends on the embedding.

Next we give a construction of Toeplitz quantization for any compact complex manifold M. First we remove a measure zero set M_0 from M and embed $M \setminus M_0$ into $\mathbb{C}P^n$. Then we pull back Hilbert space, Toeplitz operators etc from $\mathbb{C}P^n$ to $M \setminus M_0$.

In [5] we had constructed pull back coherent and squeezed states on compact smooth manifolds. We give a very simple construction of them. We use smooth embeddings into $\mathbb{C}P^d$ to pullback the Hilbert space of geometric quantization of $\mathbb{C}P^d$. Then we construct Rawnsley-type coherent and squeezed states on the manifold. We show that they satisfy reproducing kernel property, maximal likelihood property, resolution of identity property and overcompleteness. We also repeat the construction for a compact Kähler manifold M with integral Kähler forms by using a Hilbert space of geometric quantization of M.

Chapter 1

Preliminaries

We will review some material from [19] and [12].

1.1 Grassmannian and Projective space, Universal bundle, Hyperplane bundle.

Projective spaces

One of the main references in this section is [19].

In this section we review projective spaces $\mathbb{C}P^n$.

$\mathbb{C}P^n := P(\mathbb{C}^{n+1})$ which is the set of all one dimensional vector subspaces of $\mathbb{C}^{n+1}$.

In fact $\mathbb{C}P^n$ is a compact complex manifold of dimension n [19]. The charts are given as follows.

We write $\pi(z_0, z_1, ..., z_n) = [z_0, z_1, ..., z_n]$ and say that $(z_0, z_1, ..., z_n)$ are the homogeneous coordinates of $[z_0, z_1, ..., z_n]$. Two homogeneous coordinates $(z_0, z_1, ..., z_n)$ and $(z_0^{'}, z_1^{'}, ..., z_n^{'})$ of a point $[z_0, z_1, ..., z_n] \in \mathbb{C}P^n$ are related by multiplication by a non-zero complex number. Let $U_i = \{[z_0, z_1, ..., z_n] \in \mathbb{C}P^n, z_i \neq 0\}$, $i = 0, 1, ..., n$ with

$\phi_i([z_0, z_1, ..., z_n]) = (\frac{z_0}{z_i}, ... \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, ..., \frac{z_n}{z_i}) \in \mathbb{C}^n$ defining a coordinate chart of $\mathbb{C}P^n$.

It is easy to see that $\phi_i \circ \phi_j^{-1}$ is a biholomorphism from an open set of $\mathbb{C}^n$ to an open set of $\mathbb{C}^n$.

Grassmannian Manifolds

The main reference for this is [19]. Let V be a n-dimensional $\mathbb{C}$-vector space and let $G_r(V) := \{$the set of r-dimensional subspaces of $V\}$, for $r \leq dim(V)$, r fixed. Such a $G_r(V)$ is called a Grassmannian manifold. We shall use particular Grassmannian manifolds, namely $G_{r,n}(\mathbb{C}) := G_r(\mathbb{C}^n)$. The fact Grassmannian manifolds can be given a complex manifold structure and are compact can be found in [19] for instance.

The Grassmannian manifolds are clearly generalizations of the projective spaces in fact, $\mathbb{C}P^n = G_1(\mathbb{C}^{n+1})$.

Universal bundle on Grassmannians

The main reference for this is [19].

Let $U_{r,n}$ be the disjoint union of the r-dimensional $\mathbb{C}$-linear subspaces in $\mathbb{C}^n$. Let us name them W. Then there is a natural projection $\pi : U_{r,n} \longrightarrow G_{r,n}$, where $G_{r,n} = G_{r,n}(\mathbb{C})$, given by $\pi(v) = W$, if v is a vector in W and W is considered as a point in the Grassmannian manifold $G_{r,n}$.
Thus the inverse image under π of a point p in the Grassmannian is the subspace $W \subset \mathbb{C}^n$ where p is the subspace W. Thus we may regard $U_{r,n}$ as a subset of $G_{r,n} \times \mathbb{C}^n$.

We can make $U_{r,n}$ into a bundle by using the coordinate systems of $G_{r,n}$ to define transition functions. In particular, $\pi : U_{1,n} \longrightarrow G_{1,n} = \mathbb{C}P^{n-1}$, universal bundle over complex projective space.

Fubini-Study form on $\mathbb{C}P^{n-1}$

The main reference for this is [19]. We follow the notation that $G_{1,n}(\mathbb{C}) = \mathbb{C}P^{n-1}$.

Let $U_{1,n} \longrightarrow \mathbb{C}P^{n-1}$ be the universal bundle over the projective space. We see that a frame f for $U_{1,n} \longrightarrow \mathbb{C}P^{n-1}$ consists of an open set $U \subset \mathbb{C}P^{n-1}$ and smooth function $f : U \longrightarrow \mathbb{C}^n$ such that all coefficients of f are not simultaneously zero. A holomorphic frame will simply have holomorphic coefficients. We define a positive definite metric on $U_{1,n}$ by letting

$$h(f) = \bar{f}^T f$$

for any frame f for $U_{1,n}$. If g is a change of frame, then we compute that $h(fg) = \overline{(fg)}^T(fg) = \bar{g}^T \bar{f}^T fg = \bar{g}^T h(f) g$, thus we see that h defined above on frames gives a well-defined Hermitian metric on $U_{1,n}$, since the frame representation transforms correctly.

The canonical connection and curvature for $U_{1,n}$ with respect to this natural metric is as follows. If f is any holomorphic frame for $U_{1,n}$, we have

$$\theta(f) = h^{-1}(f)\partial h(f)$$

$$\Theta(f) = \bar{\partial}(h^{-1}(f)\partial h(f)).$$

Let $h = h(f)$, $\theta = \theta(f)$, $\Theta = \Theta(f)$, then

$$\Theta = h^{-1} \cdot \overline{df}^T \wedge df - h^{-1} \cdot \overline{df}^T \cdot f \cdot h^{-1} \wedge \bar{f}^T \cdot df,$$

where $h^{-1} = [\bar{f}^T f]^{-1}$.

Let W be an open subset of $\mathbb{C}^n$. With notation as in [19] if $\phi \in [\varepsilon^p(W)]^n$ and $\psi \in [\varepsilon^q(W)]^n$, then

$$< \phi, \psi >= (-1)^{pq}\bar{\psi} \wedge \phi.$$

Then the curvature form for $U_{1,n}$ becomes

$$\Theta(f) = -\frac{< f, f >< df, df > - < df, f > \wedge < f, df >}{< f, f >^2}, \tag{1.1.1}$$

where f is a holomorphic frame for $U_{1,n}$.

If we choose f to be of the form

$$f = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \tag{1.1.2}$$

where $\xi_j \in \vartheta(U)$ and $\Sigma|\xi_j|^2 = |f|^2 \neq 0$, then

$$df = \begin{bmatrix} d\xi_1 \\ d\xi_2 \\ \vdots \\ d\xi_n \end{bmatrix} \tag{1.1.3}$$

Thus

$$\overline{df}^T = (\overline{d\xi_1}, ..., \overline{d\xi_n}) = (d\bar{\xi_1}, ..., d\bar{\xi_n}), \tag{1.1.4}$$

and

$$\Theta(f) = -\frac{|f|^2 \sum_{i=1}^{n} d\xi_i \wedge d\bar{\xi}_i - \sum_{i,j=1}^{1} \bar{\xi}_i \xi_j d\xi_i \wedge d\bar{\xi}_j}{|f|^4}$$

Here ξ_1, ξ_2,...,ξ_n are functions of the local coordinates on $G_{1,n} = \mathbb{C}P^{n-1}$ and $\Theta(f)$ is a well defined 2-form on $U \subset \mathbb{C}P^{n-1}$.

We can also think of $(\xi_1, \xi_2, ..., \xi_n)$ as homogeneous coordinates on $\mathbb{C}P^{n-1}$ and get the same result.

Hyperplane Bundle on $\mathbb{C}P^n$

Dual of the universal bundle is called hyperplane bundle. For $r = 1$ hyperplane bundle is called hyperplane line bundle over projective space.

The curvature of the hyperplane line bundle is the Fubini Study form, which is negative of the curvature of the universal bundle calculated above.

In fact if $[t_0, ..., t_n]$ are coordinates on $\mathbb{C}P^n$, then $\frac{i}{2}$ times curvature of the hyperplane bundle is given by Ω, [19], page 225, as follows.

$$\Omega = \frac{i}{2} \frac{|t|^2 \sum_{\mu=0}^{n} dt_\mu \wedge d\bar{t}_\mu - \sum_{\mu,\nu=0}^{n} \bar{t}_\mu t_\nu dt_\mu \wedge d\bar{t}_\nu}{|t|^4}$$

In [19] page 189, it has been shown that this is a Kähler form on $\mathbb{C}P^n$.

A Grassmannian manifold is a coadjoint orbit.

The main reference for this section is Nakahara [12].

$U(n)$ acts transitively on $G_{k,n}$ transitively by the action induced from $\mathbb{C}^n$ and the isotropy group is isomorphic to $U(k) \times U(n-k)$. Thus $G_{k,n}(\mathbb{C}) = \frac{U(n)}{U(k) \times U(n-k)}$. In fact from calculations similar to [12], page 182, it follows that $G_{k,n}(\mathbb{C})$ is a coadjoint orbit. In particular, if we put $k = 1$ and $n = d+1$, then $G_{1,d+1}(\mathbb{C}) = \mathbb{C}P^d$. That is $\mathbb{C}P^d = \frac{U(d+1)}{U(1) \times U(d)}$

In fact one can show that $\mathbb{C}P^d = \frac{SU(d+1)}{S(U(d) \times U(1))}$.

Chapter 2

Some reviews on quantization and coherent states

2.1 Geometric quantization: a summary.

Let M be a manifold and $f \in C^{\infty}(M)$ be a smooth function on it (to be thought of an observable). We want to assign operators $\hat{f}$ (acting on a Hilbert space) to functions f such that $Q : f \to \hat{f}$ follows the three conditions below laid out by Dirac.

1) the map Q is linear over $\mathbb{R}$.

2) the map Q maps the constant function 1 to the identity operator.

3) the map Q maps Poisson brackets of two smooth functions to commutators of the corresponding operators multiplied by a constant $-i\hbar$. This conditions imply that Q is a (irreducible) representation of $C^{\infty}(M)$ on the Hilbert space $\mathcal{H}$.

Suppose we have a symplectic manifold (M, ω), with ω integral (i.e. its cohomology class is in $H^2(M, \mathbb{Z})$). Geometric quantization is a method of quantization developed by Kostant and Souriau which assigns to certain smooth functions on M an operator which acts on a section of a Hermitian line bundle, called the prequantum line bundle, such that the Poisson bracket corresponds to the commutator. The prequantum line bundle is a Hermitian line bundle with a connection whose curvature ρ is proportional to the symplectic form. The assignment of a function $f \in C^{\infty}(M)$ to an operator is as follows: $\hat{f} = -i\nabla_{X_f} + f$ acting on the Hilbert space of square integrable sections of L (the wave functions). Here locally $\nabla = d - i\theta$ where $\omega = d\theta$ locally), i.e. θ is the symplectic potential and X_f is defined by $\omega(X_f, \cdot) = -df(\cdot)$. We have taken $\hbar = 1$. The general reference for this is Woodhouse, [20].

As mentioned earlier this assignment has the property that the Poisson bracket (induced by the symplectic form), namely, $\{f_1, f_2\}_{PB}$, corresponds to an operator proportional to the commutator $[\hat{f}_1, \hat{f}_2]$ for any two functions f_1, f_2.

The Hilbert space of a pre-quantization is usually too huge for most purposes. Geometric quantization involves construction of a polarization of the symplectic manifold such that we now take polarized sections of the line bundle, yielding a finite dimensional Hilbert space in most cases. However, in general, for an arbitrary smooth function, $\hat{f}$ does not map the polarized Hilbert space to the polarized Hilbert space. Thus only a few observables from the set of all $f \in C^\infty(M)$ are quantizable.

Holomorphic quantization.

Suppose M is a compact integral Kähler manifold, ω is an integral Kähler form, and L the prequantum line bundle, one can take holomorphic square integrable sections of $L^{\otimes m}$ for $m \in \mathbb{Z}_+$ large enough as the Hilbert space of the quantization. Recall L might not have any global sections so we need to have high enough tensor product, hence we need $L^{\otimes m}$). $L^{\otimes m}$ has curvature proportional to $m\omega$. Let $\{z_1, ...z_n\}$ on M. The polarized sections ψ are given by $\nabla_{\bar{\eta}}\psi = 0$ where ∇ is the covariant derivative w.r.t. a connection θ (i.e. $m\omega = d\theta$ locally) and $\bar{\eta} = \{\frac{\partial}{\partial \bar{z}_1}, ..., \frac{\partial}{\partial \bar{z}_n}\}$. Let $\theta = \sum_{i=1}^n f_i dz_i + \sum_{i=1}^n g_i d\bar{z}_i$. Then the equation for ψ is $\frac{\partial \psi}{\partial \bar{z}_i} - i g_i \psi = 0$, $i = 1, ..., n$. One can show that the most general solution of this set of pde's is $\psi = G_0(z, \bar{z}) h(z)$, where $h(z)$ is a holomorphic section of $L^{\otimes m}$ and $G_0(z, \bar{z})$ satisfies the equation $\frac{\partial G_0}{\partial \bar{z}_i} - i g_i G_0 = 0$, $i = 1, ..., n$. Since G_0 is completely determined by θ, there is a $1 - 1$ correspondence between ψ and h i.e. the polarized sections and the holomorphic sections.

In this thesis we need to consider the geometric quantization of $\mathbb{C}P^n$ and use the holomorphic sections of $H^{\otimes m}$ where H is the hyperplane line bundle H on $\mathbb{C}P^n$. We will not explicitly need the quantum operators $\hat{f}$.

2.2 Rawnsley Coherent states and Perelomov Coherent states: a review.

Rawnsley coherent states

In [17], Rawnsley has defined coherent states on a compact Kähler manifold with an integral Kähler form which arise naturally out of geometric quantization. This goes as follows. Let $\mathcal{H}$ be space of holomorphic square integrable sections of the quantum bundle $\mathcal{L}$, the measure being $e^{-F} d\mu$ where F is a Kähler potential and $d\mu$ proportional to the volume form.

Let $\mathcal{L}_p$ be the fiber in the line bundle above $p \in M$. From $\mathcal{L}$, we remove the zero section and call it $\mathcal{L}_0$, that is $\mathcal{L}_0 = \mathcal{L} \setminus \{0\}$. Let $s \in \mathcal{H}$, then the evaluation map $s \to s(p)$ is continuous. For $q \in \mathcal{L}_0$, we obtain a continuous linear functional Δ_q on $\mathcal{H}$ by $\Delta_q(s) \cdot q = s(p)$ where $p = \pi(q)$. By Riesz representation theorem, there is a vector $e_q \in \mathcal{H}$ such that $s(p) = (s, e_q) \cdot q$. Clearly $e_{cq} = (\bar{c})^{-1} e_q$ for $c \in \mathbb{C}^*$ and $q \in \mathcal{L}_0$. Let $e : \mathcal{L}_0 \to \mathcal{H} - \{0\}$ defined by $e(q) = e_q$.

Let $P(\mathcal{H})$ denotes the projective space of $\mathcal{H}$, then there is a unique map $\epsilon : M \to P(\mathcal{H})$ such that $\Pi \circ e = \epsilon \circ \pi$.

For each $p \in M$, $\epsilon(p) \in P(\mathcal{H})$ are called **Rawnsley coherent states**.

$$\begin{array}{ccc} \mathcal{L}_0 & \xrightarrow{\quad e \quad} & \mathcal{H} - \{0\} \\ \Big\downarrow \pi & & \Big\downarrow \Pi \\ M & \xrightarrow{\quad \varepsilon \quad} & P(\mathcal{H}) \end{array}$$

Perelomov coherent states

These are defined by Perelomov as generalized coherent states in [16], page 40 as follows.

Definition(Perelomov): Let T be a representation of a Lie group G acting in the Hilbert space $\mathcal{H}$ and ψ_0 be a fixed vector in the Hilbert space. Then the system of states $\{\psi_g\}$ given by $\psi_g = T(g)\psi_0$ is called the coherent state system $\{T, \psi_0\}$. Let H be the isotropy subgroup for the state ψ_0. Then a coherent state ψ_g is determined by a point $x = x(g)$ in the coset space G/H corresponding to the element g.

Rawnsley and Perelomov coherent states for $\mathbb{C}P^n$

Let the Hilbert space for quantization of $\mathbb{C}P^n$ be identified with holomorphic sections of the hyperplane bundle, as before. Thus one can define the Rawnsley coherent states.

The Perelomov coherent states on $\mathbb{C}P^n$ can be seen as follows.

$\mathbb{C}P^n = \frac{SU(n+1)}{S(U(n)\times U(1))}$ is a coadjoint orbit. Take a fiducial vector Ψ_0 which is invariant under $S(U(n) \times U(1))$ and let the group elements act on it. This gives the Perelomov coherent states $g \cdot \Psi_0 = e^{i\alpha} \Psi_{(g \cdot p)}$, where p is an element of the coadjoint orbit and $e^{i\alpha}$ is a phase factor, [16].

Proposition 2.2.1. *(Rawnsley) The Rawnsley and Perelomov coherent states are the same for* $\mathbb{C}P^n$.

Proof. This follows from [17] as mentioned below. Let $G = SU(n+1)$, $K = S(U(n) \times U(1)) \subset G$ and ψ_0 be a non-zero vector in the Hilbert space of geometric quantization such that there exists a character $\chi : K \to \mathbb{C}^*$ such that $U_k\psi_0 = \chi(k^{-1})\psi_0$. Let $\mathcal{U}_g$ be an unitary representation of G on the Hilbert space. Then for $g \in G$, $e(g) = \mathcal{U}_g\psi_0$ are the states of the Hilbert space which are the global Perelomov states as in [17] (page 403-404) and they coincide with the Rawnsley coherent states. □

2.3 Berezin quantization for Kähler manifolds under certain conditions: a summary.

In [2] Berezin developed the theory of a quantization which is now known as Berezin quantization. We focus our attention on Kähler manifolds, though the theory is more general. Let M be a Kähler manifold. Let $\mathcal{H}$ be a Hilbert space on it (constructed through some means) which depend on a parameter $\hbar$. The Hilbert space comes equipped with a complete basis $\{e_\alpha\}$, $\alpha \in M$. We will show the existence of such basis (the overcomplete basis of coherent states) in our context later. Let $\hat{A}$ be a bounded linear operator acting on $\mathcal{H}$. Berezin defines the symbols of the operators $\hat{A}$, using expectation values in these coherent states. He gives a formula to get the operators from the symbols. Let $\hat{A}_1, \hat{A}_2$ be two such operators. He defines a (non-commutative) star product on the symbols of the operators, $A_1 * A_2$, which corresponds to the symbol of $\hat{A}_1 \circ \hat{A}_2$. There is a dependence on $\hbar$. In [2] Berezin shows that for Kähler manifolds under certain conditions, (namely, hypothesis A, B, C and D), the star products satisfy the correspondence principle which are as follows.

Let $p \in M$, given by local coordinate τ and $\{,\}_{PB}$ denote the Poisson bracket induced by the Kähler form. Then the correspondence principle says

1. $\lim_{\hbar\to 0}(A_1 \star A_2)(\tau,\bar{\tau}) = A_1(\tau,\bar{\tau})A_2(\tau,\bar{\tau})$,
2. $\lim_{\hbar\to 0}\frac{1}{\hbar}(A_1 \star A_2 - A_2 \star A_1)(\tau,\bar{\tau}) = i\{A_1, A_2\}_{PB}(\tau,\bar{\tau})$.

Chapter 3

Berezin-type quantization of smooth manifolds

In this chapter we explain some of our main results of the thesis. This chapter is mostly based on our paper [6].

3.1 Review of Berezin quantization on $\mathbb{C}P^n$: local description

This section is a review based on ideas from [2]. In [2], Berezin considered $\mathbb{C}P^n$ as a homogeneous space in order to quantized it. In this section we give an another way to the quantization using a local description.

Now we recall a local Kähler potential Φ_{FS} for the Fubini-Study Kähler form Ω_{FS} on $\mathbb{C}P^n$.

Let $U_0 \subset \mathbb{C}P^n$ given by $U_0 = \{\mu_0 \neq 0\}$ where $[\mu_0, ..., \mu_n]$ are the coordinates on $\mathbb{C}P^n$.

On U_0, let $f(\mu) = (f_0(\mu), ..., f_n(\mu)))^t$ be a frame of vectors of $\mathbb{C}^{n+1}$ on U_0. We will take $f_0(\mu) = 1, f_i(\mu) = \mu_i$ for $i = 1, ...n$, where $[1, \mu_1, ..., \mu_n] \in U_0$. Let the universal bundle be denoted by $\mathcal{U}$ (where $\mathcal{U}$ dual is the hyperplane bundle on $\mathbb{C}P^n$). Let $h(f) = \bar{f}^t f = \sum_{i=0}^n |f_i|^2$. Then

$$\Theta(f) = \bar{\partial}(h^{-1}(f)\partial h(f)) = \bar{\partial}\partial \ln(h(f)) = -\partial\bar{\partial}\ln(h(f))$$

is the curvature of the universal bundle on $\mathbb{C}P^n$, [19], page 82. We know that the curvature of the hyperplane bundle is negative of the curvature of the universal bundle and hence $\Theta(f) = -\Omega_{FS}$. Let

$$\Phi_{FS}(\mu,\bar{\mu}) = \ln\left(\sum_{i=0}^{n}|f_i(\mu)|^2\right) = \ln\left(1+\sum_{i=1}^{n}|\mu|^2\right)$$

be the Kähler potential of Ω_{FS}. The Kähler metric G is given by

$$g_{ij}^{FS} = \frac{\partial^2\Phi_{FS}}{\partial\mu_i\partial\bar{\mu}_j}.$$

The Kähler metric G and the Kähler form Ω_{FS} are related by $\Omega_{FS}(X,Y) = G(IX,Y)$. The Fubini-Study form is given by

$$\Omega_{FS} = \sum_{i,j=1}^{n}\Omega_{ij}^{FS}d\mu_i\wedge d\bar{\mu}_j.$$

The coefficients of the inverse matrix is given by (Ω_{FS}^{ij}). Then the Poisson bracket of two smooth functions t and s of $\mathbb{C}P^n$ are given by

$$\{t,s\}_{FS} = \sum_{i,j=1}^{n}\Omega_{FS}^{ij}\left(\frac{\partial t}{\partial\bar{\mu}_j}\frac{\partial s}{\partial\mu_i} - \frac{\partial s}{\partial\bar{\mu}_i}\frac{\partial t}{\partial\mu_j}\right).$$

Let

$$T = \{(\mu,\nu)\in\mathbb{C}^n\times\mathbb{C}^n|\mu\cdot\bar{\nu} = -1\}$$

and

$$S = (\mathbb{C}^n\times\mathbb{C}^n)\smallsetminus T.$$

Note that the diagonal $\Delta\subset S$.

For $(\mu,\nu)\in S$, taking a branch of the logarithm, we can define

$$\Phi_{FS}(\mu,\bar{\nu}) = \ln\left(1+\mu\cdot\bar{\nu}\right).$$

Let $H^{\otimes m}$ be the m-th tensor product of the hyperplane bundle H on $\mathbb{C}P^n$. Then $m\Omega_{FS}$ is its curvature form and $m\Phi_{FS}$ is a local Kähler potential. Let Γ_{hol} be holomorphic sections on it. Let $\{\psi_i\}_{i=1}^{N}$ be an orthonormal basis for it. On U_0, the bundles are trivial, then restriction on U_0, the sections of $H^{\otimes m}$ are functions.

Let $\hbar = \frac{1}{m}$ be a parameter. Then $\{\psi_i\}$ implicitly depend on $\hbar$. We define a volume form on $\mathbb{C}^n$ induced from $\mathbb{C}P^n$ as follows

$$\begin{aligned} dV(\mu) &= |\Omega^n_{FS}(\mu)|_{U_0}| = \mathcal{G}(\mu)\Pi^n_{i=1}|d\mu_i \wedge d\bar{\mu}_i| \\ &= \mathcal{G}(\mu)|d\mu \wedge d\bar{\mu}| = \frac{|d\mu \wedge d\bar{\mu}|}{(1+|\mu|^2)^{n+1}} \end{aligned}$$

where $\mathcal{G} = \det[g^{ij}|_{U_0}]$.

Then $V = \int_{\mathbb{C}^n} dV = \int_{\mathbb{C}^n} \frac{|d\mu\wedge d\bar{\mu}|}{(1+|\mu|^2)^{n+1}} < \infty$.

Now we give an explicit local description of Rawnsley-type coherent states on $\mathbb{C}P^n$, following and modifying [5]. Let $(\mu_1, \mu_2, ...\mu_n)$ be coordinates on $U_0 \equiv \mathbb{C}^n$ such that $[1, \mu_1, \mu_2, ..., \mu_n] \in U_0$. We know that $e^{m\Phi_{FS}(\nu,\bar{\nu})} = (1+|\nu|^2)^m$.

Let

$$(c(m))^{-1} = \int_{U_0} \frac{1}{(1+|\nu|^2)^m} dV(\nu) = \int_{U_0} e^{-m\Phi_{FS}(\nu,\bar{\nu})} dV(\nu)$$

and

$$D_{(q_1,q_2,...q_n;q)} = c(m) \int_{U_0} \frac{|\nu_1|^{2q_1}...|\nu_n|^{2q_n}}{(1+|\nu|^2)^m} dV(\nu),$$

where $q_i{}'s$ are all possible positive integers such that $q_1 + ... + q_n = q; q = 0, ..., m$.

Let

$$\Psi_{(q_1,q_2,...,q_n;q)}(\mu) = \frac{1}{\sqrt{D_{(q_1,...,q_n;q)}}} \mu_1^{q_1}...\mu_n^{q_n}$$

where $q_1 + ... + q_n = q; q = 0, ..., m$.

For shorthand, we will use I for the multi-index $I_q = (q_1, ..., q_n; q)$ which runs over the set $q_1 + ... + q_n = q; q = 0, ..., m$.

Then $D_I = c(m) \int_{U_0} \frac{|\nu|^{2I}}{(1+|\nu|^2)^m} dV(\nu)$.

Let us define an inner product on the space of functions on U_0 as

$$\begin{aligned} \langle f, g\rangle &= c(m) \int_{U_0} \frac{\overline{f(\nu)}g(\nu)}{(1+|\nu|^2)^m} dV(\nu) \\ &= c(m) \int_{U_0} \overline{f(\nu)}g(\nu)e^{-m\Phi_{FS}(\nu,\bar{\nu})} dV(\nu). \end{aligned}$$

Proposition 3.1.1. *The $\{\Psi_{(q_1,...,q_n;q)}\}$ are restriction of a basis for sections of $H^{\otimes m}$ on U_0. They are orthonormal in $\mathbb{C}^n$ with respect to the inner product defined as above.*

Proof. We know that sections of $H^{\otimes m}$ are in one to one correspondence with homogeneous polynomials of degree m in the $(n+1)$ variables $(w_0, w_1, ..., w_n)$. On U_0, we represent points as $[1, \mu_1, ..., \mu_n]$ where $\mu_i = \frac{w_i}{w_0}$. Hence we get polynomials of degree $\leq m$ in coordinates $(\mu_1, ...\mu_n)$. They are of the form $\mu_1^{q_1}...\mu_n^{q_n}$, where $q_1 + q_2 + ... + q_n = q; q = 0, 1, ..., m$. Orthonormality follows easily from the fact $\int_0^{2\pi} e^{i\theta} d\theta = 0$. □

Let N is the dimension of the Hilbert space, i.e $N = \sum_J(1)$ where J runs over the indices $J = (p_1, ..., p_n, p), p_1 + ... + p_n = p, p = 0, ..., m$ and $V = \int_{U_0} |\Omega^n|$.

Definition: The Rawnsley-type coherent states are given on U_0 by ψ_μ as follows:

$$\psi_\mu(\nu) := \sum_{q_1+q_2+...+q_n=q; q=0,1,...,m} \overline{\Psi_{(q_1,q_2,...,q_n;q)}(\mu)} \Psi_{(q_1,q_2,...,q_n;q)}(\nu).$$

In shorthand notation we have

$$\psi_\mu := \sum_I \overline{\Psi_I(\mu)} \Psi_I.$$

Proposition 3.1.2. *Reproducing kernel property.*
If Ψ is any other section, then

$$\langle \psi_\mu, \Psi \rangle = \Psi(\mu).$$

In particular,

$$\langle \psi_\mu, \psi_\nu \rangle = \psi_\nu(\mu).$$

Proof. By linearity, it is enough to check this for all basis elements. Let $\Psi = \Psi_{I_0}$ be an basis element.

$$\langle \psi_\mu, \Psi_{I_0} \rangle = \left\langle \sum_I \overline{\Psi_I(\mu)} \Psi_I, \Psi_{I_0} \right\rangle = \sum_I \Psi_I(\mu) \langle \Psi_I, \Psi_{I_0} \rangle .$$

Now we observe that $\langle \Psi_I, \Psi_{I_0} \rangle = \delta_{II_0}$. Thus $\langle \psi_\mu, \Psi_{I_0} \rangle = \Psi_{I_0}(\mu)$. □

Proposition 3.1.3. *Resolution of identity property:*

$$c(m) \int_{U_0} \langle \Psi_1, \psi_\mu \rangle \langle \psi_\mu, \Psi_2 \rangle e^{-m\Phi_{FS}(\mu,\bar{\mu})} dV(\mu) = \langle \Psi_1, \Psi_2 \rangle .$$

In particular,

$$c(m) \int_{U_0} \langle \psi_\nu, \psi_\mu \rangle \langle \psi_\mu, \psi_\nu \rangle e^{-m\Phi_{FS}(\mu,\bar{\mu})} dV(\mu) = \langle \psi_\nu, \psi_\nu \rangle .$$

Proof. By reproducing kernel property, $\langle \psi_\mu, \Psi \rangle = \Psi(\mu)$.

$$c(m) \int_{U_0} \langle \Psi_1, \psi_\mu \rangle \langle \psi_\mu, \Psi_2 \rangle e^{-m\Phi_{FS}(\mu,\bar{\mu})} dV = c(m) \int_{U_0} \overline{\Psi_1(\mu)} \Psi_2(\mu) e^{-m\Phi_{FS}(\mu,\bar{\mu})} dV$$

The above integral is $\langle \Psi_1, \Psi_2 \rangle$. □

Notation: We denote by $\mathcal{L}_m(\mu, \bar{\mu}) = \langle \psi_\mu, \psi_\mu \rangle = \psi_\mu(\mu)$ and

$\mathcal{L}_m(\mu, \bar{\nu}) = \langle \psi_\mu, \psi_\nu \rangle = \psi_\nu(\mu)$.

Recall $\mathcal{H}$ is the Hilbert space of sections of $H^{\otimes m}$ with the inner product

$$\langle f, g \rangle = c(m) \int_{U_0} \overline{f(\nu)} g(\nu) e^{-m\Phi_{FS}(\nu,\bar{\nu})} dV(\nu),$$

where dV is as before.

Let $\hat{A}$ be a bounded linear operator acting on $\mathcal{H}$. Then, following [2], we can define symbol of the operator as

$$A(\nu, \bar{\mu}) = \frac{\langle \psi_\nu, \hat{A}\psi_\mu, \rangle}{\langle \psi_\nu, \psi_\mu \rangle}.$$

We can recover the operators from their symbols by the formula [2]:

$$(\hat{A}f)(\mu) = c(m) \int_{U_0} A(\mu, \bar{\nu}) f(\nu) \mathcal{L}_m(\mu, \bar{\nu}) e^{-m\Phi(\nu,\bar{\nu})} dV(\nu).$$

We will give a proof of this in the appendix.

Let $\hat{A}_1, \hat{A}_2$ be two such operators and let $\hat{A}_1 \circ \hat{A}_2$ be their composition.

Then the symbol of $\hat{A}_1 \circ \hat{A}_2$ will be given by the star product defined as in [2]:

$$\begin{aligned} & (A_1 * A_2)(\mu, \bar{\mu}) \\ = \; & c(m) \int_{U_0} A_1(\mu, \bar{\nu}) A_2(\nu, \bar{\mu}) \frac{\mathcal{L}_m(\mu, \bar{\nu}) \mathcal{L}_m(\nu, \bar{\mu})}{\mathcal{L}_m(\mu, \bar{\mu}) \mathcal{L}_m(\nu, \bar{\nu})} \mathcal{L}_m(\nu, \bar{\nu}) dS, \end{aligned}$$

where $dS = e^{-m\Phi_{FS}(\nu,\bar{\nu})} dV(\nu)$, $\frac{1}{c(m)} = \int_{U_0} e^{-m\Phi_{FS}(\nu,\bar{\nu})} dV(\nu)$

In the appendix we will give a proof of the fact that this is the symbol of $\hat{A}_1 \circ \hat{A}_2$.

3.1.1 Some properties of the reproducing kernel and star product

Proposition 3.1.4. $\psi_\mu(\nu) = (1 + \bar{\mu} \cdot \nu)^m$.

Proof. By multinomial expansion formula we know that there exist positive real constants a_I such that

$$(1 + \bar{\mu} \cdot \nu)^m = \sum_I a_I (\bar{\mu} \cdot \nu)^I,$$

where a_I are given as follows.:

$$a_{I_p} = \binom{m}{p} \frac{p!}{p_1!...p_n!} = \frac{m!}{(m-p)!p_1!...p_n!}$$

where I_p stands for $(p_1, ..., p_n; p)$ where $p_1 + ... + p_n = p$ for each $p = 0, 1, ..., m$.

For each $p = 0, ..., m$, we denote $D_{(p_1,...,p_n;p)} = c(m) \int_{U_0} \frac{|\nu_1|^{2p_1}...|\nu_n|^{2p_n}}{(1+|\nu|^2)^m} dV(\nu)$.

Computing the integral (see appendix), we find that

$$\mathcal{I}_{I_p} = \int_{U_0} \frac{|\nu_1|^{2p_1}...|\nu_n|^{2p_n}}{(1 + |\nu|^2)^m} dV(\nu) = \frac{\pi^n}{(m+n)!} p_1!...p_n!(m-p)!.$$

Similarly,

$$(c(m))^{-1} = \mathcal{I}_{p=0} = \frac{\pi^n}{(m+n)!} m!.$$

Thus

$$D_{(p_1,...,p_n;p)} = c(m)\mathcal{I}_{I_p} = \frac{p_1!...p_n!(m-p)!}{m!}.$$

Thus

$$a_{I_p} D_{I_p} = 1,$$

and thus for all multi-index I,

$$a_I D_I = 1.$$

$$\psi_\mu(\nu) = \sum_I a_I D_I \frac{(\bar{\mu} \cdot \nu)^I}{D_I} = (1 + \bar{\mu} \cdot \nu)^m.$$

□

It is easy to check that we have the following result (called hypothesis A in [2]).

$$\mathcal{L}_m(\mu,\bar{\mu}) = \langle\psi_\mu,\psi_\mu\rangle = \psi_\mu(\mu) = e^{m\Phi_{FS}(\mu,\bar{\mu})} \tag{3.1.1}$$

and for $(\mu,\nu) \in S$,

$$\mathcal{L}_m(\mu,\bar{\nu}) = \langle\psi_\mu,\psi_\nu\rangle = \psi_\nu(\mu) = (1+\mu\cdot\bar{\nu})^m = e^{m\Phi_{FS}(\mu,\bar{\nu})}. \tag{3.1.2}$$

Let $(\mu,\nu) \in S$. Then we define

$$\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu}) = \Phi_{FS}(\mu,\bar{\nu}) + \Phi_{FS}(\nu,\bar{\mu}) - \Phi_{FS}(\mu,\bar{\mu}) - \Phi_{FS}(\nu,\bar{\nu}).$$

In fact,

$$\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu}) = \ln\Big(\frac{(1+\nu\cdot\bar{\mu})(1+\mu\cdot\bar{\nu})}{(1+|\mu|^2)(1+|\nu|^2)}\Big).$$

Proposition 3.1.5. *We have ϕ_{FS} is non-positive and has a zero and a non-degenerate critical point (as a function of ν) at $\nu=\mu$.*

Proof. We know

$$\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu}) = \ln\Big(\frac{(1+\nu\cdot\bar{\mu})(1+\mu\cdot\bar{\nu})}{(1+|\mu|^2)(1+|\nu|^2)}\Big).$$

By Cauchy-Schwartz inequality, we have ϕ_{FS} is non-positive definite. Fixing μ, a straightforward calculation shows that $\frac{\partial\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu})}{\partial\nu_i}|_{\nu=\mu} = 0$ for each i and $\frac{\partial^2\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu})}{\partial\nu_i\partial\bar{\nu}_j}|_{\nu=\mu} \neq 0$. □

Using equations (3.1.1, 3.1.2), we have

$$\frac{\mathcal{L}_m(\mu,\bar{\nu})\mathcal{L}_m(\nu,\bar{\mu})}{\mathcal{L}_m(\mu,\bar{\mu})\mathcal{L}_m(\nu,\bar{\nu})} = e^{m\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu})},$$

.

By the reproducing kernel property, we also have

$$\int_{U_0} \frac{\mathcal{L}_m(\mu,\bar{\nu})\mathcal{L}_m(\nu,\bar{\mu})}{\mathcal{L}_m(\mu,\bar{\mu})\mathcal{L}_m(\nu,\bar{\nu})} dV(\nu) = \frac{1}{c(m)}.$$

Definition:(Berezin) As in [2], we name a point $\mu \in \mathbb{C}^n$ proper if for any neighbourhood U of μ there exists an $\alpha(U) > 0$ such that $\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu}) > -\alpha(U)$ for ν not belonging to U.

Proposition 3.1.6. *Let $\mu \in \mathbb{C}^n$. Then it is a proper point.*

Proof. Let $Y = S \setminus \Delta$. Then it is easy to see that $Y = \cup_U (U \times \bar{U}^c) \cap S$ where U is an open set of $\mathbb{C}^n$. It is easy to check that $\pi_1(Y) = \mathbb{C}^n$.

Let $\mu_0 \in \mathbb{C}^n$. Then $\mu_0 \in \pi_1(Y)$.

Then there exists a ν such that $(\mu_0, \nu) \in Y$. We claim that μ_0 is proper.

Let $(\mu_0, \nu) \in Y$. Then $(\mu_0, \nu) \in (U_0 \times \bar{U_0}^c) \cap S$ for some open set U_0. Since $(U_0 \times \bar{U_0}^c) \cap S$ is an open set, there exists an open neighbourhood of (μ_0, ν) namely $\mathcal{O}$, such that $\overline{\mathcal{O}} \subset (U_0 \times \bar{U_0}^c) \cap S$. Since $\mathcal{O}$ depends on U_0, we name it $\mathcal{O}(U_0)$.

Since $\overline{\mathcal{O}}(U_0) \subset S$ we have for all $\nu \in U_0^c$, there exists an $\eta(\mathcal{O}) \neq 0$ such that $\frac{|\mu_0 \cdot \bar{\nu} + 1|^2}{(|1+|\mu_0|^2)(1+|\nu|^2)} \geq \eta(\mathcal{O}) > 0$. This η depends on U_0, so we rename it $\eta(U_0)$.

Take $\alpha(U_0) = |\ln(\eta(U_0))|$. Let $\nu \in U_0^c$.

Then $\phi_{FS}(\mu_0, \bar{\mu_0} | \nu, \bar{\nu}) = \ln(\frac{|\mu_0 \cdot \bar{\nu} + 1|^2}{|1+|\mu_0|^2)(1+|\nu|^2)}) > -\alpha(U_0)$.

Now we have to show the existence of η and α for any open neighbourhood of μ_0. Suppose $V \subset U_0$ is any neighbourhood of μ_0, then $\bar{U_0}^c \subset \bar{V}^c$. Let $\tilde{\nu}$ be such that $(\mu_0, \tilde{\nu}) \in (V \times \bar{V}^c) \cap S$. Then there is an open set $\mathcal{O}(V)$ such that $(\mu_0, \nu) \in \overline{\mathcal{O}}(V) \subset S$. Then we can find an $\alpha(V)$ which works.

Now let U_1 be a neighbourhood of μ_0 such that $U_0 \subset U_1$, the containment is strict. Then $\bar{U_1}^c \subset \bar{U_0}^c$. Since $\mu_0 \in U_0 \subset U_1$, $\mu_0 \in U_1$. If $\tilde{\nu}$ is such that $(\mu_0, \tilde{\nu}) \in (U_1 \times \bar{U}_1^c) \cap S$ then $(\mu_0, \tilde{\nu}) \in (U_0 \times \bar{U}_0^c) \cap S$. Then we can take $\alpha(U_1) = \alpha(U_0)$.

Thus μ_0 is proper. □

Then, we have the following result [2]:

Proposition 3.1.7 (Berezin)**.** *Let μ be a proper point in $\mathbb{C}^n$.*

The star product satisfies the correspondence principle:

1. $\lim_{m \to \infty} (A_1 \star A_2)(\mu, \bar{\mu}) = A_1(\mu, \bar{\mu}) A_2(\mu, \bar{\mu})$,

2. $\lim_{m \to \infty} m(A_1 \star A_2 - A_2 \star A_1)(\mu, \bar{\mu}) = i\{A_1, A_2\}_{FS}(\mu, \bar{\mu})$.

Proof. The proof follows from Lemma (2.1) in [2]. This can be seen as follows.

$$\begin{aligned}
&\lim_{m\to\infty}(A_1 * A_2)(\mu,\bar{\mu}) \\
&= c(m)\int_{U_0} A_1(\mu,\bar{\nu})A_2(\nu,\bar{\mu})\frac{\mathcal{L}_m(\mu,,\bar{\nu})\mathcal{L}_m(\nu,\bar{\mu})}{\mathcal{L}_m(\mu,\bar{\mu})\mathcal{L}_m(\nu,\bar{\nu})}\mathcal{L}_m(\nu,\bar{\nu})e^{-m\tilde{\Phi}(\nu,\bar{\nu})}dV(\nu) \\
&= c(m)\int_{U_0} A_1(\mu,\bar{\nu})A_2(\nu,\bar{\mu})e^{m\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu})}dV(\nu) \\
&= \frac{\int_{U_0} A_1(\mu,\bar{\nu})A_2(\nu,\bar{\mu})e^{m\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu})}dV(\nu)}{\int_{U_0} e^{m\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu})}dV(\nu)}
\end{aligned}$$

The star product in the limit can be written as

$$\lim_{m\to\infty}\frac{m^n\int_{U_0} A_1(\mu,\bar{\nu})A_2(\nu,\bar{\mu})e^{m\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu})}dV(\nu)}{m^n\int_{U_0} e^{m\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu})}dV(\nu)}.$$

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be the integrals on the numerator and denominator respectively.

Taking $f_{\mu,\bar{\mu}}(\nu,\bar{\nu}) = A_1(\mu,\bar{\nu})A_2(\nu,\bar{\mu})$, the limit of the integral $\mathcal{I}_1$ on the numerator can be written as

$$\lim_{m\to\infty}\mathcal{I}_1 = \lim_{m\to\infty} m^n\int_{U_0} f_{\mu,\bar{\mu}}(\nu,\bar{\nu})e^{m\phi_{FS}(\mu,\bar{\mu}|\nu,\bar{\nu})}dV(\nu).$$

Here ϕ_{FS} is non-positive and is zero only when $\nu = \mu$ and has non-degenerate critical point.

They are uniformly convergent and are of the form of the integral in Lemma (2.2) in [2]. Then by this Lemma, we get

$$\lim_{m\to\infty}\mathcal{I}_1 = A_1(\mu,\bar{\mu})A_2(\mu,\bar{\mu}) + \frac{1}{m}\Delta(A_1(\mu,\bar{\nu})A_2(\nu,\bar{\mu}))|_{\mu=\nu} + \frac{1}{m}F(\mu,\bar{\mu}) + o(\frac{1}{m}),$$

where Δ is the Laplace-Beltrami operator induced by g_{ij}^{FS} (i.e. $\Delta = \sum g_{FS}^{ij}\frac{\partial^2}{\partial\bar{\nu}_i\partial\nu_j}$) and $F(\mu,\bar{\mu})$ is a function which depends on $A_1A_2, \Delta\mathcal{G}$ etc, but does not involve any derivatives of A_1A_2. It is symmetric in A_1, A_2.

Similarly, for the integral in the denominator, $\lim_{m\to\infty}\mathcal{I}_2$ can be seen to be 1.

Then it can be shown that

$$\lim_{m\to\infty}(A_1 \star A_2)(\mu,\bar{\mu}) = A_1(\mu,\bar{\mu})A_2(\mu,\bar{\mu}).$$

The second correspondence principle can be seen as follows

$$
\begin{aligned}
& \lim_{m\to\infty} m(A_1 \star A_2 - A_2 \star A_1)(\mu,\bar{\mu}) \\
= \; & \sum g^{ij}_{FS}\Big(\frac{\partial^2(A_1(\mu,\bar{\nu})A_2(\nu,\bar{\mu}))}{\partial\bar{\nu}_i\partial\nu_j} - \frac{\partial^2(A_2(\mu,\bar{\nu})A_1(\nu,\bar{\mu}))}{\partial\bar{\nu}_i\partial\nu_j}\Big)|_{\mu=\nu} \\
= \; & i\sum \Omega^{ij}_{FS}\Big(\frac{\partial A_1(\mu,\bar{\nu})}{\partial\bar{\nu}_i}\frac{\partial A_2(\nu,\bar{\mu})}{\partial\nu_j} - \frac{\partial A_2(\mu,\bar{\nu})}{\partial\bar{\nu}_i}\frac{\partial A_1(\nu,\bar{\mu})}{\partial\nu_j}\Big)|_{\mu=\nu} \\
= \; & i\{A_1, A_2\}_{FS}(\mu,\bar{\mu}).
\end{aligned}
$$

□

3.2 Berezin-type quantization on compact even dimensional manifolds- I

Let M^{2d} be an even dimensional compact smooth manifold. We do not consider any symplectic structure or Poisson structure or group action on it. To obtain a Berezin-type quantization on it, first we embed the manifold (after perhaps removing a subset of measure zero) in $\mathbb{C}P^n$ and then induce a local Poisson structure on the embedded submanifold and induce the Berezin quantization from $\mathbb{C}P^n$. The Hilbert space of quantization is expected to be of finite dimension, for that we choose $\mathbb{C}P^n$ and not $\mathbb{C}^n$.

Let M^{2d} be a compact topological manifold. Then by [7], there exists a skeleton M_0 of dimension at most $2d-1$ such that $X = M \setminus M_0$ is homeomorphic to $\mathbb{R}^{2d}$. We assume M^{2d} is equipped with a differentiable structure such that $M \setminus M_0$ is diffeomorphic to $\mathbb{R}^{2d}$ with standard smooth structure.

Let τ be the diffeomorphism and $Y = \tau(X) = \mathbb{R}^{2d} \equiv \mathbb{C}^d$. We name the coordinates on Y as $(\tau_1, \tau_2, ..., \tau_d)$ where $\tau_j = x_j + iy_j$, $j = 1, ..., d$, where $(x_1, y_1, ..., x_d, y_d) \in \mathbb{R}^{2d}$. Let Y be given by the coordinates $(\tau_1, ..., \tau_d)$. Let U_0 be the open subset of $\mathbb{C}P^d$ given by $\{w_0 \neq 0\}$ where $[w_0, ..., w_d]$ is a local coordinate on $\mathbb{C}P^d$. Let $U_0 = \{[1, \tau_1, ..., \tau_d]\}$ where $\tau_i = \frac{w_i}{w_0}$, $i = 1, .., d$.

Let us give a metric on $X = M \setminus M_0$ by identifying it with its image $Y = \tau(X)$. The volume form is $dV = \frac{|d\tau \wedge d\bar{\tau}|}{(1+|\tau|^2)^{d+1}}$ and $V = \int_Y dV < \infty$.

3.3 Algebra of operators on $M \setminus M_0$

On $M \setminus M_0$, we define the Hilbert space of quantization to be $\tilde{\mathcal{H}}_\tau = \tau^*(\mathcal{H}_Y)$ (i.e. pulled back by the diffeomorphism τ), where the volume form on $M \setminus M_0$ is induced from $U_0 \subset \mathbb{C}P^d$. Let $\zeta \in X$. Let $\tau = \tau(\zeta)$.

Let the volume form on $M \setminus M_0$ be given as

$$h(\zeta)dS(\zeta) = dV_Y(\tau) = \frac{|d\tau \wedge d\bar{\tau}|}{(1+|\tau|^2)^{d+1}}.$$

In other words,

$$\int_X |\tau^*(s)|^2(\zeta)h(\zeta)dS(\zeta) = \int_Y |s|^2 \frac{|d\tau \wedge d\bar{\tau}|}{(1+|\tau|^2)^{d+1}} = \int_{\mathbb{C}P^d} |s|^2 \frac{|d\tau \wedge d\bar{\tau}|}{(1+|\tau|^2)^{d+1}}.$$

Last two integrals are same because $Y = \mathbb{C}P^d \setminus C_0$ where C_0 is of measure zero.

Let $\tilde{s} \in \tilde{\mathcal{H}}_\tau$ such that $\tilde{s} = \tau^*(s)$. Then we define bounded linear operators $\hat{\tilde{A}}$ to be

$$\hat{\tilde{A}}(\tilde{s})(p) \equiv \hat{A}(s)(z),$$

where $z = \tau(p) \in U_0$ and $\hat{A}$ is a bounded linear operator on $\mathcal{H}_Y$.

Then symbols and star product can be defined for $\hat{\tilde{A}}$ via $\hat{A}$ and correspondence principle follows. Now we elaborate this.

The symbol of $\hat{\tilde{A}}$ is defined to be

$$\tilde{A}(p,q) \equiv A(z,\bar{w})$$

where $z = \tau(p), w = \tau(q)$.

Suppose we have two operators $\hat{\tilde{A}}_1$ and $\hat{\tilde{A}}_2$.

Then $\tilde{A}_1 * \tilde{A}_2$ is defined on $(M \setminus M_0) \times (M \setminus M_0)$ to be

$$(\tilde{A}_1 * \tilde{A}_2)(p,p) \equiv (A_1 * A_2)(z,\bar{z})$$

in $\mathbb{C}P^d$.

In general the algebra of operators will depend on the diffeomorphism.

Then we can see that the star product satisfy the correspondence principle. The proof is exactly same as the previous section with $n = d$.

Proposition 3.3.1. *Let* $\tau \in \mathbb{C}^d$.

$(\tilde{A}_1 * \tilde{A}_2)(p, p) = (A_1 * A_2)(z, \bar{z})$ *satisfy the correspondence principle.*

1. $\lim_{m \to \infty} (A_1 \star A_2)(\tau, \bar{\tau}) = A_1(\tau, \bar{\tau}) A_2(\tau, \bar{\tau})$,

2. $\lim_{m \to \infty} m(A_1 \star A_2 - A_2 \star A_1)(\tau, \bar{\tau}) = i\{A_1, A_2\}_{FS}(\tau, \bar{\tau})$.

Proof. Use $n = d$ in the previous section. Then the proof follows essentially from Lemma (2.1) in [2] as elaborated in Proposition (3.1.7).

□

3.4 Equivalence of two Berezin quantizations:

On a smooth (complex) manifold $M^{2d} \smallsetminus M_0$, let there be a local Poisson structure and a Berezin-type quantizations defined as above induced from $\mathbb{C}P^d$. Suppose there are two diffeomorphisms (biholomorphisms, if $M \smallsetminus M_0$ is complex) which induce two such quantizations. Then there are two Hilbert spaces with reproducing kernels and star products on symbols of bounded linear operators which satisfy the correspondence principle. Suppose there exists a smooth (biholomorphic) bijective map ψ from $M \smallsetminus M_0$ to $M \smallsetminus M_0$ which preserve the local Poisson structures. If ψ induces a linear isomorphism (i.e. a bijective linear map that preserves inner product) between the two Hilbert spaces such that the reproducing kernel maps to the corresponding reproducing kernel then we shall say the two Berezin quantizations are equivalent.

3.4.1 Dependence on the diffeomorphism:

The quantization of the even dimensional manifold M depends on the diffeomorphism from $M \smallsetminus M_0$ to $\mathbb{R}^{2d}$. However if there is a complex structure on M and there is a biholomorphism from $M \smallsetminus \tilde{M}$ to $\mathbb{C}^d \smallsetminus N_0$, then we have an equivalent quantization, in the above sense. Here $\tilde{M}$ and N_0 are of lower dimension and N_0 could be empty.

First we explain why we may need to remove sets of measure zero. If $M \smallsetminus M_0$ is biholomorphic to $\mathbb{C}^d$, we do not need to remove N_0. However if there is a biholomorphism of $X = M \smallsetminus M_0$ to Δ, a polydisc in $\mathbb{C}^d$, it is not biholomorphic to $\mathbb{C}^d$. But we can show (see appendix) that there exists a set Δ_0 (of measure zero in Δ) such that $\Delta \setminus \Delta_0$ is biholomorphic to $\mathbb{C}^d \setminus N_0$, ($N_0$, a set of measure zero).

Thus there is a set X_0 such that $X \setminus X_0$ is biholomorphic to $\Delta \setminus \Delta_0$ which is biholomorphic to $\mathbb{C}^d \setminus N_0$. (Note that $X \setminus X_0 = M \setminus \tilde{M}$ where $\tilde{M} = X_0 \cup M_0$). Thus we have a biholomorphism from $M \setminus \tilde{M}$ to $\mathbb{C}^d \setminus N_0 \subset \mathbb{C}^d \equiv U_0 \subset \mathbb{C}P^d$.

Let $\tilde{\tau}$ and $\tilde{\delta}$ be two biholomorphisms from $M \setminus \tilde{M}$ to $\mathbb{C}^d \setminus N_0$. Let $\mu \in M \setminus \tilde{M}$. Let $\tilde{\tau}(\mu) = \tau = (\tau_1(\mu), ..., \tau_d(\mu))$ and $\tilde{\delta}(\mu) = \delta = (\delta_1(\mu), ..., \delta_d(\mu))$ be two charts on $M \setminus \tilde{M}$. Then $\mu = \tilde{\tau}^{-1}(\tau) = \tilde{\delta}^{-1}(\delta)$

Let $\mathcal{H}_\tau$ be generated by orthonormal basis of monomials of the form $\{\frac{\tau_I}{C_I}\}$ and $\mathcal{H}_\delta$ is generated by orthonormal monomials of the form $\{\frac{\delta_I}{C_I}\}$ mentioned before.

Note that $\tau_i = \pi_i(\tau) = \pi_i(\tilde{\tau}(\mu)) = \tau_i(\mu)$, where π_i is the projection onto the i-th coordinate. Then $\tau_I = \tau_1^{p_1}...\tau_d^{p_d}$ is given by $\tau_I(\mu) = \tau_1^{p_1}(\mu)...\tau_d^{p_d}(\mu)$.

Let f, g be two smooth complex valued functions on $M \setminus \tilde{M}$. Since $\mu = \tilde{\tau}^{-1}(\tau) = \tilde{\delta}^{-1}(\delta)$ we can define

$$\langle f, g \rangle_\tau = c(m) \int_{\mathbb{C}^d} \frac{\overline{f(\tilde{\tau}^{-1}(\tau))} g(\tilde{\tau}^{-1}(\tau))}{(1 + |\tau|^2)^{m+d+1}} |d\tau \wedge d\bar{\tau}|.$$

Similarly,

$$\langle f, g \rangle_\delta = c(m) \int_{\mathbb{C}^d} \frac{\overline{f(\tilde{\delta}^{-1}(\delta))} g(\tilde{\delta}^{-1}(\delta))}{(1 + |\delta|^2)^{m+d+1}} |d\delta \wedge d\bar{\delta}|.$$

Then $(\mathcal{H}_\tau, \langle, \rangle_\tau)$ and $(\mathcal{H}_\delta, \langle, \rangle_\delta)$ are the two Hilbert spaces corresponding to these charts. Note the integrals take the same value on $\mathbb{C}^d$ and $\mathbb{C}^d \setminus N_0$, since N_0 is of measure zero.

As showed earlier these two are Hilbert spaces with reproducing kernels constructed out of coherent states.

Now $\tau_I = \tau_1^{p_1}...\tau_d^{p_d} = \tau_1^{p_1}(\mu)...\tau_d^{p_d}(\mu)$ and $\delta_I = \delta_1^{p_1}...\delta_d^{p_d} = \delta_1^{p_1}(\mu)...\delta_d^{p_d}(\mu)$

We note that $\delta = F_{\delta\tau^{-1}}(\tau)$, where $F_{\delta\tau^{-1}} = \tilde{\delta} \circ \tilde{\tau}^{-1}$ is a biholomorphism.

$\delta_i = \pi_i(\delta) = \pi_i(F_{\delta\tau^{-1}}(\tau))$. Then $\delta_I = \pi_1(F_{\delta\tau^{-1}}(\tau))^{p_1}...\pi_d(F_{\delta\tau^{-1}}(\tau))^{p_d}$.

Then one can define a biholomorphism on the components as follows. Let $S_i : \pi_i(\tau) \mapsto \pi_i(\delta)$ be defined as $S_i(\pi_i(\tau)) = \delta_i = \pi_i(F_{\delta\tau^{-1}}(\tau))$. This induces an invertible map from $S_I : \tau_I \to \delta_I$ and thus on the reproducing kernels.

The local Poisson structure is preserved can be seen as follows.

In τ-coordinates,

$$\{t,s\}_{PB} = \sum_{i,j=1}^{d} \Omega^{ij}_{FS}(\tau,\bar{\tau}) \left(\frac{\partial t}{\partial \bar{\tau}_i} \frac{\partial s}{\partial \tau_j} - \frac{\partial s}{\partial \bar{\tau}_i} \frac{\partial t}{\partial \tau_j} \right),$$

where $[\Omega^{ij}_{FS}]$ is the inverse matrix of $[\Omega^{FS}_{ij}]$.

In δ coordinates

$$\{t,s\}_{PB} = \sum_{i,j=1}^{d} \Omega^{ij}_{FS}(\delta,\bar{\delta}) \left(\frac{\partial t}{\partial \bar{\delta}_i} \frac{\partial s}{\partial \delta_j} - \frac{\partial s}{\partial \bar{\delta}_i} \frac{\partial t}{\partial \delta_j} \right),$$

where $[\Omega^{ij}_{FS}]$ is the inverse matrix of $[\Omega^{FS}_{ij}]$.

This gives us a proposition as follows. Let all notations be as above.

Proposition 3.4.1. *If the diffeomorphism from $X = M \setminus M_0$ to $\mathbb{C}^d$ is a biholomorphism, and if there are two such biholomorphisms, then there is a one-to-one, onto map from $(\mathcal{H}_\tau, \langle , \rangle_\tau)$ to $(\mathcal{H}_\delta, \langle , \rangle_\delta)$ which maps the reproducing kernels to each other. Also the local Poisson structure is preserved. Thus the quantizations are equivalent. If, however, there is a biholomorphism from $X = M \setminus M_0$ to only a polydisc Δ, then we can have a biholomorphism from $X \setminus X_0$ to $\mathbb{C}^d \setminus N_0$, where X_0 and N_0 are sets of lower dimension. $M \setminus \tilde{M} = X \setminus X_0$ is biholomorphic to $\mathbb{C}^d \setminus N_0$ and if there are two such biholomorphisms then they induce a one-to-one, onto map from $(\mathcal{H}_\tau, \langle , \rangle_\tau)$ to $(\mathcal{H}_\delta, \langle , \rangle_\delta)$ which maps the reproducing kernels to each other. Also the local Poisson bracket is preserved. Thus the quantizations are equivalent.*

If the diffeomorphism is not a biholomorphism, then there is still a one-one, onto map between the two Hilbert spaces. However it may not take the reproducing kernel to the reproducing kernel.

Proposition 3.4.2. *Two diffeomorphisms from $M \setminus M_0$ to $\mathbb{R}^{2d}$ induce a one-to-one, onto map from $(\mathcal{H}_\tau, \langle , \rangle_\tau)$ to $(\mathcal{H}_\delta, \langle , \rangle_\delta)$. It may not preserve the reproducing kernels.*

3.5 Berezin-type quantization of a complex manifold

If M is a compact complex manifold and W be a neighbourhood of p in M. Then there is a biholomorphism from W to a polydisc Δ and a set W_0 of lower dimension in W such that $p \in W \setminus W_0$ and $W \setminus W_0$ is biholomorphic to $\mathbb{C}^d \setminus N_0$, N_0 is of lower dimension (see appendix for a proof). If there is another biholomorphism from another neighbourhood V of p to Δ then there is a lower dimensional set V_0 such that $p \in V \setminus V_0$ and $V \setminus V_0$ is biholomorphic to $\mathbb{C}^d \setminus N_0$. One can show that there is a one-to-one, onto correspondence between the respective Hilbert spaces such that the reproducing kernel maps to the reproducing kernel. This follows from an argument similar to Proposition (3.4.1). Also, the local Poisson structure is preserved. In any case, there exist star products on symbols of bounded linear operators on each Hilbert space which satisfy correspondence principle. Thus we have that on $(V \setminus V_0) \cap (W \setminus W_0)$ the respective Hilbert spaces have a correspondence with each other. Suppose there is a third neighbourhood W_1 such that $p \in \mathcal{U} = (V \setminus V_0) \cap (W \setminus W_0) \cap (W_1 \setminus W_{10})$, where W_{10} is a set of lower dimension in W_1. One can check on triple intersections that the correspondence makes sense (i.e. order does not matter).

This can be seen as follows. Let $\tilde{\tau}$ be a biholomorphism from $V \setminus V_0$ to $\mathbb{C}^d \setminus N_0$, $\tilde{\delta}$ from $W \setminus W_0$ to $\mathbb{C}^d \setminus N_0$ and $\tilde{\beta}$ from $W_1 \setminus W_{10}$ to $\mathbb{C}^d \setminus N_0$.

Let $\tilde{\tau}(\mathcal{U}) = \mathcal{V}_\tau$, $\tilde{\delta}(\mathcal{U}) = \mathcal{V}_\delta$, $\tilde{\beta}(\mathcal{U}) = \mathcal{V}_\beta$

Now as before we let $F_{\delta\tau^{-1}} : \mathcal{V}_\tau \subset \mathbb{C}^d \setminus N_0 \mapsto \mathcal{V}_\delta \subset \mathbb{C}^d \setminus N_0$ which maps $\tau \mapsto \delta$. Similarly, $F_{\tau\beta^{-1}} : \mathcal{V}_\beta \subset \mathbb{C}^d \setminus N_0 \mapsto \mathcal{V}_\tau \subset \mathbb{C}^d \setminus N_0$ such that $\beta \mapsto \tau$ and

$F_{\beta\delta^{-1}} : \mathcal{V}_\delta \subset \mathbb{C}^d \setminus N_0 \mapsto \mathcal{V}_\beta \subset \mathbb{C}^d \setminus N_0$ such that $\delta \mapsto \beta$.

On $\mathcal{V} = \mathcal{V}_\tau \cap \mathcal{V}_\delta \cap \mathcal{V}_\beta$, the order does not matter, $F_{\beta\delta^{-1}} \circ F_{\delta\tau^{-1}} = F_{\beta\tau^{-1}}$.

These induce correspondences between $(\mathcal{H}_\tau, <,>_\tau)$, $(\mathcal{H}_\delta, <,>_\delta)$ and $(\mathcal{H}_\beta, <,>_\beta)$ such that reproducing kernels are mapped to the corresponding ones respectively.

Thus there is a possibility of a global Berezin-type quantization on $M \setminus \tilde{M}$, where $\tilde{M}$ is a set of lower dimension.

Also, locally one can define an algebra of operators on $V \setminus V_0$ using the pullback of operators by τ and define the symbol and star product via operators on $\mathbb{C}P^d$ exactly as in section (3.3). Similarly one can define those on $W \setminus W_0$, using pull back by δ. There is a correspondence of these two algebras induced by the map $F_{\delta\tau^{-1}}$.

3.6 Berezin-type quantization of an even dimensional compact smooth manifold- II

In the earlier section we removed a measure zero subset of M, namely M_0 to achieve the Berezin-type quantization. In this section we give a local description without removing

M_0, i.e. a Berezin-type quantization using local coordinates.

Let Σ be a totally real submanifold of $\mathbb{C}^N$ of real dimension N. Let M be a manifold such that ϵ be an embedding of M into $\mathbb{C}^N$ where $\Sigma = \epsilon(M)$. Let $\mathbb{C}^N = (\mu_1, ..., \mu_N)$ be embedded in $\mathbb{C}P^N$ as $[1, \mu_1, ..., \mu_N]$.

Let $U_i = \{[w_0, .., w_i, ., w_N] | w_i \neq 0\}$ be an open set in $\mathbb{C}P^N$.

Let $U_0 = \{[1, \mu_1, ..., \mu_N]\}$ where $\mu_i = \frac{w_i}{w_0}$. Then U_0 be biholomorphic to $\mathbb{C}^N = \{(\mu_1, ..., \mu_N)\}$.

Let $p \in \Sigma$. Let $\chi(p) = (\mu_1(p), \mu_2(p), ..., \mu_N(p))$.

Locally, as we saw above, the Hilbert space $\mathcal{H}$ of the quantization on U_0 are given by the span of $\{\Phi_I\}$ where Φ_I is the multi index given by $\mu_1^{p_1} ... \mu_N^{p_N}$, where $p_1 + ... + p_N = p$, $0 \leq p \leq m$.

Then on M we define the Hilbert space $\mathcal{H}_1$ to be $\epsilon^*(\mathcal{H})$. Let $\hat{\tilde{A}}$ be operators defined such that $\hat{\tilde{A}}(\epsilon^*(s)) \doteq \epsilon^* \hat{A}(s)$, $\hat{A}$ acts on sections of $\mathcal{H}$. Given $\hat{\tilde{A}}$, $\hat{A}$ is unique because Σ is a totally real submanifold of dimension N, [5]. If not, suppose $\hat{\tilde{A}}(\epsilon^* s) \doteq \epsilon^*(\hat{A}s) = \epsilon^*(\hat{B}s)$ then on M, $\epsilon^*(\hat{A}s) - \epsilon^*(\hat{B}s) = 0 \Rightarrow \epsilon^*(\hat{A}s - \hat{B}s) = 0 \Rightarrow \epsilon^*((\hat{A} - \hat{B})s) = 0 \Rightarrow (\hat{A} - \hat{B})\ (s) = 0$ on $\epsilon(M) = \Sigma \Rightarrow \hat{A}s = \hat{B}s$ on Σ, since $\hat{A}(s), \hat{B}(s)$ are holomorphic and Σ is totally real and of real dimension N, by identity criterion $\hat{A} = \hat{B}$.

Then we define the symbol of $\hat{\tilde{A}}$ as $\tilde{A}(p, q) = A(z, \bar{w})$ where $z = \epsilon(p)$, $w = \epsilon(q)$ exactly as in [5]. Also, the $\mathbb{C}P^N$-star product is defined as $(\tilde{A}_1 \star \tilde{A}_2)(p, p) \doteq (A_1 \star A_2)(z, \bar{z})$ in a similar way as [5].

Let us have two smooth functions $f, g \in C^\infty(\mathbb{C}P^N)$. Recall $\{,\}_{FS}$ is the Poisson bracket induced by the Fubini Study form. Let $\tilde{f} = \epsilon^*(f)$, $\tilde{g} = \epsilon^*(g)$. Let us define the Poisson bracket of $\tilde{f}, \tilde{g}$ as $\{\tilde{f}, \tilde{g}\} \doteq \{f, g\}_{FS}$. Let $f(z) = A_1(z, \bar{z})$, $\tilde{f}(p) = f(z)$ where $z = \epsilon(p)$. Let $g(z) = A_2(z, \bar{z})$, $\tilde{g}(p) = f(z)$ where $z = \epsilon(p)$. Then, $\{\tilde{A}_1, \tilde{A}_2\} = \{A_1, A_2\}_{FS}$.

By definition it is easy to see that the $\mathbb{C}P^N$-star product on $\tilde{A}_1, \tilde{A}_2$ satisfy the correspondence principle.

3.7 Toeplitz quantization for compact complex manifolds

Let M be a compact complex manifold of dimension n. Let M_0 be a set of measure zero such that $X = M \smallsetminus M_0 \equiv \mathbb{R}^{2n} \equiv V$ (diffeomorphisms) where V is a polydisc in $\mathbb{C}^n$. Let V_0 be a measure zero set such that $V \smallsetminus V_0$ is biholomorphic to $\mathbb{C}^n \smallsetminus C_0$ and contains p,

C_0 is a measure zero set (see appendix). Let $U_0 = (1, z_1, z_2, ..., z_n)$ be a chart of $\mathbb{C}P^n$. Then $\mathbb{C}^n \setminus C_0$ is biholomorphic to $U_0 \setminus U_{00}$. So $V \setminus V_0$ is biholomorphic to $U_0 \setminus U_{00}$, which is embedded in U_0 and hence in $\mathbb{C}P^n$. Here V_0, C_0, U_{00} are all of lower dimension and hence of measure zero. Thus there is an embedding, call it ϵ, which maps $M \setminus M_0$ onto $\mathbb{C}P^n \setminus C_{00}$ where C_{00} is a set of measure zero. Here $\mathbb{C}P^n$ is endowed with Fubini-Study metric and metrics on $V \setminus V_0, U_0, \mathbb{C}^n$ are induced by the Fubini-Study metric. The volume element on $\mathbb{C}P^n$ restricted to U_0 is given as $dV(\mu) = \frac{|d\mu \wedge d\bar{\mu}|}{(1+|\mu|^2)^{n+1}}$. Let us give $M \setminus M_0$ a volume form as follows. Let $\Sigma = \epsilon(M \setminus M_0)$. Let $h(\zeta)dS(\zeta) = dV_\Sigma(\epsilon(\zeta))$, where h is such that all pullback sections are square integrable w.r.t. the measure hdS on $M \setminus M_0$.

Let H be the hyperplane line bundle on $\mathbb{C}P^n$ and H^m be the m-th tensor power of H. Let $\mathcal{H}^m$ be the Hilbert space of square integrable holomorphic sections of H^m restricted on U_0. Let $\mathcal{H}^m_X$ denotes $\epsilon^*(\mathcal{H}^m)$.

Let f, g be a smooth function on $\mathbb{C}P^n$ restricted to U_0 and let $\tilde{f}, \tilde{g}$ be the smooth functions on $M \setminus M_0$, which are pulled back by ϵ, i.e., for $\mu \in M \setminus M_0$, $\tilde{f}(\mu) \doteq f(\epsilon(\mu))$, similarly $\tilde{g}(\mu) \doteq g(\epsilon(\mu))$.

Now for $\mathbb{C}P^n$ (restricted to U_0), m-th level Toeplitz operator of f, denoted by T^m_f, defined on $\mathcal{H}^m$, defined as $T^m_f(s) = \Pi^m(fs)$, where Π^m is the projection map from square integrable section onto $\mathcal{H}^m$ and $s \in \mathcal{H}^m$. Let $\tilde{s} = \epsilon^* s$.
For $X = M \setminus M_0$, we denote $||\tilde{s}||^2 = ||\tilde{s}||^2_X = \int_X |\tilde{s}|^2 h(\zeta) dS(\zeta) = \int_\Sigma |s_{|\Sigma}|^2 dV_\Sigma(\epsilon(\zeta))$ where recall $\Sigma = \epsilon(X)$. But $\int_\Sigma |s_{|\Sigma}|^2 dV_\Sigma(\epsilon(\zeta)) = \int_{\mathbb{C}P^n} |s|^2 dV$ since $\Sigma = \mathbb{C}P^n \setminus C_{00}$ where C_{00} is a set of measure zero.

Thus we have

$$||\tilde{s}||^2 = ||s||^2 \tag{3.7.1}$$

where the first norm is in $X = M \setminus M_0$ and second norm is in $\mathbb{C}P^n$.

Now for the functions $\tilde{f}, \tilde{g}$, we define a set of operators for $M \setminus M_0$, defined on $\mathcal{H}^m_X$, denoted by $\tilde{T}^m_{\tilde{f}}, \tilde{T}^m_{\tilde{g}}$ defined as $\tilde{T}^m_{\tilde{f}}(\tilde{s}) = \tilde{\Pi}^m(\tilde{f}\tilde{s})$ where $\tilde{\Pi}^m \epsilon^* \doteq \epsilon^* \Pi^m$.

We know from Toeplitz quantization of $\mathbb{C}P^n$ (see [4]), that,

$$\lim_{m \to \infty} ||T^m_f|| = ||f||_\infty, \lim_{m \to \infty} ||m[T^m_f, T^m_g] - iT^m_{\{f,g\}}|| = 0. \tag{3.7.2}$$

Let $\{\tilde{f}, \tilde{g}\} \doteq \epsilon^* \{f, g\}$.

Proposition 3.7.1.

$$\tilde{T}^m_{\tilde{g}} \epsilon^* = \epsilon^* T^m_g$$

and

$$\tilde{T}^m_{\{\tilde{f},\tilde{g}\}}\epsilon^* = \epsilon^* T^m_{\{f,g\}}$$

Proof. To prove the first equality,

$$\begin{aligned}\tilde{T}^m_{\tilde{g}}\epsilon^* s(\mu) &= \tilde{\Pi}^m(\tilde{g}\cdot\epsilon^*(s))(\mu) = \tilde{\Pi}^m(\epsilon^* g\cdot\epsilon^*(s))(\mu) = \tilde{\Pi}^m(\epsilon^*(g\cdot s))(\mu)\\ &= (\epsilon^*\Pi^m g\cdot s)(\mu)\\ &= (\epsilon^* T^m_g)(s)(\mu)\end{aligned}$$

The second equality follows from this. □

Recall that $||\epsilon^*(s)|| = ||\epsilon^*(s)||_X$ and $||s|| = ||s||_{\mathbb{C}P^n}$. Now $||\epsilon^*(s)|| = ||s||$ by (3.7.1). This implies $||\tilde{T}^m_{\tilde{f}}|| = ||T^m_f||$ for each m.

$$\lim_{m\to\infty}||\tilde{T}^m_{\tilde{f}}|| = \lim_{m\to\infty}||T^m_f||. \tag{3.7.3}$$

Proposition 3.7.2.

$$[\tilde{T}^m_{\tilde{f}},\tilde{T}^m_{\tilde{g}}](\tilde{s}) = (\epsilon^*[T^m_f,T^m_g])(\tilde{s}).$$

Proof. Recall $\tilde{\Pi}^m\epsilon^* \doteq \epsilon^*\Pi^m$.

$\tilde{\Pi}^m(\tilde{f}\tilde{\Pi}^m\tilde{g}\tilde{s}) = \tilde{\Pi}^m(\tilde{S}_1)$ where $S_1 = f\cdot\Pi^m g\cdot s$ and $\tilde{S}_1 = \tilde{f}\tilde{\Pi}^m\tilde{g}\tilde{s} = \epsilon^*(S_1)$.

Then, since $\tilde{\Pi}^m\epsilon^* \doteq \epsilon^*\Pi^m$, we have

$$\tilde{\Pi}^m(\tilde{S}_1(\mu)) = \epsilon^*\Pi^m(S_1) = \epsilon^*\Pi^m(f\cdot\Pi^m g\cdot s)$$

Interchanging f and g we have $\tilde{\Pi}^m(\tilde{g}\tilde{\Pi}^m\tilde{f}\tilde{s}) = \epsilon^*\Pi^m(g\cdot\Pi^m f\cdot s)$

Thus $\tilde{\Pi}^m(\tilde{f}\tilde{\Pi}^m\tilde{g}\tilde{s}) - \tilde{\Pi}^m(\tilde{g}\tilde{\Pi}^m\tilde{f}\tilde{s}) = \epsilon^*\Pi^m(f\cdot\Pi^m g\cdot s) - \epsilon^*\Pi^m(g\cdot\Pi^m f\cdot s)$.

Thus we are done. □

Theorem 3.7.3.

$$\lim_{m\to\infty}||\tilde{T}^m_{\tilde{f}}|| = \lim_{m\to\infty}||T^m_f|| = ||f||_\infty$$

$$\lim_{m\to\infty}||m[\tilde{T}^m_{\tilde{f}},\tilde{T}^m_{\tilde{g}}] - i\tilde{T}^m_{\{\tilde{f},\tilde{g}\}}|| = 0.$$

Proof. As seen before $||\epsilon^*(s)|| = ||\epsilon^*(s)||_X = ||s||_{\mathbb{C}P^n} = ||s||$. This implies $||\tilde{T}_{\tilde{f}}|| = ||T_f||$.

Rest follows from the previous two propositions. □

If we have two biholomorphisms ϵ_1 and ϵ_2 from $M \setminus M_0$ to $\mathbb{C}P^n \setminus C_{00}$ we have an equivalent Toeplitz quantization because we can define an equivalence of the Hilbert spaces and the Poisson bracket is also preserved. This is by the same argument as given before proposition (3.4.1).

Chapter 4

Pullback coherent and squeezed states.

4.1 Coherent states on compact smooth manifolds

In [5] we had given a definition of coherent states on compact smooth manifolds, M^n by embedding it into $\mathbb{C}P^n$ and using pullback from $\mathbb{C}P^n$.

We can do it more straightforwardly as follows. We embed M into $\mathbb{R}^{2n}$ by the Whitney embedding theorem. We identify $\mathbb{R}^{2n}$ with $U_0 \subset \mathbb{C}P^n$. Let ϵ be the embedding and $\Sigma = \epsilon(M)$.

Let $H^{\otimes m}$ be the m-th tensor product of the hyperplane bundle on $\mathbb{C}P^n$ and $\{\Psi_I\}$ be the multinomial orthonormal basis of holomorphic sections of $H^{\otimes m}$ restricted to U_0. Let $\Phi_I = \epsilon^*(\Psi_I)$ be pullbacks of the global sections $\{\Psi_I\}$. Let dS be a volume form on M and $\mathcal{H} = \text{Span}\{\Phi_I\}$. By Gram-Schmidt orthonormalization process, we get an orthonormal basis $\{\eta_i\}_{i=1}^k$ (w.r.t. volume form dS) of $\mathcal{H}$ on M.

Define the pullback coherent states as

$$\psi_\mu = \sum_{i=1}^{k} \overline{\eta_i(\mu)}\eta_i.$$

We can see that, given any states $\Psi, \Psi_1, \Psi_2 \in \mathcal{H}$, we have completeness, the reproducing kernel property as well as the resolution of identity property.

Proposition 4.1.1. *(a) Reproducing kernel property:*

$$\langle \psi_\mu, \Psi \rangle = \Psi(\mu).$$

(b) Resolution of identity:

$$\langle \Psi_1, \Psi_2 \rangle = \int_{\Sigma} \langle \Psi_1, \psi_\mu \rangle \langle \psi_\mu, \Psi_2 \rangle \, dS(\mu).$$

(c) Overcompleteness:

$$\langle \psi_\mu, \Psi \rangle = 0, \forall \mu \textit{ implies } \Psi = 0.$$

Proof. Proofs of (a) and (b) are the same as in propositions (3.1.2) and (3.1.3). Proof of (c) follows from (a). □

The pullback coherent states depend on the embedding.

4.2 Pullback coherent and squeezed states on integral compact Kähler manifold.

Now we give a slightly different definition of pullback coherent states and squeezed states on integral compact Kähler manifold.

This is achieved by modifying the definitions in [5] and could also be generalized on a compact smooth manifold.

4.2.1 Pullback coherent states on integral compact Kähler manifold

Let $(L, \nabla, 2\pi i\Omega) \longrightarrow (M, \Omega)$ be a line bundle with Ω an integral Kähler form on M. Let h be a Hermitian metric on L^k where k is such that L^k is very ample. Let Γ be the space of global holomorphic sections of L^k.

We choose the inner product on Γ w.r.t. h, namely $< \phi_1, \phi_2 >= \int_M \overline{\phi_1} \phi_2 h dV$, to be antilinear in the first term and linear in the second (unlike Rawnsley's convention). We will retain this convention henceforth. Let $\mathcal{H}$ be the space of square integrable sections in Γ.

Let $\{\psi_i\}_{i=1}^m$ be an orthonormal basis for $\mathcal{H}$ which is basepoint free.

Let $\phi \in \mathcal{H}$, a square integrable holomorphic section of L^k, then ϕ can be expressed as a linear combination of the orthonormal basis elements $\{\psi_i\}_{i=1}^m$ as follows

$$\phi = \sum_{i=1}^{m} < \psi_i, \phi > \psi_i.$$

Let $\psi_i = f_i s_0$, where s_0 is a fixed section such that its zero set is $M_0 \subset M$ and f_i are meromorphic functions which are holomorphic on $M \setminus M_0$.

Definition 4.2.1. For $\mu \in M$, we define

$$\phi_\mu = \sum_{i=1}^{m} \overline{f_i(\mu)} \psi_i$$

where $\psi_i = f_i s_0$ where f_i is a meromorphic function on M, which is holomorphic on $M \setminus M_0$.

For $\mu \in M \setminus M_0$ one sees that

$$\phi_\mu = \frac{1}{s_0(\mu)} \sum_{i=1}^{m} \overline{\psi_i(\mu)} \psi_i.$$

Note that ϕ_μ is a smooth section of L^k.

Then

$$\phi_\mu(\nu) = \sum_{i=1}^{m} \overline{f_i(\mu)} \psi_i(\nu).$$

Under the integrality condition on the Kähler form, we have a generalization of Spera's result, Theorem (2.1) [18] as follows:

Theorem 4.2.2. *Let M be a integral compact Kähler manifold and let $\mathcal{H}$, s_0 and ϕ_μ be as defined above. Then, for all $\mu \in M$, we have*

(a) Modified reproducing kernel property.

ϕ_μ are Rawnsley-type coherent states of M.

In fact, for $\mu \in M$ the formula is $< \phi_\mu, \phi > s_0(\mu) = \phi(\mu)$.

Also, $< \phi_\mu, \phi > = \frac{\phi(\mu)}{s_0(\mu)}$ for $\mu \in M \setminus M_0$.

(b) ϕ_μ satisfy

(i) the modified maximal likelihood property, namely,

$$|\phi_\mu(\mu)|^2 \geq |\phi(\mu)|^2 |s_0(\mu)|^2 \textit{ for all } \mu \textit{ and all } \phi \in \mathcal{H} \textit{ such that } <\phi, \phi>= 1.$$

(ii) the property that

$$|\phi_\mu(\mu)|^2 \geq |\phi_{\mu'}(\mu)|^2 |s_0(\mu)|^2.$$

(iii) the modified generalized resolution of identity, namely,

$$<\psi_1, \psi_2>= \int_M <\psi_1, \phi_\mu><\phi_\mu, \psi_2> |s_0(\mu)|^2 h(\mu) dV(\mu).$$

(iv) overcompleteness, namely, $<\phi_\mu, \psi>= 0 \ \forall \mu$ *implies* $\psi = 0$.

Proof. (a) follows from a simple calculation.

(b) The proof follows by modifications of [18] as below.

(i) $|\phi(\mu)|^2 = |<\phi_\mu, \phi>|^2 |s_0(\mu)|^2 \leq |\phi_\mu|^2 |\phi|^2 |s_0(\mu)|^2 = |\phi_\mu|^2 |s_0(\mu)|^2$, since $|\phi|^2 = 1$.

(ii) This follows along similar lines as (i).

(iii) Using (a),

$$\begin{aligned} <\psi_1, \psi_2> &= \int_M \bar{\psi}_1(\mu)\psi_2(\mu) h(\mu) dV(\mu) \\ &= \int_M \overline{<\phi_\mu, \psi_1>} <\phi_\mu, \psi_2> |s_0(\mu)|^2 h(\mu) dV(\mu) \\ &= \int_M <\psi_1, \phi_\mu><\phi_\mu, \psi_2> |s_0(\mu)|^2 h(\mu) dV(\mu). \end{aligned}$$

(iv) This follows from (a).

□

4.2.2 Squeezed states on an integral compact Kähler manifold

We define squeezed states in a similar fashion as the coherent states in the previous subsection.

Let M be an integral compact Kähler manifold of dimension d. Let $q \in U \subset M$, where U is an open neighbourhood of q such that ϕ_U is a biholomorphism to an open ball $V \subset \mathbb{C}^d$ to U. Let $q = \nu = \phi_U(\nu_1 + i\nu_2)$, where $\nu_1 + i\nu_2 \in V$. Let $\zeta \in \mathbb{R}$ be such that $\nu_1 + i\zeta\nu_2$ belongs to V. Then $\nu_\zeta = \phi_U(\nu_1 + i\zeta\nu_2) \in U$. Let $q_\zeta = \nu_\zeta$.

Proposition 4.2.3. *Let M be a $2d$-dimensional compact smooth manifold. Then there exists a subset $\tilde{M}$ of dimension at most $(2d-1)$ such that $M \setminus \tilde{M}$ admits an open cover by a single open set U of the above kind.*

Proof. A $2d$-dimensional manifold admits a cell-decomposition as a single $2d$-dimensional cell glued to a skeleton of dimension at most $(2d-1)$, [7]. Let $\tilde{M}$ be this skeleton. Then $M \setminus \tilde{M}$ is homeomorphic to $\mathbb{C}^d$. Thus one open set U is enough to cover $M \setminus \tilde{M}$. □

Let $\{\psi_i\}_{i=1}^m$ be the orthonormal basis for the Hilbert space of geometric quantization as described in the previous section, with inner product $< \cdot, \cdot >$. Let $\mu = \phi_U(\mu_1 + i\mu_2)$ and $\mu_\zeta = \phi_U(\mu_1 + i\zeta\mu_2)$.

Let us define the squeezed states as follows. Let s_0 be a fixed holomorphic section of the prequantum bundle whose vanishing set is $M_0 \subset M$.

Then for $\mu_\zeta \in M \setminus (\tilde{M} \cup M_0)$, we define the squeezed states as follows.

$$\phi_\mu^\zeta(\nu) = \frac{1}{\overline{s_0(\mu_\zeta)}} \sum_{i=1}^m \overline{\psi_i(\mu_\zeta)}\psi_i(\nu).$$

We have the theorem below.

Theorem 4.2.4. *The ϕ_μ^ζ satisfy the following properties.*

1. *The modified reproducing kernel property.*

$$< \phi_\mu^\zeta, \phi > s_0(\mu_\zeta) = \phi(\mu_\zeta)$$

.

2. *The modified maximal likelihood property, namely,*

$$|\phi_\mu^\zeta(\mu_\zeta)|^2 \geq |\phi(\mu_\zeta)|^2 |s_0(\mu_\zeta)|^2 \text{ for all } \mu_\zeta \in U \text{ and all } \phi \in \mathcal{H} \text{ such that } < \phi, \phi >= 1.$$

3. *Modified resolution of identity:*

$$< \phi_1, \phi_2 >= \int_M < \phi_1, \phi_\mu^\zeta >< \phi_\mu^\zeta, \phi_2 > |s_0(\mu_\zeta)|^2 h(\mu_\zeta) dV(\mu_\zeta).$$

4. *Overcompleteness:*

$$< \phi_\mu^\zeta, \phi >= 0 \text{ for all } \mu \text{ iff } \phi = 0.$$

Proof. The proof is similar to Theorem (4.2.2). We have to use that $\tilde{M} \cup M_0$ is of measure zero and removing this set does not affect the integrals. $\square$

Chapter 5

An Example: Symmetric Product of Riemann surface of genus $g > 1$

In this chapter we do not construct anything new but we show an example illustrating the theory above. We use an embedding of the symmetric product (see Appendix) of a compact Riemann surface of genus $g > 1$ into $\mathbb{C}P^N$, which is developed in [3] and [1]. In [3], the authors construct an embedding of the symmetric product into a Grassmannian manifold and finally into a projective space using the Plücker embedding (see Appendix).

We closely follow Biswas and Romão [3] and Aryasomayajula and Biswas [1].

Let Σ be a compact connected Riemann surface of genus $g > 1$. Let $T^{\star}_{\Sigma}$ = Holomorphic cotangent bundle of Σ. ${T^{\star}_{\Sigma}}^{\otimes k}$ = k-times tensor product of the holomorphic cotangent bundle of Σ.

Let $M = Sym^d(\Sigma)$.

$\Omega = T^{\star}_{\Sigma}$ is the cotangent bundle on Σ. A section of ${T^{\star}_{\Sigma}}^{\otimes k}$ is of the form $f(z)dz^{\otimes k}$.

When lifted to the upper half plane (universal cover of Σ), $f(z)$ transforms as a cusp form under the $SL(2,\mathbb{Z})$ transformation $\gamma(z) = \frac{az+b}{cz+d}$. This is because $f(\gamma(z))d\gamma(z)^{\otimes k} = f(z)dz^{\otimes k}$ implies that $f(\gamma(z)) = \frac{f(z)}{(cz+d)^{2k}}$, by a simple calculation.

For any $k > 0$, let $S^{2k}(\Gamma)$ denote the complex vector space of weight-2k cusp forms on Σ.

In [3] Biswas and Romão have shown that $Sym^d(\Sigma)$ can be embedded in a certain Grassmannian using certain sections of $H^0(\Sigma, T^{\star \otimes k}_{\Sigma})$.

Let us focus on $L = T^{\star\otimes k}_{\Sigma} = \Omega^{\otimes k}$

Let $\omega,\eta \in H^0(X, T_\Sigma^{\star\otimes k})$. Then $\omega(z) = f(z)dz^{\otimes k}$, $\eta(z) = g(z)dz^{\otimes k}$ and

$$\langle\omega,\eta\rangle_{hyp} = \int y^{2k} f(z)\overline{g(z)}\mu_\Sigma^{hyp}(z).$$

Complex vector space $H^0(X, T_\Sigma^{\star\otimes k})$ have dimension n_k, where $n_k = (2k-1)(g-1)$, $k \geq 2$ and $n_1 = g$.

Let $\{e_1, e_2, ..., e_{n_k}\}$ denote a basis of $H^0(X, T_\Sigma^{\star\otimes k})$.

Set $r_k = n_k - d$, (where k is large enough such that $r_k \geq 1$). Let $Gr(r_k, n_k)$ denote the Grassmannian parametrizing r_k dimensional subspaces of n_k dimensional complex vector space $H^0(X, T_\Sigma^{\star^{\otimes k}})$.

Let $H^0(\Sigma, T_\Sigma^{\star^{\otimes k}} \otimes O_\Sigma(-x_1 - x_2... - x_d))$ consist of all the holomorphic sections (cusp forms) of $H^0(X, T_\Sigma^{\star^{\otimes k}})$ which vanishes at the points $x_1, x_2, ..., x_d$.
Clearly $H^0(\Sigma, T_\Sigma^{\star^{\otimes k}} \otimes O_\Sigma(-x_1 - x_2... - x_d))$ is a vector subspace of $H^0(X, T_\Sigma^{\star^{\otimes k}})$.

Now in [3] and [1], the authors define a map Φ: $Sym^d(\Sigma) \longrightarrow Gr(r_k, n_k)$ by mapping $(x_1, x_2, ..., x_d)$ to the vector space $H^0(\Sigma, T_\Sigma^{\star^{\otimes k}} \otimes O_\Sigma(-x_1 - x_2... - x_d))$, where $x_1, x_2, ..., x_d$'s are (not necessarily distinct points) in Σ.

For any k with $(2k-1)(g-1) > d$, we have dim $H^0(\Sigma, T_\Sigma^{\star^{\otimes k}} \otimes O_\Sigma(-x_1 - x_2... - x_d)) = n_k - d = r_k$.

So $H^0(\Sigma, T_\Sigma^{\star^{\otimes k}} \otimes O_\Sigma(-x_1 - x_2... - x_d))$ is a $(n_k - d)$ ie, r_k dimensional vector subspace of $H^0(X, T_\Sigma^{\star^{\otimes k}})$.
Therefore $H^0(\Sigma, T_\Sigma^{\star^{\otimes k}} \otimes O_\Sigma(-x_1 - x_2... - x_d))$ is an element of $Gr(r_k, n_k)$.
It is known that Φ is holomorphic embedding.

5.1 Embedding into $\mathbb{C}P^N$.

In this section we use the Plücker embedding (see Appendix) of $Gr(r_k, n_k)$ into $\mathbb{C}P^N$. Here P is the Plücker embedding (where $N = \binom{n_k}{r_k} - 1$) which we use to embed $Sym^d(\Sigma)$ into $\mathbb{C}P^N$.

Recall, the embedding $Gr(r_k, n_k)$ into projective space goes as follows. First we identify $H^0(\Sigma, L^{\otimes k})$ locally with $\mathbb{C}^{n_k}$ and denote orthonormal basis of $\mathbb{C}^{n_k}$ by $\{e_1, e_2, ..., e_{n_k}\}$ (using same notation for basis elements).

Let $\mathbb{C}^{n_k}$ be given by a basis $\{e_1, ..., e_{n_k}\}$ and $N + 1 = \binom{n_k}{r_k}$.

Let $q \in Gr(r_k, n_k)$, that is q is a r_k-dimensional vector subspace of $\mathbb{C}^{n_k}$.

$V = \wedge^{r_k}\mathbb{C}^{n_k} = \mathbb{C}^{N+1}$ has basis $\mathcal{B} = \{e_{i_1} \wedge e_{i_2} \wedge .. \wedge e_{i_{r_k}}\}_{1 \le i_1 < i_2 < ... < i_{r_k} \le n_k}$

Let P be the Plücker embedding. Then $P(q) = [p] \in P(V)$ where

$$p = \sum_{1 \le i_1 < i_2 < ... < i_{r_k} \le n_k} c_{i_1 i_2 ... i_{r_k}}(q) e_{i_1} \wedge e_{i_2} \wedge .. \wedge e_{i_{r_k}} \in V.$$

We can identify $q \in Gr(r_k, n_k)$, with a point $[p]$ in the projective space $\mathbb{C}P^N$, namely the class of $\{c_{i_1 i_2 ... i_{r_k}}(q)\}$ (ordering is maintained).

One can embed $Sym^d(\Sigma)$ into $\mathbb{C}P^N$ by composing $\epsilon = P \circ \Phi$.

5.2 Berezin and Toeplitz quantization and pullback coherent states on $Sym^d(\Sigma)$.

Let $M = Sym^d(\Sigma)$, where Σ is a compact Riemann surface of genus > 1. Let $x = (x_1, ..., x_d)$ be a point on $Sym^d(\Sigma)$ and let $\{e_i(x)\}$ be a basis of cusp forms on Σ which vanish at x and $\{e_I\} = \{e_{i_1} \wedge e_{i_2} \wedge .. \wedge e_{i_{r_k}}\}$ be the basis of $\mathbb{C}^{N+1} = V$. Let $\{c_{i_1 i_2 ... i_{r_k}}(q)\}$ be the coordinates of $[p] = \epsilon(x)$ in $\mathbb{C}P^N$. Suppose $[p] \in U_0 \subset \mathbb{C}P^N$ (w.l.g). Let us write $\{c_{i_1 i_2 ... i_{r_k}}(q)\}$ as $[p] = [1, \mu_1, ..., \mu_N]$ on U_0.

Proposition 5.2.1. *Let Σ be a compact Riemann surface of genus > 1. Then $M = Sym^d(\Sigma)$ has an algebra of operators acting on a Hilbert space which have a star product on their $\mathbb{C}P^N$-symbol which satisfy the correspondence principle. Here N could be as defined in the previous section.*

Proof. Let ϵ be the above embedding of M into $\mathbb{C}P^N$. Then we can repeat what was done in section (3.6) and get a Berezin-type quantization. □

Proposition 5.2.2. *There exists a measure zero set M_0 such that $M \setminus M_0$ has a set of Toeplitz operators which satisfy the Toeplitz quantization conditions as in (3.7).*

Proof. This follows from section (3.7). □

Proposition 5.2.3. *M has pull back coherent states and squeezed states which satisfy overcompleteness, reproducing kernel property, maximum likelihood property and resolution of identity.*

Proof. We have en embedding ϵ of M into $\mathbb{C}P^N$. Then one can define the pull back coherent states and squeezed states as in the chapter (4). □

Chapter 6

Conclusion and Further work

6.1 Compact integral Kähler manifold

Suppose M is a compact integral Kähler manifold and suppose we embed it into $\mathbb{C}P^n$ and induce the Berezin quantization by the embedding as in [6]. Suppose M has a Berezin quantization as in [2] under the conditions mentioned in [2]. Then we try to answer the question how would the two Berezin quantizations differ.

6.2 Local Berezin-type quantization for compact odd-dimensional smooth manifolds.

We can embed compact odd-dimensional smooth manifolds into $\mathbb{C}P^n$ and try to execute the Berezin quantization.

6.3 Berezin-type quantization for even dimensional non-compact smooth manifolds.

1) If the volume of the even dimensional non-compact smooth manifold is finite, (with respect to some Riemannian metric) then we embed the manifold into $\mathbb{C}P^n$ and pull back the Hilbert space of quantization from there, as we expect a finite dimensional Hilbert space.

2) If the even dimensional non-compact smooth manifold is of infinite volume (with respect to some Riemannian metric) we embed it into $\mathbb{C}^n$ and induces the quantization from there. We will obtain an infinite dimensional Hilbert space but this is expected.

Appendix A

Two Proofs

In this section we prove two statements in Berezin [2] whose proof is not provided in the paper.

Let $\hat{A}$ be a bounded linear operator acting on $\mathcal{H}$. Then, as in [2], one can define a symbol of the operator as

$A(\mu, \bar{\nu}) = \frac{\langle \psi_\nu, \hat{A}\psi_\mu \rangle}{\langle \psi_\nu, \psi_\mu \rangle}$.

We can recover the operator $\hat{A}$ from the symbols as in the next proposition.

Let $\frac{1}{c(m)} = \int_{U_0} \frac{\mathcal{L}_m(\bar{\mu},\nu)\mathcal{L}_m(\bar{\nu},\mu)}{\mathcal{L}_m(\bar{\mu},\mu)\mathcal{L}_m(\bar{\nu},\nu)} dV(\nu)$.

Proposition A.0.1 (Berezin)**.**

$$(\hat{A}f)(\mu) = c(m) \int_{U_0} A(\nu, \bar{\mu}) f(\nu) \mathcal{L}_m(\mu, \bar{\nu}) e^{-m\tilde{\Phi}(\nu,\bar{\nu})} dV(\nu).$$

Proof. Since $\hat{A}$ is linear, we need to prove this only for $f = \Psi_I$, a basis vector.

$$\begin{aligned}
R.H.S. &= c(m) \int_{U_0} \frac{\langle \psi_\mu, \hat{A}\psi_\nu \rangle}{\langle \psi_\mu, \psi_\nu \rangle} \Psi_I(\nu) \mathcal{L}_m(\mu, \bar{\nu}) e^{-m\tilde{\Phi}(\nu,\bar{\nu})} dV(\nu) \\
&= c(m) \int_{U_0} \langle \psi_\mu, \hat{A}\psi_\nu \rangle \Psi_I(\nu) e^{-m\tilde{\Phi}(\nu,\bar{\nu})} dV(\nu) \\
&= c(m) \int_{U_0} (\hat{A}\psi_\nu)(\mu) \Psi_I(\nu) e^{-m\tilde{\Phi}(\nu,\bar{\nu})} dV(\nu) \\
&= c(m) \int_{U_0} (\hat{A}(\sum_J \overline{\Psi_J(\nu)} \Psi_J)(\mu) \Psi_I(\nu) e^{-m\tilde{\Phi}(\nu,\bar{\nu})} dV(\nu) \\
&= c(m)(\hat{A}\Psi_J)(\mu) \sum_J \int_{U_0} \overline{\Psi_J(\nu)} \Psi_I(\nu) e^{-m\tilde{\Phi}(\nu,\bar{\nu})} dV(\nu) \\
&= (\hat{A}\Psi_J)(\mu) \sum_J \delta_{IJ} \\
&= (\hat{A}\Psi_I)(\mu) = L.H.S.
\end{aligned}$$

□

Proposition A.0.2 (Berezin)**.** *Let $\hat{A}_1, \hat{A}_2$ be two bounded linear operators on the Hilbert space $\mathcal{H}$ and let A_1, A_2 be their symbols. Then $A_1 * A_2$ is the symbol of $\hat{A}_1 \circ \hat{A}_2$.*

Proof. The proof below holds in general, but we show it only for Berezin star product in $\mathbb{C}P^n$.

Let $\tilde{A}$ be the symbol of $\hat{A}_1 \circ \hat{A}_2$. It is given by

$$\tilde{A}(\mu, \bar{\mu}) = \frac{\langle \psi_\mu, \hat{A}_1 \circ \hat{A}_2 \psi_\mu \rangle}{\langle \psi_\mu, \psi_\mu \rangle}.$$

In other words by reproducing kernel property, we get

$$\tilde{A}(\mu, \bar{\mu}) = \frac{(\hat{A}_1 \circ \hat{A}_2 \psi_\mu)(\mu)}{\psi_\mu(\mu)}.$$

Let $\Psi_I, \Psi_J, \Psi_P, \Psi_Q$ be different notations for the basis elements of $\mathcal{H}$ where I, J, P, Q run over the same multi-indices.

Let $\hat{A}_2 \Psi_I = \sum_I C_I^J \Psi_J$ and $\hat{A}_1 \Psi_P = \sum B_P^Q \Psi_Q$, C_I^J, B_P^Q are constants. Using the definition of ψ_μ, we get

$$\hat{A}_1 \psi_\nu = \sum_{P,Q} B_P^Q \Psi_Q(\mu) \overline{\Psi_P(\nu)},$$

$$(\hat{A}_1 \circ \hat{A}_2 \psi_\mu)(\mu) = \sum_{I,J,K} C_I^J B_J^K \Psi_K(\mu) \overline{\Psi_I(\mu)}.$$

On the other hand, we have, [2],

$$
\begin{aligned}
&(A_1 * A_2)(\mu,\bar{\mu})\\
=\ & c(m)\int_{U_0} A_1(\mu,\bar{\nu})A_2(\nu,\bar{\mu})\frac{\mathcal{L}_m(\mu,\bar{\nu})\mathcal{L}_m(\nu,\bar{\mu})}{\mathcal{L}_m(\mu,\bar{\mu})\mathcal{L}_m(\nu,\bar{\nu})}\mathcal{L}_m(\nu,\bar{\nu})e^{-m\tilde{\Phi}(\nu,\bar{\nu})}dV(\nu)\\
=\ & c(m)\int_{U_0}\frac{(\hat{A}_1\psi_\nu)(\mu)(\hat{A}_2\psi_\mu)(\nu)}{\psi_\mu(\mu)\psi_\nu(\nu)}\frac{|d\nu\wedge d\bar{\nu}|}{(1+|\nu|^2)^{d+1}}\\
=\ & \frac{c(m)}{\psi_\mu(\mu)}\int_{U_0}\left(\sum_{P,Q}B_P^Q\Psi_Q(\mu)\overline{\Psi_P(\nu)}\right)\left(\sum_{I,J}C_I^J\overline{\Psi_I(\mu)}\Psi_J(\nu)\right)e^{-m\Phi_{FS}(\nu,\bar{\nu})}\frac{|d\nu\wedge d\bar{\nu}|}{(1+|\nu|^2)^{d+1}}\\
=\ & \frac{1}{\psi_\mu(\mu)}\left(\sum_{P,Q}B_P^Q\Psi_Q(\mu)\right)\left(\sum_{I,J}C_I^J\overline{\Psi_I(\mu)}\right)\delta_{PJ}\\
=\ & \frac{1}{\psi_\mu(\mu)}\left(\sum_{J,Q,I}B_J^QC_I^J\Psi_Q(\mu)\overline{\Psi_I(\mu)}\right)\\
=\ & \frac{1}{\psi_\mu(\mu)}\left(\sum_{J,K,I}B_J^KC_I^J\Psi_K(\mu)\overline{\Psi_I(\mu)}\right)\\
=\ & \tilde{A}(\mu,\bar{\mu}).
\end{aligned}
$$

□

Appendix B

The integral

Let $0 \leq p \leq m$, p is an integer. Let p_i be integers such that $p_1 + ... + p_n = p$. Let $I_p = I_{p_1,..,p_n;p}$.

Let $\mathcal{I}_{\mathcal{I}_p} = \int_{\mathbb{C}^n} \frac{|\mu|^{2I_p}|d\mu\wedge d\bar{\mu}|}{(1+|\mu|^2)^{m+n+1}} = \int_{\mathbb{C}^n} \frac{|\mu_1|^{2p_1}...|\mu_n|^{2p_n}|d\mu\wedge d\bar{\mu}|}{(1+|\mu|^2)^{m+n+1}}$

Writing $\mu_j = r_j e^{i\theta_j}$, $|d\mu \wedge d\bar{\mu}| = \Pi_{j=1}^n r_j dr_j d\theta_j$ and letting $u_j = r_j^2$ we get after performing the θ_j integrals,

$\mathcal{I}_{\mathcal{I}_p} = \frac{(2\pi)^n}{2^n} \int_0^\infty ... \int_0^\infty \frac{u_1^{p_1}...u_n^{p_n} du_1...du_n}{(1+u_1+...+u_n)^{m+n+1}}$.

We will use the integral

$$\int_0^\infty e^{-ax} x^k dx = \frac{k!}{a^{k+1}} \tag{B.0.1}$$

as follows.

We take $a = (1 + u_1 + ... + u_n)$ in the above integral and write

$$\frac{1}{(1+u_1+...+u_n)^{m+n+1}} = \frac{1}{(m+n)!} \int_0^\infty e^{-ax} x^{m+n} dx.$$

Then inserting the the auxiliary variable x-integral inside $\mathcal{I}_{I_p}$, we get

$$\begin{aligned} \mathcal{I}_{I_p} &= \frac{\pi^n}{(m+n)!} \int_0^\infty ... \int_0^\infty u_1^{p_1}...u_n^{p_n} e^{-(1+u_1+...+u_n)x} x^{m+n} du_1...du_n dx \\ &= \frac{\pi^n}{(m+n)!} \int_0^\infty u_1^{p_1} e^{-u_1 x} u_2^{p_2} e^{-u_2 x} ... u_n^{p_n} e^{-u_n x} e^{-x} x^{m+n} du_1...du_n dx \end{aligned}$$

We will perform the x-integral last. Again using equation (B.0.1) we get

$$\int_0^\infty u_j^{p_j} e^{-u_j x} du_j = \frac{p_j!}{x^{p_j+1}}.$$

Then using $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ we get

$$\mathcal{I}_{I_p} = \frac{\pi^n}{(m+n)!} p_1! \ldots p_n! \int_0^\infty e^{-x} x^{m-p} dx = \frac{\pi^n}{(m+n)!} p_1! \ldots p_n! (m-p)!.$$

There is another integral of importance as follows.

$$(c(m))^{-1} = \mathcal{I}_{p=0} = \frac{\pi^n}{(m+n)!} m!$$

where $p_1 = p_2 = \ldots = p_n = 0$.

Appendix C

Removing sets of measure zero

Proposition C.0.1. *Let Δ be a polydisc in $\mathbb{C}^n$. There exists a set of measure zero, namely, $\Delta_0 \subset \Delta$ such that $\Delta \setminus \Delta_0$ is biholomorphic to $\mathbb{C}^n \setminus N_0$ where N_0 is a set of measure zero.*

Proof. There is a biholomorphism from 1-dimensional disc $D \setminus \{0\}$ to $\mathbb{C} \setminus [-1,1]$ given by $w(z) = \frac{1}{2}(z + \frac{1}{z})$, the Joukowski map. We claim that there is a set Δ_0 of lower dimension such that $\Delta \setminus \Delta_0 = (D \setminus \{0\})^d$. We know that $\Delta = D \times D \times .... \times D$. Δ_0 is then union of sets of the type $D \times D \times .... \times \{0\} \times ... \times D$. This can be seen as follows. $x = (x_1, ..., x_d) \in (D \setminus \{0\})^d$ iff $x \in \Delta$ such that none of the x_i are zero, for $i = 1, ..., d$. In other words, Δ_0 is the measure zero set characterized by the set such that at least one of the $x_i = 0$.

Thus we have $\Delta \setminus \Delta_0$ is biholomorphic to $(\mathbb{C} \setminus [-1,1])^d$. But $(\mathbb{C} \setminus [-1,1])^d = \mathbb{C}^d \setminus N_0$ where N_0 is the union of sets of the type $\mathbb{C} \times \mathbb{C} \times .. \times [-1,1] \times ... \times \mathbb{C}$.

□

Appendix D

Modular Forms

Poincaré upper half plane

Poincaré upper half plane $\mathbb{H} = \{z \in \mathbb{C} | Im(z) \geq 0\}$, which is a complex manifold of dimension 1 on which we can talk of holomorphic functions.

The group $SL(2,\mathbb{R})$ acts on H by linear fractional transformations:

$g(z) := \frac{az+b}{cz+d}$, $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R})$.

Modular forms

The subgroup of $SL(2,\mathbb{R})$ consisting of matrices with integer coefficients is by definition $SL(2,\mathbb{Z})$. It is called the "Full modular group".

A modular form $f : \mathbb{H} \longrightarrow \mathbb{C}$ of weight k for $SL(2,\mathbb{Z})$ is a function on $\mathbb{H}$ with the following properties:

1. $f : \mathbb{H} \longrightarrow \mathbb{C}$ is a holomorphic function (meromorphic function in case of a modular function).

2. $f(\frac{az+b}{cz+d}) = (cz+d)^k \ f(z)$, $\forall z \in \mathbb{H}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$

Note: For $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in SL(2,\mathbb{Z})$ we have $f(z+1) = f(z)$. Hence these functions will have a Fourier expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi \iota n z} \tag{D.0.1}$$

3. $a_n = 0, \forall n < 0$, i.e., $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi\iota nz} = \sum_{n=0}^{\infty} a_n q^n, q = e^{2\pi\iota z}$

Note: If $a_0 = 0$, then f is called a **cusp form**.

Remark:

Condition 3 is equivalent to say that f remains bounded at $\iota\infty$, i.e. $lim_{t\to\infty} f(\iota t) < \infty$, and the **cuspidality** condition is equivalent to say that $f(\iota\infty) = 0$.

Remarks:

1. $z \to e^{2\pi\iota z} = q$ gives rise to the identification: $\mathbb{H}/\mathbb{Z} \cong D^* = \{q||q| < 1, q \neq 0\}$.

2. If f is a modular form, f considered as a function on $D^\star$ can be extended to a holomorphic function on D. It is a cusp form if it extends to D holomorphically and vanishes at $q = 0$.

3. Since $SL(2,\mathbb{Z}) = \langle S, T\rangle$, where $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, for **modularity** of a function f on $\mathbb{H}$, we have to check $f(z+1) = f(z)$ and $f(-1/z) = z^k f(z)$.

4.Modular forms are an interplay between continuous and discrete as they are continuous functions with discrete symmetries.

5. The set of all **modular forms** of weight k for $SL(2,\mathbb{Z})$ is a finite dimensional vector space, and is denoted by $M_k(SL(2,\mathbb{Z}))$. Similarly, the set of all **cusp forms** of weight k for $SL(2,\mathbb{Z})$ is also a vector space, and is denoted by $S_k(SL(2,\mathbb{Z}))$.

Appendix E

Plücker embedding

Here we mention a well known result, see for example [19], page 11.

Proposition E.0.1. *(Plücker) The Grassmannian manifolds $G_{r,n}(\mathbb{C})$ are projective algebraic manifolds.*

The proof is given considering the following map: $\tilde{F} : M_{r,n}(\mathbb{C}) \longrightarrow \wedge^k(\mathbb{C}^n) \smallsetminus \{0\}$ defined by

$$\tilde{F}(A) = \tilde{F}(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix}) = a_1 \wedge \ldots \wedge a_r \tag{E.0.1}$$

We embed the Grassmannian $G_{r,n}(\mathbb{C})$ in $P(\wedge^r\mathbb{C}^n)$ by completing the following diagram.

$$\begin{array}{ccc} M_{r,n}(\mathbb{C}) & \xrightarrow{\quad \tilde{F} \quad} & \wedge^r\mathbb{C}^n \smallsetminus \{0\} \\ \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_P} \\ Gr_{r,n}(\mathbb{C}) & \xrightarrow{\quad F \quad} & P(\wedge^r\mathbb{C}^n) \end{array}$$

where π_G, π_P are the usual projections. We can show that F is well defined. In fact $\pi_G(A) = \pi_G(B)$ implies that $A = gB$ for $g \in GL(r,\mathbb{C})$, $\pi_P(a_1 \wedge \ldots \wedge a_r) = \pi_P(\mathrm{det(g)}(\mathrm{b}_1 \wedge \ldots \wedge \mathrm{b}_r)) = \pi_\mathrm{P}(\mathrm{b}_1 \wedge \ldots \wedge \mathrm{b}_r)$, and so the map F is well defined. One can show it is an embedding. This is called the Plücker embedding.

Appendix F

Symmetric Product

Let X be a topological space (or manifold), then the **n-th symmetric product of** X is the space denoted by $Sym^n(X)$, defined as $Sym^n(X) = X^n/S_n$, that is, the orbit space given by the quotient of the n-fold Cartesian product of X by the natural action of the symmetric group S_n that permutes the entries of an element.

The n-th symmetric product of X consists of unordered n-tuples of points on X, not necessarily distinct.

If X is a compact Riemann surfaces, $Sym^d(X)$ is d-dimensional complex manifold. For a proof see [14].

Example: $Sym^n(\mathbb{C}P^1)$ is biholomorphic to $\mathbb{C}P^n$

We present a known proof of a well known result that $Sym^n(S^2)$ is homeomorphic to $\mathbb{C}P^n$, [15]. Regard S^2 as the complex projective line $\mathbb{C}P^1$. The Cartesian product $(S^2)^n$ represents ordered n-tuples of complex homogeneous parameters $\lambda_1, \lambda_2, ..., \lambda_n$. The homogeneous polynomial of degree n having these as roots determines, by its coefficients, a point in n-dimensional complex projective space, $\mathbb{C}P^n$. This gives a continuous map $(S^2)^n \longrightarrow \mathbb{C}P^n$. Permuting the roots $\lambda_1, \lambda_2, ..., \lambda_n$ does not alter the polynomial, so this map factors through the n-th symmetric product of S^2. The map from symmetric product $Sym^n(S^2)$ to $\mathbb{C}P^n$ is certainly one-to-one, and since every such complex polynomial has n complex roots, it is onto as well. Since $(S^2)^n$ is compact, hence $Sym^n(S^2)$ is compact (as a quotient space of a compact space is compact) and $\mathbb{C}P^n$ is Hausdorff, the symmetric product $Sym^n(S^2)$ is homeomorphic to $\mathbb{C}P^n$.

Note: Here we think of $S^2 = \mathbb{C} \cup \{\infty\}$ as the Riemann sphere $\mathbb{C}P^1$. The identification sends a point in $\mathbb{C}P^n$ (thought of as the coefficients of a homogeneous polynomial of degree n), to the roots of this polynomial. If the degree of the polynomial is $k < n$, then there are only k complex roots and the remaining entries will be assigned the point at ∞. Thus the correspondence $\mathbb{C}P^n \longrightarrow Sym^n(S^2)$ is $[a_k : ... : a_1 : a_0] \longmapsto <\xi_1, \xi_2, ..., \xi_k, \infty, ..., \infty>$, where ξ_i's are the roots of $a_k X^k + ... + a_1 X + a_0$ is well-defined and bijective, hence a homeomorphism.

In fact it is a well known fact that $Sym^n(\mathbb{C}P^1)$ is also a Kähler manifold and is thus biholomorphic to $\mathbb{C}P^n$ by the following thereom, [9], [21].

Theorem F.0.1 (Hirzebruch, Kodaira, Yau)**.** *If a Kähler manifold M is homeomorphic to $\mathbb{C}P^n$ then M is biholomorphic to it.*

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