On Gauge-String Dualities and String Amplitudes

A thesis submitted to the

Tata Institute of Fundamental Research
Mumbai, India - 400005

for the degree of

Doctor of Philosophy
in Physics

by

Pronobesh Maity

Advisor: Rajesh Gopakumar

International Centre for Theoretical Sciences
Tata Institute of Fundamental Research
Bengaluru, India - 560089

(Final Version Submitted in October, 2023)

DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Prof. Rajesh Gopakumar, at the International Centre for Theoretical Sciences -Tata Institute of Fundamental Research, Bengaluru.

Pronobesh Maity

Pronobesh Maity

Date: October 27, 2023

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Rajesh Gopakumar

Rajesh Gopakumar

Date: October 27, 2023

Dedicated to

the song "Dernière Danse" which resonates with my PhD life in so many ways!

Acknowledgements

First and foremost, I would like to thank my advisor Rajesh Gopakumar for his guidance, care and encouragement during my Ph.D. years. His magical style of doing physics has had a profound impact on me. Despite getting involved in intense administrative work, his support and shared time were always highly beneficial for me.

I thank R. Loganayagam for teaching me the art of doing research through the research methodology course, and for emphasizing the supremacy of independent scientific voice, even (sometimes) against the authorities. I thank Shota Komatsu, for sincerely guiding me for a large portion of my PhD and clarifying countless misconceptions I had. I thank David Gross, for giving me the opportunity to work closely with him during my stay at KITP. It was very special to work with Ashoke Sen, and he remains to me a rare model of being both a legend and a humblest persona, the ideals to which one can only aspire.

I thank all my collaborators for teaching me so many topics and making physics a fun project.

I thank all my friends (alphabetically in surnames): Faruk Abdulla, Nirnoy Basak, Soummyadip Basak, Faizan Bhat, Tuneer Chakraborty, Joydeep Chakravarty, Rajarshi Chattopadhyay, Chandramouli Chowdhury, Sarthak Duary, Victor Godet, Ashik H, Vijay Kumar, Basudeb Mondal, Godwin Martin, Amiya Mishra, Sugan Durai Murugan, Aswin P M, Srikanth Pai, Priyadarshi Paul, Somnath Porey, Shivam Sharma, Pabitra Ray, Tamoghna Ray, Baishali Roy, Shashank Kumar Roy, Sandip Sahoo, Saikat Santra, Debmalya Sarkar, Alan Sherry, Omkar Shetye, and Prashant Singh, for sharing me with quality discussions on multiple academic and non-academic topics. I hardly would believe in friendship unless Alankar Dutta, Shreya Kumbhakar, and Samudra Sur showed me what it is, during my initial PhD years. I also thank Faizan Bhat and Sarthak Duary for listening to me when I had no success, and the world appeared to get ripped off from me!

If I have any analytical ability, that was deeply modeled by my math tutor Barun Kumar Das when I was a school kid, and I want to thank him for this.

I especially thank my parents (*Baba-Ma*) for their attempts to protect me from anything vile, and for always encouraging me to higher studies, and my elder sister (*Didi*) for sincerely developing moral values in me when I was a mere kid.

Finally, I must thank myself for enduring so much pain and suffering throughout these years and still surpassing all of them in the dream of writing a thesis acknowledgment line like this!

Abstract

String theory has been a powerful enterprise in theoretical physics, which, apart from proposing a theory of everything, provides us a handful of dualities. One of the most surprising of them is the Gauge-string duality. This goes back to 't Hooft's original idea of associating two-dimensional surfaces (similar to worldsheet of a string) to the large N expansion of a gauge theory, which then has been concretized by Maldacena's discovery of the famous AdS/CFT correspondence. But questions remain unanswered on the underlying mechanism and generality (e.g applicability to non-supersymmetric gauge theory) of this correspondence.

In the first part of this thesis, we will take some initial steps towards these aspects, in particular, we will demonstrate an explicit realization of the underlying mechanism of gauge-string duality for tensionless limit of AdS_3/CFT_2 and (a particular sector of) AdS_5/CFT_4 correspondence. We will then move on to put forward a string theory dual to large N (non-supersymmetric) pure two-dimensional Yang Mills theory.

In the second part, we will probe into independent questions of consistency of string theory itself. In particular, we will comment on the positivity of the coefficients of string amplitudes when expanded into the partial waves at the massive poles in $D = 4$. This positivity is demanded by unitarity, which is a fundamental criteria for physical consistency of any theory. We will comment on the similar positivity of the Coon amplitude in $D = 4$, the latter being an amplitude interpolating between a string amplitude and a scalar particle amplitude.

Contents

Chapter 1

Introduction

String theory has been a powerful enterprise in the realm of theoretical physics for the last fifty years. Apart from proposing a 'theory of everything' (TOE), which claims strings as fundamental building blocks of the universe, it also gave us a handful of new dualities. One of the most surprising dualities of this kind is the one between Gauge fields and Strings. Unlike the TOE approach which supports the standard reductionist attitude of scientific endeavour, in the Gauge-String duality, string emerges as a natural geometry from a large N gauge theory.

So it is natural and important to ask

- whether string theory is a consistent theoretical framework,
- what are the deeper implications and rationale behind the Gauge-String duality?

On Gauge-String dualities

We will begin with the second question first. This goes back to 't Hooft's original idea [1] of associating two-dimensional surfaces (similar to worldsheet of a string) to the large N expansion of a gauge theory (N being the rank of the gauge group of the theory):

$$\log \mathcal{Z}(N, g_{\mathrm{YM}}) = \sum_{g=0}^{\infty} N^{2-2g}\, F_g(g_{\mathrm{YM}}^2 N)$$

where $\mathcal{Z}(N, g_{\mathrm{YM}})$ is the partition function of the gauge theory, g_{YM} is the coupling const., and g denotes the minimum genus of the compact Riemann surface on which we can draw the loop diagrams without any crossing. Thus, in the 't Hooft limit of $N \to \infty$ and fixed $\lambda = g_{\mathrm{YM}}^2 N$, the $1/N$ expansion of gauge-invariant observables (e.g $\log \mathcal{Z}(N, g_{\mathrm{YM}})$) of the gauge theory can be seen as a topological expansion in terms of topology of the corresponding Feynman diagrams.

This idea then has been concretized by Maldacena's discovery [2], through the physics of D-branes, of the famous AdS/CFT correspondence. The AdS/CFT correspondence claims a non-perturbative duality between certain string theories living in the bulk of certain AdS geometry and a specific large N gauge theory living at boundary of it. It provided us a precise dictionary [3, 4] between observables of a set of quantum field theories to those of a dual string theory, while the latter admits a weakly coupled gravity description.

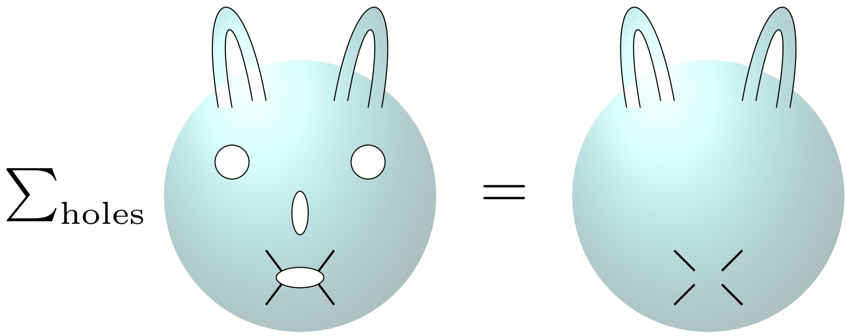

Figure 1.1: Open-closed string duality: we sum over all the holes in an open string configuration, backreaction of D-branes alters the background, and we get a closed string theory.

But even after two decades, two aspects of such gauge-string dualities remain mysterious:

1. What is the underlying mechanism of Gauge-String dualities? In particular, how does a (large N) gauge theory reorganise itself into a string theory?
2. How generic are such dualities? Does it hold for non-supersymmetric gauge theories as well?

In the first part of this thesis, we take some initial steps towards answering these questions. We elaborate on these in the following:

$\boxed{1}$ Mechanism of the Gauge-String Duality

The D-brane physics of Maldacena's correspondence indicates the Open-Closed string duality as the underlying reason of the Gauge-String duality, where we sum over all the holes of a open string configuration, and backreaction of D-branes alters the background making it a closed string theory, see figure 1.1. But this is difficult to be seen explicitly in the limit of large 't Hooft coupling $\lambda = g_s N$, where we have effectively infinite number of holes to sum over. Instead we could focus on the opposite regime of small λ, which corresponds to weakly coupled field theories, and by the AdS/CFT dictionary, it describes dual tensionless string theory on the bulk (i.e strings in a highly curved bulk background). In this regime, we have only finite number of the holes to sum over and we still posses the usual genus expansion (or 1/N expansion of the gauge theory) as indicated in the vertical axis indexed with $g_s \sim \frac{\lambda}{N}$ along the lamp-post in figure 1.2.

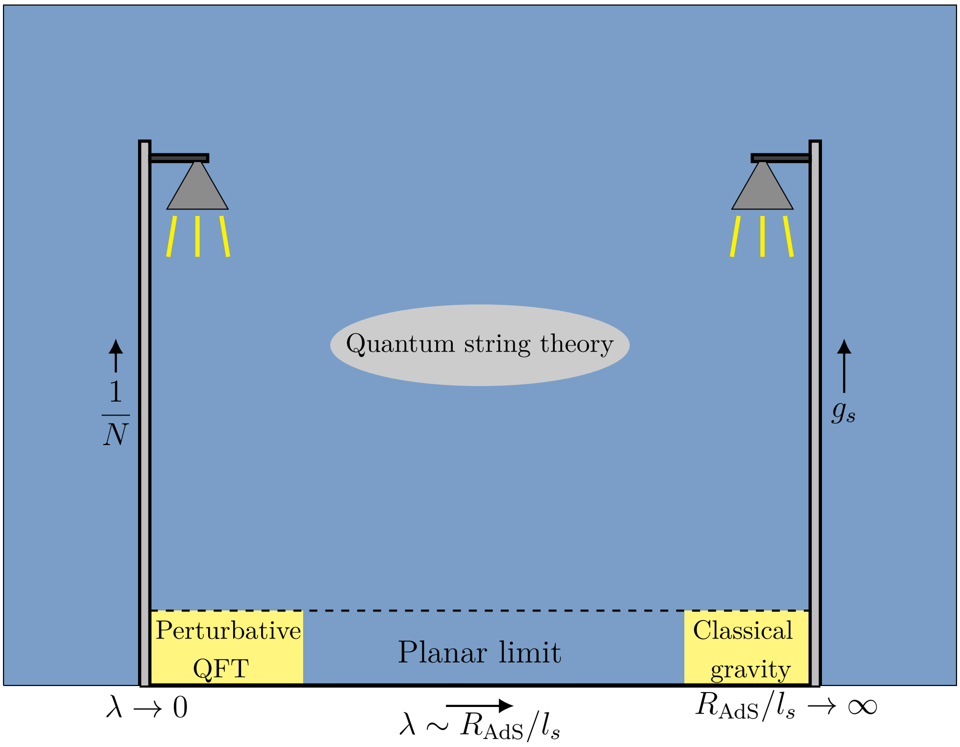

Figure 1.2: The lamp post figure describing parameter space of the Gauge-String duality. AdS/CFT dictionary (for the case of $\mathcal{N}=4$ super Yang-Mills) implies: $N = \left(\frac{R_{\text{AdS}}}{l_P}\right)^4$, $g_s = g_{YM}^2$, and $\lambda = \left(\frac{R_{\text{AdS}}}{l_s}\right)^4$, where N: rank of the gauge group of the gauge theory at the boundary of AdS, g_s: string coupling, g_{YM}: coupling constant of the gauge theory, R: radius of AdS, l_P: planck length. Conventionally AdS/CFT is studied in the regime $\lambda \to \infty$ under the second lamp post, which describes weakly coupled supergravity background, and correspondingly strongly coupled theory on the boundary. Instead we will focus on the other extreme with $\lambda \to 0$, where boundary field theory is well understood as an expansion in the perturbative Feynman graphs, and bulk corresponds to a string theory in highly curved background (i.e in the tensionless limit). We will demonstrate an explicit mechanism of the duality in this regime. We will always stay in the classical regime $N \to \infty$, or equivalently $g_s \to 0$.

But in general, both sides (boundary and bulk) of the AdS/CFT correspondence are not decipherable simultaneously in the proposed examples of Maldacena, making it harder to portray the mechanism behind it. A possible exception is for AdS_3/CFT_2 correspondence, where the bulk worldsheet description has been developed [5–7] a long time ago and recently the exact boundary dual of the superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ with $k = 1$ unit of NS-NS flux (i.e in the tensionless limit of the bulk string theory) has been shown to be the free symmetric product orbifold CFT [8–10]. Sometime later, the worldsheet description of the string theory dual to free $\mathcal{N} = 4$ super Yang-Mills theory has also been proposed in [84].

These tensionless duality examples therefore offer a unique and rare opportunity to decipher into the underlying mechanism of Gauge-String duality, and indeed from the worldsheet perspective, there have been striking localization of the string path integral on special points of the string moduli space [10, 91] which admit a direct correspondence with the boundary theory computation of gauge-invariant correlators. This could be seen as a demonstration of how bulk stringy physics captures the boundary field theory computation. It is then natural to turn round and ask for the workings behind fields (boundary) to the strings (bulk).

Some time ago, a prescription [13] (which was further refined in [14] for relatively simpler gaussian matrix models) was put forward by Gopakumar on how the distinct worldline trajectories of a Feynman graph that contributes to the free correlator of a field theory, could be reorganised into a sum over distinct worldsheets which are glued up versions of the original double line graphs. This was then described as the underlying mechanism of such a field-string correspondence. In chapter 2 and 3, we will see explicit realisations of this proposed prescription in the tensionless limit of the AdS_3/CFT_2 and some sector of AdS_5/CFT_4 correspondence. We outline the key findings in the relevant chapters in the following;

Chapter 2 : From Symmetric Product CFTs to AdS_3

Symmetric product orbifold CFTs are special kinds of CFTs whose correlators are given by a finite sum of admissible branched covers of the two-dimensional target space-time. Determining these covering maps is in general a hard problem. In chapter 2, we will take a large twist (which corresponds to large dimension) limit of all operators involved in the correlator of this CFT and show that branched covers are then given in terms of the spectral curve of a Penner-like matrix model, which rather surprisingly, turns out to be directly related to the Strebel differential on the covering space. Using a mathematical result on the Strebel differential, this then enables to write the spacetime CFT correlator as an integral over the entire worldsheet moduli space weighted by a Nambu-Goto-like action. This provides an explicit realisation of the underlying mechanism of Gauge-String

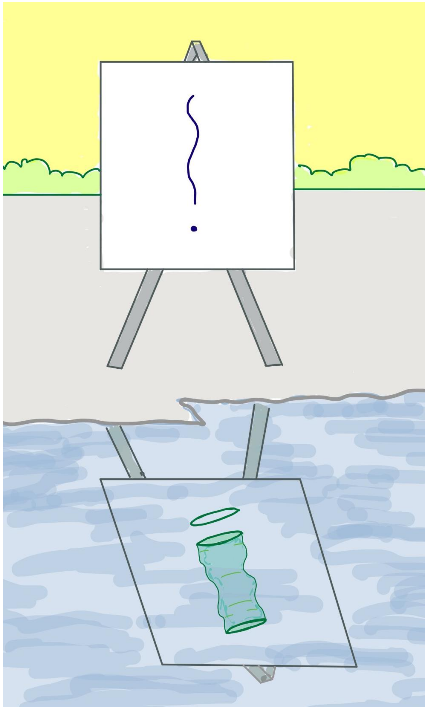

Figure 1.3: An artist's impression of the Gauge-String duality

duality in this set-up of the AdS/CFT correspondence. A flowchart is given in figure 1.4 with general logic of how a field theory correlator map onto a worldsheet integral over string moduli space. This chapter is based on the work [15].

Chapter 3 : Twistor Coverings and Feynman Diagrams

A worldsheet dual to free $\mathcal{N} = 4$ Super Yang-Mills has been proposed in [84] in terms of twistor variables for AdS_5. As we discussed for the case of AdS_3 dual of symmetric orbifold CFT, holomorphic covering maps play a central role in determining correlators associated to Feynman diagrams. In chapter 3, we will recast these maps in terms of the worldsheet twistor variables for AdS_3, which we will then generalise for AdS_5. We will propose stringy incidence relations and appropriate reality conditions for the twistor covering maps. We will then exhibit an explicit construction of such covering map for a special kinematic configuration of correlator. Closed string worldsheet corresponding to this map is related to a gauge theory Feynman diagram by the Strebel construction, similar to the AdS_3/CFT_2 case. Strikingly the regularised Strebel area of the worldsheet reproduces the Feynman propagator of the free field theory. This chapter is based on the work [16].

$\boxed{2}$ On a String dual of (non-supersymmetric) two-dimensional Yang Mills

Next we concentrate on the other question of generality of the correspondence:

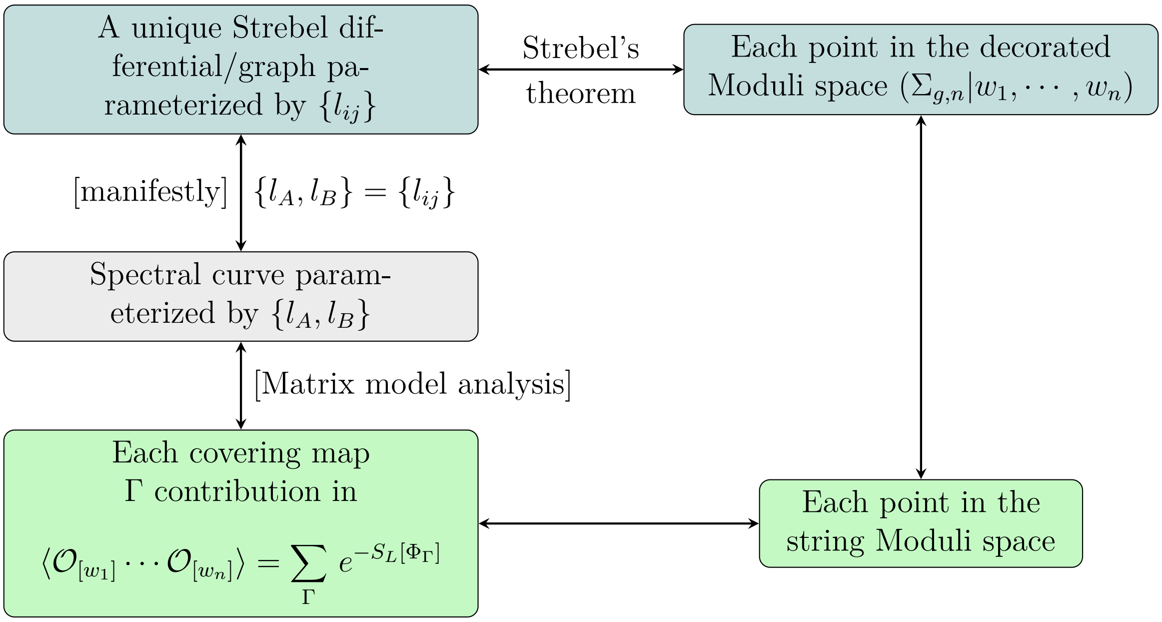

Figure 1.4: Flowchart explaining the general logic of how a field theory correlator maps onto a worldsheet integral over string moduli space. Here $\mathcal{O}_{[w]}$ is a twist field of symmetric product orbifold CFT, $\{l_A, l_B\}$ corresponds to the A and B-cycle periods of the spectral curve, and $\{l_{ij}\}$ are lengths between zeros of the Strebel differential measured with the Strebel metric.

Despite giving a strong evidence of interpreting large N gauge theory as a string theory, AdS/CFT correspondence would truly be efficient for the explanation of natural phenomena if we can enlarge it's applicability further and include non-supersymmetric gauge theory for a potential string dual. Indeed non-supersymmetric four dimensional Quantum Chromodynamics is what describes the physics of hardrons, and finding the string dual of it is an important open problem in theoretical physics. It could help us provide an analytical understanding of the infrared dynamics of the theory, where we lack sufficient analytical control and mainly rely on lattice gauge theory computations based on Monte Carlo methods. One toy model for this is two-dimensional Yang-Mills (2d YM) theory, which is exactly solvable on an arbitrary Riemann surface. Gross and Taylor, in their seminal paper [17], interpreted the partition function of 2d YM in terms of branched coverings from an auxiliary surface to the 2d target spacetime. Yet the explicit worldsheet action was still unknown. Attempts on this include [50, 82, 129]. We will take this problem seriously in the chapter 4:

Chapter 4 : String Dual of Two-dimensional Yang-Mills

In chapter 4, we will propose a bosonic worldsheet model dual to the chiral sector of pure 2d YM theory in the large N limit. In particular, we will reproduce the torus partition function of large N 2d YM from the worldsheet, and also match the three point amplitude of winding states in 2d YM with a certain three-point string amplitude. Our worldsheet theory consists of the standard $\beta - \gamma$ system deformed by an exact chiral version of the familiar Polchinski-Strominger term. This chapter is based on the upcoming work [18].

On String Amplitudes

In the second part of the thesis, we probe independent questions of consistency of string theory itself. In particular, unitarity which is a fundamental criteria for physical consistency of any theory, demands that under critical space-time dimensions (for bosonic string, it is 26), the residues of tree-level string amplitudes $\mathcal{M}(s,t)$ at massive poles ($s = n$) must have non-negative coefficients when expanded in terms of Gegenbauer polynomials $C_l^{(\alpha)}(x)$:

$$-\mathrm{Res}_{s=n}\mathcal{M}(s,t) = \sum_{l=0}^{n+1} a_{n,l}\, C_l^{(\alpha)}\left(1 + \frac{2t}{n+4}\right).$$

then

$$a_{n,l} \geq 0, \quad \text{for} \quad D \leq 26$$

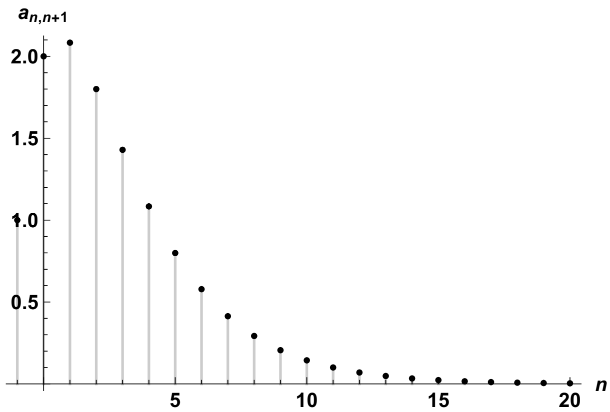

Figure 1.5: Numerical plot of the coefficients on the leading Regge trajectory $a_{n,n+1}$ of the Veneziano amplitudes in $D = 4$. Our task is to analytically show $a_{n,n+1} \geq 0 \; \forall \; n \geq -1$.

Unitarity is of course implied in String theory, rather indirectly, from the famous no-ghost theorem [19], but it seems a curious question whether and how the above positivity condition holds at the level of scattering amplitudes.

In chapter 5, we demonstrate this positivity for poles with maximum spin at a given mass (see fig. 1.5), and then with a numerical evidence, we will argue this hints positivity of the famous Veneziano amplitude in $D = 4$. Subsequently in chapter 6, we will show positivity of the Coon amplitude for D = 4, the latter being an amplitude interpolating between the Veneziano amplitude and a scalar particle amplitude. These are based on the works [20, 21].

Chapter 2

From Symmetric Product CFTs to AdS_3

How exactly do quantum field theories reassemble themselves into string theories (or M-theory generalisations) in the large N limit? This question has been with us ever since 't Hooft showed that Feynman diagrams of large N gauge theories admit a genus expansion [1] which hinted at a dual string theory description. Maldacena's discovery [2], through the physics of D-branes, of the string dual for a large class of supersymmetric gauge theories gave this question fresh impetus. It provided us with a set of quantum field theories where one had a precise dictionary [3, 4] between observables, to those of a dual string theory, with the latter often admitting a weakly coupled gravity description.

More specifically, the duality gives rise to a map between (single trace) gauge invariant operators, and vertex operators for perturbative string states (w and h label the state, while z is a worldsheet coordinate)

$$\mathcal{O}_h^{(w)}(x) \quad \longleftrightarrow \quad \mathcal{V}_h^w(x;z) \ . \tag{2.1}$$

The relation then between n-point (Euclidean) correlators is,

$$\left\langle \mathcal{O}_{h_1}^{(w_1)}(x_1)\mathcal{O}_{h_2}^{(w_2)}(x_2)\dots\mathcal{O}_{h_n}^{(w_n)}(x_n)\right\rangle_{\mathrm{S}^d}\Big|_g = \int_{\mathcal{M}_{g,n}} d\mu \left\langle \mathcal{V}_{h_1}^{w_1}(x_1;z_1)\mathcal{V}_{h_2}^{w_2}(x_2;z_2)\dots\mathcal{V}_{h_n}^{w_n}(x_n;z_n)\right\rangle_{\Sigma_{g,n}} \ , \tag{2.2}$$

where g is the genus of the world-sheet, while on the CFT side (i.e. the LHS) it captures a certain contribution in the $\frac{1}{N}$ expansion. Both sides of this equality are autonomously defined, and it should be possible to decipher the mechanism behind this remarkable correspondence. However, up to now this question has remained unanswered in any precise sense — the miraculous nature of the equality between the LHS and RHS of (2.2) has remained so even after two decades.

Some time ago, a proposal was made for what this underlying mechanism might be [13, 22, 23], in an expansion around the free field point on the LHS. This weak coupling limit of the quantum field theory translates into a tensionless (or high curvature) limit of the dual string theory, and this is also the regime of the Feynman diagram expansion which underlies the original 't Hooft analysis. The basic idea behind the proposal of [13, 22, 23] was to reorganise the sum over distinct worldline trajectories that Feynman diagrams represent, into a sum over distinct world-sheets which are glued up versions of the original double line graphs. In other words, it is a prescriptive procedure for how to go from the LHS to the RHS of (2.2). Note that this is a reverse engineering problem in that one aims to reconstruct an integrand on moduli space, and there is *a priori* no unique way of doing so. That does not, however, obviate the possibility of a canonical or natural way of going between the two sides of (2.2), and indeed the proposal of [13, 22, 23] gives rise to a particular integrand on the RHS.

The first step in this reorganisation was to group together genus g Feynman diagrams in the free theory that contribute to a gauge invariant n-point correlator — the LHS of (2.2) — into 'skeleton graphs' [23]. This essentially involved gluing together homotopically equivalent edges of the Feynman graph. These skeleton graphs capture, in a sense, the inequivalent topologies of the worldlines. It was then seen that these skeleton graphs are precisely in correspondence with a simplicial decomposition of the (decorated) moduli space $\mathcal{M}_{g,n} \times \mathbb{R}^n_+$. This can be thought of as a refinement of 't Hooft's association of a genus to Feynman diagrams. The next step [13] was to make a one-to-one correspondence between individual diagrams and individual world-sheets, i.e. the points on the moduli space on the RHS of (2.2). This exploited the mathematics of Strebel differentials which underlies the above cell decomposition of moduli space.

Strebel differentials $\phi_S(z)dz^2$ are meromorphic quadratic differentials on a Riemann surface $\Sigma_{g,n}$ with double poles at the n punctures. Moreover, the $(6g-6+3n)$ independent Strebel lengths between any two of the $(4g+2n-4)$ (generically distinct) zeroes are always real,

$$\int_{a_k}^{a_m} \sqrt{\phi_S(z)} = l_{km} \in \mathbb{R}_+ \ . \tag{2.3}$$

A theorem due to Strebel tells us that there is a unique such Strebel differential for every point on $\mathcal{M}_{g,n}$ if one fixes the n residues at the poles (which are necessarily real since they are linear combinations of the l_{km}). This is a bijective correspondence, i.e. if one specifies the $(6g-6+3n)$ Strebel lengths then one lands on a unique point in $\mathcal{M}_{g,n} \times \mathbb{R}^n_+$ and vice versa. Thus, if we fix the residues, the remaining $(6g-6+2n)$ independent real Strebel lengths l_{km} can be viewed as a particular parametrisation of the string moduli space $\mathcal{M}_{g,n}$. Furthermore, there is a so-called critical graph associated with the Strebel differential which has n faces (with the topology of a disk), each of which contains exactly one double pole of the Strebel differential; in addition it has $(4g+2n-4)$ vertices which

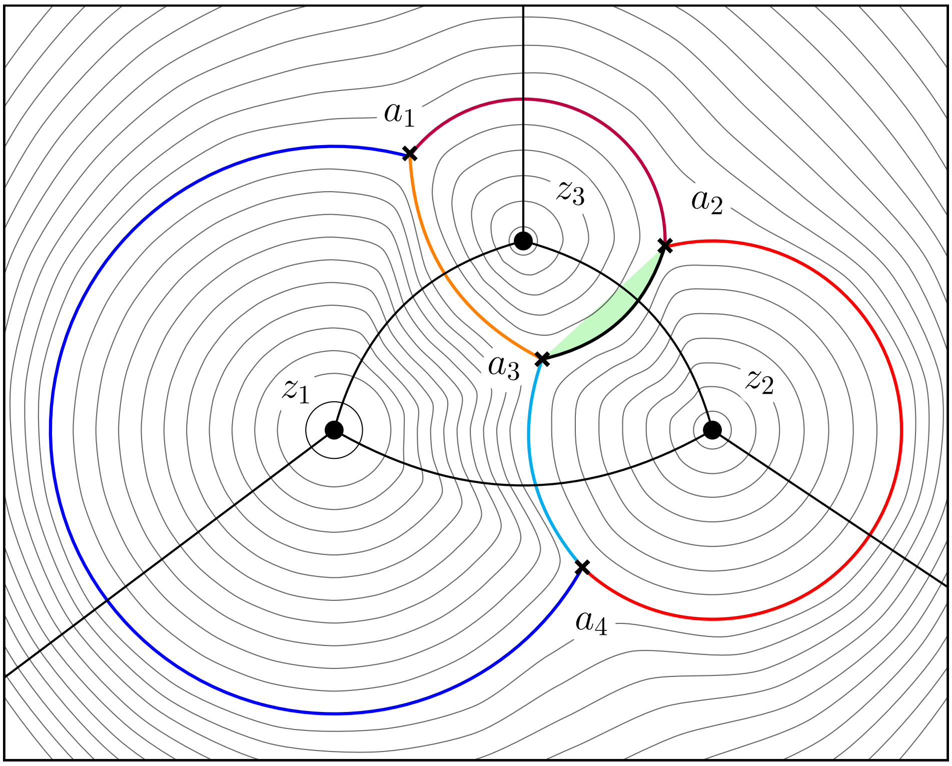

Figure 2.1: The grey lines are the horizontal trajectories of a Strebel differential, see eq. (2.59), while the coloured lines describe the critical horizontal trajectories that make up the critical Strebel graph, see Section 2.5 for more details. The (double) poles z_i of the Strebel differential are denoted by black dots ($z_4 = \infty$), while the zeros a_i are represented by black crosses, see eq. (2.5). Finally, the solid black lines between the poles describe the dual edges to the critical Strebel graph, and therefore correspond to the edges of the skeleton graph of the field theory.

are the zeroes of the Strebel differential, and $(6g + 3n - 6)$ edges which connect the vertices, see Fig 2.1. The idea of [13] was then that this critical graph is to be identified with the dual graph to the skeleton diagram of the field theory; the skeleton graph itself is then represented by the solid black lines in Fig 2.1. In fact, this construction can be seen as a way of implementing open-closed string duality, with the open string ribbon graphs of the field theory being glued along the critical Strebel graph to form the closed string worldsheet [13].

In [13], an additional identification was proposed between the (inverse) Schwinger proper times of the Feynman diagrams and the Strebel lengths (2.3). This was a concrete way to promote the field theory answer into a dual world-sheet correlator; in particular it leads to a specific candidate integrand on moduli space — the RHS of (2.2). However, it was pointed out in [24] that this prescription had the disadvantage of not manifestly preserving the global spacetime special conformal symmetry of the putative world-sheet correlators. An alternative prescription was put forward by Razamat [14] which associated the number n_{ij} of homotopic Wick contractions between a pair of vertices, to the corresponding Strebel lengths (2.3) of the dual edges. This discrete prescription was particularly well suited for the zero-dimensional Gaussian matrix model (where the correlators do not carry any spacetime dependence), and was explored in [14, 25–28].

To summarise, the broad thrust of [13, 14, 22, 23] was a definite prescription[1] by which the mechanism of open-closed string duality is realised. It associates to individual Feynman diagrams of the field theory, specific points in the moduli space of the dual closed string theory, thus giving a constructive method to go from the LHS to the RHS in (2.2).

In this chapter, we will check this proposal in an example of the AdS/CFT correspondence which can serve as a concrete testbed for understanding the precise working of the duality and its dictionary. It has recently been understood that string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ with one unit ($k = 1$) of NS-NS flux is dual to the free symmetric orbifold CFT $\mathrm{Sym}^K(\mathbb{T}^4)$ in the large K limit. The full perturbative string spectrum exactly agrees with that of the symmetric orbifold [9] (earlier work on the $k = 1$ theory includes [8, 94–97]). Furthermore, it was shown in [10] how correlators in the world-sheet string theory localise on moduli space to special points which admit a covering map of the AdS_3 boundary S^2, thereby manifestly reproducing the symmetric orbifold correlators that can be calculated using such a covering map approach [39, 40]. This was generalised beyond genus zero on the world-sheet in [90], and to geometries with other boundaries in [42]. Recently, in [91] the crucial property of world-sheet localisation was shown to follow from a twistorial incidence relation in a free field realisation of the $k = 1$ world-sheet sigma model. In this example we therefore understand in detail how the RHS of (2.2) gives rise to the LHS.

This example therefore allows us to analyse whether the reconstruction proposal of [13, 22, 23] will indeed reproduce the correct world-sheet theory.[2] To this end, we will start with an n-point twisted sector correlator in the orbifold CFT — i.e. the LHS of (2.2) —

$$\langle \sigma_{w_1}(x_1) \cdots \sigma_{w_n}(x_n) \rangle \ , \tag{2.4}$$

and try to rewrite it as a world-sheet integral. The key idea is to study (2.4) in a special Gross-Mende like limit [46], where the twist[3] labels w_i of the operators are taken large (but small compared to K). The nice feature of this limit is that the covering maps which contribute to the correlator actually become dense on the moduli space of the covering surface. If we concentrate on the case where the covering surface (which will be identified with the world-sheet) is of genus zero, the computation of correlators via the covering map [39, 40] simplifies significantly. In fact, we will be able to map the problem of finding

[1]This prescription was generalised to theories with fundamental matter in [29] and also applied to a number of different correlators, see [24, 30–34] for a partial list.

[2]It would be interesting to see how to generalise the approach here to theories whose bulk duals are not string theories, see recent work on reconstructing the bulk and thus deriving the AdS/CFT correspondence for vector-like large N models in [44, 45] and references therein.

[3]This is not to be confused with $\tau = \Delta - J$. In our context 'twist' refers to the twisted sectors of the symmetric orbifold.

the different covering maps to that of solving for the large N limit[4] of a special class of matrix models with Penner-like logarithmic potentials.

The inequivalent covering maps for an n-point correlator will turn out to be parametrised by a set of $(2n-6)$ real parameters. These parameters determine the so-called spectral curve of the matrix model at leading order in $\frac{1}{N}$. This defines a meromorphic differential $y_0(z)dz$ whose square has double poles at the points z_i on the covering space which correspond to the pre-images of the branch points x_i in (2.4), see eq. (2.57)

$$y_0^2(z)dz^2 = \frac{\alpha_n^2\, dz^2}{\prod_{i=1}^n (z-z_i)^2} \prod_{k=1}^{2n-4} (z-a_k) = -4\pi^2 \phi_S(z)\, dz^2 \ . \tag{2.5}$$

The $(2n-6)$ real parameters that specify the solution also determine the zeroes and poles of this differential, and are nothing other than the period integrals of $y_0(z)dz$ along inequivalent homology cycles on the covering surface.

The upshot of this analysis is therefore, quite remarkably, that the solution (2.5) to the matrix model which determines the covering maps is (minus) a Strebel differential $\phi_S(z)dz^2$ on the covering space — it naturally comes baked into the problem! In the process we will also see that there is a one-to-one mapping between these solutions of the matrix model, and the Feynman diagrams of the orbifold theory that can be associated to every admissible covering map [107]; this is exactly as envisaged in the proposal of [13, 23]. The Strebel lengths (2.3), which are proportional to the periods, take arbitrary real values which are equal to $\frac{n_{ij}}{2N}$ to leading order in $\frac{1}{N}$, where n_{ij} is the number of lines between vertices. This is similar to what was proposed in [14], although it will also be clear that such a simple relation will not hold when one takes into account subleading orders in $\frac{1}{N}$. Thus at large N the sum over admissible covering maps goes over to an integral, as the set of allowed points on moduli space becomes dense in $\mathcal{M}_{0,n}$.

Having seen how the moduli space of world-sheets on the RHS of (2.2) emerges naturally from the different Feynman diagram contributions, one can then address the second aspect of the program of [13]. Namely, to determine the integrand on moduli space on the RHS of (2.2) from the field theory. This can, again, be read off from the symmetric orbifold calculation of Lunin-Mathur, see eq. (2.12). In the large N limit, the dominant contribution turns out to be the Liouville action (2.11), and this weighting factor has multiple fascinating interpretations. On the one hand, it is proportional to a Nambu-Goto action for the Strebel metric — which is the natural metric on the world-sheet one can associate to the quadratic differential $\phi_S(z)dz^2$, see eqs. (2.67) and (2.68). Another striking form for this action arises from the observation that, to leading order in

[4]This N should not be confused with the parameter of the 't Hooft genus expansion of the orbifold theory — that was denoted above by K, the rank of the symmetric group, and will be taken to ∞ as we will always be at genus zero. N is more like a charge, proportional to the large twists w_i, see eq. (2.15), which is separately taken to be large.

$\frac{1}{N}$, the Strebel differential in (2.5) is nothing other than the Schwarzian of the covering map. The action is therefore also proportional to the absolute value of the Schwarzian, see eq. (2.71), and this leads to a suggestive AdS_3 generalisation of the universal AdS_2 expression for the spacetime action [47]. Finally, there is a third interpretation of the action from the spacetime S^2 point of view which is very reminiscent of the action for a rigid string that had been proposed in pre-holographic days to describe large N gauge theories [48–50], see eq. (2.70).

In [10] the above Liouville action on the symmetric orbifold was shown to agree with the classical action of the AdS_3 string theory, evaluated on the covering maps onto which the world-sheet path integral localises. This therefore essentially completes the circle by showing how the the broad procedure of [13, 14, 23] applied to free field orbifold CFT correlators can reconstruct not only the string moduli space in a precise way, but also what the world-sheet correlators are, at least in the large twist limit. It is a proof of concept that it is possible to go in both directions between the two sides of (2.2).

The chapter is organised as follows. After a brief recap of the covering space approach to computing symmetric orbifold correlators in Section 2.1, we set up the problem of finding the relevant covering maps in the large twist limit in Section 2.2. Section 2.3 is the technical heart of the chapter where the equations determining the covering map are mapped onto the saddle point equations of a Penner-like matrix model. We use conventional matrix model technology to find the saddle point solutions, but then interpret the parameters in a slightly different way than for usual matrix models. We also make some comments on the finite N generalisation in Section 2.3.3. With these results in hand we are then ready to understand their consequences: in Section 2.4 we use the connection between covering maps and Feynman diagrams to see that the free parameters that enter into the matrix model solution are proportional to the number of Wick contractions, to leading order in $\frac{1}{N}$. Further, in Section 2.5 we see that (the square of) the spectral curve of the matrix model is (minus) the Strebel differential, again to leading order in $\frac{1}{N}$, and hence that the sum over covering maps goes over to an integral over the string moduli space. We comment on the relation of the Strebel differential to the Schwarzian, at finite N, in Section 2.5.1. Section 2.6 then goes on to reconstruct the weights associated with each point in moduli space, and briefly discusses their different avatars. Finally, Section 2.7 contains a somewhat extended list of open questions, categorised by theme. There are a couple of appendices with some additional technical sidelights.

2.1 Correlators in the symmetric orbifold CFT

In this section we give a brief review about how correlation functions of symmetric orbifold theories can be computed in terms of covering maps; this approach goes back to the work of Lunin & Mathur [39, 40]. In the following we shall mainly concentrate on the twisted sector ground states, although the method is also believed to be applicable to descendant states, see eq. (2.12) below.

Let us start by recapitulating some basic features of symmetric orbifold theories. Let $\mathcal{S}$ be a conformal field theory — the case we primarily have in mind is that $\mathcal{S}$ is the superconformal field theory associated to $\mathbb{T}^4$, but this will not matter for the following — then the symmetric orbifold theory of $\mathcal{S}$ is obtained by considering the K-fold tensor product theory $\mathcal{S}^{\otimes K}$ and taking the orbifold by S_K, where S_K permutes the K copies. As is familiar from general orbifold constructions, the orbifold theory consists of the so-called untwisted sector — these are the states in $\mathcal{S}^{\otimes K}$ that are invariant under the action of all generators in S_K — as well as twisted sectors. These twisted sectors are in one-to-one correspondence with the conjugacy classes of S_K. Every permutation in S_K can be written in cycle decomposition, and the cycle shape is invariant under conjugation; thus the conjugacy classes of S_K are in one-to-one correspondence with the cycle shapes, and thus with partitions of K. In the following we shall exclusively consider the 'single cycle' conjugacy classes, i.e. those permutations that are conjugate to a single cyclic permutation, say of length w. Note that under the AdS/CFT correspondence the states from these single cycle twisted sectors correspond to single string states in AdS. We will always be working in the large K limit so that there is no restriction on the range of w; this corresponds to the limit in which the string coupling constant g_s of the dual string theory is small.

Let σ_w be the ground state of the single cycle twisted sector of length w, and let π_w be a representative of the corresponding conjugacy class of permutations, i.e. $\pi_w = (j_1 \cdots j_w)$, where $j_m \in \{1, \ldots, K\}$ and $j_m \neq j_n$ for $m \neq n$. The twisted sector state σ_w has the property that as we analytically continue a field ϕ_l from the l'th copy of $\mathcal{S}^{\otimes K}$ around σ_w, it gets transmuted into the field $\phi_{\pi_w(l)}$ associated to the $\pi_w(l)$'th copy,

$$V(\phi_l, e^{2\pi i} z)\, \sigma_w(0) = V(\phi_{\pi_w(l)}, z)\, \sigma_w(0) \ . \tag{2.6}$$

The main quantity we want to calculate are correlation functions of such twisted sector ground states σ_{w_i},

$$\langle \sigma_{w_1}(x_1) \cdots \sigma_{w_n}(x_n) \rangle \ , \tag{2.7}$$

where x_i are coordinates on the sphere S^2. Such a correlation function is only non-zero if there exist representatives π_{w_i} — recall that the twisted sectors are associated to conjugacy classes, so there are many different w-cycle permutations $\pi_w = (j_1 \cdots j_w)$ — such that

$$\pi_{w_1} \cdots \pi_{w_n} = \mathrm{id} \ . \tag{2.8}$$

Let us assume this to be the case, and let us take σ_{w_i} to be the twisted sector field associated to this representative permutation π_{w_i}. (Note that for the calculation of the actual correlation function one also needs to include combinatorial factors, see e.g. [?].)

As is familiar from general CFT considerations, it is often useful to consider not just the correlation function (2.7) itself, but to insert in addition chiral fields ϕ_l from $\mathcal{S}^{\otimes K}$. (If $\mathcal{S}$ is the $\mathbb{T}^4$ theory, ϕ_l could, for example, be one of the 4 free bosons associated to the l'th copy of $\mathcal{S}^{\otimes K}$.) It follows from (2.6) that with respect to these individual fields the correlation function is not single valued since the fields ϕ_l change identity as they go around the different twisted sector insertions.

The basic idea of Lunin & Mathur [39, 40] is to use the conformal symmetry to lift these correlation functions via the holomorphic covering map $\Gamma : \Sigma \to \mathrm{S}^2$ to single-valued functions on the covering surface Σ. Here Σ is another Riemann surface — we shall mainly be interested in the situation where Σ is also a sphere, although in general Σ may have higher genus — and the condition that the correlation functions become single valued on Σ is that near $\Gamma^{-1}(x_i) = z_i$ we have

$$\Gamma(z) = x_i + a_i^\Gamma (z - z_i)^{w_i} + \cdots \quad \text{for } z \sim z_i \qquad (i = 1, \dots, n) \ . \tag{2.9}$$

(Thus the covering surface looks like a w_i-fold multi-storey car park near z_i, and hence the fields ϕ_l become single valued near z_i.) We note that the genus g of Σ is determined by the Riemann-Hurwitz formula in terms of the degree N of the covering map, i.e. the number of preimages of a generic point $x \in \mathbb{CP}^1$, as

$$g = 1 - N + \frac{1}{2} \sum_{j=1}^{n} (w_j - 1) \ . \tag{2.10}$$

From the symmetric orbifold perspective N is the number of active colors, i.e. the number of different j_n appearing in all the permutations π_{w_i} in eq. (2.8).

Returning to the correlator, if the σ_{w_i} are the twisted sector ground states, they disappear entirely from the discussion once one has gone to the covering surface, i.e. they are invisible to the chiral fields ϕ_l. As a consequence, the resulting correlation function on the covering surface is the vacuum correlator, which is therefore equal to unity. Thus the covering map transforms the correlation function (2.7) to a trivial amplitude, and (2.7) is just equal to the conformal factor associated to this covering map transformation. This conformal factor can be calculated using the Liouville action

$$S_{\mathrm{L}}[\Phi] = \frac{c}{48\pi} \int d^2 z \sqrt{g} \Big(2\, \partial\Phi\, \bar{\partial}\Phi + R\, \Phi \Big) \ , \tag{2.11}$$

where c is the central charge of the 'seed theory' $\mathcal{S}$, e.g. if $\mathcal{S}$ is the $\mathbb{T}^4$ or K3 theory $c = 6$, and we have explicitly [39, 40]

$$\langle \mathcal{O}_{w_1}(x_1)\cdots\mathcal{O}_{w_n}(x_n)\rangle = \sum_{\Gamma} W_\Gamma \prod_{i=1}^{n} |a_i^\Gamma|^{-2(h_i-h_i^0)} e^{-S_{\rm L}[\Phi_\Gamma]} \ , \tag{2.12}$$

where

$$\Phi_\Gamma = \log\, \partial_z\Gamma(z) + \log\, \partial_{\bar z}\bar\Gamma(\bar z) \ . \tag{2.13}$$

Here we have written the expression for the more general case where $\mathcal{O}_{w_i}$ is some operator of conformal dimension h_i in the w_i-twisted sector, i.e. not necessarily equal to the ground state σ_{w_i} whose conformal dimension equals $h^0 = c\frac{w^2-1}{24w}$, with c the central charge of $\mathcal{S}$. The parameters a_i^Γ are the coefficients appearing in (2.9), and the W_Γ are expected to be constants independent of Γ [51]. Here the sum is over all possible covering maps — for a given choice of w_i only a finite number of branched coverings exist. (In general the different covering maps will involve covering surfaces Σ of different genus g, though.)

While this is, in principle, a very powerful method for the calculation of these twisted sector correlators, it requires finding the corresponding covering maps. In general, this is a difficult problem, but as we are about to explain, it actually simplifies in the limit in which the w_i become large.

2.2 The large twist limit for branched covers

As we have seen in the previous section, the computation of correlators in the symmetric orbifold theory reduces to finding all the branched coverings of the n-punctured sphere S^2 of the spacetime CFT. We will restrict to the case where the genus in (2.10) equals $g = 0$, i.e. we will consider covering maps

$$\Gamma \ :\ \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \ , \tag{2.14}$$

for which the degree N is given by, see eq. (2.10)

$$N = 1 + \frac{1}{2}\sum_{j=1}^{n}(w_j - 1) \ . \tag{2.15}$$

The number of covering maps up to equivalence, i.e. up to the composition with Möbius transformations, is finite; for example for $n = 4$ and fixing the x_i (but allowing for a Möbius transformation to act on the z_i), it equals [52]

$$\#(\text{branched coverings}) = \text{Min}_j\, w_j(N + 1 - w_j) \ . \tag{2.16}$$

An explicit formula for the general case of $n > 4$ does not seem to be known, but it follows from our analysis in Section 2.4 below that the corresponding number scales as N^{2n-6}

for large w_i, see eq. (2.54). The problem of explicitly determining these branched covers for given branching data $\{z_i, w_i\}$ is difficult, even for a four-point function ($n = 4$). We will show in this section that there is a systematic way in which to compute genus-zero covering maps for connected n-point functions of single-cycle twist fields, if we consider the limit of large twist for all the operators.[5] More specifically, we will always assume that the 'rank' K of the symmetric group S_K is taken to infinity first, before we take the twists $w_i \leq K$ to infinity.[6]

Before we continue let us pause to mention that this limit is the analogue, in the dual AdS_3 space, of a Gross-Mende like limit [46] in that we are looking at the scattering of states with high (spacetime) conformal dimension or energy in AdS_3. (This follows from the fact that $h^0 = c\,\frac{w^2-1}{24w}$ scales as w in the large twist limit.) Our twisted sector ground state operators are the analogue of tachyonic or massless states in flat space. Since we are keeping the positions x_i fixed, we are considering the analogue of fixed angle scattering. (This is to be contrasted with a Regge-type limit, which it would also be interesting to explore.)

It follows from eq. (2.16) (or eq. (2.54) for $n > 4$) that we have a large number of covering maps in this limit, and one of our main results is that the sum over all of these covering maps becomes, in a precise manner, an integral over the moduli space of the covering space (which is an n-punctured sphere with $n-3$ moduli). Furthermore, the weight with which these different covering maps contribute equals the Nambu-Goto action, with the world-sheet metric in Strebel gauge. Alternatively, one may write this action in terms of the modulus of the Schwarzian of the covering map, see Section 2.6 below. We expect this action to have a Gross-Mende saddle point as in flat space.

Let us return to the problem at hand and determine the covering maps that contribute to the correlator (2.7) in the large twist limit. We shall assume that infinity is a generic point of the covering map, i.e. that $x_i \neq \infty$ for $i = 1, \ldots, n$, and we shall denote by λ_a, $a = 1\ldots, N$, the preimages of ∞, i.e. the poles of $\Gamma(z)$. In what follows, it will be convenient to use the Möbius invariance on the covering space Σ to fix $z_1 = 0$, $z_2 = 1$ and $z_n = \infty$. Then the covering map takes the form of a rational function of degree N,

$$\Gamma(z) = \frac{p_N(z)}{q_N(z)} = \frac{p_N(z)}{\prod_{a=1}^{N}(z-\lambda_a)} \ , \tag{2.17}$$

where both $p_N(z)$ and $q_N(z)$ are polynomials of degree N. We have chosen the latter, without loss of generality, to be a monic polynomial with N distinct zeroes corresponding

[5]While there have been many analyses of correlators of heavy operators in AdS/CFT, a Gross-Mende like limit for extremal correaltors in 4d based on the prescription of [13] was attempted in [33]. Recently a very interesting 'large p' limit of half-BPS operators was taken with a view to study the analogue of the Gross-Mende regime(s) in $AdS_5 \times S^5$ [53].

[6]This will ensure that the states we are considering correspond to perturbative string states in the dual theory and not, for example, to D-branes which would backreact on the geometry.

to the poles λ_a of $\Gamma(z)$.

Following [114] we now observe that the poles and zeros of $\partial\Gamma(z)$ are determined as follows.[7] From (2.9) it is clear that the only zeros of $\partial\Gamma(z)$ occur at $z = z_i$ with order $(w_i - 1)$. On the other hand, the only poles appear at $z = \lambda_a$, and they are all double poles. Thus $\partial\Gamma(z)$ necessarily has the form

$$\partial\Gamma(z) = M_\Gamma \frac{\prod_{i=1}^{n-1}(z - z_i)^{w_i - 1}}{\prod_{a=1}^{N}(z - \lambda_a)^2} \ , \tag{2.18}$$

where M_Γ is a non-zero constant. Note that $\partial\Gamma$ then also has the right branching behaviour at infinity since $\partial\Gamma(z) \sim z^{-w_n - 1}$ as $z \to \infty$, see eq. (2.15). It is furthermore clear that the residue of $\partial\Gamma(z)$ at $z = \lambda_a$, i.e. the coefficient of the simple poles $\frac{1}{z-\lambda_a}$, must vanish (since $\Gamma(z)$ does not contain a logarithmic term). This leads to the N "scattering equations" [114][8]

$$\sum_{i=1}^{n-1} \frac{w_i - 1}{\lambda_a - z_i} = \sum_{b \neq a}^{N} \frac{2}{\lambda_a - \lambda_b} \ , \qquad (a = 1, \ldots, N) \ . \tag{2.19}$$

On the face of it, this seems to give rise to N equations for the N unknowns λ_a, up to permutations. However, there are actually only $(N-1)$ independent equations here since the sum of all the residues vanishes necessarily. Thus we can solve, for example, for λ_a for $a = 1, \ldots, N-1$ in terms of λ_N. Once one has determined these $(N-1)$ λ_a, one is left with three undetermined parameters for the covering map: λ_N, M_Γ and the constant of integration in going from $\partial\Gamma(z)$ to $\Gamma(z)$. These are fixed, for instance, by requiring that $\Gamma(z_i) = x_i$ (for $i = 1, 2, n$), with $x_j \neq \infty$.

Our key observation here is that in the large N limit — because of (2.15) this is the limit when we take the w_i large — this equation has a natural matrix model interpretation: the λ_a can be thought of as the eigenvalues of a matrix model, with the RHS being the usual eigenvalue repulsion while the LHS is playing the role of an external potential determined by the z_i. We will exploit this connection in what follows, working mainly at large N, but commenting on the finite N case at various points.

[7]We should alert the reader that [114] takes one of the branch points to be $x_n = \infty$, which implies a coalescence of the poles and the zeroes of $\partial\Gamma(z)$. As a result, the specific expressions in [114] have a different power of z in the denominator.

[8]This is, again, very similar to the Gross-Mende equations of [46].

2.3 A Penner-like matrix model and its solution

In this section we use matrix model techniques in order to solve the eigenvalue problem (2.19), which we rewrite as

$$\frac{1}{2}\sum_{i=1}^{n-1}\frac{\alpha_i}{\lambda_a - z_i} = \frac{1}{N}\sum_{b\neq a}^{N}\frac{1}{\lambda_a-\lambda_b}\,, \qquad \left[\alpha_i = \frac{w_i-1}{N}\right]\,, \tag{2.20}$$

where α_i is held fixed in the large twist limit. Note that (2.15) implies that

$$\frac{1}{2}\sum_{i=1}^{n}\alpha_i = 1-\frac{1}{N} \quad\Longrightarrow\quad \sum_{i=1}^{n-1}\alpha_i = 2-\alpha_n-\frac{2}{N}\,. \tag{2.21}$$

As noted in the previous section we actually have $N-1$ independent equations here leaving one additional undetermined parameter in the solution. We will see later, see the discussion around eq. (2.49), how this works for the large N solution.

We want to interpret eq. (2.20) as the saddle-point equation for the large N matrix integral

$$\mathcal{Z} = \int [dM] e^{-N\,\mathrm{Tr}W(M)} = \int\prod_{a=1}^{N}\frac{d\lambda_a}{2\pi}\,\Delta^2(\{\lambda_b\})\,e^{-N\sum_{a=1}^{N}W(\lambda_a)}\,, \tag{2.22}$$

where the potential has a logarithmic Penner-like form

$$W(z) = \sum_{i=1}^{n-1}\alpha_i\log\left(z-z_i\right)\,, \tag{2.23}$$

and therefore

$$W'(z) = \sum_{i=1}^{n-1}\frac{\alpha_i}{(z-z_i)}\,. \tag{2.24}$$

Furthermore $\Delta(\lambda_a)$ is the usual Vandermonde determinant. The covering map is related to the matrix model as

$$\frac{1}{N}\log\left[M_\Gamma^{-1}\partial\Gamma(z)\right] = W(z) - \frac{2}{N}\sum_{a=1}^{N}\log(z-\lambda_a)\,, \tag{2.25}$$

where we have used (2.18).

We should remark that the matrix integral is, at least at this stage, just a convenient prop for solving the equations (2.19). In particular, in our problem, the λ_a (and z_i) are complex and so are not, strictly speaking, eigenvalues of a Hermitian matrix model. What the mathematical correspondence to saddle point equations for matrix integrals suggests is a method to solve them in the large N limit. In fact, as we will indicate below, our question will dictate a slightly different angle and interpretation of the problem from the more traditional matrix model point of view.

The class of matrix models with logarithmic potential as in (2.23) are known as

Penner-like models and have appeared in a number of physics contexts, including recently in the connection between AGT and topological strings [55–59]. We will draw upon some of these results in what follows though, as mentioned, we will have a somewhat different interpretation. It will, nevertheless be interesting to see whether the appearance of the same matrix models, in the context of topological strings and Liouville theory in the above references as well as in the present context, is more than a technically fortuitous coincidence.

In any case, we proceed by introducing a (normalised) density of “eigenvalues"

$$\rho(\lambda) = \frac{1}{N}\sum_{a=1}^{N}\delta(\lambda-\lambda_a)\ . \tag{2.26}$$

We expect that in the large N limit this will go over to a smooth function which has support over some set of curves (which we denote below by $\mathcal{C}$) on the complex plane. The equation (2.20) determining the covering maps becomes then (for $\lambda \in \mathcal{C}$)

$$\frac{1}{2}W'(\lambda) = P\int_{\mathcal{C}}\frac{\rho(\lambda')d\lambda'}{\lambda-\lambda'}\ , \tag{2.27}$$

where P denotes the principal value. This integral equation for $\rho(\lambda)$, including its support, can now be solved using conventional matrix model technology. We will do so in the next subsection using the method of loop equations. The advantage of this method is that it generalises beyond the leading large N limit. There is also an equivalent method for solving the large N equations in terms of a Riemann-Hilbert problem, which we outline in Appendix 2.A.

Either way, we note that we expect to find a family of solutions corresponding to the large multiplicity of covering maps in the large twist limit, see eqs. (2.16) and (2.54). For any of these solutions for $\rho(\lambda)$, the corresponding covering map will be determined, at leading order in N, by (2.25)

$$\frac{1}{N}\log\left[M_\Gamma^{-1}\partial\Gamma(z)\right] = \sum_{i=1}^{n}\alpha_i\log(z-z_i) - 2\int_{\mathcal{C}}d\lambda\,\rho(\lambda)\log(z-\lambda)\ . \tag{2.28}$$

We will also see in Section 2.3.3 that it will be natural to go somewhat beyond this leading large N answer.

2.3.1 Solving the Matrix Model

We can convert the saddle point equations for the eigenvalue density into a set of equations for the resolvent — which are known as loop equations [60] and can actually be written down for finite N. These are then solved in terms of an auxiliary spectral curve, see [61] for a very nice exposition of these techniques. For the multi-Penner potentials the corresponding solutions and spectral curves have been explicitly studied to leading order in $\frac{1}{N}$ in [58].

To obtain the spectral curve we first define the resolvent via

$$u(z) = \frac{1}{N}\sum_{a=1}^{N}\frac{1}{z-\lambda_a} = \int_{\mathcal{C}}\frac{\rho(\lambda)d\lambda}{z-\lambda}\ , \tag{2.29}$$

from which we can deduce the eigenvalue density (in the large N limit) by the discontinuity across the support $\mathcal{C}$

$$\rho(\lambda) = -\frac{1}{2\pi i}\Big[u(\lambda+i\epsilon)-u(\lambda-i\epsilon)\Big] \qquad (\lambda\in\mathcal{C})\ . \tag{2.30}$$

The loop equations are now obtained as follows. First we rewrite (2.20) as

$$\frac{1}{2}W'(\lambda_a) = \frac{1}{N}\sum_{b\neq a}\frac{1}{\lambda_a-\lambda_b}\ , \tag{2.31}$$

multiply both sides by $\frac{1}{(\lambda_a-z)}$, and then sum over a, see for example [61]. A short calculation[9] then leads to the so-called loop equation that the resolvent must obey,

$$u^2(z) - W'(z)u(z) + \frac{1}{N}u'(z) + R(z) = 0\ . \tag{2.32}$$

Here we have introduced the function $R(z)$, see eq. (2.88),

$$R(z) = \frac{1}{N}\sum_{a=1}^{N}\frac{W'(\lambda_a)-W'(z)}{(\lambda_a-z)} = \sum_{i=1}^{n-1}\frac{\alpha_i u(z_i)}{z-z_i}\ . \tag{2.33}$$

As the derivation in Appendix 2.B shows, the loop equations (2.32) actually hold at any finite N. They are, moreover, in a form which is suitable for a large N expansion.

Next we introduce the so-called quantum corrected spectral curve, which is defined in terms of the resolvent via

$$y(z) = W'(z) - 2u(z)\ . \tag{2.34}$$

We can then rewrite (2.32) as an equation that determines the spectral curve

$$y^2(z) - \frac{2}{N}y'(z) = \Big(W'(z)\Big)^2 - \frac{2}{N}W''(z) - 4R(z)\ . \tag{2.35}$$

It is natural to view $y(z)dz$ as a differential on the underlying Riemann surface that emerges in the large N limit, as we will see below.

We should note that in terms of our original problem, the function $y(z)$ defined by

[9] For the convenience of the reader, this is reproduced in Appendix 2.B.

(2.34) equals

$$y(z)=\sum_{i=1}^{n-1}\frac{\alpha_i}{(z-z_i)}-\frac{2}{N}\sum_{a=1}^{N}\frac{1}{(z-\lambda_a)}=\frac{1}{N}\frac{\partial^2\Gamma(z)}{\partial\Gamma(z)}=\frac{1}{N}\partial\ln\partial\Gamma, \tag{2.36}$$

where we have used the expression (2.18) for $\partial\Gamma(z)$, as well as the relation $N\alpha_i=(w_i-1)$ from eq. (2.20). This identification (2.36), which will be important in what follows, is also true at finite N.

For the rest of this subsection we shall concentrate on the leading large N limit, and discuss what happens for finite N in Section 2.3.3. Starting from eq. (2.35) let us replace $y(z)$ by its leading term in the large N limit, which we denote by $y_0(z)$. Then (2.35) reduces to

$$y_0^2(z)=\Big(W'(z)\Big)^2-4R_0(z)\ , \tag{2.37}$$

where we have dropped the terms with the explicit factors of $\frac{1}{N}$, and replaced the function $R(z)$ by its leading piece in the large N limit, which we have denoted by $R_0(z)$; using (2.33) and (2.34) this can be written as

$$R_0(z)=\sum_{i=1}^{n-1}\frac{\alpha_i u_0(z_i)}{z-z_i}=\frac{1}{2}\sum_{i=1}^{n-1}\frac{\alpha_i}{z-z_i}\Big(W'(z)-y_0(z)\Big)\Big|_{z\to z_i}\ . \tag{2.38}$$

Together with the form of (2.24), this now allows us to rewrite eq. (2.37) as

$$y_0^2(z)=\frac{\tilde W_{n-2}^2(z)-\prod_{i=1}^{n-1}(z-z_i)\tilde R_{n-3}(z)}{\prod_{i=1}^{n-1}(z-z_i)^2}\equiv\frac{Q_{2n-4}(z)}{\prod_{i=1}^{n-1}(z-z_i)^2}\ . \tag{2.39}$$

Here $\tilde W_{n-2}(z)$ is a polynomial of degree $(n-2)$ that is determined by $W'(z)$, and $\tilde R_{n-3}(z)$ is similarly determined by $R_0(z)$ after rationalisation. Naively one may have thought that $\tilde R_{n-3}(z)$ is of degree $(n-2)$, but it actually has degree $(n-3)$ since the leading $\frac{1}{z}$ coefficient of $R_0(z)$ is proportional to $\sum_i\alpha_i u_0(z_i)$, which vanishes because of eq. (2.89).

Note that while $\tilde R_{n-3}(z)$ implicitly depends on $y_0(z_i)$ via (2.38), we will think of its $n-2$ parameters as initially unknown, i.e. not directly related to $y_0(z)$. Then we can think of eq. (2.39) as determining $y_0(z)$ in terms of these unknown parameters. We will explain below (in Section 2.3.2) how these parameters can subsequently be fixed by self-consistency.

Returning to eq. (2.39) we now note that the numerator defines a hyperelliptic curve with $(n-2)$ cuts (and genus $(n-3)$),

$$\hat y^2(z)=Q_{2n-4}(z)=\Big(\tilde W_{n-2}(z)\Big)^2-\prod_{i=1}^{n-1}(z-z_i)\tilde R_{n-3}(z)\equiv\alpha_n^2\prod_{j=1}^{2(n-2)}(z-a_j)\ . \tag{2.40}$$

The one form $y_0(z)dz$ is then a meromorphic differential on this Riemann surface, and it has poles at $z = z_i$ with residue α_i. We furthermore require that the residue at $z_n = \infty$ equals α_n, and this then fixes the overall coefficient in (2.40). We also see from (2.34) that the leading order solution for the resolvent, $u_0(z) = \frac{1}{2}(W'(z) - y_0(z))$, has branch cuts and therefore discontinuities giving the eigenvalue densities as in (2.30). An alternative form for $y_0(z)$ and the resolvent $u_0(z)$, exhibiting their single pole structure, is given in (2.75) and (2.74), respectively.

2.3.2 Independent parameters

In the usual matrix model treatment of this problem, the final step is to obtain from the spectral curve the complete solution by determining the unknown polynomial $\tilde{R}_{n-3}(z)$ in (2.39), see the comment above eq. (2.40). Often this is done by specifying the $(n-3)$ "filling fractions" of the eigenvalue density along the $(n-3)$ independent A-cycles,[10] i.e. the $(n-3)$ periods

$$\frac{1}{4\pi i}\oint_{A_i} y_0(z)dz = \int_{\mathcal{C}_i\equiv[a_{2i-1},a_{2i}]} d\lambda\, \rho(\lambda) = \nu_i \ , \qquad i = 1, \ldots, n-3 \ . \tag{2.41}$$

Note that we actually only need to specify $(n-3)$ such periods since the overall coefficient of $\tilde{R}_{n-3}$ is fixed by the overall normalisation of $Q_{2n-4}(z)$ in (2.40). More specifically, the coefficient of z^{2n-4} in $(\tilde{W}_{n-2}(z))^2$ equals $(\sum_{i=1}^{n-1}\alpha_i)^2 = (2-\alpha_n)^2$, where we have used (2.21), and the z^{n-3} coefficient of $\tilde{R}_{n-3}(z)$ is thus determined by the condition that the coefficient of z^{2n-4} in $Q_{2n-4}(z)$ equals α_n^2.

As a consequence the $(n-3)$ conditions in (2.41) are enough to solve for the polynomial $\tilde{R}_{n-3}(z)$, and this then also fixes the locations a_j of the branch points of the cuts in (2.40). In this approach, the other set of periods (over the B-cycles) are determined in terms of the effective action of the matrix model

$$\frac{1}{4\pi i}\oint_{B_i} y_0(z)dz \sim \frac{\partial \mathcal{S}_{\text{eff}}(\{z_j,\nu_j\})}{\partial \nu_i} \ . \tag{2.42}$$

Note that from the matrix model perspective, the z_j are given *ab initio* — they define the matrix model potential $W(z)$ as in (2.23). Thus we solve for the spectral curve and thus the eigenvalue density (including its support) in terms of the input data of the (α_i, z_i, ν_i).

As we will see more fully in Section 2.4, our present problem dictates a slightly different perspective: for us the z_i are not specified initially, since they live on the auxiliary covering space which we are constructing by solving for the covering map. Instead, we will think of equations like (2.42) as determining the z_i in terms of the B-cycle periods.

More specifically, in our context it will be natural, instead, to specify the $2(n-3)$ independent periods of the spectral curve — over *both* the A- and B-cycles. We can then

[10]There are alternative conditions one could impose to determine these unknowns, such as demanding that eigenvalues do not tunnel, see [61].

use these period integrals to determine the $(n-3)$ independent parameters of $\tilde{R}_{n-3}(z)$ together with the $(n-3)$ cross ratios of the z_i. (Recall that we have used the Möbius invariance to fix three of the z_i). This is the sense in which our approach exhibits a slightly different angle and interpretation of the matrix model solution.

2.3.3 The solution at finite N

Having determined the spectral curve and thus the solution to the covering map to leading order in N, we will now study the solution at finite N; much of Sections 2.4 and 2.5 will be independent of the considerations here, and readers can return to this subsection at a later stage if they wish. We start with the loop equation (2.35) that was derived without taking the large N limit (and therefore also holds at finite N)

$$y^2(z) - \frac{2}{N}y'(z) = \Big(W'(z)\Big)^2 - \frac{2}{N}W''(z) - 4R(z) \ . \tag{2.43}$$

It is striking that the combination that appears on the LHS is, in terms of the covering map, see eq. (2.36), precisely

$$\begin{aligned} y^2(z) - \frac{2}{N}y'(z) &= \frac{1}{N^2}\left[\left(\frac{\Gamma''}{\Gamma'}\right)^2 - 2\frac{\Gamma'''}{\Gamma'} + 2\left(\frac{\Gamma''}{\Gamma'}\right)^2\right] && (2.44)\\ &= -\frac{2}{N^2}\left[\frac{\Gamma'''}{\Gamma'} - \frac{3}{2}\left(\frac{\Gamma''}{\Gamma'}\right)^2\right] = -\frac{2}{N^2}S[\Gamma] \ , && (2.45)\end{aligned}$$

where $S[\Gamma]$ is the Schwarzian derivative of $\Gamma(z)$. We furthermore rewrite the RHS of (2.43) in terms of the tree level curve $y_0(z)$ in (2.37) as

$$-\frac{2}{N^2}S[\Gamma] = y_0^2(z) - \frac{2}{N}W''(z) - \frac{4}{N}R_1(z) \ , \tag{2.46}$$

where $R(z) = R_0(z) + \frac{1}{N}R_1(z)$. Next we observe that the arguments leading up to (2.39) equally apply without taking the large N limit — in particular, the leading $\frac{1}{z}$ coefficient of $R(z)$ is proportional to $\sum_i \alpha_i u(z_i)$, and thus also vanishes because of eq. (2.89) — and we can therefore write (2.43) as

$$\tilde{y}^2(z) \equiv y_0^2(z) - \frac{2}{N}W''(z) - \frac{4}{N}R_1(z) = \frac{\tilde{Q}_{2n-4}(z)}{\prod_{i=1}^{n-1}(z-z_i)^2} \ . \tag{2.47}$$

We can therefore interpret $\tilde{y}^2(z)$ is the $\frac{1}{N}$-corrected version of $y_0^2(z)$. Note that, because of the $-\frac{2}{N}W''(z)$ correction term, its "residue" at z_i is

$$\tilde{y}^2(z) \sim \frac{w_i^2-1}{N^2}\frac{1}{(z-z_i)^2} \ , \tag{2.48}$$

whereas the coefficient of the double pole of $y^2(z)$ at $z = z_i$ equals $\frac{(w_i-1)^2}{N^2}$. This shift is required in order for it to agree with the Schwarzian of the covering map, see eq. (2.46). The two correction terms in (2.47) will also affect the periods (2.41) and (2.42).

As we will see below in Section 2.5, $(-y_0^2)(z)dz^2$ defines a Strebel differential at leading order in $\frac{1}{N}$. As a consequence, the same is therefore true for the Schwarzian of the covering map. This is quite a remarkable relation between the Schwarzian derivative and Strebel differentials, and as far as we are aware, this connection had not been noticed before. Obviously, the Schwarzian derivative of the covering map can also be evaluated at finite N, and one could ask whether the Schwarzian at finite N also defines a Strebel differential. Similarly, one could analyse this question in the large N limit, but including subleading $\frac{1}{N}$ corrections. It would be interesting to explore these directions further.

Let us close this section by explaining how the matrix model approach leads to the correct number of solutions, see the discussion below (2.19). As we have explained above in Section 2.3.2, we can determine the leading order spectral curve $y_0(z)$, as well as the function $R_0(z)$, by specifying the periods. This argument also applies to the exact loop equation, and thus the periods determine the RHS of (2.46). This leads to a differential equation determining the covering map $\Gamma(z)$ which is of Sturm-Liouville type, see e.g. [62].

At finite N, the LHS is the Schwarzian derivative of the covering map $\Gamma(z)$, and the equation therefore only determines $\Gamma(z)$ up to a Möbius transformation in the x-space. This is a consequence of the transformation property of the Schwarzian [62]

$$S[f \circ g] = (S[f] \circ g)(g')^2 + S[g] \ . \tag{2.49}$$

Indeed, if we take $g = \Gamma$ and let f be a Möbius transformation, we see that the Schwarzian is invariant under Möbius transformations in the x-space,[11]

$$S[f \circ \Gamma] = S[\Gamma] \ , \qquad \text{if } f \text{ is a Möbius transformation.} \tag{2.50}$$

Thus (2.46) only allows us to determine the covering map up to Möbius transformations in x, which means that we have the freedom to specify three of the branch points x_i as expected.

This should be contrasted with the leading order analysis, for which the loop equations determine $y^2(z)$. This is related to the covering map as in (2.36), but the RHS of (2.36) is *not* invariant under replacing $\Gamma \mapsto f \circ \Gamma$, where f is a Möbius transformation. Thus to leading order in $\frac{1}{N}$ the resulting solution would not allow us to specify three of the branch points x_i at will. In retrospect, the fact that this subtlety is related to a $\frac{1}{N}$ effect is also

[11]This follows from $S[f] = 0$. Note that this should not be confused with the transformation of the Schwarzian under a Möbius transformation in the z-space, see eq. (2.51) below.

clear from the original discussion below eq. (2.19): in that context it was important that there are actually only $(N-1)$, rather than N independent scattering equations (which, in the large N limit, is 'a $\frac{1}{N}$ effect').

Finally we note that the RHS of (2.36) does not transform covariantly under Möbius transformations, but it is only the combination $y^2(z) - \frac{2}{N}y'(z) = -\frac{2}{N^2}S[\Gamma](z)$ that appears in (2.46) which has a nice transformation behaviour. In fact, $\Gamma(z)$ transforms as a quadratic differential

$$S[\Gamma(f(z))] = f'(z)^2\, S[\Gamma(z)] \,, \tag{2.51}$$

where $f(z)$ is a Möbius transformation, and we have used (2.49). Obviously, the difference between $y^2(z)$, and $y^2(z) - \frac{2}{N}y'(z)$, is subleading in $\frac{1}{N}$, and hence to leading order it does not matter which one considers. But the considerations of the last two paragraphs suggest that it is more natural to include the $-\frac{2}{N}y'(z)$ correction and consider the LHS of (2.43) instead of just $y^2(z)$.

2.4 The spectral curve and Feynman diagrams

In the previous section we have outlined the method for solving the equations (2.19) and hence for determining the branched covers for correlators with large twist. The result is encoded in the spectral curve in the form of eq. (2.37), which in turn determines the resolvent, see eq. (2.34), and thereby also the discontinuities ("eigenvalue densities") see eq. (2.30). The branched covers themselves are then obtained via eq. (2.36), or more accurately via eq. (2.46), see the discussion at the end of Section 2.3.3.

As we discussed at the end of Section 2.3.2 it is natural to fix the independent parameters (including the z_i) by specifying the period integrals over both the A- and the B-cycles. It is the aim of this section to explain in more detail why this is so. In the process of doing so we will also exhibit the meaning of the periods themselves.

Let us begin by describing the covering map using the diagrammatic picture of [**?**]. They associate to each covering map a diagram by considering a Jordan curve passing through the n points x_i (in some prescribed fixed ordering) on the spacetime sphere, and enclosing $x = \infty$.[12] This curve then has a pre-image in the covering space, where it defines a graph with the vertices being the branch points z_i with $\Gamma(z_i) = x_i$. The poles λ_a of the covering map $\Gamma(z)$ are, on the other hand, the pre-images of $x = \infty$, and since the Jordan curve encloses $x = \infty$, they are associated with the N faces (or so-called "coloured loops") of the resulting configuration.

The complement of the above Jordan curve defines another set of N faces (the so-called "dashed loops"). The construction of [**?**] therefore associates to each covering

[12] More precisely, they actually introduce a "bifundamental" double line curve, with the inner solid curve enclosing $x = \infty$, and an outer dashed curve, see Fig. 2.2.

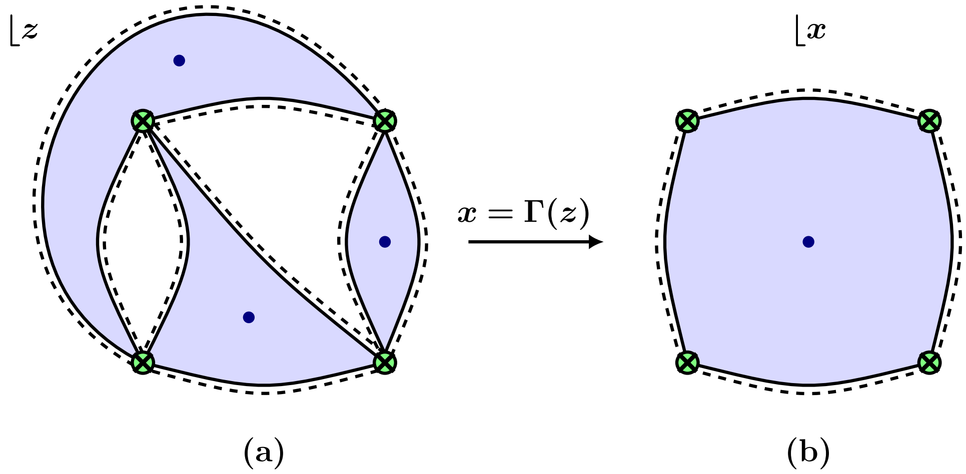

Figure 2.2: An illustration of the Feynman graph of a symmetric orbifold correlator, constructed via the preimage of a Jordan curve under Γ. The case with $n = 4$ and all $w_i = 2$ is depicted, for which $N = 3$.

map a Feynman-like double line diagram comprising of "Wick contractions" between the n vertices, see Fig. 2.2. These diagrams are the analogue of the free field 't Hooft diagrams for n-point correlators in Yang-Mills theory.[13] As in that case, we associate a genus to the diagram which is that of the covering space that is triangulated by the above faces.

Since we are considering here genus zero covering spaces, our double-line diagrams are planar. Following [23] we associate to each such allowed diagram a *skeleton graph* whereby we glue all homotopically equivalent edges. It is easy to verify, using Euler's formula for genus zero,

$$n_V - n_E + n_F = 2 \ , \tag{2.52}$$

that the resulting skeleton graph $\mathcal{G}$ will have $n_E = (3n-6)$ edges and $n_F = (2n-4)$ faces (which are generically triangular), as well as obviously $n_V = n$ vertices. It will also be useful to consider the dual graph $\mathcal{G}^D$ which then has n faces, $(2n-4)$ vertices and $(3n-6)$ dual edges which are transverse to the edges of the original skeleton graph.

Given that we have 'collapsed' homotopic Wick contractions, a given skeleton graph $\mathcal{G}$ accounts, however, for many inequivalent double-line diagrams. The *additional* data that is needed to reconstruct the double-line diagram from the skeleton graph is simply the number of edges between pairs of vertices (i,j) in the original double-line diagram, which we denote by $n_{ij} = n_{ji}$. These integers n_{ij} are constrained only by positivity and the requirement that at each vertex (labelled by i) they satisfy

$$\sum_{j \neq i} n_{ij} = 2w_i \ , \qquad (\forall i = 1, \ldots, n) \ . \tag{2.53}$$

Since there can only be $(3n-6)$ edges (and therefore as many non-zero n_{ij}), and taking into account the n constraints from (2.53), we see that only $(2n-6)$ of the n_{ij} are

[13]Note that in the present case we have $2w_i$ double lines emerging from each vertex with twist w_i.

independent. So far, everything we have said is true for finite w_i.

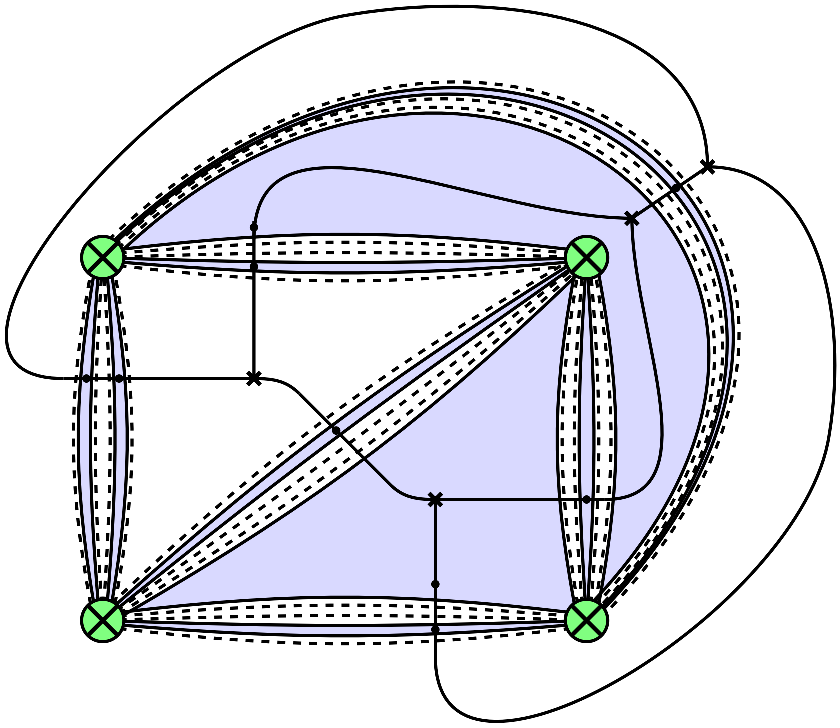

Figure 2.3: The Feynman graph for a four-point correlator with $w_i = 5$ and therefore $N = 9$. The critical points are denoted by $\otimes$. The dual of its skeleton graph, $\mathcal{G}^D$, is described by the black solid lines (and its vertices are denoted by crosses); it corresponds to the graph of critical horizontal trajectories of the Strebel differential as discussed in Section 2.5.

We are interested in the regime where we scale the twists as $w_i \sim \alpha_i N$ with N large, and then the n_{ij}, when they are non-zero, also generically scale as N. We note in passing that the number of covering maps scales in this limit as

$$\#(\text{branched coverings}) \sim N^{2n-6} \ , \tag{2.54}$$

as follows from the argument below eq. (2.53); for $n = 4$, this is in agreement with eq. (2.16). In the original double-line diagram we had one pole λ_a for each of the N coloured faces, but in the skeleton graph only $(2n-4)$ (generically triangular) faces remain. This implies that, at large N, most of the N poles are associated with the two-edged faces formed from homotopic Wick contraction, i.e. with the faces that disappear when we glue the double-line diagram to form the skeleton graph, see Fig. 2.3. In fact, as we have learnt in Section 2.3, these poles coalesce in the large N limit into a system of cuts $\mathcal{C}$, which are transverse to the original edges, and are now seen to build up the edges of the dual skeleton graph $\mathcal{G}^D$. Thus we can identify the different cuts in the cut-system $\mathcal{C}$ with the edges of the dual skeleton graph $\mathcal{G}^D$, and the $(2n-4)$ vertices of $\mathcal{G}^D$ with the end-points of the cuts, i.e. the $(2n-4)$ a_j from eq. (2.40). Furthermore, the number of poles associated to the dual edge $\widehat{(ij)}$, i.e. the edge of $\mathcal{G}^D$ that is transverse to the edge

(ij) of $\mathcal{G}$, is approximately $\frac{n_{ij}}{2}$ in the leading large N limit.[14] Finally, the n faces of $\mathcal{G}^D$ each contain one of the n vertices z_i of $\mathcal{G}$; these can be identified with the simple poles of the spectral curve $y_0(z)$ as in (2.79).

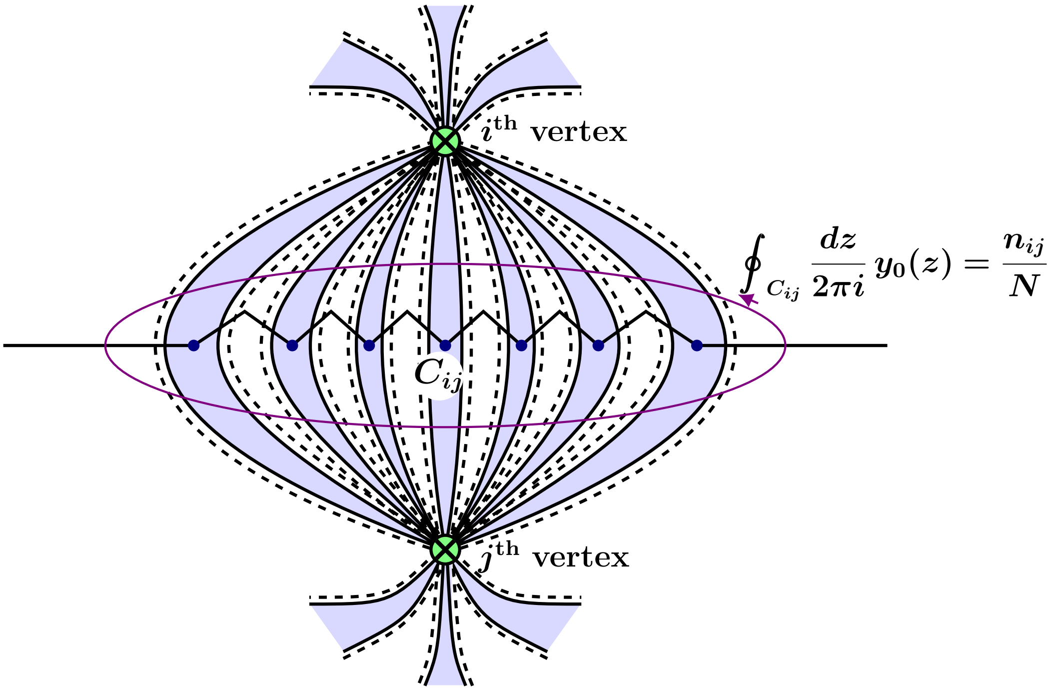

Figure 2.4: The period integral along the cut that corresponds to the dual edge $\widehat{(ij)}$.

It should now be clear how the matrix model solution characterised by the spectral curve $y_0(z)$, is related to the covering map described in terms of the skeleton graph $\mathcal{G}$ and the numbers n_{ij}. Starting from the spectral curve, we identify the cut system $\mathcal{C}$ with the dual skeleton graph $\mathcal{G}^D$. The discontinuity of $y_0(z)$ across a cut counts the fraction of λ_a poles associated to this cut, and thus the period integral of $y_0(z)$ along this cut is exactly twice the fraction $\frac{n_{ij}}{2N}$ one associates to the corresponding dual edge $\widehat{(ij)}$ of $\mathcal{G}^D$, see eqs. (2.34), (2.30) and Fig. 2.4. As we have seen above, see eq. (2.53), there are precisely $(2n-6)$ independent such n_{ij} from the viewpoint of $\mathcal{G}^D$. In terms of the spectral curve, this follows from the fact that the integral of the spectral curve $y_0(z)$ around the edges of a given face of $\mathcal{G}^D$, say the one that contains z_i, equals the residue of $y_0(z)$ at z_i, i.e. $\alpha_i = \frac{w_i}{N}$. This leads to n constraints among the $(3n-6)$ integrals of $y_0(z)$ along the edges of $\mathcal{G}^D$ (or the cuts of the cut-system $\mathcal{C}$), and hence reduces the number of independent period integrals to $2n-6$. If we denote an appropriate set of $(2n-6)$ independent n_{ij}

[14]Note that, at large N, only approximately half of the Wick contractions n_{ij} correspond to poles in the coloured loops, see Fig. 2.4.

by $n^{(l)}$ and $\tilde{n}^{(l)}$, with $l = 1, \dots, n-3$, we have[15]

$$\frac{1}{4\pi i}\oint_{A_l} y_0(z)dz \equiv \nu_l = \frac{n^{(l)}}{2N} , \qquad \frac{1}{4\pi i}\oint_{B_l} y_0(z)dz \equiv \mu_l = \frac{\tilde{n}^{(l)}}{2N} . \tag{2.55}$$

This therefore determines the dual skeleton graph $\mathcal{G}^D$ and the ratios $\frac{n_{ij}}{N}$ in terms of the spectral curve.

Conversely, the ratios $\frac{n_{ij}}{N}$ along the edges of the dual skeleton graph $\mathcal{G}^D$ fix the periods of the spectral curve, and hence determine it by the arguments of the previous section. Note that the discrete family of covering maps, labelled by the different double-line diagrams, goes over, in the large twist limit, to a continuous family labelled by the periods (ν_l, μ_l) as in (2.55).

Let us illustrate this for the simplest nontrivial case of $n = 4$. The skeleton planar graph $\mathcal{G}$ is a tetrahedron with six edges, and so is the dual graph $\mathcal{G}^D$. The four constraints (2.53) at the vertices of $\mathcal{G}$ imply that there are only two independent sets of n_{ij}. We can take them to be, say, n_{12} and n_{13}, and then all other n_{ij} are determined in terms of these. The spectral curve determining the branched cover is of genus one, and thus there are four a_j in eq. (2.40), which correspond to the four vertices of $\mathcal{G}^D$. Generically there are therefore six period integrals taken along the six different cuts (or edges of $\mathcal{G}^D$), but only two of them are independent. If we choose these to be the cuts transverse to the edges (12) and (13) of $\mathcal{G}$, the corresponding periods in (2.55) are proportional to n_{12} and n_{13}. Fixed values of these periods correspond to a particular graph with specified values of n_{12} and n_{13}, and each of them gives rise to a distinct covering map.

2.5 The spectral curve and the Strebel differential

In the previous section we have explained how the matrix model results are related to the Feynman diagrams of symmetric orbifold correlators that capture the different covering map contributions. In this section we want to show that, in the large twist limit, the sum over these discrete contributions becomes an integral over the moduli space of the covering space. The key observation that makes this possible is a remarkable relation between the spectral curve of the matrix model, and the Strebel differential on the moduli space of the covering space. As we will explain, our system therefore realises very explicitly the mechanism put forward some time ago in [13, 22, 23] by means of which the string world-sheet path integral emerges from the dual CFT correlators.

Recall from Section 2.3 that, to leading order in the large N limit, the spectral curve

[15] Since $y_0(z)$ is meromorphic with simple poles at z_i (2.73), we have to consider the nontrivial homology cycles of the hyperelliptic curve with n punctures. Since the residues around the poles are fixed $(= \alpha_i)$, we have a choice in picking $(2n-6)$ independent periods. This precisely corresponds to the freedom of picking different independent n_{ij}.

$y_0(z)$ has the form, see eqs. (2.39) and (2.40)

$$y_0^2(z) = \frac{\alpha_n^2}{\prod_{i=1}^{n-1}(z-z_i)^2} \prod_{j=1}^{2n-4} (z-a_j) \ . \tag{2.56}$$

We also noted in Section 2.3.3 that it agrees in the large N limit with the Schwarzian of the covering map. Since the latter is a quadratic differential, see eq. (2.51), it follows that also

$$4\pi^2 \phi_S(z) dz^2 \equiv -y_0^2(z) dz^2 = -\frac{\alpha_n^2}{\prod_{i=1}^{n-1}(z-z_i)^2} \prod_{k=1}^{2n-4} (z-a_k) \tag{2.57}$$

defines a quadratic differential. It is clear from this explicit form that the quadratic differential $\phi_S(z)dz^2$ has double poles at z_i, as well as at $z_n = \infty$, and it follows from (2.75) that the "residues" at these double poles are $-\alpha_i^2$, which are therefore real and negative. We also see from eqs. (2.40) and (2.39) that the only zeros are at $z = a_k$. Finally, the discussion of the previous subsection implies that all the periods around pairs of branch points a_k, see eq. (2.55), are *real* and positive (with the appropriate orientation of the integral).

These properties are precisely what characterises a *Strebel differential* on the n-punctured sphere.[16] Recall that at any point on the moduli space of n-punctured Riemann surfaces $\mathcal{M}_{g,n}$ there exists a Strebel differential, i.e. a unique meromorphic quadratic differential with only double poles (and specified negative residues) at the n punctures, such that all the "lengths" between zeroes are real,

$$l_{km} = \int_{a_k}^{a_m} \sqrt{\phi_S(z)} \in \mathbb{R}_+ \ . \tag{2.58}$$

Each such Strebel differential $\phi_S(z)$ defines a critical graph on the Riemann surface, the so-called *Strebel graph*, whose vertices are the zeros of the Strebel differential, and whose edges are the *critical horizontal trajectories*. Here *horizontal* means that the curve $z(t)$ satisfies

$$\phi_S\big(z(t)\big)\left(\frac{dz}{dt}\right)^2 > 0 \ , \tag{2.59}$$

and a horizontal trajectory is *critical* if it is not closed, but rather connects two zeros of the Strebel differential, see Fig. 2.1. These edges divide the Riemann surface into n ring domains (faces with the topology of a disc), each of which contains exactly one double pole of ϕ_S. This Strebel graph is therefore nothing other than $\mathcal{G}^D$ — the dual to the skeleton graph $\mathcal{G}$ of the previous section.

The reason why this is significant is that Strebel differentials are, on the other hand, known to parameterise the (decorated) string moduli space. This is known as Strebel's

[16] For a physicist friendly introduction to Strebel differentials see, for example, Section 3.2 of [63], Section 3 of [13], Section 2 of [64], or Appendix A of [14].

Theorem [108]: for every Riemann surface $\Sigma_{g,n}$ with $n > 0$ and $2g + n > 2$, and any n specified positive numbers $(p_1, \ldots, p_n)$, three exists a unique Strebel differential. This Strebel differential is holomorphic everywhere on $\Sigma_{g,n}$, except at the n marked points where it has double poles with "residue" equal to $-p_i^2$ at the i'th pole.

As a consequence, each covering map is a contribution from a single point on moduli space since it is uniquely specified by the Strebel lengths in (2.55). Furthermore, the sum over all the branched covers defining the symmetric product correlator in (2.12) goes over, in the large twist limit, to an integral over the moduli space of the n-punctured sphere $\mathcal{M}_{0,n}$, where Strebel's Theorem guarantees that we cover the moduli space exactly once.

Thus we have rewritten the symmetric orbifold correlators in the large twist limit as a world-sheet integral, with the world-sheet being the covering surface of the symmetric orbifold correlator. This therefore ties in very nicely with the proposal of [39**?** , 40] that the covering surface should be identified with the world-sheet of the dual string theory; this was recently confirmed by an explicit world-sheet calculation [10, 91]. It also realises beautifully the general picture about how field theory diagrams combine into a world-sheet integral that was put forward by one of us some time ago [13, 22, 23].

2.5.1 Finite N generalisation

We should note that the Strebel lengths are, in our large N limit, proportional to the positive integers n_{ij} which count the number of edges between vertices in the Feynman diagrams. This is somewhat reminiscent of the relation proposed by Razamat [14] as an alternative to identifying the Schwinger parameters of the field theory [13] with the Strebel lengths. Such a discrete relation was also seen to be very natural for the zero-dimensional Gaussian matrix model where one does not have any spacetime dependence in the correlators [14, 25]. In fact, the connection between Gaussian correlators and Belyi branched covering maps [65] (see also [66, 67]), and between the latter and integer length Strebel differentials [109], made this relation compelling. It also suggested a candidate dual closed topological string theory [26–28].

This raises the natural question of what exactly the picture at finite N should be in our case. For finite twist, there are only a finite number of covering maps, see e.g. eq. (2.16), and thus the symmetric orbifold correlator only gets contributions from isolated points in the moduli space, see also [10, 91]. It would be very interesting to understand what characterises the corresponding Strebel differentials. Note that this localisation fixes the cross-ratios of the world-sheet coordinates z_i in terms of those of the spacetime CFT, i.e. the x_i, and that the values of these cross ratios vary smoothly as we vary the x_i. In other words, the discrete points in the moduli space of the covering space (i.e. the world-sheet) are continuous functions of x_i, and cannot just be labelled by the discrete n_{ij}, unlike in the case of the Gaussian matrix model. The answer is presumably some discrete interpolation between the proper time prescription of [13] and the integer length prescription of [14]. We leave this important question for future investigation.

It is also rather striking that to leading order in N, the Strebel differential is the same as another natural quadratic differential one can associate to the covering map $\Gamma(z)$, namely the Schwarzian $S[\Gamma]$. In fact, we see from (2.46) that

$$4\pi^2\phi_S(z) = -y_0^2(z) = \frac{2}{N^2}S[\Gamma] - \frac{2}{N}W''(z) - \frac{4}{N}R_1(z)\ . \tag{2.60}$$

Despite appearances, the first term on the RHS is actually of $\mathcal{O}(1)$ in the large N limit, whereas the other two terms are down by factors of $\frac{1}{N}$. In fact, the $\frac{1}{N}$ corrections are relatively mild: as described in Section 2.3.3, the second term on the RHS of (2.60) only corrects the residue of the double pole at $z = z_i$ from being proportional to $(w_i - 1)^2$ to $(w_i^2 - 1)$, while the last term in (2.60) shifts the coefficient of the subleading simple pole at $z = z_i$. One may therefore think that, in some sense, the Strebel differential is essentially the Schwarzian of the covering map to all orders in $\frac{1}{N}$, although maybe not at finite N.

2.6 Reconstructing the world-sheet

In the previous section we have seen that one part of the program of [13, 23] is beautifully realised for the case of the symmetric orbifold correlators, at least in the limit of large twists: the sum of all Feynman diagram contributions to (free field) CFT correlators gives rise to an integral over the moduli space of Riemann surfaces of the dual string theory,

$$\sum_{\{n_{ij}\}} \quad \longrightarrow \quad \int \prod_{l=1}^{n-3} [d\nu_l d\mu_l] = \int_{\mathcal{M}_{0,n}} |\omega^{(n-3)}(z_i)|^2\ , \tag{2.61}$$

where the flat measure in terms of the periods goes over to a top form in terms of the conventional z_i parametrising the n-punctured sphere. Here we have used that the discrete sum over covering maps is indexed by the independent parameters n_{ij}, and that the sum over all such contributions goes over to an integral over the $(2n-6)$ independent periods or Strebel lengths in (2.55) in the large twist limit. The second equality uses the Jacobian of the transcendental relation between the Strebel lengths and the conventional moduli z_i, see [31] for a complete form of the relation for the $n = 4$ case.

The second part of the program of [13, 23] is to obtain the *integrand* on moduli space from the dual CFT correlators, i.e. to reconstruct the actual world-sheet correlators from the spacetime perspective. As we have seen in (2.12) the spacetime CFT correlators are of the form

$$\langle \sigma_{w_1}(x_1) \cdots \sigma_{w_n}(x_n)\rangle = \sum_{\Gamma} W_\Gamma \prod_{i=1}^{n} |a_i^\Gamma|^{-2(h_i - h_i^0)}\, e^{-S_{\mathrm{L}}[\Phi_\Gamma]}\ , \tag{2.62}$$

where the coefficients a_i^Γ are defined via,

$$\partial\Gamma(z) \sim a_i^\Gamma w_i \,(z - z_i)^{w_i - 1} \;, \qquad \text{as } z \to z_i. \tag{2.63}$$

Here the sum over Γ denotes the sum over all branched coverings, and this will become the integral over moduli space as in (2.61). We are therefore interested in the large N limit of the different summands in (2.62).

Let us start with the first term, the one involving a_i^Γ. Using the form of $\partial\Gamma(z)$ in (2.18) and (3.14), it follows that at leading order in $\frac{1}{N}$, a_i^Γ behaves as

$$a_i^\Gamma = \frac{1}{w_i} M_\Gamma \prod_{j(\neq i)}^{n} (z_i - z_j)^{w_j - 1} e^{-2N \int_{\mathcal{C}} d\lambda\, \rho(\lambda) \log(z_i - \lambda)} \;. \tag{2.64}$$

We also note that the factors of $(h_i - h_i^0) \sim \mathcal{O}(1)$. Therefore the contribution from

$$\left|a_i^\Gamma\right|^{-2(h_i - h_i^0)} \sim e^{-N} \;. \tag{2.65}$$

This will turn out to be subdominant at large N compared to the Liouville term which we analyse next. To evaluate the Liouville action, we note that the conformal factor Φ_Γ is given by (2.13)

$$\Phi_\Gamma(z, \bar{z}) = \ln\Big(|\partial\Gamma|^2\Big) \;,$$

which implies, using (2.36), that

$$\frac{1}{N}\partial\Phi_\Gamma = \frac{1}{N}\partial \log \partial\Gamma = y(z) \;, \tag{2.66}$$

where $y(z)$ is the spectral curve. To leading order in N, $y(z) \cong y_0(z) = i\sqrt{\phi_S(z)}$, and thus the classical Liouville action becomes, to leading order in $\frac{1}{N}$

$$\mathrm{S}_L[\Gamma] = \frac{c}{48\pi}\int d^2z \left(|\partial\Phi_\Gamma(z)|^2 + 2R\Phi_\Gamma\right) = \frac{cN^2}{48\pi}\int d^2z \left(|\phi_S(z)| + \frac{2}{N^2}R\Phi_\Gamma\right) . \tag{2.67}$$

As explained in [39] the $R\Phi_\Gamma$ term is only needed to regularise the contribution from infinity (in z-space), but does not otherwise contribute. If we ignore this regulator term the action is just given in terms of the Strebel differential, which defines an almost flat metric on the world-sheet with a line element given by

$$ds^2 = |\phi_S(z)| dz d\bar{z} \to \sqrt{\det\{g\}} = |\phi_S(z)| \;. \tag{2.68}$$

Thus eq. (2.67) is essentially the Nambu-Goto area action for the world-sheet in "Strebel gauge". Note that this Strebel gauge is characterised by the property that all its curvature is localised at the punctures or insertions of vertex operators, z_i, as well as at the zeroes

a_k of $\phi_S(z)$. The latter can be viewed as the interaction vertices of the string [13]. This gauge had already appeared in the putative dual to the Gaussian model as was observed in [27].

We should emphasise that the Strebel metric in (2.68) is *distinct* from the induced metric on the covering space [40], which is given by the pullback from the boundary S^2 of the covering map

$$ds^2_{\text{pull}} = \Big|\partial\Gamma(z)\Big|^2 dz d\bar{z} \ . \tag{2.69}$$

The Strebel metric is instead to be thought of as an induced metric from the full dual AdS_3 geometry. Indeed, the Liouville action (2.67) has been shown to arise as the classical on-shell action on AdS_3 with the conformal factor Φ being identified with the radial direction [10]. From the AdS_3 perspective, the relevant world-sheet is pinned to the insertion points at the boundary, but extends into the interior of AdS_3, and (2.67) should describe the 'area' of this surface, i.e. the Strebel metric should be induced from this AdS embedding.[17] It would be very interesting to work this out in more detail.

In this context it is very curious to note that there is also another way of viewing the on-shell Liouville action, which connects to old ideas of the rigid string [48, 49]. If we substitute (2.66) in (2.67) and recall that the covering map $\Gamma(z)$ is nothing other than $X(z)$, which parametrises the boundary S^2, then the Liouville action can be suggestively recast as (dropping as before the curvature term)

$$\mathrm{S}_L[X] = \frac{c}{48\pi}\int d^2z\, \frac{1}{\partial X \bar{\partial}\bar{X}}\, \partial^2 X(z)\, \bar{\partial}^2\bar{X}(\bar{z}) \ . \tag{2.70}$$

This is an action for the purely two dimensional modes X, $\bar{X}$ of the large N CFT. It is a four derivative action of the form very reminiscent of that which appears for rigid strings where one adds an extrinsic curvature term, compare with, say, eq. (8) of [48]. Here, the curvature term may be some effective way of incorporating the extra (radial) dimension, and one would also have to incorporate the $B_{\rm NS}$-field that is ultimately responsible for the field $X(z)$ to be holomorphic.

Finally, there is yet another interesting way in which we can cast the Liouville action, again to leading order in $\frac{1}{N}$. Using (2.60) we can write, again dropping terms down by $\frac{1}{N}$ as well as the regulator term

$$\mathrm{S}_L[\Gamma] = \frac{c}{48\pi}\int d^2z\, |\partial\Phi_\Gamma(z)|^2 = \frac{c}{24\pi}\int d^2z\, \Big|S[\Gamma](z)\Big| \ . \tag{2.71}$$

This suggests a direct spacetime description somewhat analogous to what appears in the near AdS_2 dual of the SYK models. Indeed, it was realised there that in terms of a bulk

[17] Since the AdS_3 background also has a $B_{\rm NS}$-field, this will not just be the geometrical area. More generally, this picture also ties in with the general philosophy of [13, 23], namely that it is a signature of the holographic nature of the dual world-sheet action.

description such as the JT gravity action, the Schwarzian action for the reparametrisation of the boundary S^1 captures the low energy physics [47]. This can also be viewed as arising from a coadjoint orbit quantisation of the Virasoro group [69]. On the other hand, the broken conformal symmetry of the SYK model [70–72], also dictates a Schwarzian action for the low energy modes [73]. In the present case, the presence of the correlators can be viewed as slightly breaking the 2d conformal symmetry and perhaps similar arguments can explain the universal nature of the Schwarzian action that governs the physics of the almost topological $k = 1$ AdS_3 string theory.

2.7 Discussion and outlook

Let us conclude by making a number of comments, and suggesting interesting directions for further research. To make this somewhat longish list of ideas more readable, we have organised the different points according to themes.

The Gross-Mende like limit

- The large twist limit seems to be a fruitful regime to investigate correlators in the symmetric orbifold CFT, given how difficult it is to explicitly compute these even for small twist. It would be interesting, for instance, to study the four-point function specifically and connect with some of the techniques already employed in, for instance, [39, 51**?**]. Can we understand the limiting geometries of these covers better? Can we get a handle on systematic $\frac{1}{N}$ corrections?

- Another question is the extension of the considerations here to the case where the covering space/world-sheet is of higher genus. Is there a corresponding set of scattering equations and a role for a large N matrix model reformulation? Gross-Mende saddle points in flat space, at higher genus, [46] were very simply related to those at genus zero. Is there also such a relation here? See also the discussion below on world-sheet correlators.

- Relatedly, eq. (2.19) essentially gives rise to the flat space Gross-Mende equations if one also includes the poles λ_a of the covering map as dynamical quantities. Is there a sense in which these are building up the world-sheet in our setup?

- We have looked at the correlators at fixed x_i. It would be interesting to look at Regge-like limits, perhaps in Mellin space.

- Recently, a 'large p' limit has been considered for half-BPS operators in 4d $\mathcal{N} = 4$ Super Yang-Mills theory at strong coupling, with a similar aim of studying a Gross-

Mende like limit [53]. The analogous scenario at weak coupling, informed by the approach of [13, 23], will be discussed in chapter 3.

The Relation to Matrix Models

- We used matrix model technology to study the large N limit of (2.19). Is it possible to extend this to finite N as well? Note that studying solutions of (2.19) at finite N is not the same as studying the finite N matrix integral in (2.22). The loop equations (2.32) and (2.35), on the other hand, do hold at finite N, and we have taken some steps towards including the subleading effects. It would be good to do this more systematically and apply it to studying covering maps. In this context it is likely that the Schwarzian of the covering map, which naturally appears in the finite N equations, see eq. (2.45), will play a significant role.

- Relatedly, the fact that the equations (2.19) can be viewed as the saddle point equations for a matrix integral cries out for a deeper explanation. The fact that these are the same Penner-like potentials which appeared in the AGT context, in a triad with Seiberg-Witten theory and 2d Liouville CFT, might perhaps be a clue. Note that Strebel differentials have also made an appearance in Seiberg-Witten theory, see for example, [64, 74].

- Even apart from the connection to Seiberg-Witten theory, one can ask the question whether these Penner-like matrix models have a meaning in the present context of the $\text{AdS}_3/\text{CFT}_2$ duality. Note that in the language of the matrix integral (2.22), the potential (2.23) corresponds to insertions of powers of the operator $\det(z_i - M)$. These are suggestive of D-brane like operators, and perhaps one has an alternative open string description here along the lines of the open-closed-open string trialities of [75], see also [76, 77].

Feynman Diagrams and Strebel differentials

- We saw that a Strebel differential was naturally associated to the covering maps (and the corresponding Feynman diagrams) at large N. The corresponding Strebel lengths (2.55) took continuous real values signifying that one was covering all of moduli space. This leads to a measure on moduli space as in (2.61). Can we get a handle on this top form in any natural geometric way? This should play an important role in understanding the world-sheet correlator.

- This leads to the important question of how all of these considerations are modified for finite N when we will only have a few discrete points on moduli space. What is the distribution of these points on moduli space? What characterises the Strebel differentials and the corresponding discrete Strebel lengths? In which way does it deform away from the large N result of eq. (2.55)? Is the finite N Strebel differential given by the Schwarzian of the covering map? The latter has all the right features but it is not clear if it has real periods.

- While finite N is maybe too ambitious, it would also be interesting to consider the large N limit, but to include all subleading $\frac{1}{N}$ corrections. Can we understand the structure of the Schwarzian of the covering map in this limit, and does this define a Strebel differential?

- It is crucial for future generalisations to understand exactly the dictionary between the Strebel lengths and the position dependent field theory amplitudes, extending in some way the proposals of [13, 14]. One possible way to obtain the relation between the Strebel lengths and the insertion points x_i, at finite N, is to see how this translates into a world-sheet OPE (along the lines of [31–33]) which should then match that of the world-sheet correlators obtained in [10, 91]. More generally, it will be interesting to understand in more detail the connection this approach suggests between the world-sheet OPE and the spacetime OPE, see [78, 79].

Reconstructing world-sheet correlators

- In this chapter we have found a very concrete realisation of the idea of [13, 23] for how to reconstruct the world-sheet theory describing strings on AdS, from the dual conformal field theory perspective. However, for the case of AdS_3 that is relevant for this chapter, we actually knew already the world-sheet beforehand: the relevant string theory has minimal pure NS-NS flux, and it can be described in terms of an $\mathfrak{sl}(2,\mathbb{R})$ (or $\mathfrak{psu}(1,1|2)$) WZW model [9, 10, 91]. It would therefore be interesting to confirm that the 'reconstructed' world-sheet theory actually agrees with this WZW model. To a large extent this is already manifest — in particular, the world-sheet correlators have exactly the expected form — but there are some aspects that it would be interesting to check further. In particular, the analysis of Section 2.6 allows one to reconstruct the weight W_Γ with which the different world-sheet configurations contribute to the correlator, see eq. (2.62), and it would be interesting to rederive this from first principles from the WZW model perspective. (From that viewpoint, these weights should be fixed by crossing symmetry.) There is again an interesting interplay here between the spacetime CFT and the worldsheet CFT, and their respective bootstrap conditions [80].

- Continuing in this vein, it was already clear from [10] that the underlying Liouville action, see eq. (2.67), agrees with the on-shell sigma model action for the exact semi-classical branched cover solutions of AdS_3. However, in this chapter we found different microscopic interpretations for it: we could either think of it as the Nambu-Goto action but in a special world-sheet gauge where the metric is that given by the Strebel differential. Alternatively, we saw that we could view this action purely in terms of the boundary S^2 — the naive target space of the string theory — where it takes a form similar to that of rigid strings [48, 49]. And finally, we could write it in terms of a Schwarzian action that is very reminiscent of what happens for the AdS_2 duals of SYK models. It would be very interesting to understand these different viewpoints more directly. In each of these approaches, there is a signature of an extra dimension, and it would be very interesting to see how one can reverse engineer the entire *off-shell* AdS_3 sigma model from these different viewpoints.

- In any of these different ways of phrasing the action to leading order in $\frac{1}{N}$, it would be very interesting to identify the relevant saddle point that dominates at large N. For example in the Schwarzian approach, this amounts to identifying the covering map which minimises the Schwarzian functional, see eq. (2.71), as we vary the z_i over the n-punctured sphere, subject to the branching conditions at these insertions. Can we find this saddle point, and is there a nice geometric interpretation to it? Can we also understand the role of the finite N corrections to this saddle point?

- If we can find this Gross-Mende like saddle point for the genus zero covering space, we can ask about the corresponding saddles at higher genus. Are they related in some simple way through covers of the genus zero solution as in the flat space case [46]? Can we do a resummation of the pertubative expansion of the string theory in this limit?

- The reader might have noticed that we have not used much information about the seed CFT whose symmetric product we are taking. We have also not used supersymmetry anywhere in our analysis. It is likely that there are further criteria that must be obeyed by the integrands on moduli space to be correlators of a *bona fide* string background. It will be important, going ahead, to understand what these are, and to see how they arise on the field theory side.

Other Questions

- It might be timely to revisit the dual of the Gaussian matrix model along the lines of [14, 26]. In [26, 27], based on the association of Feynman diagrams for the Gaussian correlators with Belyi maps [65], a dual A-model closed topological string with target $\mathbb{P}^1$ was proposed. Recall that Belyi maps are also holomorphic covering maps of $\mathbb{P}^1$ with special branching, and thus the setting is very similar to the present one. In fact, in [27, 28] n-point correlators were compared on both sides, with there being a precise matching in the limit of large operators, quite analogous to the large twist limit considered here. It would be instructive to see this matching at the level of world-sheet correlators like in our present example.

- It would be very nice to use the present approach to get a more complete understanding of the string dual to 2d Yang-Mills theory [17, 81]. This, once again, is formulated in terms of branched covers of a target space by the world-sheet. The proposals of [50, 82] predate holography, and it is likely that we can lift the topological rigid string theory of [50] to higher dimensions, as suggested by the considerations of Section 2.6. One obstacle to overcome is to generalise the considerations here to the partition function and non-local observables like Wilson loops.

- Finally, to generalise these considerations to the all important case of four-dimensional gauge theories, it would be fruitful to make a connection of this approach with the hexagon program and integrability, perhaps along the lines already put forward in [83].

Appendix

2.A Riemann-Hilbert solution to the Penner-like models

In this appendix we present an alternative way of solving the matrix model to leading order in $\frac{1}{N}$. The idea is to solve the Riemann-Hilbert problem that is defined by the integral equation (2.27)

$$W'(\lambda) = -\Big[u(\lambda + i\epsilon) + u(\lambda - i\epsilon)\Big] \qquad (\lambda \in \mathcal{C}) \ , \tag{2.72}$$

which we have rewritten in terms of the resolvent (2.29). Note that the eigenvalue density itself is then given by the discontinuity of the resolvent to leading order in large N, see eq. (2.30). The general solution to this Riemann-Hilbert problem is of the form, see e.g. [61]

$$u_0(z) = \frac{1}{2}\oint_{\mathcal{C}} \frac{dv}{2\pi i}\,\frac{W'(v)}{z-v}\sqrt{\prod_{j=1}^{2\ell}\frac{z-a_j}{v-a_j}} \ , \tag{2.73}$$

where ℓ is the number of cuts $[a_{2i-1}, a_{2i}]$ with $i = 1, \ldots, \ell$.

For the case that is of interest to us, the potential $W(z)$ is given by eq. (2.23). The integrand of the contour integral in (2.73) has poles at $v = z$, as well as at $v = z_i$, while there is no pole at infinity (since $W'(v) \sim \frac{1}{v}$). Carrying out the integral we therefore find

$$u_0(z) = \frac{1}{2}\left[W'(z) - \sum_{i=1}^{n-1}\frac{\alpha_i}{(z-z_i)}\sqrt{\prod_{k=1}^{2\ell}\frac{z-a_k}{z_i-a_k}}\ \right] \ , \tag{2.74}$$

and the spectral curve (2.34) equals

$$y_0(z) = \sum_{i=1}^{n-1}\frac{\alpha_i}{(z-z_i)}\sqrt{\prod_{k=1}^{2\ell}\frac{z-a_k}{z_i-a_k}} \ . \tag{2.75}$$

The positions of the branch cuts, i.e. the a_i, are now determined by the requirement that the resolvent (2.74) goes as $\frac{1}{z}$ as $z \to \infty$. This is not obvious since the second term in the bracket goes as $\sim z^{\ell-1} + \cdots$. Requiring that these higher powers of z vanish (and that the coefficient of the z^{-1} is equal to 1), leads to the $\ell+1$ equations

$$\sum_{i=1}^{n-1} \tilde{\alpha}_i z_i^m = \alpha_0\, \delta_{m,\ell}\ , \qquad m = 0, \ldots, \ell\ , \tag{2.76}$$

where we have defined

$$\tilde{\alpha}_i = \frac{\alpha_i}{\sqrt{\prod_{k=1}^{2\ell}(z_i - a_k)}}\ , \qquad \text{and} \qquad \alpha_0 = \sum_{i=1}^{n-1} \alpha_i - 2 = -\alpha_n - \frac{2}{N}\ , \tag{2.77}$$

and we have used eq. (2.21) for the final equation. We would expect a maximum of $\ell = n-2$ cuts because there are $(n-2)$ critical points of W, $W'(z_j^*) = 0$, $j = 1, \ldots, n-2$, and usually the branch cuts emerge from the different critical points. For $\ell = n-2$, the system of $\ell+1$ homogeneous equations in (2.76) is invertible $\tilde{\alpha}_i$,

$$\tilde{\alpha}_i = \frac{\alpha_i}{\sqrt{\prod_{k=1}^{2n-4}(z_i - a_k)}} = \frac{a}{\prod_{j\neq i}(z_i - z_j)}\ , \qquad i = 1, \ldots, (n-1)\ , \tag{2.78}$$

where a is a constant. Plugging this solution into eq. (2.75) the spectral curve then simplifies to

$$y_0(z) = \frac{\alpha_n}{\prod_{i=1}^{n-1}(z - z_i)} \sqrt{\prod_{k=1}^{2n-4}(z - a_k)}\ . \tag{2.79}$$

Here we have used the fact that the residue at $z_n = \infty$ is α_n. We see that we have arrived at the same leading order spectral curve as we did through the loop equations — see eqs. (2.39) and (2.40).

Note that in this way of solving the matrix model, we use the $(n-1)$ resolvent equations in (2.76), as well as the $2(n-3)$ independent period equations of the spectral curve — over both the A- and B-cycles — in (2.55). These are then $(3n-7)$ conditions for the $(2n-4)$ a_k's, as well as the $(n-3)$ cross ratios of the z_i. (Recall we have used the Möbius invariance to fix three of the z_i). Thus the counting also works out as expected.

2.B Deriving the loop equations

In this appendix we derive the loop equations (2.32). We begin with the scattering equations rewritten as (2.20)

$$\frac{1}{2} W'(\lambda_a) = \frac{1}{N} \sum_{b\neq a} \frac{1}{\lambda_a - \lambda_b}\ . \tag{2.80}$$

Following [61], we now multiply both sides by $\frac{1}{(\lambda_a - z)}$ and sum over a to obtain

$$\frac{1}{2}\sum_{a=1}^{N}\frac{W'(\lambda_a)}{(\lambda_a - z)} = \frac{1}{N}\sum_{a\neq b}\frac{1}{(\lambda_a-\lambda_b)(\lambda_a - z)} \equiv \mathcal{S}\ . \tag{2.81}$$

We can rewrite $\mathcal{S}$ in terms of partial fractions as

$$\mathcal{S} = \frac{1}{N}\sum_{a\neq b}\frac{1}{(\lambda_a-\lambda_b)(\lambda_a - z)} = \frac{1}{N}\sum_{a\neq b}\frac{1}{(\lambda_b - z)}\left[\frac{1}{(\lambda_a-\lambda_b)} - \frac{1}{(\lambda_a - z)}\right]\ . \tag{2.82}$$

The first term on the right-hand-side equals $-\mathcal{S}$ — it is obtained from $\mathcal{S}$ in (2.81) upon exchanging the dummy variables $a \leftrightarrow b$ — and thus we deduce that

$$\begin{aligned}\mathcal{S} &= -\frac{1}{2N}\sum_{a\neq b}\frac{1}{(\lambda_a - z)(\lambda_b - z)} = -\frac{1}{2N}\Big(\sum_{a=1}^{N}\frac{1}{(z-\lambda_a)}\Big)^2 + \frac{1}{2N}\sum_{a=1}^{N}\frac{1}{(z-\lambda_a)^2}\\ &= -\frac{N}{2}u^2(z) - \frac{1}{2}u'(z)\ ,\end{aligned} \tag{2.83}$$

where $u(z)$ is the resolvent of eq. (2.29), which for finite N becomes

$$u(z) = \frac{1}{N}\sum_{a=1}^{N}\frac{1}{(z-\lambda_a)}\ . \tag{2.84}$$

Plugging this back into eq. (2.81) we therefore deduce that

$$\frac{1}{2N}\sum_{a=1}^{N}\frac{W'(\lambda_a)}{\lambda_a - z} = -\frac{1}{2}u^2(z) - \frac{1}{2N}u'(z)\ . \tag{2.85}$$

If we introduce the function $R(z)$ via

$$R(z) = \frac{1}{N}\sum_{a=1}^{N}\frac{W'(\lambda_a) - W'(z)}{(\lambda_a - z)}\ , \tag{2.86}$$

then (2.85) becomes

$$u^2(z) - W'(z)u(z) + \frac{1}{N}u'(z) + R(z) = 0\ , \tag{2.87}$$

see eq. (2.32). We note that for our Penner like potential

$$R(z) = \frac{1}{N}\sum_{a=1}^{N}\frac{W'(z) - W'(\lambda_a)}{z-\lambda_a} = -\frac{1}{N}\sum_{i=1}^{n-1}\frac{\alpha_i}{z-z_i}\sum_a\frac{1}{\lambda_a - z_i} = \sum_{i=1}^{n-1}\frac{\alpha_i u(z_i)}{z-z_i}\ , \tag{2.88}$$

where we have used the definition (2.29) for the resolvent. We also note that

$$\sum_{i=1}^{n-1} \alpha_i u(z_i) = -\frac{1}{N} \sum_{i=1}^{n-1} \sum_a \frac{\alpha_i}{\lambda_a - z_i} = -\frac{2}{N^2} \sum_a \sum_{b \neq a} \frac{1}{\lambda_a - \lambda_b} = 0 \ , \tag{2.89}$$

where we have used the saddle point equation (2.20) in the second last equality, and antisymmetry under (a, b) exchange in the last. These relations again hold at finite N.

Chapter 3

Twistor Coverings and Feynman Diagrams

3.1 Introduction

Understanding the string dual to free $\mathcal{N} = 4$ super Yang-Mills theory would give a new vantage point from which one could set out to decipher the AdS/CFT correspondence; this being the diametrically opposite regime from that described by supergravity on the dual $\mathrm{AdS}_5 \times S^5$ spacetime. Recently, in [84, 85], a proposal for a worldsheet description of the dual tensionless string theory was made[1]. This builds on the success of the description of the corresponding tensionless limit for strings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ dual to the free symmetric product orbifold CFT, $\mathrm{Sym}^N(\mathbb{T}^4)$[9, 10, 15, 90–93][2]. Both tensionless string descriptions are in terms of a set of free twistor variables, subject to a gauge constraint, with spectrally flowed sectors in the Hilbert space. These spectrally flowed sectors are crucial in accounting for the rich spectrum of single trace operators of the dual large N CFT. While a first principles quantisation of the worldsheet theories is yet to be carried out in both cases, one can argue, based on a few plausible assumptions on the physical state conditions, that the spectrum nontrivially matches on both sides [9, 84, 85].

Additional support, in the free super Yang-Mills example, comes from the fact that the proposal gives a covariant version of the BMN [101] organisation of the large N spectrum. Furthermore, this proposal is a closed string cousin of the ambitwistor open string description [102, 103] of tree level $\mathcal{N} = 4$ Yang-Mills gluon amplitudes [104]. The worldsheet realisation of higher spin symmetries in this proposal has also been studied in [105].

In the AdS_3/CFT_2 case, one can, in fact, go beyond the agreement of the spectrum

[1] A somewhat different topological string approach to free super Yang-Mills is proposed in [86–89].

[2] The tensionless limit for AdS_3 i.e. with massless higher spin fields in the spectrum, occurs for $k = 1$ units of NS-NS flux - see [8, 94–96] for some of the precursor works at this special point. For works exploring the $k < 1$ region, the transition to the $k > 1$ region and their CFT duals, see [97–100].

and argue for the (manifest) equality of correlation functions on both sides, giving a *de facto* derivation of the correspondence [10, 90, 91]. On the worldsheet this followed from the localisation, on the string moduli space, of physical correlators onto those discrete points which admit finite degree holomorphic covering maps from the worldsheet to the spacetime (more accurately, its S^2 boundary) with specified branching data. This localisation was later seen to follow from a stringy incidence relation obeyed by the twistorial worldsheet fields [91, 92]. This striking property on the worldsheet is precisely what is needed to match the Lunin-Mathur computation [106] of correlators in the free symmetric orbifold CFT [10]. The latter computation is in terms of the same covering maps from an auxiliary Riemann surface (now identified with the worldsheet, cf. [107]) to the boundary S^2.

The tensionless AdS_3/CFT_2 example also gives a rather explicit realisation [15] of a general mechanism for implementing open-closed string duality [13, 22, 23]. This mechanism proceeds via a recasting of individual Feynman diagrams of the field theory into worldsheets of the dual closed string theory using the Strebel parametrisation of moduli space [108]. The upshot of [15] was that, at least for correlators with large twist, the localisation of the previous paragraph, was precisely to the points on moduli space which admit an *integer* Strebel differential. Strebel differentials are special meromorphic quadratic differentials $\phi_S(z)dz^2$ which are completely characterised by real "lengths" l_{ij} (defined as $\int_{a_i}^{a_j} dz\sqrt{\phi_S(z)}$ between zeroes (a_i, a_j)); the Strebel lengths give a real parametrisation of the string moduli space. For an integer Strebel differential these lengths are proportional to (positive) integers[3].

In the large twist limit of [15], this integer Strebel differential arises naturally as the Schwarzian of the covering map. As a consequence, one can identify the integer Strebel lengths directly with the number of "Wick contractions" between vertices of the Feynman diagram that is associated to each covering map [107] contribution to the field theory correlator. In other words, there is a one to one correspondence between the Feynman diagrams of the field theory and the individual closed string worldsheets that the string correlator localises onto. This gives a precise realisation of the proposal of [13], as refined by [14], where the Strebel differential is the bridge between the Feynman diagrams of the field theory and the worldsheets of the dual closed string. In fact, in this approach, the Strebel construction of the closed string worldsheet by gluing up strips with fixed Strebel lengths is the mathematical underpinning of open-closed string duality, with the strips being identified with the open string Feynman diagrams. See Fig. 3.1.1. In the AdS_3/CFT_2 case, it was moreover seen that the closed string weight associated to each Feynman diagram was the natural Nambu-Goto area for the worldsheet metric in Strebel gauge.

[3] These are very special points in the moduli space which correspond to arithmetic Riemann surfaces (see for instance [109]) and also arise in the worldsheet dual to the Gaussian matrix model [14, 26, 27].

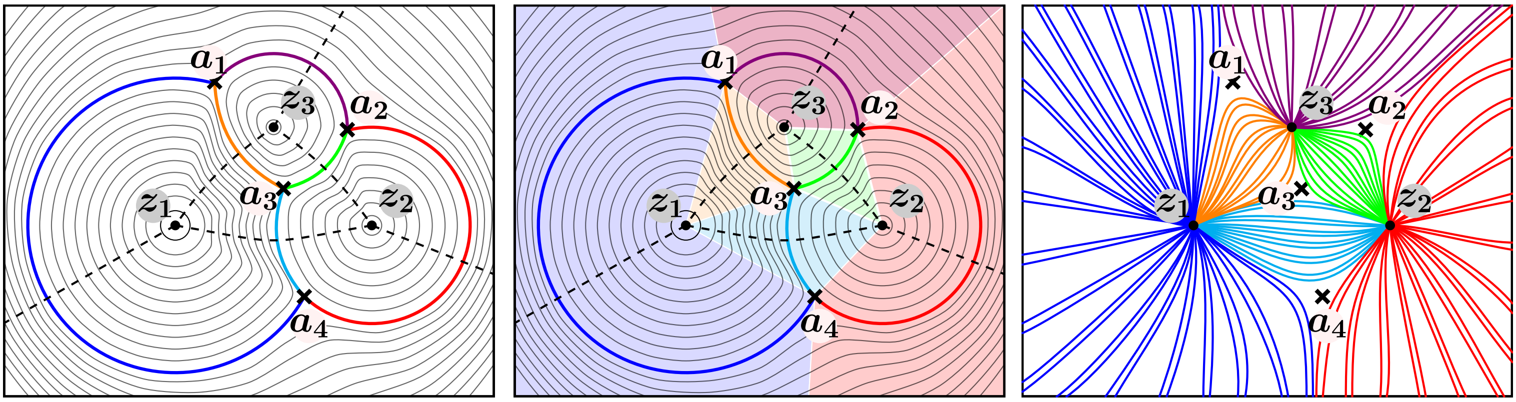

Figure 3.1.1: Open-Closed String Duality and the Strebel Construction. (Left) A foliation of a closed string Riemann Surface $\Sigma_{g,n}$ by the horizontal trajectories of the associated Strebel differential. The coloured lines are 'critical' horizontal trajectories with a_k the zeroes of the differential. They form the Strebel graph. The z_i are double poles and $z_4 = \infty$ (not shown). (Middle) The Strebel graph gives a canonical decomposition ('gluing') of $\Sigma_{g,n}$ into (shaded) regions which are conformal to infinite strips. The Strebel lengths are the widths of these strips. (Right) These strips are viewed as open string diagrams (Feynman-'tHooft double lines) which give rise to the free field Wick contractions. The width of the strips is identified with the number of contractions. Note that the skeleton Feynman graph (from gluing together homotopic edges), denoted by the dashed line in the middle and left, is the dual to the Strebel graph.

In this chapter, we will take steps towards generalising the above considerations and building a geometric picture in terms of twistor maps. We begin with AdS_3 and write down the classical twistor configurations which correspond to the holomorphic covering maps $\Gamma(z)$ that are the (exact) saddle points of the (genus zero) worldsheet path integral. These twistor covers are elegantly given in terms of the simple polynomials which enter into $\Gamma(z)$ (see Eqs.(3.22), (3.21), (3.14)). This is consistent with the fact that only a finite number of wedge modes (for each twist/spectral flow label w) of the twistor fields are physical (Sec. 5 of [85]) and therefore excited in the classical configuration[4]. We also find that the worldsheet stress tensor $T(z)$ of the twistor theory (Eq. (3.26)), evaluated on these twistor configurations is exactly the Schwarzian (and thus the Strebel differential, for large w) $S[\Gamma](z)$. This rounds off the picture for AdS_3 in giving the twistor versions of the classical string configurations that the tensionless worldsheet path integral localises onto [10]. It will be interesting to see how the localisation onto these configurations arises directly from a worldsheet path integral for the twistor fields.

We next proceed to consider the analogues of these solutions for AdS_5. As per the proposal of [84, 85], the worldsheet theory dual to the tensionless $\mathrm{AdS}_5 \times \mathrm{S}^5$ now consists of the ambitwistor fields

$$\begin{aligned} Z^I &= (\lambda^\alpha, \mu^{\dot\alpha}; \psi^a) \\ Y_J &= (\mu^\dagger_\beta, \lambda^\dagger_{\dot\beta}; \psi^\dagger_a) \end{aligned} \tag{3.1}$$

subject to the ambitwistor constraint $Y_I Z^I = 0$. We will drop, from now on, the fermionic

[4]Recall that the wedge modes are those with mode number r ($-\frac{w-1}{2} \leq r \leq \frac{w-1}{2}$). They behave as generalised zero modes in the wth spectrally flowed sector.

fields which parametrise the S^5 and focus on the bosonic twistor variables $(\lambda^{\alpha}, \mu^{\dot{\alpha}})$ and $(\mu^{\dagger}_{\beta}, \lambda^{\dagger}_{\dot{\beta}})$. These fields can be viewed, on the one hand, as parametrising the ambitwistor space of the conformal boundary of AdS_5 or, on the other hand, the twistor space of AdS_5. We employ here the twistor description of (euclidean) AdS_5 and its incidence relations following [110] (for our notation and conventions, see Appendix 3.A and [111]). We are thereby led to propose a set of stringy incidence relations that the twistor worldsheet configurations should obey in the tensionless limit.

$$\begin{aligned}
\mu^{\dagger}_{\alpha}(z) + X_{\alpha}{}^{\dot{\beta}}(z,\bar{z})\lambda^{\dagger}_{\dot{\beta}}(z) &= \lambda^{\beta}(z)\epsilon_{\beta\alpha} \\
\mu^{\dot{\alpha}}(z) - X^{\dot{\alpha}}{}_{\beta}(z,\bar{z})\lambda^{\beta}(z) &= \frac{1}{2}R^2(z,\bar{z})\epsilon^{\dot{\alpha}\dot{\beta}}\lambda^{\dagger}_{\dot{\beta}}(z)\ .
\end{aligned} \tag{3.2}$$

Here, $X^{\dot{\alpha}}{}_{\beta}(z,\bar{z})$ are the stringy coordinates along the $d = 4$ boundary (in bispinor notation) while $R(z,\bar{z})$ is the radial profile. One may view these stringy incidence relations as a general solution of the (bosonic) ambitwistor constraint (see Eq.(3.44)) which must be imposed in the worldsheet theory as a gauge constraint. While the relations can be thus motivated, we expect them to be directly derived from the properties of worldsheet correlators as in [91]. This is currently under investigation [112].

Imposing reality conditions on the $X^{\dot{\alpha}}{}_{\beta}(z,\bar{z})$ appropriate to the euclidean signature implies a natural set of reality conditions on the twistor fields. This is a direct indication that the (physical) left and right moving modes on the worldsheet are not independent as already suggested in [84, 85]. These reality conditions also enable us to invert the incidence relations and express the spacetime string configurations in terms of the twistor configurations. We will often restrict to configurations where the worldsheet is essentially at the boundary of the AdS_5, as was the case for AdS_3. However, unlike for AdS_3, the boundary spacetime string configurations are in general, non-holomorphic. Nevertheless, there is a hidden holomorphy inherited from that of the underlying twistor fields. Thus, we immediately observe from the structure of Eq.(3.2) that the matrix $\bar{\partial}X_{\alpha}{}^{\dot{\beta}}(z,\bar{z})$ has one zero eigenvalue.

We then specialise to a special kinematic regime where the dual field theory operators are inserted at points x_i all of which lie in a two dimensional plane (note that this is not a restriction for 2-, 3- and 4-point correlators; recall that for 4 points we can use the conformal symmetry of $\mathcal{N} = 4$ SYM). In this case, the holomorphic eigenvector picks out (together with the radial direction) an AdS_3 subspace. We can write down the twistor configuration again in terms of polynomials, whose ratio gives a *bona fide* holomorphic covering map. The polynomial nature of the configuration is again a reflection of the fact that only the wedge modes of the twistor fields are physical, as proposed in [84, 85].

The simplest such polynomial corresponds to the two point function of a BPS highest weight state in the SYM theory. In this case, from the explicit form of the covering map, we see that its Schwarzian, in the natural coordinates, is a constant which agrees with the Strebel differential in this case. The Strebel length is proportional to the number of Wick contractions. Further, we compute the Strebel area i.e. the area computed with the worldsheet metric given by $ds^2 = |\phi_S(z)||dz|^2$ – "Strebel gauge" [26]. This is formally infinite but needs to be evaluated with a careful regularisation. When we do that we find that a Nambu-Goto weight with the Strebel area precisely reproduces the free Feynman propagator associated to the Feynman diagrams! This picture can be extended to a multipoint correlator by gluing together the Strebel differentials in different patches, which we then identify with the Schwarzian of the covering map. The *additive* nature of the Strebel areas for each strip directly leads to the *multiplicative* form of the individual Feynman propagators.

We view this as evidence that, in parallel to the AdS_3/CFT_2 example, there exists a closed string picture in terms of twistor covering maps underlying Feynman diagrams in the free field limit. In particular, the fact that twistor covers of AdS_5 reproduce the Feynman propagator, through the Strebel construction, buttresses the twistor proposal [84, 85] for the worldsheet description of free $\mathcal{N} = 4$ super Yang-Mills theory. Admittedly, we are working in a special kinematic regime, for the 4-point and higher correlator, but we expect to be able to overcome this restriction in the future.

The plan of this chapter is as follows: Sec. 3.2 focusses on the AdS_3 case, where we exhibit the twistor configurations corresponding to a covering map. After a brief review of the twistor geometry for AdS_5 in Sec. 3.3, we describe the euclidean reality conditions that we impose and some of its consequences. Sec. 3.4 uses these ingredients to propose stringy incidence relations for the worldsheet twistors. We then specialise to the classical configurations that correspond to mapping worldsheets into an AdS_3 subspace. We employ the explicit form of these maps in Sec. 3.5 to flesh out the connection to Feynman diagrams in the gauge theory. We describe the resulting picture, which agrees with the Strebel construction, moreover reproducing the free propagator from the regularised Strebel area. Sec. 3.6 has brief concluding remarks. Appendix 3.A sets out our twistor conventions and the nature of the twistor correspondence, with or without the Euclidean reality conditions.

3.2 Twistor covering maps in AdS_3

As we discussed in the last chapter, the correlators in the free symmetric product CFT_2 are given in terms of covering maps from an auxiliary covering space to the spacetime S^2, with specified branching data. In [10] it was shown that this is precisely mirrored in the tensionless string theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ (i.e. with $k = 1$ unit of NS-NS flux). Indeed, as anticipated in [106, 107] the covering space is the worldsheet of the dual string theory. Thus the corresponding correlators in the worldsheet theory have the remarkable property that they are delta function supported on those points on the worldsheet moduli space which admit covering maps consistent with the branching data. The fundamental origin of this localisation was later seen to be a twistorial ward identity for correlators in the free field realisation of the $\mathfrak{psu}(1,1|2)_1$ worldsheet theory [91, 92].

In [10] it was also shown that when there is an admissible covering map, there is a semiclassical solution for the string worldsheet which exhibits this covering of the boundary S^2 of the AdS_3 part of the spacetime. As will be reviewed below, since the worldsheet is embedded in AdS_3 it also has a nontrivial radial profile even though it is essentially at the boundary. This gives a geometric picture, even in this highly stringy regime, of the string worldsheet exhibiting a nontrivial wrapping of the spacetime boundary. Since the underlying worldsheet theory is free in this limit, as can be seen in terms of the original sigma model (see Eq. (2.4) and below of [10]) or from the free field realisation of the $\mathfrak{psu}(1,1|2)_1$ theory, these solutions are semiclassically exact. In fact, one can show [10] that the semiclassical action associated to these solutions gives precisely the Lunin-Mathur weight associated to each covering map.

In this section, we will translate the solutions of [10], which were given in terms of the AdS_3 coordinates, into classical solutions for the twistor fields that describe the $\mathfrak{psu}(1,1|2)_1$ theory. We will see that the solutions have a nice form which is also consistent with the behaviour of quantum correlators of these fields as found in [91]. We will also show that the classical stress tensor evaluated on these solutions agrees with the Schwarzian derivative of the covering map. Thus the stress tensor of the free field realisation is closely related to the Strebel differential due to the connection between the latter and the Schwarzian (at least in the limit of large twist). These solutions will give some guidance when we generalise to the case of the twistor classical solutions for the worldsheet theory of tensionless strings in AdS_5 in the next section.

The conventional AdS_3 sigma model action, in first order form, is given by

$$S_{\mathrm{AdS}_3} = \frac{k}{4\pi}\int \mathrm{d}^2 z\ \left(4\partial\Phi\bar{\partial}\Phi + \bar{\beta}\partial\bar{\gamma} + \beta\bar{\partial}\gamma - \mathrm{e}^{-2\Phi}\beta\bar{\beta} - k^{-1}R\Phi\right) . \tag{3.3}$$

Here $\gamma(z)$ (and its conjugate) represent holomorphic coordinates for the spacetime S^2's with which we foliate euclidean AdS_3. The radial coordinate $\Phi(z,\bar{z})$ is related to the usual poincare coordinate as $r = e^{-\Phi}$, such that the boundary is at $r = 0 \leftrightarrow \Phi = \infty$. In terms of these coordinates one finds classical solutions by considering the holomorphic $\mathfrak{sl}(2,\mathbb{R})$ currents, as given by the Wakimoto form and imposing the right boundary conditions at the insertions together with the condition on the spacetime energies for the vertex operators (the J_0^3 eigenvalue near each insertion).

$$\begin{aligned}
J^+ &= \beta \\
J^3 &= -\partial\Phi + k\beta\gamma \\
J^- &= -2\gamma\partial\Phi + \beta\gamma\gamma - \partial\gamma\ .
\end{aligned} \tag{3.4}$$

Here we have put the level $k = 1$ in the usual (classical) Wakimoto representation. Thus the classical solution for an n-point correlator of vertex operators for the ground states of twisted sectors of the dual symmetric orbifold CFT, is [10]

$$\gamma(z) = \Gamma(z) \ , \qquad \Phi(z, \bar{z}) = -\log \epsilon - \frac{1}{2} \ln |\partial\Gamma|^2 \ , \qquad \beta(z) = -\frac{(\partial\Phi)^2}{\partial\Gamma} \ . \tag{3.5}$$

$\Gamma(z)$ is the covering map, from the worldsheet to the spacetime, with branch points at the points z_i where the vertex operators (in spectrally flowed sectors w_i) are inserted, with the behaviour as $z \to z_i$,

$$\Gamma(z) = x_i + a_i^{\Gamma}(z - z_i)^{w_i} + \ldots, \qquad (i = 1 \ldots n) \ . \tag{3.6}$$

The covering map is uniquely specified by the branching data (z_i, w_i) and specifying three of the x_i's. Equivalently, if the n locations x_i are specified, the covering map exists only for discrete choices of $(n - 3)$ of the z_i i.e. on a discrete set of points on the moduli space $\mathcal{M}_{0,n}$. The radial field $\Phi(z, \bar{z})$ has an infrared cutoff $\epsilon \to 0$ which indicates that the string is stuck at the boundary ($\Phi = \infty$). We also see that $\partial\Phi$ is independent of ϵ and exhibits a nontrivial profile which will play an important role. Finally, the last relation in Eq. (3.5) for $\beta(z)$ follows from the on-shell condition for the worldsheet stress tensor which implies (for these ground state correlators) that

$$J^+(z)J^-(z) = (J^3(z))^2 \ . \tag{3.7}$$

We now note that the $\mathfrak{sl}(2, \mathbb{R})$ currents in Eq. (3.4) are bilinears of the free twistor fields of AdS_3 (see Eq. (2.2) of [91]). Thus in terms of the pairs of symplectic bosons $\xi^\pm$ and $\eta^\pm$ of $\mathfrak{u}(1, 1|2)_1$ we have

$$J^3(z) = -(\eta^+\xi^-)(z) \ , \quad J^\pm(z) = (\eta^\pm\xi^\pm)(z) \tag{3.8}$$

where we additionally have used the ambitwistor constraint which generates the quotient $\mathfrak{psu}(1, 1|2)_1$

$$\xi^+\eta^- = \xi^-\eta^+ \ . \tag{3.9}$$

We notice that the on-shell constraint Eq. (3.7) is automatically satisfied in terms of the twistor variables.

We also note that this free field representation has the gauge freedom in which we rescale

$$\xi^\pm(z) \to \lambda(z)\xi^\pm(z) \ , \qquad \eta^\pm(z) \to \lambda(z)^{-1}\eta^\pm(z) \ . \tag{3.10}$$

Together with the ambitwistor constraint Eq. (3.9), this means one can choose a gauge in which $\xi^\pm(z) = -\eta^\pm(z)$.

We can then solve for ξ^+ and η^+ using the expression for $J^+(z)$ in Eq. (3.4) and equating it to that in Eq. (3.8), together with the classical solution for $\beta(z)$ given in Eq. (3.5). Then the expression for $J^3(z)$ allows us to solve for $\xi^-(z)$ and $\eta^-(z)$. We thus find the classical twistor solutions (in our gauge) to be

$$\xi^+ = -\eta^+ = -\frac{\partial\Phi}{\sqrt{\partial\Gamma}} = \frac{1}{2}\frac{\partial^2\Gamma}{(\partial\Gamma)^{\frac{3}{2}}}, \quad \xi^- = -\eta^- = \frac{\Gamma\partial\Phi + \partial\Gamma}{\sqrt{\partial\Gamma}} = -\frac{\Gamma}{2}\frac{\partial^2\Gamma}{(\partial\Gamma)^{\frac{3}{2}}} + \sqrt{\partial\Gamma} \ . \tag{3.11}$$

Note that the classical solutions obey stringy twistor "incidence relations" in terms of the boundary variables.

$$\xi^- + \Gamma\xi^+ = \sqrt{\partial\Gamma} = \sqrt{\rho}, \qquad \eta^- + \Gamma\eta^+ = -\sqrt{\partial\Gamma} = -\sqrt{\rho} \ . \tag{3.12}$$

The right hand side is not zero but rather proportional to the radial profile. Thus we have defined a holomorphic radial profile $\rho(z)$, using Eq.(3.5),

$$r^2(z,\bar{z}) = e^{-2\Phi(z,\bar{z})} \equiv \epsilon^2\rho(z)\bar{\rho}(\bar{z}) \ . \tag{3.13}$$

The RHS of Eq.(3.12) vanishes as $(z-z_i)^{\frac{w_i-1}{2}}$ as one approaches any of the inserted vertex operators, where the string worldsheet is pinned to the boundary.

Given a covering map

$$\Gamma(z) = \frac{P_N(z)}{Q_N(z)} \tag{3.14}$$

where $P_N(z)$ and $Q_N(z)$ are degree N polynomials, define the Wronskian

$$W = P_N'(z)Q_N(z) - Q_N'(z)P_N(z) = C\prod_{i=1}^{n}(z-z_i)^{w_i-1} \ . \tag{3.15}$$

In the second equality we have used that $\partial\Gamma(z) = \frac{W(z)}{Q_N^2(z)}$ and vanishes as $(z-z_i)^{w_i-1}$ near each of the z_i. Comparing the degrees of the polynomials in Eq. (3.15), we see that this determines the Wronskian upto the overall constant C.

Then the AdS_3 twistors Eq. (3.11) can be expressed nicely as

$$\begin{aligned}
\xi^+ &= -\eta^+ = \frac{1}{2W(z)^{\frac{3}{2}}}(W'(z)Q_N(z) - 2W(z)Q_N'(z)) \\
\xi^- &= -\eta^- = -\frac{1}{2W(z)^{\frac{3}{2}}}(W'(z)P_N(z) - 2W(z)P_N'(z))
\end{aligned} \tag{3.16}$$

We also see that

$$W'(z) = W(z)\sum_{i=1}^{n}\frac{(w_i-1)}{z-z_i} = W(z)\frac{\tilde{R}_{n-1}(z)}{\prod_{i=1}^{n}(z-z_i)} \tag{3.17}$$

where $\tilde{R}_{n-1}(z) = \sum_{i=1}^{n}(w_i - 1)\prod_{j\neq i}^{n}(z - z_j)$ is a polynomial of degree $(n-1)$ determined solely by the covering map data (z_i, w_i). Using this, we can rewrite

$$\begin{aligned} W'(z)Q_N(z) - 2W(z)Q_N'(z) &= W(z)\frac{\tilde{R}_{n-1}(z)Q_N(z) - 2\prod_{i=1}^{n}(z-z_i)Q_N'(z)}{\prod_{i=1}^{n}(z-z_i)} \\ &= \frac{W(z)\tilde{Q}_{N+n-1}(z)}{\prod_{i=1}^{n}(z-z_i)} \end{aligned} \tag{3.18}$$

where $\tilde{Q}_{N+n-1}(z)$ is a polynomial of degree $(N+n-1)$ that is completely determined by $Q_N(z)$ and the covering map data (z_i, w_i). In fact, notice that

$$-\frac{1}{2}\frac{\tilde{Q}_{N+n-1}(z)}{\prod_{i=1}^{n}(z-z_i)^{\frac{(w_i+1)}{2}}} = \frac{d}{dz}\left[\frac{Q_N(z)}{\prod_{i=1}^{n}(z-z_i)^{\frac{(w_i-1)}{2}}}\right] . \tag{3.19}$$

We can similarly define

$$\tilde{P}_{N+n-1}(z) = \tilde{R}_{n-1}(z)P_N(z) - 2\prod_{i=1}^{n}(z-z_i)P_N'(z) . \tag{3.20}$$

Then we can express the twistor classical solutions in a nice symmetric form as

$$\xi^+ = -\eta^+ = \frac{\tilde{Q}_{N+n-1}(z)}{2\prod_{i=1}^{n}(z-z_i)^{\frac{(w_i+1)}{2}}} = -\frac{d}{dz}\left[\frac{Q_N(z)}{\prod_{i=1}^{n}(z-z_i)^{\frac{(w_i-1)}{2}}}\right] \tag{3.21}$$

and similarly,

$$\xi^- = -\eta^- = -\frac{\tilde{P}_{N+n-1}(z)}{2\prod_{i=1}^{n}(z-z_i)^{\frac{(w_i+1)}{2}}} = \frac{d}{dz}\left[\frac{P_N(z)}{\prod_{i=1}^{n}(z-z_i)^{\frac{(w_i-1)}{2}}}\right] . \tag{3.22}$$

The takeaway from the final form of the twistor solutions in Eqs. (3.21, 3.22) is that they are rational functions which are determined by the two polynomials Q_N, P_N and the covering data. This fits in with our expectation for the quantum theory that only the wedge modes of the twistors are excited on-shell – see the discussion around Eq. (2.26) of [85]. We see that these twistors have a singularity as $z \to z_i$

$$\xi^{\pm}, \eta^{\pm} \sim \frac{1}{(z-z_i)^{\frac{(w_i+1)}{2}}} . \tag{3.23}$$

This is consistent with what we know about the OPE of these fields in the quantum correlators. In particular, we know that at the origin $(x = 0)$ the fields behave as in Eqs. (2.36-39) of [91]. However, at a generic point, the OPE of these fields with the spectrally flowed vertex operators is given by Eq. (2.42) of [91]. And in that notation $\xi^{\pm(x)}, \eta^{\pm(x)}$ behave as $(z-z_i)^{-\frac{(w_i+1)}{2}}$. See also Eq. (4.12) of [91]. The combination which appears in the incidence relation then has a regular OPE which is also reflected in the vanishing of the right hand side of Eq. (3.12) as $z \to z_i$.

For the generalisation and comparison to the higher dimensional case we define the ambitwistor variables

$$Z^I = \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix}, \quad Y_I = \begin{pmatrix} -\eta^- \\ \eta^+ \end{pmatrix} \tag{3.24}$$

and the ambitwistor constraint Eq. (3.9) reads as

$$Y_I Z^I = \xi^- \eta^+ - \xi^+ \eta^- = 0 \; . \tag{3.25}$$

We also note for later reference that the stress tensor of the worldsheet theory is given by

$$\begin{aligned} T(z) &= \frac{1}{2}\epsilon_{\alpha\beta}(\xi^\alpha \partial \eta^\beta + \eta^\alpha \partial \xi^\beta) \\ &= -\eta^- \partial \xi^+ + \eta^+ \partial \xi^- = Y_I \partial Z^I \end{aligned} \tag{3.26}$$

where we have used Eq. (3.9) in the second line. Evaluating this in terms of the classical solution Eq. (3.11) leads to

$$\begin{aligned} T(z) &= \frac{1}{2}\left[\partial\left(\frac{\partial^2\Gamma}{\partial\Gamma}\right) - \frac{1}{2}\left(\frac{\partial^2\Gamma}{\partial\Gamma}\right)^2\right] \\ &= \frac{1}{2} S\left[\Gamma(z)\right] \end{aligned} \tag{3.27}$$

In [15], it was shown that, in the large twist limit, the quadratic differential defined by the Schwarzian of the covering map could be identified with the integer Strebel differential which defines the corresponding closed string worldsheet. We therefore see that there is a close relation between the worldsheet stress tensor and the Strebel diffferential - a relation which we think should be generic in open-closed string duality[5].

3.3 The Twistor space of AdS_5

We want to generalise the twistor solutions of the previous section to the case of AdS_5. As a preliminary, in this section, we will describe the twistor space of AdS_5 and its relation to the ambitwistor space of its conformal boundary, following the discussion in [110]. We then review the twistor incidence relations in AdS_5 as well as the reality condition that can be imposed in Euclidean signature. We will find that imposing these reality conditions enables one to explicitly solve for the usual spacetime coordinates of AdS_5 in

[5]See remark 5.3 on p. 101 of [113] for a reason why the two might be related in a 'heavy' limit. We thank Edward Mazenc for drawing our attention to this.

terms of the twistors.

We begin with the complex projective space $\mathbb{CP}^5$ with homogeneous co-ordinates X^{IJ} represented as an antisymmetric 4×4 matrix with the identification $X \sim \lambda X$ with $\lambda \in \mathbb{C}^*$. We can then define a holomorphic metric on $\mathbb{CP}^5$ which has the property of being invariant under local scalings $X_{IJ} \to \lambda(X^2)X_{IJ}$,

$$ds^2 = -\frac{dX^2}{X^2} + \left(\frac{X \cdot dX}{X^2}\right)^2 \tag{3.28}$$

where the contraction of indices is performed w.r.t ϵ_{IJKL}. This is the metric on complexified AdS$_5$.

The conformal boundary of this spacetime is at $X^2 = 0$ i.e.

$$M = \{X^2 = 0 \,|\, X \in \mathbb{CP}^5\} \ . \tag{3.29}$$

Using the scaling freedom we can always parametrise the points in M as, [110] [6]

$$X_b^{IJ} = \begin{bmatrix} \epsilon^{\alpha\beta} & x^{\alpha\dot{\beta}} \\ -x^{\dot{\alpha}\beta} & \frac{1}{2}x^2\epsilon^{\dot{\alpha}\dot{\beta}} \end{bmatrix} \ . \tag{3.30}$$

This form ensures $X_b^2 = 0$ i.e $\det(X_b) = 0$. It immediately gives

$$(X_b)_{IJ} = \begin{bmatrix} \frac{1}{2}x^2\epsilon_{\alpha\beta} & -x_{\alpha\dot{\beta}} \\ x_{\dot{\alpha}\beta} & \epsilon_{\dot{\alpha}\dot{\beta}} \end{bmatrix} \ . \tag{3.31}$$

Going away from the boundary, in terms of,

$$I^{IJ} = \begin{bmatrix} 0 & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{bmatrix} \ , \tag{3.32}$$

we can parametrise a generic point as

$$X^{IJ} = (X_b)^{IJ} + \frac{r^2}{2}I^{IJ} = \begin{bmatrix} \epsilon^{\alpha\beta} & x^{\alpha\dot{\beta}} \\ -x^{\dot{\alpha}\beta} & \frac{1}{2}(x^2 + r^2)\epsilon^{\dot{\alpha}\dot{\beta}} \end{bmatrix} \ . \tag{3.33}$$

We then have

$$X_{IJ}X^{IJ} = 2r^2 \ . \tag{3.34}$$

It is not difficult to verify that this parametrisation, when plugged into Eq. (3.28), gives the usual Poincare metric, with r^2 being the radial coordinate. We refer the reader to [110] for further details.

[6] For our conventions, especially with regard to spinor indices, we refer the reader to Appendix 3.A.

We now define the twistors

$$\begin{aligned} Z^I &= (\lambda^\alpha, \mu^{\dot\alpha}) = (\lambda^1, \lambda^2, \mu^1, \mu^2) \\ Y_J &= (\mu^\dagger_\beta, \lambda^\dagger_{\dot\beta}) = (\mu^\dagger_1, \mu^\dagger_2, \lambda^\dagger_1, \lambda^\dagger_2) \end{aligned} \tag{3.35}$$

These will play the role of ambitwistor variables for the conformal boundary of AdS_5 but will more generally be viewed as twistor variables for the bulk. We next describe the incidence relations that they obey which define the twistor correspondence with spacetime on the boundary as well as the bulk.

3.3.1 The incidence relation on the boundary

On the boundary the above twistor variables (Z^I, Y_J) can be defined as being in the kernel of $(X_b)_{IJ}$ and its dual X_b^{IJ}, respectively [110]. Note that the kernel is non-empty precisely at the boundary where, as we noted $\det(X_b) = 0$. Then we see that the relation $(X_b)_{IJ} Z_b^J = 0$ yields

$$\begin{aligned} x_{\dot\alpha\beta}\lambda^\beta + \epsilon_{\dot\alpha\dot\beta}\,\mu^{\dot\beta} &= 0 \\ \frac{1}{2}x^2\epsilon_{\alpha\beta}\lambda^\beta - x_{\alpha\dot\beta}\,\mu^{\dot\beta} &= 0 \end{aligned} . \tag{3.36}$$

In fact, the second equation is a consequence of the first, using the first identity in (3.106). The latter can also be expressed as

$$\mu^{\dot\alpha} = x^{\dot\alpha}{}_\beta \lambda^\beta \ . \tag{3.37}$$

The dual kernel relation $X_b^{IJ} Y_J^b = 0$ yields

$$\begin{aligned} \epsilon^{\alpha\beta}\mu^\dagger_\beta + x^{\alpha\dot\beta}\lambda^\dagger_{\dot\beta} &= 0 \\ -x^{\dot\alpha\beta}\mu^\dagger_\beta + \frac{1}{2}x^2\epsilon^{\dot\alpha\dot\beta}\lambda^\dagger_{\dot\beta} &= 0 \ . \end{aligned} \tag{3.38}$$

Again the second equation arises from the first and the latter can be neatly expressed as

$$\mu^\dagger_\alpha = -x_\alpha{}^{\dot\beta}\lambda^\dagger_{\dot\beta} \ . \tag{3.39}$$

We also note that the incidence relation Eq. (3.37) and its dual Eq. (3.39) together imply that the (Z_b^I, Y_J^b) obey the ambitwistor constraint

$$\mathcal{C}_b \equiv Z_b^I Y_I^b = 0 \ , \tag{3.40}$$

justifying the terminology.

Using the incidence relations (3.37) and (3.39), we can represent a point on the boundary $M_{\mathbb{C}}$ via any of the two bitwistors:

$$(X_b)^{IJ} = -\frac{Z_1^{[I}Z_2^{J]}}{\langle\lambda_1\lambda_2\rangle} \quad \text{and} \quad (X_b)_{IJ} = -\frac{(Y_1)_{[I}(Y_2)_{J]}}{[\lambda_1^\dagger\lambda_2^\dagger]} \tag{3.41}$$

This is the usual twistor correspondence which associates a line in $\mathbb{PT}$ (determined by two points Z_1 and Z_2) to a point in complexified Minkowski space $M_{\mathbb{C}}$. The second relation above is the analogous correspondence for the dual twistor space. We refer the reader to Figs. 3.A.1 and 3.A.2 in appendix 3.A which describes the twistor correspondence as a double fibration, on both the boundary and in the bulk, in the complexified case as well as after imposing the (euclidean) reality conditions of Sec. 3.3.3.

3.3.2 The incidence relation in the bulk

The above incidence relations in the boundary arise from a careful limit of incidence relations in the bulk. When $r \neq 0$, we no longer have a nontrivial kernel for X_{IJ}. Instead we impose the natural twistorial incidence relation [110]

$$\begin{aligned} &Z^I = X^{IJ}Y_J \\ \Rightarrow &\begin{bmatrix} \lambda^\alpha \\ \mu^{\dot\alpha} \end{bmatrix} = \begin{bmatrix} \epsilon^{\alpha\beta} & x^{\alpha\dot\beta} \\ -x^{\dot\alpha\beta} & \frac{1}{2}(x^2+r^2)\epsilon^{\dot\alpha\dot\beta} \end{bmatrix} \begin{bmatrix} \mu^\dagger_\beta \\ \lambda^\dagger_{\dot\beta} \end{bmatrix}\,. \end{aligned} \tag{3.42}$$

This gives the following two independent equations:

$$\begin{aligned} \mu^\dagger_\alpha + x_\alpha{}^{\dot\beta}\lambda^\dagger_{\dot\beta} &= \lambda^\beta\epsilon_{\beta\alpha} \\ \mu^{\dot\alpha} - x^{\dot\alpha}{}_\beta\lambda^\beta &= \frac{1}{2}r^2\epsilon^{\dot\alpha\dot\beta}\lambda^\dagger_{\dot\beta} \end{aligned} \tag{3.43}$$

where the second relation is obtained using the second identity in (3.106). We also note that the bulk incidence relation in the form $Z^I = X^{IJ}Y_J$ immediately implies the quadric (or ambitwistor) relation

$$\mathcal{C} = Z^I Y_I = 0 \tag{3.44}$$

since the X^{IJ} is antisymmetric in its indices. Thus the incidence relations can be viewed as disentangling the ambitwistor constraint.

To recover the boundary incidence relations in Eqs. (3.37) and (3.39) as $r \to 0$ we need to take the limit where we scale $Y_J \to \frac{Y_J^b}{r}$ whereas $Z^I \to Z_b^I$. In other words, $(\mu^\dagger_\beta, \lambda^\dagger_{\dot\beta})$ scale as $\frac{1}{r}$ as we approach the boundary, while $(\mu^{\dot\alpha}, \lambda^\beta)$ scale as r^0. Note that the overall scale is immaterial for the ambitwistor variables on the boundary. Once we do this rescaling, we see that we recover the boundary incidence relations as $r \to 0$.

3.3.3 Reality conditions

Our considerations thus far apply to a complexified spacetime and the corresponding twistors. When we specialise to the case of a real slice of the spacetime, we need to impose a corresponding set of reality conditions on the twistors. We will take the AdS_5 spacetime to be euclidean since we aim to focus on euclidean correlators in the dual CFT - there are likely to be additional subtleties in lorentzian signature. One of the simplifications afforded by euclidean signature is that, given a point in twistor space, the corresponding spacetime point is determined, as we will see explicitly below. See Figs. 3.A.1 and 3.A.2 for the fibration picture of this bijection. This is unlike lorentzian signature where a point in twistor space corresponds to a null ray in spacetime. For a discussion of reality conditions for twistors in different signatures, see for instance [111] or, in the AdS_5 context, [110].

In the Euclidean signature, the spacetime coordinates of complexified Minkowski space obey

$$\widehat{x}^{\dot\alpha}{}_{\beta} = x^{\dot\alpha}{}_{\beta} \ , \tag{3.45}$$

where $\widehat{x}^{\dot\alpha}{}_{\beta}$ is defined through Eq. (3.108) (with bars referring to ordinary complex conjugation). Taking the complex conjugate of the boundary incidence relation Eq. (3.37) and using Eq. (3.45) gives

$$\mu^{\dot\alpha} = x^{\dot\alpha}{}_{\beta}\lambda^{\beta} \quad \Rightarrow \quad \hat\mu^{\dot\alpha} = x^{\dot\alpha}{}_{\beta}\hat\lambda^{\beta} \tag{3.46}$$

Similarly, we also find the complex conjugate relation to Eq. (3.39)

$$\hat\mu^{\dagger}_{\alpha} = -x_{\alpha}{}^{\dot\beta}\hat\lambda^{\dagger}_{\dot\beta} \tag{3.47}$$

Here we have defined the hatted twistor variables

$$\begin{aligned} \widehat{Z}^I &\equiv (\hat\lambda^{\alpha}, \hat\mu^{\dot\alpha}) \equiv (-\bar\lambda^2, \bar\lambda^1, -\bar\mu^2, \bar\mu^1) \\ \widehat{Y}_J &\equiv (\hat\mu^{\dagger}_{\beta}, \hat\lambda^{\dagger}_{\dot\beta}) \equiv (-\bar\mu^{\dagger}_2, \bar\mu^{\dagger}_1, -\bar\lambda^{\dagger}_2, \bar\lambda^{\dagger}_1) \end{aligned} \tag{3.48}$$

In other words, the hatted twistors are essentially complex conjugates[7] and also obey the same incidence relations as the original twistors.

It is easy to verify that if we take the radial coordinate r to be real, then we are extending the above reality conditions on the boundary into the bulk (euclidean AdS_5). The reality conditions on the twistors given above also ensure that the bulk incidence

[7] Note, however, that the conjugation operation defined here is not the identity when applied twice. See below Eq. (1.17) in [111].

relations Eq. (3.43) are satisfied in terms of the hatted twistors. More succinctly,

$$\widehat{Z}^I = X^{IJ}\widehat{Y}_J \ . \tag{3.49}$$

The reality constraints can be used to obtain the spacetime coordinate corresponding to a point in twistor space. First we consider a point on the boundary. Since both $Z^I_b, \widehat{Z}^I_b$ obey the incidence relation with the same $x^{\dot\alpha}{}_\beta$, we can use the general relation Eq. (3.41) to write

$$(X_b)^{IJ} = -\frac{Z_b^{[I}\widehat{Z}_b^{J]}}{\langle\lambda\hat\lambda\rangle} \quad \text{and} \quad (X_b)_{IJ} = -\frac{Y^b_{[I}\widehat{Y}^b_{J]}}{[\lambda^\dagger\hat\lambda^\dagger]} \tag{3.50}$$

Using these, the condition $(X_b)^2 = 0$ imposes additional constraints on the boundary ambitwistors:

$$(X_b)_{IJ}(X_b)^{IJ} = 0 \Rightarrow Z_b \cdot \widehat{Y}^b = \widehat{Z}_b \cdot Y^b = 0 \tag{3.51}$$

where we have used the ambitwistor condition $Z_b \cdot Y^b = 0$. From Eq. (3.50), we can read off the components

$$x^{\alpha\dot\beta} = \frac{\hat\mu^{\dagger\,\alpha}\lambda^{\dagger\,\dot\beta} - \mu^{\dagger\,\alpha}\hat\lambda^{\dagger\,\dot\beta}}{[\hat\lambda^\dagger\lambda^\dagger]} \tag{3.52}$$

$$= \frac{\lambda^\alpha\hat\mu^{\dot\beta} - \hat\lambda^\alpha\mu^{\dot\beta}}{\langle\hat\lambda\lambda\rangle} \tag{3.53}$$

In the second relation above we have used $x^{\alpha\dot\beta} = x^{\dot\beta\alpha}$ We also find

$$x^2 = 2\frac{[\hat\mu\mu]}{\langle\hat\lambda\lambda\rangle} = 2\frac{\langle\hat\mu^\dagger\mu^\dagger\rangle}{[\hat\lambda^\dagger\lambda^\dagger]} \ . \tag{3.54}$$

We use a similar logic to determine the coordinates in AdS$_5$ given the twistors (Z^I, Y_J) and their complex conjugates. One can write a general ansatz for X^{IJ}:

$$X^{IJ} = \alpha\frac{Z^{[I}\widehat{Z}^{J]}}{\langle\lambda\hat\lambda\rangle} + \beta\epsilon^{IJKL}\frac{Y_{[K}\widehat{Y}_{L]}}{[\lambda^\dagger\hat\lambda^\dagger]} \tag{3.55}$$

Note that X^{IJ} is real under the conjugation operation. The ansatz implies

$$X_{IJ} = \frac{1}{2}\alpha\epsilon_{IJKL}\frac{Z^{[K}\widehat{Z}^{L]}}{\langle\lambda\hat\lambda\rangle} + 2\beta\frac{Y_{[I}\widehat{Y}_{J]}}{[\lambda^\dagger\hat\lambda^\dagger]} \tag{3.56}$$

where we have used the identity $\epsilon_{IJKL}\epsilon^{KLMN} = 2\delta_I^{[M}\delta_J^{N]}$. The bulk incidence relations contain enough information to determine α and β:

$$Z^I = X^{IJ} Y_J = \alpha Z^I \frac{\widehat{Z} \cdot Y}{\langle \lambda \hat{\lambda} \rangle} \Rightarrow \alpha = \frac{\langle \lambda \hat{\lambda} \rangle}{\widehat{Z} \cdot Y} \tag{3.57}$$

$$Y_I = -\frac{2}{r^2} X_{IJ} Z^J = -\frac{4}{r^2} \beta \, Y_I \frac{Z \cdot \widehat{W}}{[\lambda^\dagger \hat{\lambda}^\dagger]} \Rightarrow \beta = -\frac{r^2}{4} \frac{[\lambda^\dagger \hat{\lambda}^\dagger]}{Z \cdot \widehat{Y}} \tag{3.58}$$

where we have used $X_{IJ} X^{IJ} = 2r^2$ Eq. (3.34).

From the conjugate incidence relation, we also find the reality constraint on the quadric

$$\widehat{Z}^I = X^{IJ} \widehat{Y}_J = -\alpha \widehat{Z}^I \frac{Z \cdot \widehat{Y}}{\langle \lambda \hat{\lambda} \rangle} \Rightarrow \widehat{Z} \cdot Y = -Z \cdot \widehat{Y} . \tag{3.59}$$

We will denote this constraint as $\mathcal{D}$,

$$\mathcal{D} \equiv \ \widehat{Z} \cdot Y + Z \cdot \widehat{Y} = 0 \ . \tag{3.60}$$

Note that this is a weaker constraint than the one that obeyed by the boundary twistors Eq. (3.51).

Using these results, we get a (many-to-one - see Fig. 3.A.2) correspondence between the twistors and a point in the bulk AdS_5 spacetime

$$X^{IJ} = \frac{1}{\widehat{Z} \cdot Y} \left[Z^{[I} \widehat{Z}^{J]} + \frac{r^2}{4} \epsilon^{IJKL} Y_{[K} \widehat{Y}_{L]} \right] \ . \tag{3.61}$$

To find r^2 and $x^{\alpha\dot{\beta}}$ we look at the upper diagonal block of X^{IJ} in Eq. (3.33) and equate that with r.h.s of Eq. (3.61), which immediately gives

$$r^2 = -2 \frac{\widehat{Z} \cdot Y + \langle \lambda \hat{\lambda} \rangle}{[\lambda^\dagger \hat{\lambda}^\dagger]} \ . \tag{3.62}$$

Similarly equating with any of the non-diagonal blocks in Eq. (3.33) gives

$$x^{\alpha\dot{\beta}} = \frac{1}{\widehat{Z} \cdot Y} \left[(\lambda^\alpha \hat{\mu}^{\dot{\beta}} - \hat{\lambda}^\alpha \mu^{\dot{\beta}}) + \frac{r^2}{2} (\hat{\mu}^{\dagger\,\alpha} \lambda^{\dagger\,\dot{\beta}} - \mu^{\dagger\,\alpha} \hat{\lambda}^{\dagger\,\dot{\beta}}) \right] \ , \tag{3.63}$$

which can be rewritten, using the expression of r^2, as

$$x^{\alpha\dot{\beta}} = \frac{\langle \hat{\lambda} \lambda \rangle}{\widehat{Z} \cdot Y} \left[\frac{\lambda^\alpha \hat{\mu}^{\dot{\beta}} - \hat{\lambda}^\alpha \mu^{\dot{\beta}}}{\langle \hat{\lambda} \lambda \rangle} - \frac{\hat{\mu}^{\dagger\,\alpha} \lambda^{\dagger\,\dot{\beta}} - \mu^{\dagger\,\alpha} \hat{\lambda}^{\dagger\,\dot{\beta}}}{[\hat{\lambda}^\dagger \lambda^\dagger]} \right] + \frac{\hat{\mu}^{\dagger\,\alpha} \lambda^{\dagger\,\dot{\beta}} - \mu^{\dagger\,\alpha} \hat{\lambda}^{\dagger\,\dot{\beta}}}{[\hat{\lambda}^\dagger \lambda^\dagger]} \ . \tag{3.64}$$

On the boundary, this is consistent with the expression for $x^{\alpha\dot{\beta}}$ which we obtained in Eq. (3.52) since the first term in square brackets above vanishes. As another consistency check, we can reproduce (3.43) with these expressions of r^2 and $x^{\alpha\dot\beta}$. For example

$$x^{\alpha\dot\beta}\lambda^\dagger_{\dot\beta} = (\lambda^\alpha - \mu^{\dagger\,\alpha}) - \frac{1}{\widehat{Z}\cdot Y}(\hat\lambda^\alpha\mu^{\dot\beta}\lambda^\dagger_{\dot\beta} + \lambda^\alpha\hat\lambda^\beta\mu^\dagger_\beta + \mu^{\dagger\,\alpha}\langle\lambda\hat\lambda\rangle) \tag{3.65}$$

where we have used the expression of r^2. Now using the ambi-twistor condition, third and fourth terms of the above expression become

$$\hat\lambda^\alpha\mu^{\dot\beta}\lambda^\dagger_{\dot\beta} + \lambda^\alpha\hat\lambda^\beta\mu^\dagger_\beta = (-\hat\lambda^\alpha\lambda^\beta + \lambda^\alpha\hat\lambda^\beta)\mu^\dagger_\beta = -\epsilon^{\alpha\beta}\langle\lambda\hat\lambda\rangle\mu^\dagger_\beta \tag{3.66}$$

This cancels the last term in (3.65) and we get back the first equation in (3.43).

3.4 Twistor space covering maps in AdS_5

We have described the twistor space geometry of AdS_5 and the reality conditions which fix a point in the bulk in terms of the twistor variables. We want to use this geometrical picture to describe string configurations which capture the dual $\mathcal{N}=4$ super Yang-Mills theory as recently proposed in [84, 85]. We will not give a complete picture here – this will be described elsewhere – but instead focus on a special configuration which is a direct generalisation of the AdS_3 case as described in Sec. 3.3.

The basic idea is that we will consider holomorphic worldsheet twistor fields $Y^I(z), Z_I(z)$, together with the corresponding right moving antiholomorphic twistor fields $\widehat{Z}^I(\bar z), \widehat{Y}_J(\bar z)$, which are related to the holomorphic fields by the conjugation operation defined in Eq. (3.48). Here the conjugation is assumed to act on the worldsheet variables on which the fields now depend, in the standard way $z \to \bar z$. Thus the right movers are not independent degrees of freedom and determined in terms of the left movers by Eq. (3.48). This is consistent with the observation in [84, 85] that we only need one copy of the wedge modes (i.e. not both left and right movers) of the twistor fields to account for the correct spectrum of the free super Yang-Mills theory[8].

To connect with the Feynman diagrams for the dual gauge theory operators, we will need to translate twistor string configurations into AdS spacetime configurations. We will determine the latter by requiring that the twistors obey the AdS_5 twistor incidence relations, together with the reality conditions. In other words, we now require the stringy

[8]It may be that we only need to impose this reality condtion on the finite number of wedge modes of the worldsheet twistor fields. Since these are the only ones excited in the classical solutions, we will not be able to distinguish the two possibilities in the present discussion. The right prescription will presumably arise from a complete worldsheet analysis. We thank Matthias Gaberdiel for discussions on this point and ongoing collaborations [112].

versions of Eq. (3.43).

$$\begin{aligned} \mu^{\dagger}_{\alpha}(z) + X_{\alpha}{}^{\dot{\beta}}(z,\bar{z})\lambda^{\dagger}_{\dot{\beta}}(z) &= \lambda^{\beta}(z)\epsilon_{\beta\alpha} \\ \mu^{\dot{\alpha}}(z) - X^{\dot{\alpha}}{}_{\beta}(z,\bar{z})\lambda^{\beta}(z) &= \frac{1}{2}R^2(z,\bar{z})\epsilon^{\dot{\alpha}\dot{\beta}}\lambda^{\dagger}_{\dot{\beta}}(z)\ . \end{aligned} \tag{3.67}$$

Here we have denoted the string configurations in AdS_5 by $X^{\dot{\alpha}}{}_{\beta}\lambda^{\beta}(z,\bar{z})$ and the radial $R(z,\bar{z})$. As we will see shortly, unlike in AdS_3, the longitudinal coordinates $X^{\dot{\alpha}}{}_{\beta}$ are now no longer purely holomorphic in the worldsheet coordinates. Ultimately, these relations will not need to be postulated; rather, we expect they will arise from an analysis of the ward identities of the worldsheet CFT as in [91]. This is currently under investigation [112].

In fact, since the bulk incidence relations Eq. (3.67) hold pointwise on the worldsheet, we will have the same expressions for the bulk coordinates $R(z,\bar{z}), X^{\dot{\alpha}}{}_{\beta}(z,\bar{z})$, in terms of the twistor fields, as given in Eqs. (3.62), (3.63). Since these depend on both the twistor fields and their conjugates, we clearly see that they cannot be holomorphic (or anti-holomorphic). Note that since the twistor fields obey the bulk incidence relation it also immediately follows that they obey, pointwise, the ambitwistor constraint

$$\lambda^{\dagger}_{\dot{\alpha}}(z)\mu^{\dot{\alpha}}(z) + \mu^{\dagger}_{\alpha}(z)\lambda^{\alpha}(z) = 0\ . \tag{3.68}$$

One may alternately view the ambitwistor constraint as primary and then the requirement of the bulk incidence relations in Eq. (3.67) follows. The latter are a general way of satisfying the ambitwistor constraints – see the discussion below Eq. (3.43). The ambitwistor constraint arose as a fundamental gauge constraint requirement in the worldsheet proposal of [84, 85].

We will further assume that as in AdS_3, the worldsheet is localised near the boundary i.e. that $R(z,\bar{z}) \propto \epsilon \to 0$. This is also physically motivated by the fact that the dual free theory is at the UV fixed point and hence essentially glued to the boundary of AdS_5, using the relation between scale and the radial direction. We expect, however, there to be a nontrivial radial (Liouville) profile $\partial \ln R$ which we will not determine here.

As discussed in Sec. 3.3.2, when the radial coordinate goes to zero, the bulk incidence relations reduce to the boundary incidence relations. Thus for string configurations near the boundary we must have the analogues of Eqs (3.37),(3.39), namely,

$$\begin{aligned} \mu^{\dot{\alpha}}(z) &= X^{\dot{\alpha}}{}_{\beta}(z,\bar{z})\lambda^{\beta}(z) \\ \mu^{\dagger}_{\alpha}(z) &= -X_{\alpha}{}^{\dot{\beta}}(z,\bar{z})\lambda^{\dagger}_{\dot{\beta}}(z)\ . \end{aligned} \tag{3.69}$$

We will mostly work with these (stringy) incidence relations on the boundary given our physical assumption of the worldsheet being localised there.

As mentioned at the beginning of the section, we are also imposing the reality conditions pointwise in $(z,\bar{z})$. Therefore we can use the above incidence relations and the corresponding conjugate relations to obtain the stringy generalisation of Eq. (3.52)

$$X^{\alpha\dot{\beta}}(z,\bar{z}) = \frac{\lambda^{\alpha}(z)\hat{\mu}^{\dot{\beta}}(\bar{z}) - \hat{\lambda}^{\alpha}(\bar{z})\mu^{\dot{\beta}}(z)}{\langle\hat{\lambda}(\bar{z})\lambda(z)\rangle} \tag{3.70}$$

$$= \frac{\hat{\mu}^{\dagger\,\alpha}(\bar{z})\lambda^{\dagger\,\dot{\beta}}(z) - \mu^{\dagger\,\alpha}(z)\hat{\lambda}^{\dagger\,\dot{\beta}}(\bar{z})}{[\hat{\lambda}^{\dagger}(\bar{z})\lambda^{\dagger}(z)]} \tag{3.71}$$

One sees, somewhat reassuringly, that the spacetime configurations $X^{\alpha\dot{\beta}}(z,\bar{z})$ are not purely holomorphic or anti-holomorphic even though the twistor fields are. Nevertheless, the holomorphic twistor fields obey the incidence relations with $X^{\alpha\dot{\beta}}(z,\bar{z})$ in Eq. (3.69), as can be seen from the explicit form in Eq. (3.70).

The mixed z dependence here is unlike the AdS$_3$ case where there was a split of the boundary spacetime coordinates into what we had denoted as $\Gamma(z)$ and its conjugate. Despite that, there is a sense in which there is a kind of local holomorphy in the boundary spacetime. This is reflected in the fact that even though $X^{\alpha\dot{\beta}}(z,\bar{z})$ is not holomorphic, the matrix $\bar{\partial}X^{\alpha\dot{\beta}}(z,\bar{z})$ (as also $\partial X^{\alpha\dot{\beta}}(z,\bar{z})$) has a zero eigenvector.

This follows directly by applying $\bar{\partial}$ on both sides of the two equations in Eq. (3.69). We obtain

$$\begin{aligned} &\bar{\partial}X^{\dot{\alpha}}{}_{\beta}(z,\bar{z})\lambda^{\beta}(z) = 0 \ , \\ &\bar{\partial}X_{\alpha}{}^{\dot{\beta}}(z,\bar{z})\lambda^{\dagger}_{\dot{\beta}}(z) = 0 \ . \end{aligned} \tag{3.72}$$

In other words, we see that $\lambda^{\beta}(z)$ is the zero eigenmode of $\bar{\partial}X^{\dot{\alpha}}{}_{\beta}(z,\bar{z})$ while $\lambda^{\dagger}_{\dot{\beta}}(z)$ is the zero eigenvector of the transposed matrix $\bar{\partial}X_{\alpha}{}^{\dot{\beta}}$. One immediate consequence of this, as can also be easily verified directly using the expressions in Eq. (3.70), is that

$$\bar{\partial}X_{\alpha\dot{\beta}}\bar{\partial}X^{\alpha\dot{\beta}} = 0 \ . \tag{3.73}$$

For $\partial X^{\alpha\dot{\beta}}$, we can similarly write down the zero eigenvectors in terms of the conjugate twistor fields and the analogue of Eq. (3.73). We interpret the presence of this zero eigenvector of $\bar{\partial}X^{\dot{\alpha}}{}_{\beta}(z,\bar{z})$ as indicating that, locally, on the worldsheet and hence in space time, we can always view the string configuration $X^{\dot{\alpha}}{}_{\beta}(z,\bar{z})$ as a holomorphic embedding into the boundary of an AdS$_3$ subspace of the bulk spacetime. We will exploit this interpretation in what follows.

We also note that if we consider the more general stringy incidence relations, we can arrive at a similar conclusion. Namely, applying $\bar{\partial}$ on the first equation of Eq. (3.67) we find that $\lambda^{\dagger}_{\dot{\beta}}(z)$ continues to be the zero eigenvector of $\bar{\partial}X_{\alpha}{}^{\dot{\beta}}(z,\bar{z})$. However, we see from the non-holomorphic dependence of $R^2(z,\bar{z})$ in the RHS of the second equation of

Eq. (3.67) that $\lambda^{\beta}(z)$ is no longer the zero eigenmode of $\bar{\partial} X^{\dot{\alpha}}{}_{\beta}(z, \bar{z})$.

3.4.1 Maps to an AdS_3 subspace

In this subsection, we will consider the subset of solutions where the string lies entirely within an AdS_3 subspace, or equivalently the boundary insertion points x_i lie in a two dimensional plane. Note that this can always be done for the case of two, three and four point functions (using conformal symmetry of $\mathcal{N} = 4$ SYM for the latter). For five point functions and higher this is a special configuration. We will work in this special kinematic setup as it will enable us to be very explicit. We will also be able to illustrate, in the next section, how the corresponding covering maps reproduce the boundary free field Feynman diagram answers.

Since $\bar{\partial} X^{\dot{\alpha}}{}_{\beta}(z, \bar{z})$ has one zero eigenvalue, we will choose it to be in diagonal form (which can always be done locally, but here we are doing so globally)

$$\bar{\partial} X^{\dot{\alpha}}{}_{\beta}(z, \bar{z}) = \begin{bmatrix} 0 & 0 \\ 0 & -\bar{\partial} \bar{V}(\bar{z}) \end{bmatrix} . \tag{3.74}$$

Recall from the general form of $X^{\dot{\alpha}}{}_{\beta}$ (see Eq. (3.110)) that this implies we have taken $\bar{\partial} U = \bar{\partial} \bar{U} = 0$ i.e. $U = 0$ without loss of generality. We also have $\bar{\partial} V = 0$ i.e. $V = V(z)$ and thus the purely antiholomorphic dependence in $\bar{V}(\bar{z})$. In other words, we have the string configuration

$$X^{\dot{\alpha}}{}_{\beta}(z, \bar{z}) = \begin{bmatrix} -V(z) & 0 \\ 0 & -\bar{V}(\bar{z}) \end{bmatrix} . \tag{3.75}$$

As mentioned, the string configuration is entirely in the (x_1, x_2) plane.

The zero eigenvector $\lambda^{\beta}(z)$ of Eq. (3.74) then has the form

$$\lambda^{\beta}(z) = \begin{bmatrix} \lambda^1(z) \\ 0 \end{bmatrix} . \tag{3.76}$$

Together with Eq. (3.75) and Eq. (3.69) we have

$$\mu^{\dot{\alpha}}(z) = - \begin{bmatrix} V(z)\lambda^1(z) \\ 0 \end{bmatrix} . \tag{3.77}$$

Thus $\mu^1(z) = -V(z)\lambda^1(z)$. Since $V(z)$ is a (finite degree) covering map from the genus zero worldsheet to the S^2 boundary of the AdS_3, it will be a ratio of polynomials. As in the case of AdS_3, we can achieve this if the twistor fields are given by rational functions. In other words, we are only exciting finitely many modes around each worldsheet insertion of the vertex operators. We will therefore consider fields of the form

$$\lambda^1(z) = \frac{R_{n-1}(z)Q^1_N(z)}{\prod_{i=1}^n (z-z_i)^{\frac{w_i}{2}}} \ , \quad \mu^1(z) = \frac{R_{n-1}(z)P^1_N(z)}{\prod_{i=1}^n (z-z_i)^{\frac{w_i}{2}}} \ . \tag{3.78}$$

This is a generalisation of the expressions Eqs. (3.21),(3.22). We note that the order of the poles in the denominator in the above is shifted. If w_i^{2d}, w_i^{4d} denote the spectral flow parameters of the AdS_3 and AdS_5 theories, (or equivalently, the twisted sector and the number of Yang-Mills bits in the dual CFT) respectively, then we have a shift (see Eq. (5.11) of [85])

$$w_i^{4d} = w_i^{2d} + 1 \ . \tag{3.79}$$

At the risk of creating confusion we will drop the superscripts, with the context hopefully making clear which label is being considered and Eq. (3.79) being the dictionary between the two.

We see from Eq.(3.78) that the twistor fields $\lambda^1(z), \mu^1(z)$ near $z = z_i$ have a singularity of order $\frac{1}{(z-z_i)^{\frac{w}{2}}}$. This is consistent with the fact that the twistors have spin half and only have wedge modes with mode number $r \leq \frac{w_i - 1}{2}$ excited. At the same time we expect a branching behaviour of order $(w_i - 1)$ near each such vertex operator insertion since $r \geq -\frac{w_i-1}{2}$ (cf. the shift of Eq (3.79))[9]. In other words, from Eq. (3.78), we can define the covering map in position space

$$-V(z) = \frac{\mu^1(z)}{\lambda^1(z)} = \frac{P_N(z)}{Q_N(z)} \ . \tag{3.80}$$

We expect that $V(z) \sim V_i + a_i(z-z_i)^{w_i - 1}$, where $V_i = (x_{(i)1} + ix_{(i)2})$.

Note also that we have the condition $N + n - 1 = \frac{1}{2}\sum_i w_i$ for the fields in Eq. (3.78) to be rational functions on the worldsheet with the prescribed poles. Therefore, $\frac{1}{2}\sum_i w_i$ must be an integer. Indeed this is consistent with the fact that the number of free field wick contractions in a Feynman diagram for a correlator built of operators with $\{w_i\}$ fields, $\langle \prod_{i=1}^n \mathcal{O}^{(w_i)}(x_i)\rangle$, is given precisely by $\frac{1}{2}\sum_i w_i$. Finally, since the branching behaviour for a w_i spectrally flowed operator is $(w_i - 1)$, the degree N of the polynomials, Q and P are given by the Riemann-Hurwitz formula as in AdS_3 with N being the nett degree of the branched cover. The polynomial $R_{n-1}(z)$ is a common overall factor in Eq. (3.78) and does not affect the branching behaviour.

Thus when the points x_i are all in a plane, the twistor solutions are a generalisation of the ones we saw in AdS_3. The main difference is that for a string at the boundary, the incidence relations for AdS_5 Eq. (3.67) reduce to Eq. (3.69). Hence the analogue of the terms on the RHS of Eq. (3.12) are absent here. As a result, in Eq. (3.78) we do not

[9] This is true for a generic vertex operator. However, for the BPS highest weight state the branching is of order w_i due to additional mode annihilation. We thank M. Gaberdiel for bringing this to our attention.

have the analogue of the second term on the RHS of Eq. (3.20).

We have not discussed the twistor fields $\lambda^{\dagger}_{\dot\beta}(z)$ thus far. For the choice of $\bar\partial X^{\dot\alpha}{}_{\beta}(z,\bar z)$ in Eq. (3.74), $\lambda^{\dagger}_{\dot\beta}(z)$ is also proportional to the vector in Eq (3.76), with a non-zero component $\lambda^{\dagger}_{1}(z)$. This is because it is a zero eigenvector of the transpose of the matrix, $\bar\partial X^{\dot\alpha}{}_{\beta}(z,\bar z)$. However, given the second incidence relation in Eq. (3.69), we see that $\mu^{\dagger}_{1}(z) = V(z)\lambda^{\dagger}_{1}(z)$. Since the branching behaviour at the insertions, which is determined by the spectral flow, is the same for both sets of twistor fields $(\lambda^{\dagger}_{\dot\beta}(z), \lambda^{\beta}(z))$, this implies that they are determined by the same covering map. Thus the same polynomials $P_N(z)$ and $Q_N(z)$ determine $\mu^{\dagger}_{1}(z)$ and $\lambda^{\dagger}_{1}(z)$ respectively. Therefore these are not independent degrees of freedom.

We expect the general solution to be determined by double the number of degrees of freedom of the special kinematic setup. In other words, there will now be two $\lambda^{\beta}(z)$ and the corresponding two $\mu^{\dot\alpha}(z)$. And we expect the $\lambda^{\dagger}_{\dot\beta}(z)$ and $\mu^{\dagger}_{\alpha}(z)$ not to be independent of the former. This will be studied elsewhere [112]. In the next section, we will use the restricted covering maps of this section to show how it reproduces the Yang-Mills propagators in a very natural way which parallels that in the AdS_3/CFT_2 case [15].

3.5 Relation to Feynman diagrams in gauge theories

Consider the correlation function of n gauge invariant operators $\mathcal{O}^{(w_i)}(x_i)$ (made out of w_i 'letters' or singletons) in free $SU(N)$ Yang-Mills theory, in the large N limit. For concreteness, we will assume we are working with $\mathcal{N}=4$ SYM and the $\mathcal{O}^{(w)}$ are built from w scalar fields. Then

$$G_n(\{x_i\}) = \langle \mathcal{O}^{(w_1)}(x_1)\cdots\mathcal{O}^{(w_n)}(x_n)\rangle_{planar} = \sum_{\{n_{ij}\}} C_{\{n_{ij}\}} \prod_{(i,j)} \left(\frac{1}{x_{ij}^2}\right)^{n_{ij}} \tag{3.81}$$

is simply given in terms of sums of products of the individual propagators $\left(\frac{1}{x_{ij}^2}\right)^{n_{ij}}$, where $n_{ij}=n_{ji}$ denote the number of (homotopic) Wick contractions between the pair of vertices at positions (x_i, x_j). Here $C_{\{n_{ij}\}}$ are combinatorial factors counting various planar diagrams, whose precise form is inessential for what we are about to describe. We also have the constraints on the $\{n_{ij}\}$

$$\sum_j n_{ij} = w_i \ , \tag{3.82}$$

holding for each vertex (i). Each of the contributions in Eq. (3.81) corresponds to a planar Feynman graph with n vertices and n_{ij} homotopic edges between the pair (ij) of vertices. Thus there are a total of $\frac{1}{2}\sum_{i,j} n_{ij} = \frac{1}{2}\sum_i w_i$ edges in the graph.

In AdS$_3$, the correlator of twisted sector fields corresponding to Eq. (3.81) was given by a sum over contributions associated to different covering maps in the Lunin-Mathur description [106]. These, in turn were associated [107] in a one to one way with a set of Feynman diagrams which captured the covering map data, as we discussed in the last chapter 2. Here we want to do something parallel, but in the opposite direction, in a sense. Namely, we would like to associate a covering map from the worldsheet for each familiar Feynman diagram contribution. We will see that, at least for the restricted kinematic configurations of the previous subsection, we will have an analogous picture[10]. Each graph will therefore correspond to a certain point in the moduli space of the closed string theory (which admits this covering map with specified branching data). We will see, via a connection of the Schwarzian of this covering map to the unique Strebel differential at that point in the moduli space, that this association of Feynman graphs to closed string worldsheets is as per the Strebel prescription of [13, 14]. This is also in parallel to the AdS_3/CFT_2 case as in [15]. And very strikingly, the weight, associated to each Feynman diagram, coming from the propagators in Eq. (3.81) can be reproduced in terms of the area of the worldsheet in the so-called Strebel gauge [26], again in parallel to the AdS$_3$ case [15]. Let us now describe how this comes about.

3.5.1 Two point function

We start with the simplest correlator, a two point function. Then we can always choose the two points (x_i, x_j) to lie in a specified plane on the boundary of the AdS$_5$. Since we are considering scalar fields, we can take these to be the highest weight BPS state in the SYM theory corresponding to the w-spectrally flowed vacuum sector $|0\rangle_w$ [84, 85] (together with the conjugate field). We will view the corresponding states on the worldsheet to be inserted at $z = (0, \infty)$, respectively, without loss of generality. For such a correlator, as discussed in the previous section, the classical twistor field will have branching of order w at $z = (0, \infty)$ (see footnote 9). The corresponding covering map in Eq. (3.80) is then fixed (after absorbing a constant into a rescaling of z) to take the form

$$V(z) = \frac{V_j\, z^w + V_i}{z^w + 1} \ , \tag{3.83}$$

where V_i is the complex coordinate in the $(1, 2)$ plane corresponding to x_i (see below Eq. (3.80)). This corresponds to the polynomials in Eq. (3.78) and Eq. (3.80) taking the values

$$P_w(z) = V_j\, z^w + V_i \ ; \quad Q_w(z) = z^w + 1 \ . \tag{3.84}$$

From the point of the view of the twistor fields Eq. (3.78), taking into account the weight half of the spinor fields, we see that we are exciting only the wedge modes with $r = \pm\frac{(w-1)}{2}$

[10] As mentioned, placing all the n-points (x_i) in a 2d plane is always possible for $n = 2, 3, 4$, so this is really a special choice only for five and higher point functions.

as appropriate for the highest weight BPS state (and its conjugate).

As we will shortly see, it will be convenient to view the covering map in coordinates $z = e^{2\pi i \frac{u}{w}}$ which maps the vertical strip $(0 < \mathrm{Re}\, u \leq w)$ onto the sphere such that $z = (0, \infty)$ are images of $u = \pm i\infty$, respectively, on the strip. In terms of these coordinates the covering map takes the form

$$\Gamma(u) \equiv V(z(u)) = \frac{V_i + V_j}{2} + \frac{V_i - V_j}{2i} \tan(\pi u) \ . \tag{3.85}$$

In this form, we now observe that this is essentially the unique map for which the Schwarzian is a constant:

$$S[\Gamma(u)] = \frac{\Gamma'''}{\Gamma'} - \frac{3}{2}\left(\frac{\Gamma''}{\Gamma'}\right)^2 = 2\pi^2 \ . \tag{3.86}$$

At the same time, we know that the unique Strebel quadratic differential on the strip with poles only at $u = \pm i\infty$ is also just du^2. So we see that, as in [15] (cf. Eq. (6.5) there), the Strebel differential on the worldsheet is identified with the Schwarzian of the covering map.

$$\phi_S(u) du^2 = \frac{1}{2\pi^2} S[\Gamma(u)] du^2 \ . \tag{3.87}$$

Note that the Schwarzian also transforms as a quadratic differential and hence this is a coordinate independent statement. Indeed, in terms of the coordinate z, we have the Strebel differential

$$du^2 = -\frac{w^2}{4\pi^2} \frac{dz^2}{z^2} \ , \tag{3.88}$$

which has double poles, as expected at $z = 0, \infty$ with "residues" $\propto w$.

This identification enables us to interpret this worldsheet as that corresponding to the Feynman diagram associated to this two point function. Then the strip of width w is nothing other than the w double line edges glued together. This Strebel prescription of [13, 14, 26] was seen to be also realised in the large w_i limit in the case of AdS_3/CFT_2 in the tensionless regime.

We can go further and look at the Nambu-Goto area of the worldsheet in the metric induced by the Strebel differential. This had also entered into the weight associated with the correlator in AdS_3 [15]. The area is simplest to compute in the u coordinates. It is just the area of the strip. The width is proportional to w. However, the length is formally infinite. This is a reflection of the UV divergence of the field theory. To exhibit this, we put cutoffs $-L \leq \mathrm{Im}\, u \leq L$ with $L \gg 1$. We now denote

$$|V_i - \Gamma(u = iL)| = |V_j - \Gamma(u = -iL)| = \epsilon \tag{3.89}$$

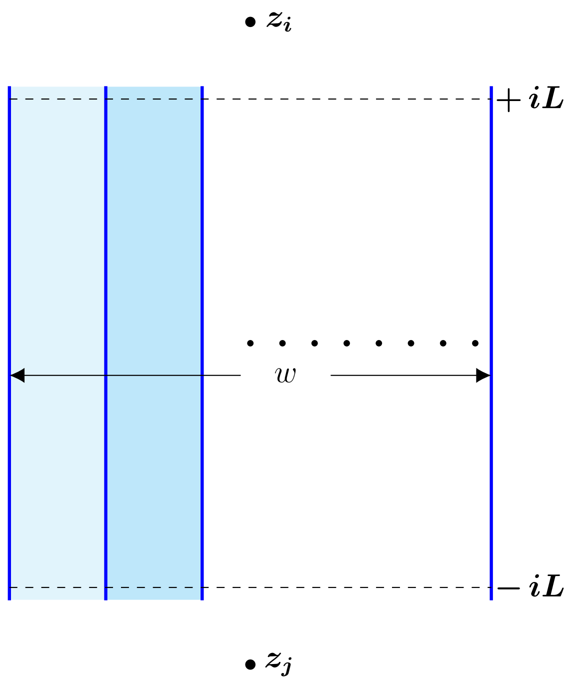

Figure 3.5.1: A vertical strip of width w in the u-plane. Regulators at $u = \pm iL$ are shown.

where ϵ is a short distance cutoff in spacetime. Using the behaviour

$$i \tan u = \mp(1 - 2\, e^{\pm 2iu}), \quad u \to \pm i\infty, \tag{3.90}$$

we find for the covering map Eq. (3.85) that

$$\epsilon = |V_i - \Gamma(u = iL_i)| = |V_i - V_j| \exp(-2\pi L), \quad L \to \infty\ . \tag{3.91}$$

In other words,

$$L = \frac{1}{4\pi} \ln\left(\frac{x_{ij}^2}{\epsilon^2}\right)\ , \tag{3.92}$$

where we have used $|V_i - V_j|^2 = (x_{ij})^2$. We then see that the (regulated) area of the strip in Strebel gauge is $A_S = 2Lw$. Hence, the natural Nambu-Goto weight

$$e^{-2\pi A_S} = \epsilon^{2w}\left(\frac{1}{x_{ij}^2}\right)^w\ . \tag{3.93}$$

We reproduce the propagator of the free gauge theory simply via the area of the string worldsheet in the special Strebel gauge we are working in. Note that the ϵ dependence in the answer is just a multiplicative renormalisation that can be absorbed as usual.

3.5.2 Multi point correlator

We have seen that for the two point function, we can reproduce the correct propagator Eq. (3.93) from the Strebel area of the Strebel diffferential, given by the Schwarzian of the covering map Eq. (3.83) as in Eq. (3.88). To generalise to a multi-point correlator, we will use the results of [15], restricting to the special kinematic configuration where all the n-points are in the same plane. In the regime of large w_i, which we are considering, the covering maps with specified branching data (x_i, w_i) were explicitly characterised in terms of a spectral curve for a Penner-like matrix model. The spectral curve, and thus the covering map, was determined in terms of a set of integers n_{ij} obeying the constraint Eq. (3.82). Each such covering map was associated with a Feynman diagram with n vertices with w_i double lines emanating from each of the vertices. The n_{ij} were the number of edges between the pair of vertices (i, j).

Furthermore, it was argued in [15] that to leading order (i.e. for large w_i), this spectral curve, given by the Schwarzian of the covering map, was, rather remarkably, proportional to the unique Strebel differential on the worldsheet with specified Strebel lengths n_{ij}. The Feynman-'t Hooft diagram associated to the covering map was then simply the Strebel construction of the closed string worldsheet, with n punctures, formed from from gluing strips together as in the general prescription of [13, 14, 26].

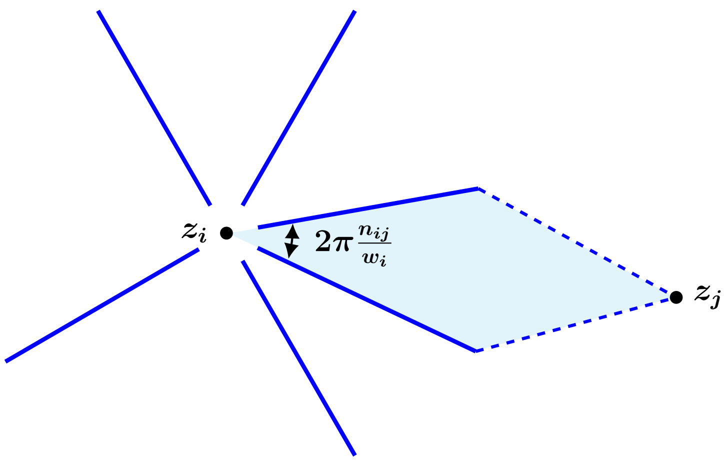

Figure 3.5.2: The local neighbourhood near a double pole of the Strebel differential - where strips such as in Fig. 3.5.1 are glued together. Compare with the middle picture of Fig. 3.1.1.

All these considerations go through in our present context of a restricted kinematic configuration. We will therefore use the connection between the Strebel differential and the Schwarzian of the covering map to locally describe the covering map. Thus, we will consider the worldsheet, which admits a covering map, to be comprised of vertical strips of width n_{ij} which connect the pair of worldsheet points (z_i, z_j). The Strebel differential has no poles or zeroes on this strip, except at the boundaries and hence we can find a local coordinate $u_{(ij)}$ on this strip in which it is just $du^2_{(ij)}$. The Schwarzian of the associated covering map $\Gamma(u_{(ij)})$ is a constant. Given that the covering map must interpolate (as $u_{(ij)} \to \pm i\infty$) between the space time points with planar coordinates V_i and V_j, this fixes the covering map in this coordinate chart to be as in Eq. (3.85).

$$\Gamma(u_{(ij)}) = \frac{V_i + V_j}{2} + \frac{V_i - V_j}{2i} \tan\Big(\pi u_{(ij)}\Big) \ . \tag{3.94}$$

Note that this is the covering map on a local patch of the Riemann surface. To obtain the full covering map, one will have to glue together these maps in different patches. There is a Strebel differential $\phi_S(z)dz^2$ (with given Strebel lengths n_{ij}) which takes the form $(\frac{du_{(ij)}(z)}{dz})^2 dz^2$ in each strip and

$$\phi_S(z)dz^2 \sim -\frac{w_i^2}{4\pi^2}\frac{dz^2}{(z-z_i)^2} \tag{3.95}$$

in the vicinity of the insertions z_i. The covering map is then obtained by gluing together the different patches $u_{(ij)}(z)$. See fig. 3.5.2.

However, for the purpose of obtaining the Strebel area, we will not really need the global covering map or Strebel differential. We simply add up the areas of each of these strips, appropriately regularised. Given that in each strip the covering map is given by Eq. (3.94), we just repeat the considerations of the previous subsection. We thus obtain a regularised strip of width n_{ij} - the Strebel length - and vertical size

$$2L_{ij} = \frac{1}{2\pi} \ln\left(\frac{x_{ij}^2}{\epsilon^2}\right) , \tag{3.96}$$

with ϵ again being a short distance spacetime regulator as in Eq. (3.89). Thus the Strebel area ($A_{ij} = 2L_{ij}n_{ij}$) of each such strip gives a Nambu-Goto weight

$$e^{-2\pi A_{ij}} = \epsilon^{2n_{ij}} \left(\frac{1}{x_{ij}^2}\right)^{n_{ij}} \ . \tag{3.97}$$

As before, we absorb the regulator dependence in a multiplicative renormalisation of the corresponding YM operators.

We therefore conclude that the worldsheets weighted with the Nambu-Goto weight in Strebel gauge give rise to the individual propagator contributions to the Feynman diagram for the Yang-Mills correlator in Eq. (3.81). The sum over the different terms in Eq. (3.81) is now interpreted as a sum over different points in the moduli space which admit covering maps. These are specified by different numbers $\{n_{ij}\}$ of Wick contractions or equivalently, from the closed string point of view, the integer Strebel lengths characterising these points on moduli space.

3.6 Concluding remarks

In this work, we have begun the task of pushing the proposal of [84, 85] beyond that of the agreement of the spectrum. Our goal here was to develop a geometric picture of the twistor description and see how far this can reproduce the correlators of free super Yang-Mills. This needs to be taken further and in a more systematic way at the level of the quantum correlators. Nevertheless, we already see many striking elements, through our analysis of classical configurations, which we expect will hold exactly in the full description. In developing this picture, we have also fleshed out the covering map description of AdS_3/CFT_2 in terms of the worldsheet twistor fields. The resulting classical twistor configurations are seen to have only the wedge modes excited as expected from the fact that these are the physical modes of the string theory after gauge fixing (as in Sec. 5 of [85]).

We then saw that this geometric picture admits a natural generalisation to AdS_5. We described the natural set of incidence relations that the classical worldsheet ambitwistor configurations should satisfy. Imposing euclidean reality conditions allowed us to solve for the AdS_5 spacetime string configurations in terms of the ambitwistor fields. Restricting to boundary correlators which lie on a two plane, we could write down the solutions, as in AdS_3, in terms of holomorphic covering maps with the right branching behaviour. The corresponding twistor fields are then essentially polynomials, again supporting the identification of the physical modes with the finite number of wedge modes [84, 85]. We take this as a robust indication that the wedge twistor modes are indeed the crucial physical degrees of freedom in the tensionless limits of strings on both AdS_3 and AdS_5. It will, of course, be important to corroborate this conclusion from a first principles worldsheet analysis.

Furthermore, for this kinematic configuration, we found that the explicit covering map is associated with the special integer points on the moduli space ('arithmetic curves'). This is because the Schwarzian of the covering map turned out to be the Strebel differential (as for AdS_3/CFT_2). The corresponding Strebel graph (or more precisely, its dual) is associated to the Feynman diagram of the free Yang-Mills theory as per [13, 14]. This realises the picture of open-closed string duality of [13, 22, 23], associating individual 'tHooft-Feynman diagrams (open strings) with closed string worldsheets. One of the new things we learnt is that the Nambu-Goto weight (the area in Strebel gauge) associated with each such worldsheet is precisely the free Yang-Mills propagator (after proper regularisation). This is, to us, compelling evidence that the Strebel construction paves the path between perturbative field theories and closed string worldsheet theories in broad generality.

There are a number of questions which merit further investigation, which have mostly been mentioned already. Perhaps the most important of them is to place the stringy

AdS_5 twistor incidence relations on a firm footing, from the standpoint of worldsheet correlators, as in [91]. The exact nature of the spacetime covering maps when one goes away from our special kinematic configuration is another important point to understand better. This will begin to kick in at the level of 4-point functions. As mentioned, the maps $X^{\alpha\dot{\beta}}(z, \bar{z})$ are now holomorphic in a subtle way, in there being locally a holomorphic eigenvector. Can we still use the power of holomorphy and constrain these maps? The radial profile of these configurations, which are infinitesimally near the boundary, would also be interesting to understand and should correspond to some kind of effective Liouville mode on the worldsheet. We note that the picture of the worldsheet configurations is somewhat like that described in [89]. The connection to the Feynman diagrams for these more general configurations also needs to be worked out. Can we, for instance, see the detailed spin dependent numerators of the Feynman weights, for a general correlator, from the Strebel construction for the twistor worldsheet? It would be very nice to develop a general dictionary for the usual Feynman rules, that we can expect to hold in any open-closed string duality in the perturbative field theory limit.

Appendix

3.A Conventions

The spin group $SO(4,\mathbb{C})$ of complexified Minkowski space (denoted as $M_{\mathbb{C}}$) is locally isomorphic to $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Given a four-vector $T^\mu = (T^0, T^1, T^2, T^3)$, we can represent it in terms of Pauli matrices ${\sigma_\mu}^{\alpha\dot{\alpha}}$ as

$$T^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} T^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{bmatrix} T^0 + T^3 & T^1 - iT^2 \\ T^1 + iT^2 & T^0 - T^3 \end{bmatrix} \tag{3.98}$$

where un-dotted and dotted spinor indices $\{\alpha = 0,1 | \dot{\alpha} = 0,1\}$ are in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, known as negative and positive chirality spinors respectively. Here and below we will be largely following the conventions of [111].

For lowering and raising the spinor indices, we can use the $SL(2,C)$-invariant tensors, i.e the usual Levi-Civita symbols

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.99}$$

With this convention for inverse marices $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$,

$$\epsilon^{\alpha\beta}\epsilon_{\gamma\beta} = \delta^\alpha_\gamma \quad \text{and} \quad \epsilon^{\alpha\beta}\epsilon_{\alpha\beta} = 2 \tag{3.100}$$

and similarly for dotted indices. Our convention for lowering and raising of spinor indices will follow the slogan, '*lower to the right, raise to the left*':

$$u_\alpha = u^\beta \epsilon_{\beta\alpha} \quad \text{and} \quad v^\alpha = \epsilon^{\alpha\beta} v_\beta \ , \tag{3.101}$$

and similarly for dotted indices. The epsilon tensors define $SL(2,\mathbb{C})$ invariant inner products in the spaces of both positive and negative chirality spinors separately. We define them as

$$\langle u\, v\rangle := u^\alpha\, v_\alpha = u^\alpha v^\beta \epsilon_{\beta\alpha} \quad \text{and} \quad [\tilde{u}\, \tilde{v}] := \tilde{u}^{\dot{\alpha}} \tilde{v}_{\dot{\alpha}} = \tilde{u}^{\dot{\alpha}} v^{\dot{\beta}} \epsilon_{\dot{\beta}\dot{\alpha}}. \tag{3.102}$$

We note that these inner products are skew-symmetric, i.e $\langle u\,v\rangle = -\langle v\,u\rangle$ (and similarly for the dotted spinors).

The space-time vector x^μ in $M_{\mathbb{C}}$ has the form, as in Eq. (3.98)

$$x^{\alpha\dot\beta} = \frac{1}{\sqrt{2}}\begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix} . \tag{3.103}$$

Lowering and raising different indices, this becomes

$$x_\alpha{}^{\dot\beta} = x^{\gamma\dot\beta}\epsilon_{\gamma\alpha} = \frac{1}{\sqrt{2}}\begin{bmatrix} -(x^1 + ix^2) & -(x^0 - x^3) \\ x^0 + x^3 & x^1 - ix^2 \end{bmatrix} ; \tag{3.104}$$

$$x^{\dot\alpha}{}_\beta = x^{\dot\alpha\gamma}\epsilon_{\gamma\beta} = \frac{1}{\sqrt{2}}\begin{bmatrix} -(x^1 + ix^2) & x^0 + x^3 \\ -(x^0 - x^3) & x^1 - ix^2 \end{bmatrix} . \tag{3.105}$$

This leads to the following identities

$$x_{\alpha\dot\gamma}x^{\dot\gamma}{}_\beta = \frac{1}{2}x^2\epsilon_{\alpha\beta} \quad \text{and} \quad x^{\dot\alpha\gamma}x_\gamma{}^{\dot\beta} = -\frac{1}{2}x^2\epsilon^{\dot\alpha\dot\beta} . \tag{3.106}$$

So far our discussion applies to $M_{\mathbb{C}}$. Taking an euclidean slice of this complexified space amounts to setting

$$x^{\alpha\dot\beta} = \widehat{x}^{\,\alpha\dot\beta}, \tag{3.107}$$

where

$$\widehat{x}^{\,\alpha\dot\beta} = \frac{1}{\sqrt{2}}\begin{bmatrix} \bar{x}^0 - \bar{x}^3 & -\bar{x}^1 + i\bar{x}^2 \\ -\bar{x}^1 - i\bar{x}^2 & \bar{x}^0 + \bar{x}^3 \end{bmatrix} . \tag{3.108}$$

This is because Eq. (3.107) implies $\bar{x}^0 = x^0$, $\bar{x}^i = -x^i$ $(i = 1, 2, 3)$ so that $x^0 = y^0$, $x^i = -iy^i$ $(i = 1, 2, 3)$ with $y^0, y^i \in \mathbb{R}$ giving the euclidean metric $ds^2 = (dy^0)^2 + \sum_{i=1}^3 (dy^i)^2$. Thus $x^{\dot\alpha}{}_\beta$ in this Euclidean real slice becomes

$$x^{\dot\alpha}{}_\beta = \frac{1}{\sqrt{2}}\begin{bmatrix} -y^2 + iy^1 & y^0 - iy^3 \\ -y^0 - iy^3 & -y^2 - iy^1 \end{bmatrix} . \tag{3.109}$$

The stringy generalization of this expression is

$$X^{\dot\alpha}{}_\beta(z, \bar{z}) = \begin{bmatrix} -V(z, \bar{z}) & U(z, \bar{z}) \\ -\bar{U}(z, \bar{z}) & -\bar{V}(z, \bar{z}) \end{bmatrix} . \tag{3.110}$$

where we have defined $U(z, \bar{z}) = (y^0 - iy^3)(z, \bar{z})$ and $V(z, \bar{z}) = (y^2 - iy^1)(z, \bar{z})$.

In Fig. 3.A.1 we have illustrated the double fibration that underlies the twistor correspondence on the Minkowski boundary of AdS_5. On the left hand side is the complexified case while the RHS is after imposing the euclidean reality condition. In the latter, case the second fibration becomes a bijection between the twistor space and the spin bundle over real Euclidean space. In Fig. 3.A.2, the analogous figures are shown for the complexified AdS$_5$ spacetime and after imposing euclidean reality conditions.

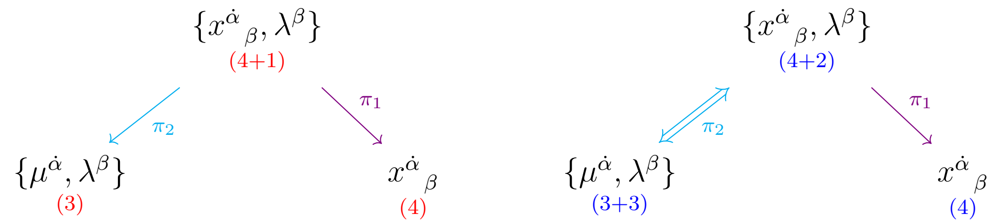

Figure 3.A.1: (Left) Double fibration of the projective spinor bundle $\{x^{\dot\alpha}{}_\beta, \lambda^\beta\} \cong M_{\mathbb{C}} \times \mathbb{CP}^1$ over the twistor space $\{\mu^{\dot\alpha}, \lambda^\beta\} \cong \mathbb{CP}^3$ and the space-time on the boundary of AdS_5, $\{x^{\dot\alpha}{}_\beta\} \cong M_{\mathbb{C}}$. Numbers in red are the complex dimensions of the corresponding spaces. (Right) The same fibration after imposing the euclidean reality conditions. Numbers in blue now show the real dimensions of the corresponding spaces. Note that π_2 is an isomorphism in this signature.

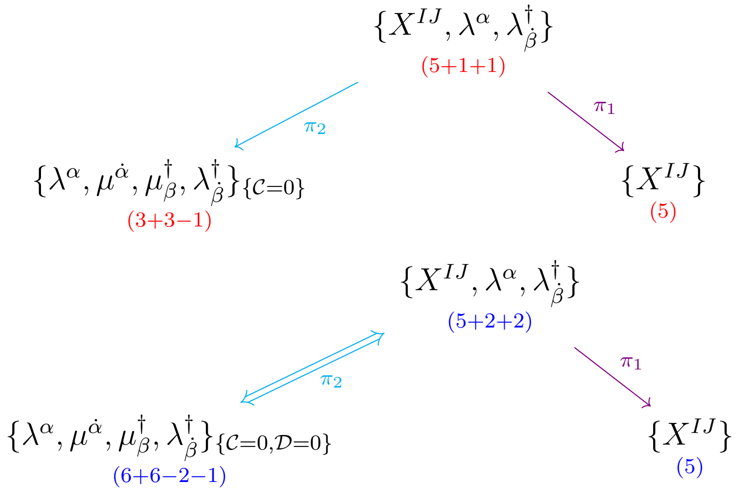

Figure 3.A.2: (Top) Double fibration of the projective spinor bundle $\{X^{IJ}, \lambda^\alpha, \lambda^\dagger_{\dot\beta}\} \cong \mathbb{CP}^5 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ over the Ambi-twistor space $\{\lambda^\alpha, \mu^{\dot\alpha}, \mu^\dagger_\beta, \lambda^\dagger_{\dot\beta}\}_{\{\mathcal{C}=0\}} \cong (\mathbb{CP}^3 \times \mathbb{CP}^3)_{\{\mathcal{C}=0\}}$ and the "space-time" $\{X^{IJ}\} \cong \mathbb{CP}^5$ in the complexified AdS_5 bulk. $\mathcal{C} = 0$ refers to the ambitwistor constraint Eq. (3.40). Numbers in red are the complex dimensions of the corresponding spaces. (Bottom) The same fibration after imposing the euclidean reality conditions which leads to the additional constraint $\mathcal{D} = 0$ – see Eq. (3.60). Numbers in blue are the real dimensions of the corresponding spaces. Note that π_2 is an isomorphism in this signature.

Chapter 4

String Dual of two-dimensional Yang-Mills

After 't Hooft's observation of associating two-dimensional surfaces to the large N expansion of a gauge theory, this connection was concretized by Maldacena in the famous AdS/CFT correspondence [2]. It gave us a precise duality between $\mathcal{N}=4$ Super Yang Mills theory at large N and IIB string theory in the bulk $AdS_5 \times S^5$. But finding the string dual to four-dimensional non-supersymmetric QCD (which describes our real world) remained an important and unsolved problem of theoretical physics. It could help understand infrared behaviour of the theory, which is otherwise strongly coupled from field theoretic perspective and we hardly have any analytical control there.

As a first step in accomplishing this grand goal, we could ask for the string dual of two dimensional Yang Mills which is exactly solvable on an arbitrary Riemann surface even at finite N. Also 2d QCD indeed lead to color confinement along with infinite no. of resonances when quarks are present [115].

Indeed in their seminal paper [17], Gross and Taylor expanded the partition function in $1/N$ and interpreted as a sum over branched covering surfaces from an auxiliary worldsheet to the target space-time manifold. But the explicit worldsheet action was yet unknown! There have been attempts [50, 82, 129] which either captures topological sector of the theory, or the worldsheet is too complicated to perform any realistic calculation.

In this chapter, we will propose a bosonic worldsheet model dual to the chiral sector[1] of pure 2d YM theory in the large N limit. In particular, we will reproduce the torus partition function of large N 2d YM from the worldsheet, and also match three point amplitude of winding states in 2d YM with a certain three-point string amplitude. Our worldsheet theory consists of the standard $\beta-\gamma$ system deformed by an exact chiral

[1]At large N, the partition function of 2d YM factorizes into chiral and anti-chiral parts:

$$\mathcal{Z}_{\mathcal{M}} \to \mathcal{Z}_{\mathcal{M}}^{\text{Chiral}} \times \mathcal{Z}_{\mathcal{M}}^{\text{Anti-chiral}}$$

version of the familiar Polchinski-Strominger term [120].

4.1 Worldsheet action on the cylinder

4.1.1 The action

We propose the following worldsheet action on the cylinder:

$$S = \int \frac{d^2z}{2\pi}\left[\beta\,\bar{\partial}X^+ + \bar{\beta}\,\partial X^- + \lambda\pi\left(\partial X^+\bar{\partial}X^- - \bar{\partial}X^+\partial X^-\right) - \frac{\kappa}{2}\left(2\,\partial\phi\,\bar{\partial}\phi + \hat{R}\,\phi\right)\right] \quad (4.1)$$

where the $\{X^\pm\}$ are two massless scalar fields, which can be seen as the light-cone coordinates of the target space:

$$X^+ = X^1 + X^0, \quad X^- = X^1 - X^0 \quad (4.2)$$

and the Liouville field ϕ is “on-shell" as

$$\phi = \log\left(\partial X^+\,\bar{\partial}X^-\right) \quad (4.3)$$

Along with this matter sector, we also have familiar b, c-ghost system:

$$S_{\text{ghost}} = \int \frac{d^2z}{2\pi}\,b\,\bar{\partial}c \quad (4.4)$$

Here λ is the 't Hooft coupling of the dual 2d Yang Mills theory and κ is still undetermined constant which we will determine from the demand of no conformal anomaly (hence total central charge of the matter and ghosts is zero). We've assumed that X^1 is compactified to a circle of radius R, i.e

$$X^1\left(\sigma^1 + 2\pi, \sigma^2\right) = X^1(\sigma^1, \sigma^2) + 2\pi R \quad (4.5)$$

Note that e.o.m of $(\beta, \bar{\beta})$ imposes the chirality conditions:

$$X^+ \equiv X^+(z), \quad X^- \equiv X^-(\bar{z}) \quad (4.6)$$

We will also choose $\beta \equiv \beta(z)$ and $\bar{\beta} \equiv \bar{\beta}(\bar{z})$. Note that $\beta(z)$ and $X^+(z)$ have conformal dimensions $(1,0)$ and $(0,0)$ respectively. The string tension is clearly $\lambda/2$ from 4.1, which is precisely what was argued/predicted in [81].

(Anti-) Holomorphicity of (X^-) X^+ makes one of the tensionful terms vanish: $\bar{\partial}X^+\partial X^- = 0$. But it is important to realize that in the presence of verterx operators (which depends on β) this holomorphicity breaks down and we have the anti-symmetric combination in the tensionful term.

Imposing holomorphicity of $X^{\pm}$, the kinetic term for ϕ becomes

$$\partial\phi\,\bar{\partial}\phi = \frac{\partial^2 X^+}{\partial X^+}\,\frac{\bar{\partial}^2 X^-}{\bar{\partial} X^-} \tag{4.7}$$

which makes the action a non-standard one.

4.1.2 Computation of the Stress tensor

Liouville part

For the computation of the stress tensor, we need to covariantize the action. First we consider the Liouville action (without the factor $-\kappa/2$ which we will multiply at the end)

$$S = \underbrace{\int \frac{d^2 z}{2\pi}\, 2\,\partial\phi\,\bar{\partial}\phi}_{S_1} + \underbrace{\int \frac{d^2 z}{2\pi}\, \hat{R}\phi}_{S_2} \tag{4.8}$$

For the first kinetic piece:

$$S_1 = \int \frac{d^2 z}{2\pi}\, 2\,\partial\phi\,\bar{\partial}\phi \;\rightarrow\; S_1 = \int \frac{d^2 \sigma}{2\pi}\, \sqrt{-g}\, g^{ab}\partial_a\Phi\,\partial_b\Phi \tag{4.9}$$

where we can determine the covariant form of ϕ, $\Phi(z,\bar{z})$ in the following way:

The worldsheet metric in the Euclidean signature is

$$ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2, \quad (\eta)_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; a,b = 1,2 \tag{4.10}$$

In complex co-ordinates $(z,\bar{z})$ defined by

$$z = \sigma^1 + i\sigma^2,\; \bar{z} = \sigma^1 - i\sigma^2, \quad ds^2 = dz\,d\bar{z}, \quad (\eta')_{ab} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \; a,b = z,\bar{z} \tag{4.11}$$

Next we need to covariantize the chiral combination $\partial X^+ \bar{\partial} X^-$:

$$\partial X^+\bar{\partial}X^- = \frac{1}{4}\left(\partial_1 X^+\partial_1 X^- + \partial_2 X^+\partial_2 X^-\right) + \frac{i}{4}\left(\partial_1 X^+\partial_2 X^- - \partial_1 X^-\partial_2 X^+\right) \tag{4.12}$$

we rewrite it in the following way:

$$\partial X^+\bar{\partial}X^- = \frac{1}{4}\eta^{ab}\partial_a X^+\partial_b X^- + \frac{i}{4}\,\epsilon^{ab}\partial_a X^+\partial_b X^- \tag{4.13}$$

where ϵ^{ab} is the Levi-Civita tensor defined by

$$\sqrt{|g|}\,\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \tilde{\epsilon}^{ab} \tag{4.14}$$

Note that in (σ^1, σ^2) coordinate system, ϵ^{ab} is identical to the Levi-Civita symbol $\tilde{\epsilon}^{ab}$ (the latter is not a tensor[2]) since $|\eta| = 1$. Thus the covariant form of the chiral combination $\partial X^+ \bar{\partial} X^-$

$$\partial X^+ \bar{\partial} X^- = \frac{1}{4}\, g^{ab} \partial_a X^+ \partial_b X^- + \frac{i}{4}\, \epsilon^{ab} \partial_a X^+ \partial_b X^- \tag{4.16}$$

and hence that of Φ is the following:

$$\boxed{\Phi = \log\left[\frac{1}{4}\, g^{ab} \partial_a X^+ \partial_b X^- + \frac{i}{4}\, \epsilon^{ab} \partial_a X^+ \partial_b X^-\right]} \tag{4.17}$$

Note that to get the chiral term $\partial X^+ \bar{\partial} X^-$, we need to introduce the B-field (which is unity here). This B-field plays a crucial role in the entire analysis of this note.

We can check the above formula in $(z, \bar{z})$-coordinate in the following way: We put $g = \eta'$ as defined above. Then we determine $(\epsilon^{ab})_{z,\bar{z}}$ using tensorial rules:

$$\epsilon^{z\bar{z}} = \frac{\partial z}{\partial \sigma^1} \frac{\partial \bar{z}}{\partial \sigma^2}\, \epsilon^{12} + \frac{\partial z}{\partial \sigma^2} \frac{\partial \bar{z}}{\partial \sigma^1}\, \epsilon^{21} = -i \times 1 + i \times (-1) = -2i \tag{4.18}$$

and thus

$$(\epsilon^{ab})_{z,\bar{z}} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \tag{4.19}$$

Using these, the above form gives,

$$\begin{aligned} \frac{1}{4} g^{ab} \partial_a X^+ \partial_b X^- + \frac{i}{4}\, \epsilon^{ab} \partial_a X^+ \partial_b X^- &= \frac{1}{4}\left(2 \partial X^+ \bar{\partial} X^- + 2 \bar{\partial} X^+ \partial X^-\right) + \frac{i}{4}\left(-2i \partial X^+ \bar{\partial} X^- + 2i \bar{\partial} X^+ \partial X^-\right) \\ &= \partial X^+ \bar{\partial} X^-. \end{aligned} \tag{4.20}$$

Thus

$$T_1(z) = -\frac{4\pi}{\sqrt{-g}} \frac{\delta S_1}{\delta g^{zz}} = -2\,(\partial\phi)^2 \tag{4.21}$$

---

[2] $\tilde{\epsilon}^{ab}$ transforms in the following way:

$$\tilde{\epsilon}'_{a'b'} = \left|\frac{\partial x'^{a'}}{\partial x^a}\right| \tilde{\epsilon}_{ab}\, \frac{\partial x^a}{\partial x'^{a'}} \frac{\partial x^b}{\partial x'^{b'}} \tag{4.15}$$

which can be shown easily from the definition of determinant in terms of epsilon symbol.

Note that variation along δg^{zz} from the field $\Phi(z,\bar{z})$[3] and from the determinant $\sqrt{-g}$ don't contribute using the e.o.m, $\partial X^- = 0$, and using $\delta(\sqrt{g}) = \frac{1}{2}\delta g^{cd}\, g_{cd}$ with $g_{zz} = g_{\bar{z}\bar{z}} = 0$ in the conformal gauge, respectively. Similarly for the curvature part of the Liouville action,

$$S_2 = \int \frac{d^2 z}{2\pi}\, \hat{R}\,\phi \to S_2 = \int \frac{d^2\sigma}{\pi}\sqrt{-g}\,\hat{R}\,\phi \tag{4.22}$$

Thus

$$T_2(z) = -\frac{4\pi}{\sqrt{-g}}\frac{\delta S_2}{\delta g^{zz}} = +4\,\partial^2\phi \tag{4.23}$$

where we've used[4] $(\delta\hat{R}/\delta g^{zz}) = -\partial^2$ in the last line. Thus from the Liouville part (without the multiplicative factor $-\kappa/2$), we get

$$T_{\text{Liouville}}(z) = (T_1 + T_2)(z) = -2\,(\partial\phi)^2 + 4\,\partial^2\phi = 4\,\{X^+, z\} \tag{4.24}$$

where we've used

$$\boxed{\begin{aligned}\{X^+, z\} &= \partial^2\phi - \frac{1}{2}(\partial\phi)^2 \\ &= \partial^2 \log\partial X^+ - \frac{1}{2}(\partial\log\partial X^+)^2 \\ &= \frac{\partial^3 X^+}{\partial X^+} - \frac{3}{2}\left(\frac{\partial^2 X^+}{\partial X^+}\right)^2\end{aligned}} \tag{4.25}$$

Tensionful part

We can easily determine that the tensionful part of the worldsheet action (4.1) is actually anti-symmetric B-field term:

$$\left(\partial X^+\bar{\partial}X^- - \bar{\partial}X^+\partial X^-\right) = \frac{i}{2}\left(\partial_1 X^+\partial_2 X^- - \partial_1 X^-\partial_2 X^+\right) = \frac{i}{2}\,\epsilon^{ab}\partial_a X^+\partial_b X^- \tag{4.26}$$

So that if we vary with respect to g^{zz} (or $g^{\bar{z}\bar{z}}$), this won't contribute to the stress tensor.

We could easily compute the stress tensor for the rest of the action. Adding all these up, the stress energy tensor of the worldsheet model takes the following form;

$$\begin{aligned} T(z) &= -\beta\,\partial X^+(z) - 2\,\kappa\,\{X^+, z\} \\ &= -\beta\,\partial X^+(z) - 2\kappa\,\partial^2\log\partial X^+(z) + \kappa\,(\partial\log\partial X^+(z))^2 \end{aligned} \tag{4.27}$$

[3] which is proportional to $\eta_{\mu\nu}\partial X^\mu\partial X^\nu = \partial X^+\partial X^-$

[4] This can be shown in the following way: In 2d, $R_{\mu\nu} = \frac{\hat{R}}{2}\,g_{\mu\nu}$. In the conformal gauge ($g_{zz} = g_{\bar{z}\bar{z}} = 0$), clearly then $R_{zz} = R_{\bar{z}\bar{z}} = 0$. Now using $\delta(\hat{R}) = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}$ and the relation $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\sigma[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})]$, we could easily get $\delta\hat{R}/\delta g^{zz} = -\partial_z^2$ in the conformal gauge of the metric.

For future reference, we mention the covariant form of the Liouville action

$$\Gamma[\phi] = -\frac{\kappa}{2}\int \frac{d^2z}{2\pi}\left(2\,\partial\phi\,\bar{\partial}\phi + \hat{R}\phi\right) \tag{4.28}$$

to be

$$\boxed{(\Gamma[\phi])_{\text{covariant}} = -\frac{\kappa}{2}\int \frac{d^2\sigma}{2\pi}\sqrt{-g}\left[g^{ab}\,\partial_a\phi\,\partial_b\phi + 2\hat{R}\,\phi\right]} \tag{4.29}$$

OPE of stress tensors

We note the following OPEs:

$$\begin{aligned} &\beta(z)\,X^+(w) \sim -\frac{1}{z-w}, \quad \bar{\beta}(\bar{z})\,X^-(\bar{w}) \sim -\frac{1}{\bar{z}-\bar{w}} \\ &\beta(z)\,\beta(w) \sim -2\kappa\,\partial_z\partial_w\left[\frac{1}{(z-w)^2}\,\frac{1}{\partial_z X^+(z)\,\partial_w X^+(w)}\right], \quad \text{it's cc} \end{aligned} \tag{4.30}$$

Stress tensors OPE is

$$\begin{aligned} T(z)\,T(w) =& :\left(-\beta\,\partial X^+ - 2\kappa\,\partial^2\log\partial X^+ + \kappa\,(\partial\log\partial X^+)^2\right)(z) : \\ & :\left(-\beta\,\partial X^+ - 2\kappa\,\partial^2\log\partial X^+ + \kappa\,(\partial\log\partial X^+)^2\right)(w) : \end{aligned} \tag{4.31}$$

- First let's consider the OPE

$$\begin{aligned} :\beta(z)\partial_z X^+(z) :: \beta(w)\partial_w X^+(w) := &-2\kappa\,\partial_z\partial_w\left[\frac{1}{(z-w)^2}\,\frac{1}{\partial_z X^+(z)\partial_w X^+(w)}\right]\partial_z X^+(z)\partial_w X^+(w) \\ &-\frac{\partial_z X^+(z)\beta(w) + \beta(z)\partial_w X^+(w)}{(z-w)^2} + \frac{1}{(z-w)^4} \end{aligned} \tag{4.32}$$

  Note that this is the only place where $\beta - \beta$ OPE enters.

  The last two terms of the above expression gives

$$\frac{1}{(z-w)^4} - \frac{2\,\beta(w)\partial_w X^+(w)}{(z-w)^2} - \frac{\partial_w\beta(w)\,\partial_w X^+(w) + \beta(w)\partial_w^2 X^+(w)}{z-w} \tag{4.33}$$

  which is the standard OPE of $\beta - X^+$ system (just like in $\beta - \gamma$ system).

- From ∂^2 piece of the stress tensor, we would have to compute the following OPEs:

$$+2\kappa\,\beta(z)\partial X^+(z)\,\partial_w^2\log\partial X^+(w) + 2\kappa\,\partial_z^2\left[\log\partial X^+(z)\right]\beta(w)\partial X^+(w) \tag{4.34}$$

For the first term above

$$\begin{aligned} +2\kappa\,\beta(z)\partial X^+(z)\,\partial_w^2\log\partial X^+(w) &= +2\kappa\,\partial X^+(z)\,\partial_w^2\left[\beta(z)\,\log\partial X^+(w)\right] \\ &= -2\kappa\,\partial X^+(z)\,\partial_w^2\left[\frac{1}{\partial X^+(w)}\frac{1}{(z-w)^2}\right] \end{aligned} \tag{4.35}$$

where for computing $\beta(z)\,\log\partial X^+(w)$ OPE, we used the following strategy:

$$\begin{aligned} \beta(z)\,f(\partial X^+(w)) &\sim \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{(n-1)!}\,[\partial X^+(w)]^{n-1}\ :\beta(z)\,\partial X^+(w): \\ &\sim -\frac{1}{(z-w)^2}\,f'(\partial X^+(w)) \end{aligned} \tag{4.36}$$

which is strictly not valid for log function, but we used it to define the 'OPE' formally. Similarly

$$+2\kappa\,\partial_z^2\,[\log\partial X^+(z)]\,\beta(w)\partial X^+(w) = -2\kappa\,\partial X^+(w)\,\partial_z^2\left[\frac{1}{\partial X^+(z)}\frac{1}{(z-w)^2}\right] \tag{4.37}$$

- Next we consider $(\partial\log\partial X^+)^2$ piece of the stress tensor, for that we need to compute

$$-\kappa\,\beta(z)\,\partial X^+(z)\,(\partial_w\log\partial_w X^+(w))^2 - \kappa\,\beta(w)\partial X^+(w)\,(\partial_z\log\partial_z X^+(z))^2 \tag{4.38}$$

For the first term, as before

$$\begin{aligned} -\kappa\,\beta(z)\,\partial_z X^+(z)\,(\partial_w\log\partial_w X^+(w))^2 &= -2\kappa\,\partial_z X^+(z)\,\partial_w\left[\beta(z)\,\log\partial_w X^+(w)\right]\,\partial_w\log\partial_w X^+(w) \\ &= 2\kappa\,\partial_z X^+(z)\,\partial_w\left[\frac{1}{\partial_w X^+(w)}\frac{1}{(z-w)^2}\right]\,\partial_w\log\partial_w X^+(w) \end{aligned} \tag{4.39}$$

Similarly,

$$-\kappa\,\beta(w)\,\partial X^+(w)\,(\partial_z\log\partial_z X^+(z))^2 = 2\kappa\,\partial_w X^+(w)\,\partial_z\left[\frac{1}{\partial_z X^+(z)}\frac{1}{(z-w)^2}\right]\,\partial_z\log\partial_z X^+(z) \tag{4.40}$$

- Also

$$+\kappa^2\,\partial_z^2\log\partial X^+(z)\,\partial_w^2\log\partial X^+(w) = 0 \tag{4.41}$$

Making a Laurent expansion around $z=w$ and adding all these up, we get

$$T(z)\,T(w) = \frac{1-12\kappa}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \tag{4.42}$$

Thus requiring total central charge from the matter to be 26 requires

$$\kappa = -1 \tag{4.43}$$

Note that we can add the standard (b, c) ghost system with $c_{ghost} = -26$ to the action just as in Gomis-Ooguri [118].

Aside:

We could have started with the following $\beta\beta$ OPE (note the parameter q):

$$\beta(z)\,\beta(w) \sim q\,\kappa\,\partial_z\partial_w\left[\frac{1}{(z-w)^2}\,\frac{1}{\partial_z X^+(z)\,\partial_w X^+(w)}\right], \quad \text{it's cc} \tag{4.44}$$

then the "OPE structure" would remain intact:

$$T(z)\,T(w) = \frac{1-6\kappa(q+4)}{(z-w)^4} + \frac{2\tilde{T}(w)}{(z-w)^2} + \frac{\partial_w \tilde{T}(w)}{z-w} \tag{4.45}$$

where $\tilde{T}(w)$ is:

$$\tilde{T}(w) = -\beta\,\partial_w X^+(w) - \kappa(q+4)\,\{X^+(w), w\} \tag{4.46}$$

It reduces to $T(w)$ only for

$$\boxed{q = -2}$$

4.1.3 Relation to non-relativistic string of Gomis-Ooguri

Other than the κ-dependent piece, the action (4.1) is precisely the non-crtitical worldsheet model introduced [118] by Gomis and Ooguri:

$$S^1_{G-O} = \frac{1}{4\pi\alpha'}\int d^2z\left[\partial X^+\partial X^- + \partial X^-\bar{\partial}X^+ - 2\pi\alpha' B\left(\partial X^+\partial X^- - \partial X^-\bar{\partial}X^+\right)\right] \tag{4.47}$$

taking the low-energy and near critical field limit

$$\alpha' \to 0, \quad 2\pi\alpha' B = 1 - 2\pi\lambda\alpha' \tag{4.48}$$

This can be seen once we rephrase (4.47) by introducing two auxiliary fields $(\beta, \bar{\beta})$:

$$S^2_{G-O} = \int \frac{d^2z}{2\pi}\left(\beta\,\bar{\partial}X^+ + \bar{\beta}\,\partial X^- - \frac{2\alpha'}{1-2\pi\alpha' B}\,\beta\bar{\beta} + \frac{1-2\pi\alpha' B}{2\alpha'}\,\partial X^+\bar{\partial}X^-\right) \tag{4.49}$$

and take the limit in (4.48).

Polchinski-Strominger term for $D = 2$

We repeat here the covariant form of the Liouville action $\Gamma[\phi]$ (4.29):

$$(\Gamma[\phi])_{\text{covariant}} = -\frac{\kappa}{2}\int \frac{d^2\sigma}{2\pi}\sqrt{-g}\left[g^{ab}\,\partial_a\phi\,\partial_b\phi + 2\hat{R}\,\phi\right] \tag{4.50}$$

which matches precisely with the Polchinski-Strominger term at $D = 2$ [120, 121] :

$$S_{\text{Polchinski-Strominger}} = \frac{26-D}{24}\int \frac{d^2\sigma}{2\pi}\sqrt{-g}\left(\frac{1}{2}g^{ab}\partial_a\phi\,\partial_b\phi - \hat{R}\,\phi\right) \tag{4.51}$$

4.2 Torus partition function

4.2.1 Worldsheet action on the torus

On the target-space torus, the worldsheet sigma model take the form:

$$S_{\mathbb{T}^2} = \int \frac{d^2z}{2\pi}\left(\beta\,\bar{\partial}X + \bar{\beta}\,\partial\bar{X} + \lambda\pi\,\partial X\bar{\partial}\bar{X} - \kappa\,\partial\phi\,\bar{\partial}\phi\right) \tag{4.52}$$

Note that we have set the Ricci scalar $\hat{R} = 0$ on the torus, which is always possible (both locally and globally) for $\mathbb{T}^2$. To clarify the notations, here target space $\mathbb{T}^2$ has been represented by the complex plane quotiented by a complex lattice:

$$\mathbb{T}^2 \equiv C/(\mathbb{Z}\oplus\zeta\,\mathbb{Z}) \tag{4.53}$$

where the complex modulus $\zeta \in \mathcal{H}_+$. We can co-ordinatize this target space T^2 by $(X, \bar{X})$, where

$$X = X^1 + iX^2, \quad \bar{X} = X^1 - iX^2 \tag{4.54}$$

with the following identification under periodicity conditions:

$$(X^1, X^2) \equiv (X^1 + 2\pi R, X^2), \quad (X^1, X^2) \equiv (X^1 + 2\pi R\zeta^1, X^2 + 2\pi R\zeta_2) \tag{4.55}$$

The worldsheet torus is parametrized by complex coordinates $(z, \bar{z})$, with $z = \sigma^1 + i\sigma^2$, where (σ^1, σ^2) has similar periodicity as for the target space:

$$(\sigma^1, \sigma^2) \equiv (\sigma^1 + 2\pi, \sigma^2), \quad (\sigma^1, \sigma^2) \equiv (\sigma^1 + 2\pi\tau_1, \sigma^2 + 2\pi\tau_2) \tag{4.56}$$

Note that we have ignored the other tensionful term $\bar{\partial}X\partial\bar{X}$ which vanishes anyway for holomorphicity reason in the worldsheet path integral.

4.2.2 Partition function of 2d YM

The partition function of 2d Yang Mills theory on a torus can be written as an expansion in large N:

$$\mathcal{F}(\sigma) = \log\left[\mathcal{Z}(\sigma)\right] = \sum_{g=1}^{\infty} \left(\frac{1}{N}\right)^{2g-2} \mathcal{F}_{g\to 1}(\sigma) \tag{4.57}$$

where

$$\sigma = i\lambda\,\mathrm{Area}/(2\pi)$$

For the chiral 2d Yang Mills, we have the following expressions for the free energies [81]:

$$\begin{aligned}
\mathcal{F}_{1\to 1} &= -\log\left[\eta(\sigma)\right] = \sum_{n=1}^{\infty} e^{-n\lambda\,\mathrm{Area}}\,\frac{1}{2n}\sum_{a.d=n}(a+d) \\
\mathcal{F}_{2\to 1} &= \frac{(\lambda\,\mathrm{Area})^2}{2^5\cdot 3^4\cdot 5}\left(10E_2^3 - 6E_2E_4 - 4E_6\right)(\sigma)
\end{aligned} \tag{4.58}$$

Here $\eta(\sigma) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ with $q = \exp[2\pi i\sigma]$ and $E_{2k}(\sigma)$ is the holomorphic Eisenstein series of weight $2k$:

$$E_{2k}(\tau) = \frac{1}{2\,\zeta_R(2n)} \sum_{m,n\,\in\mathbb{Z}-\{0\}\}} \frac{1}{(m\tau+n)^{2k}} \tag{4.59}$$

4.2.3 Worldsheet partition function

The torus partition function involves a sum over maps $(X,\bar{X})$ from the worldsheet torus to the target space torus. Such map takes the form

$$\begin{aligned}
\frac{1}{R}X^1(z,\bar{z}) &= \frac{(m^1-\bar{\tau}w^1)+\zeta^1(m^2-\bar{\tau}w^2)}{\tau-\bar{\tau}}\,z + \frac{(\tau w^1-m^1)+\zeta^1(\tau w^2-m^2)}{\tau-\bar{\tau}}\,\bar{z} \\
\frac{1}{R}X^2(z,\bar{z}) &= \frac{\zeta^2(m^2-\bar{\tau}w^2)}{\tau-\bar{\tau}}\,z + \frac{\zeta^2(\tau w^2-m^2)}{\tau-\bar{\tau}}\,\bar{z}
\end{aligned} \tag{4.60}$$

which satisfy the winding boundary conditions:

$$\begin{aligned}
X^1(z+2\pi,\bar{z}+2\pi) &= X^1(z,\bar{z}) + 2\pi R w^1 + 2\pi R\zeta_1 w^2, \\
X^2(z+2\pi,\bar{z}+2\pi) &= X^2(z,\bar{z}) + 2\pi R\zeta_2 w^2 \\
X^1(z+2\pi\tau,\bar{z}+2\pi\bar{\tau}) &= X^1(z,\bar{z}) + 2\pi R m^1 + 2\pi R\zeta_1 m^2, \\
X^2(z+2\pi\tau,\bar{z}+2\pi\bar{\tau}) &= X^2(z,\bar{z}) + 2\pi R\zeta_2 m^2
\end{aligned} \tag{4.61}$$

(m^1,m^2,w^1,w^2) label different winding sectors of the string configuration in computing the partition function path integral.

The worldsheet partition function is

$$\mathcal{Z} = Z_{\beta,X}\,Z_{ghost} \tag{4.62}$$

The bc-ghost partition function Z_{ghost} cancels the contribution from the non-zero modes of (β, X) in $Z_{\beta,X}$, i.e familiar theta function won't appear here.

Thus the partition function is simply given by contributions from maps:

$$\mathcal{Z}(R_0, R_1, \zeta, \bar{\zeta}) = 4\pi^2 R_0 R_1 \zeta_2 \sum_{\{m^a, w^a\}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\, e^{-S^{X,\bar{X}}_{\{m^a,w^a\}}}\, \mathcal{F}_{\beta_0}(\{m^a, w^a\}) \tag{4.63}$$

The prefactor $4\pi^2 R_0 R_1 \zeta_2$ comes from the zero mode integral of X which gives the volume (i.e area here) of the target space, and $\mathcal{F}_{\beta_0}(\{m^a, w^a\})$ is the contribution from path integral over $(\beta_0, \bar{\beta}_0)$.

We will now compute $\mathcal{F}_{\beta_0}(\{m^a, w^a\})$ and observe a localization scenario for (anti-) holomorphicity constraints on $(\bar{X})$ X field. For that we note

$$\bar{\partial} X = R \frac{(\tau w^1 - m^1) + \zeta(\tau w^2 - m^2)}{\tau - \bar{\tau}} \tag{4.64}$$

Since β can be written as $\beta = \beta_0 + \partial[\text{periodic}]$ with β_0 being the zero mode, the worldsheet action of this zero mode gives

$$\int \frac{d^2 z}{2\pi}\, \beta_0\, \bar{\partial} X = -2\pi i \beta_0 R\,(w^1 + \zeta w^2) \left[\tau - \frac{m^1 + \zeta m^2}{w^1 + \zeta w^2}\right] \tag{4.65}$$

where note that $\int d^2 z = 2 \int d^2\sigma = 8\pi^2 \tau_2$. Similarly from the part with $\bar{\beta}_0$, we would obtain

$$\int \frac{d^2 z}{2\pi}\, \bar{\beta}_0\, \partial \bar{X} = 2\pi i \bar{\beta}_0 R\,(w^1 + \bar{\zeta} w^2) \left[\bar{\tau} - \frac{m^1 + \bar{\zeta} m^2}{w^1 + \bar{\zeta} w^2}\right] \tag{4.66}$$

Performing the path-integral over $(\beta_0, \bar{\beta}_0)$ in the path integral, we will get two-dimensional delta function:

$$\mathcal{F}_{\beta_0}(\{m^a, w^a\}) = \frac{1}{4\pi^2 R_1^2 |w^1 + \zeta w^2|^2}\, \delta^{(2)} \left[\tau - \frac{m^1 + \zeta m^2}{w^1 + \zeta w^2}\right] \tag{4.67}$$

which fixes the worldsheet modulus τ at

$$\tau^* = \frac{m^1 + \zeta m^2}{w^1 + \zeta w^2} \tag{4.68}$$

The map (4.60) suggests holomorphic ∂X which is essentially independent of $(z, \bar{z})$, and it is what we expect on the torus where there is only one abelian differential which is just identity (on a genus g surface, there are g such differentials). With this map the kinetic piece of Liouville term in the action (4.52) vanishes:

$$\partial\phi \bar{\partial}\phi = \frac{\partial^2 X^+}{\partial X^+} \frac{\bar{\partial}^2 X^-}{\bar{\partial} X^-} \to 0, \quad \text{under the map (4.60)} \tag{4.69}$$

We will now exploit a trick [125] to simplify the computation of sum over windings $\{m^a, w^a\}$ which can be represented as the sum over all possible matrices

$$\begin{pmatrix} m^2 & m^1 \\ w^2 & w^1 \end{pmatrix}$$

We define K to be the determinant of this matrix $K = (m^2 w^1 - w^2 m^1)$. We can split the sum over windings in (4.63) into sum over K and matrices of the above form with fixed determinant K:

$$\mathcal{Z}(R, \zeta, \bar{\zeta}) = \sum_K \mathcal{Z}_K(R, \zeta, \bar{\zeta}) \tag{4.70}$$

We will consider non-zero winding here and separate two cases of winding numbers being $+K$ and $-K$ ($K \in \mathbb{Z}_+$) in what follows.

Such an matrix with fixed determinant can been obtained by applying an element of $SL(2, \mathbb{Z})$ on any one of the matrices

$$\mathcal{S}_{\pm K} = \left\{ \begin{pmatrix} a & b \\ 0 & \pm d \end{pmatrix}, \quad a, b, d \in \mathbb{Z}, \quad ad = K, \quad 0 \le b < d, \quad d > 0 \right\} \tag{4.71}$$

when the determinant is $\pm K$ ($K \in \mathbb{Z}_+$).

Using the above $SL(2, \mathbb{Z})$ translates, we can unfold the fundamental domain of τ integral in (4.63) to the whole upper half plane $\mathcal{H}_+$ and replace the sum over $\{m^a, w^a\}$ as a sum over matrices of the form (4.71) with fixed $(\pm K)$, i.e

$$\mathcal{Z}(R, \zeta, \bar{\zeta}) = \sum_{K=1}^{\infty} \mathcal{Z}_{\pm K}(R, \zeta, \bar{\zeta}) \tag{4.72}$$

with

$$\mathcal{Z}_{\pm K}(R, \zeta, \bar{\zeta}) = \zeta_2 \sum_{\mathcal{S}_{\pm K}} \frac{1}{d^2} \int_{\mathcal{H}_+} \frac{d^2\tau}{\tau_2}\, e^{-S^{\pm}_{\{a,b,d\}}}\, \delta^{(2)}\left(\tau \mp \frac{b + a\zeta}{d}\right) \tag{4.73}$$

We quickly make the observation that since $a, b, d > 0$ in (4.71) and ζ being target space torus modulus stays in the fundamental domain $\mathcal{F} \subset \mathcal{H}_+$, for $\mathcal{Z}_{-K}$ localization points

$$\tau^*_- = -\frac{b + a\zeta}{d} \notin \mathcal{H}_+$$

and so

$$\boxed{\mathcal{Z}_{-K}(R, \zeta, \bar{\zeta}) = 0} \tag{4.74}$$

Next to compute $\mathcal{Z}_{+K}(R, \zeta, \bar{\zeta})$, we need to evaluate $S^+_{\{a,b,d\}}$:

$$S^+_{\{a,b,d\}} = \lambda \frac{\pi^2 R^2 d^2}{\tau_2} \left| \tau - \frac{b + a\bar{\zeta}}{d} \right|^2 \tag{4.75}$$

at the points $\tau_+^* = \frac{b+a\zeta}{d}$, where it gives

$$S^+_{\{a,b,d\}}(\tau_+^*) = \lambda\,(4\pi^2 R^2 \zeta_2)\, ad = \lambda\, K\, \text{Area} \tag{4.76}$$

Thus

$$\boxed{\mathcal{Z}_{+K}(R_0, R_1, \zeta, \bar\zeta) = \frac{1}{K}\sum_{S_{+K}} e^{-\lambda\, K\, \text{Area}} = \frac{1}{K}\, e^{-\lambda\, K\, \text{Area}} \sum_{a.d=K}\sum_{b=0}^{d-1} 1 = e^{-\lambda\, K\, \text{Area}}\, \frac{1}{K}\sum_{a.d=K} d} \tag{4.77}$$

and so we conclude from (6.37),

$$\mathcal{Z}_{worldsheet} = \sum_{K=1}^{\infty} e^{-\lambda\, K\, \text{Area}}\, \frac{1}{2K}\sum_{a.d=K}(a+d) = -\log\eta(\sigma) \tag{4.78}$$

where we used

$$\sum_{a.d=K}(a+d) = 2\sum_{a.d=K} d$$

The result (4.78) matches precisely with the leading expression of free energy (4.58) in the chiral 2d YM.

4.3 String amplitudes on the cylinder

4.3.1 Vertex Operator

Our winding string vertex operator is

$$V_k(z_k, \bar z_k) =: \exp\left[i w_k R \int^{z_k} dz' \left(\beta(z') + \lambda\pi\, \partial' X^-\right) + i w_k R \int^{\bar z_k} d\bar z' \left(\bar\beta(\bar z') + \lambda\pi\, \bar\partial' X^+\right)\right] : \tag{4.79}$$

where we've removed the cocycle factors which could only add an extra sign. We can show that it has indeed a winding monodromy when we drag $X^+(z,\bar z)$ around it following the OPE:

$$\begin{aligned} X^+(z,\bar z)\, V_k(z_k,\bar z_k) &\sim i w_k R \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\left(i w_k R \int^{z_k}\beta\right)^{n-1} [\int^{z_k} X^+(z)\beta(z'')\, dz'']\, e^{i w_k R \int^{\bar z_k}\bar\beta} \\ &\sim -i w_k R\, \log(z - z_k)\, V_k(z_k, \bar z_k) \end{aligned} \tag{4.80}$$

4.3.2 Correlation function on the worldsheet

The n-point correlation function with the above vertex operators is

$$G_n = \langle \prod_{k=1}^{n} V_k(z_k, \bar{z}_k) \rangle \sim \int \mathfrak{D}X^+ \mathfrak{D}X^- \mathfrak{D}\beta\, \mathfrak{D}\bar{\beta}\, \exp[-S_{\text{Action}} + S_{\text{Vertex-Operator}}] \tag{4.81}$$

where the exponent is given by:

$$\begin{aligned} &- S_{\text{Action}} + S_{\text{Vertex-Operator}} \\ &= -\int \frac{d^2z}{2\pi} \left[\beta\, \bar{\partial} X^+ + \bar{\beta}\, \partial X^- + \lambda\pi \left(\partial X^+ \bar{\partial} X^- - \bar{\partial} X^+ \partial X^- \right) - \frac{\kappa}{2} \left(2\, \partial\phi\, \bar{\partial}\phi + \hat{R}\, \phi \right) \right] \\ &\quad + i \sum_{k=1}^{n} w_k R \int^{z_k} dz' \left(\beta(z') + \lambda\pi\, \partial' X^- \right) + i \sum_{k=1}^{n} w_k R \int^{\bar{z}_k} d\bar{z}' \left(\bar{\beta}(\bar{z}') + \lambda\pi\, \bar{\partial}' X^+ \right) \end{aligned} \tag{4.82}$$

Now we will <u>integrate out $(\beta, \bar{\beta})$</u> in the above worldsheet path integral:

We can concentrate on the terms with β in the exponent:

$$\int \frac{d^2z}{2\pi}\, \beta(z) \bar{\partial} X^+ - i \sum_{k=1}^{n} w_k R \int^{z_k} \beta = \int \frac{d^2z}{2\pi}\, \beta(z)\, \bar{\partial} \left[X^+ + i \sum_{k=1}^{n} w_k R\, \log(z - z_k) \right] \tag{4.83}$$

where we've used the following steps for the second term (using $\delta^{(2)}(z - z') = \frac{1}{2\pi}\, \bar{\partial} \left(\frac{1}{z - z'} \right)$):

$$\begin{aligned} -i \sum_{k=1}^{n} w_k R \int^{z_k} \beta &= -i \sum_{k=1}^{n} w_k R \int d^2z\, \beta(z) \int^{z_k} dz'\, \delta^{(2)}(z - z') \\ &= \int \frac{d^2z}{2\pi}\, \beta(z) \left[-i \sum_{k=1}^{n} w_k R\, \bar{\partial} \left(\int^{z_k} \frac{dz'}{z - z'} \right) \right] \\ &= \int \frac{d^2z}{2\pi}\, \beta(z)\, \bar{\partial} \left[+i \sum_{k=1}^{n} w_k R\, \log(z - z_k) \right] \end{aligned} \tag{4.84}$$

Thus integrating out $\beta, \bar{\beta}$, we would be left with delta functions:

$$G_n \sim \int \mathfrak{D}X^+ \mathfrak{D}X^- \delta \left(X^+ - i \sum_{k=1}^{n} w_k R\, \log(z - z_k) \right) \delta \left(X^- - i \sum_{k=1}^{n} w_k R\, \log(\bar{z} - \bar{z}_k) \right) e^{-S_0 - \Gamma[\phi]} \tag{4.85}$$

where

$$S_0 = \int \frac{d^2z}{2\pi}\, \lambda\pi \left(\partial X^+ \bar{\partial} X^- - \bar{\partial} X^+ \partial X^- \right) + i\lambda\pi \sum_{k=1}^{n} w_k R\, X^-(z_k, \bar{z}_k) + i\lambda\pi \sum_{k=1}^{n} w_k R\, X^+(z_k, \bar{z}_k) \tag{4.86}$$

and

$$\Gamma[\phi] = -\int \frac{d^2z}{2\pi} \frac{\kappa}{2} \left(2\, \partial\phi\, \bar{\partial}\phi + \hat{R}\, \phi \right), \quad \phi = \log \left[\partial X^+ \bar{\partial} X^- \right]$$

We notice that $X^{\pm}(z,\bar z)$ fields localize on the Mandelstam maps $(\rho,\bar\rho)$ in the path integral:

$$\begin{aligned} X^+(z,\bar z) &\to \rho(z) := -i\sum_{k=1}^{n} w_k R\,\log(z-z_k),\\ X^-(z,\bar z) &\to \bar\rho(\bar z) := -i\sum_{k=1}^{n} w_k R\,\log(\bar z-\bar z_k) \end{aligned} \tag{4.87}$$

and the correlation function is simply given by

$$G_n \sim \left[e^{-S_0-\Gamma[\phi]}\right]_{X^+(z,\bar z)=\rho(z),X^-(z,\bar z)=\bar\rho(\bar z)} \tag{4.88}$$

We can evaluate the Liouville action contribution easily (see appendix A):

$$[e^{-\Gamma[\phi]}]_{X^+=\rho,X^-=\bar\rho} = \prod_{k=1}^{n}(w_k\,R)^{-2\kappa}\left|\sum_{k=1}^{n} w_k R\,z_k\right|^{4\kappa}\left|\prod_{I=1}^{n-2}\partial^2\rho(Z_I)\right|^{-\kappa} \tag{4.89}$$

Here $\{Z_I\}$ s are $(n-2)$ string interaction points defined by

$$\partial\rho(Z_I) = -i\sum_{k=1}^{n}\frac{w_k R}{Z_I-z_k} = 0 \tag{4.90}$$

And the contribution from the tensionful terms vanishes (see Appendix C):

$$[e^{-S_0}]_{X^+=\rho,X^-=\bar\rho} = 1 \tag{4.91}$$

We also have to take into account the ghost contribution $\langle\prod_{k=1}^{n} c(z_k)\,\bar c(\bar z_k)\rangle$ to the world-sheet correlator.

4.3.3 Three point String amplitude

Let's compute the correlator for three-point functions $n=3$. From (4.90) for $n=3$, we find the only interaction point $I=1$:

$$Z_1 = -\frac{p_1^+ z_2z_3 + p_2^+ z_1z_3 + p_3^+ z_1z_2}{p_1^+ z_1 + p_2^+ z_2 + p_3^+ z_3} \tag{4.92}$$

With this,

$$\partial^2\rho(Z_1) = -i\,\frac{(\sum_{i=1}^{3} p_i^+ z_i)^4}{\left(\prod_{i=1}^{3} p_i^+\right)\prod_{i<j}^{3} z_{ij}^2} \tag{4.93}$$

With these, we clearly obtain

$$[e^{-\Gamma[\phi]}]_{X^+=\rho,X^-=\bar\rho} = R^3\,w_1w_2w_3\prod_{i<j}^{3}|z_{ij}|^{-2} \tag{4.94}$$

Using the standard form of ghost correlator on the worldsheet,

$$\langle\prod_{i=1}^{3} c(z_i)\,\bar{c}(\bar{z}_i)\rangle \sim \prod_{i<j}^{3} |z_{ij}|^2 \tag{4.95}$$

The worldsheet three-point function G is simply (setting $R = 1$):

$$G \sim w_1 w_2 w_3 \tag{4.96}$$

4.4 Amplitudes of 2d Yang Mills

In the hamiltonian formulation of pure $U(N)$ Yang Mills on the cylinder, the hamiltonian is given by

$$H_{U(N)} = \frac{\lambda L}{2}\left(\sum_{m=1}^{\infty} a_m^\dagger a_m + \frac{1}{N}\sum_{n,m=1}^{\infty}(a_{m+n}^\dagger a_m a_n + a_m^\dagger a_n^\dagger a_{m+n})\right) \tag{4.97}$$

A permutation $|\sigma\rangle$ with k_i cycles of length i ($i = 1, 2, \cdots$) corresponds to a closed string state with k_i strings winding i times around the circle in a particular orientation (say anti-clockwise). In terms of the bosonic creation operators, we can write it as

$$|\sigma\rangle = \prod_i \left(a_i^\dagger\right)^{k_i} |\Omega\rangle \tag{4.98}$$

where $|\Omega\rangle$ is the vacuum in the full interacting theory, i.e $a_k|\Omega\rangle = 0 \quad \forall\, k$. The winding operators satisfy the relation

$$[a_k, a_l^\dagger] = k\,\delta_{k,l} \tag{4.99}$$

Let us assume that at $t = t_0$, the interaction and Heisenberg pictures become equivalent. We denote fields in the interaction picture as $O_I(\theta, t)$ and in the Heisenberg picture as $O(\theta, t)$. Then

$$O(\theta, t) = e^{iH(t-t_0)}\,O(\theta)\,e^{-iH(t-t_0)}, \quad O_I(\theta, t) = e^{iH_0(t-t_0)}\,O(\theta)\,e^{-iH_0(t-t_0)} \tag{4.100}$$

where $O(\theta) \equiv O(\theta, t_0) \equiv O_I(\theta, t_0)$. It then easily follows that

$$O(\theta, t) = U^\dagger(t, t_0)\,O_I(\theta, t)\,U(t, t_0), \quad U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{4.101}$$

We want to compute the three point amplitude of strings with winding n_1, n_2, n_3:

$$\langle\Omega|a_{n_1}(\infty)\,a_{n_2}(\infty)\,a_{n_3}^\dagger(-\infty)|\Omega\rangle \tag{4.102}$$

If we simply demand the obvious relation for winding modes

$$a_n(t) = e^{iH(t-t_0)} a_n e^{-iH(t-t_0)} \tag{4.103}$$

H being the full hamiltonian and $a_n \equiv a_n(t_0)$, then 4.102 becomes

$$\begin{aligned}&\langle\Omega|a_{n_1}(\infty)\, a_{n_2}(\infty)\, a^\dagger_{n_3}(-\infty)|\Omega\rangle \\ &\sim \langle 0|U(\infty,t_0)e^{iH(\infty-t_0)}\, a_{n_1}a_{n_2}\, e^{-iH(\infty-t_0)}e^{iH(\infty-t_0)}a^\dagger_{n_3}(t_0)e^{-iH(\infty-t_0)}U(t_0,-\infty)|0\rangle\end{aligned} \tag{4.104}$$

Now

$$\langle 0|U(\infty,t_0) \sim \langle 0|e^{-iH(\infty-t_0)}, \quad \text{since } H_0|0\rangle = 0$$

and similarly

$$U(t_0,-\infty)|0\rangle \sim e^{iH(\infty-t_0)}|0\rangle$$

With these, we obtain

$$\begin{aligned}&\langle\Omega|a_{n_1}(\infty)\, a_{n_2}(\infty)\, a^\dagger_{n_3}(-\infty)|\Omega\rangle \\ &\sim \langle 0|a_{n_1}a_{n_2}\exp\left[-i\frac{\lambda L}{2}\left(\sum_{m=1}^{\infty} a^\dagger_m a_m + \frac{1}{N}\sum_{n,m=1}^{\infty}(a^\dagger_{m+n}a_m a_n + a^\dagger_m a^\dagger_n a_{m+n})\right)\right] a^\dagger_{n_3}|0\rangle\end{aligned} \tag{4.105}$$

Performing this, we obtain

$$\langle\Omega|a_{n_1}(\infty)\, a_{n_2}(\infty)\, a^\dagger_{n_3}(-\infty)|\Omega\rangle \sim n_1 n_2 n_3\, \delta_{n_1+n_2,n_3} \tag{4.106}$$

This matches exactly with the worldsheet three point string amplitude (4.96).

4.5 Comments

In this chapter, we proposed an exact worldsheet description dual to the chiral pure two-dimensional Yang Mills theory. The duality is buttressed by two strands: matching of the leading large N torus partition function, and the scattering amplitudes of winding string states from both sides. An immediate open question is to add dynamical boundaries to the worldsheet, and try to reproduce the famous 't Hooft meson spectrum [115] from this. This then could be a toy version of the grand dream of computing the meson spectrum of 4d QCD, which describes our real world, from it's potential string dual. We should also be able to reproduce the instaton contributions to 2d YM from D-instanton computation on the worlsheet side. On a different note, [134] studied the spatial entanglement entropy in the two-dimensional Yang Mills, and tried to argue a stringy picture of this. Given our explicit worlsheet model, it would be very exciting to understand how their picture translates in our worldsheet model in a precise manner.

Appendix

4.A Derivation of the Mandelstam formula

The Liouville part of the action reads

$$\Gamma[\phi] = -\int \frac{d^2 z}{2\pi} \frac{\kappa}{2} \left(2\,\partial\phi\,\bar{\partial}\phi + \hat{R}\,\phi\right) \tag{4.107}$$

whose covariant form (4.29) is

$$(\Gamma[\phi])_{\text{covariant}} = -\frac{\kappa}{2} \int \frac{d^2\sigma}{2\pi} \sqrt{-g}\left[g^{ab}\,\partial_a\phi\,\partial_b\phi + 2\hat{R}\,\phi\right] \tag{4.108}$$

with the Liouville field $\phi(z, \bar{z})$ related to the conformal factor

$$\phi(z, \bar{z}) = \log \partial X^+(z) + \log \bar{\partial} X^-(\bar{z}) \tag{4.109}$$

We want to evaluate $\Gamma[\phi]$ when (X^+, X^-) are localized on the Mandelstam maps $(\rho, \bar{\rho})$:

$$\rho(z) = \sum_{k=1}^{n} \alpha_k \, \log(z - z_k) \quad \bar{\rho}(\bar{z}) = \sum_{k=1}^{n} \bar{\alpha}_k \, \log(\bar{z} - \bar{z}_k) \tag{4.110}$$

So under this localization,

$$[\phi(z, \bar{z})]_{X^+=\rho, X^-=\bar{\rho}} = \log \partial\rho(z) + \log \bar{\partial}\bar{\rho}(\bar{z}) \tag{4.111}$$

4.A.1 Kinetic term

Let's first compute the Liouville kinetic term. We rewrite it as

$$(\Gamma[\phi])_{\text{kinetic}} = -\frac{\kappa}{2} \int \frac{d^2\sigma}{2\pi}\, \partial_\alpha\phi\, \partial^\alpha\phi = +\frac{\kappa}{4\pi} \int d^2\sigma\, \phi\, \partial_\alpha \partial^\alpha \phi - \frac{\kappa}{4\pi} \int_\partial ds\, \phi\, \partial_n \phi \tag{4.112}$$

where we've used $d^2 z = 2\, d^2\sigma$ and $4\partial\phi\,\bar{\partial}\phi = \partial_\alpha\phi\,\partial^\alpha\phi$ and the stokes theorem: $\int d^2\sigma\, \partial_\alpha \left(\phi\, \partial^\alpha \phi\right) = \int ds\, \phi\, \partial_n \phi$. It's clear $\partial_\alpha\partial^\alpha\phi = 0$ in the bulk of the worldsheet and so only contribution comes from the boundary of our surface, which are discs around the zeros $\{Z_I\}_{I=1}^{n-2}$, and poles $\{z_i\}_{i=1}^{n}$ of $\partial\rho(z)$ (and also $\bar{\partial}\bar{\rho}(\bar{z})$) and around infinity $(z_\infty, \bar{z}_\infty)$ that we will cut off in the following.

$$(\Gamma[\phi])_{\text{kinetic}} = -\frac{\kappa}{4\pi}\int_\partial ds\,\phi\,\partial_n\phi = -\frac{\kappa}{4\pi}\left(\int_{\cup_I \partial D_{Z_I}} + \int_{\cup_k \partial D_{z_k}} + \int_{D_{z_\infty}}\right) ds\,\phi\,\partial_n\phi \tag{4.113}$$

Normal derivative ∂_n at the boundary of any disc[5] is

$$\partial_n = -\frac{1}{|z|}(z\,\partial + \bar{z}\,\bar{\partial}) \tag{4.114}$$

so that

$$\int_\partial ds\,\phi\,\partial_n\phi = i\oint dz\,\phi\partial\phi - i\oint d\bar{z}\,\phi\bar{\partial}\phi \tag{4.115}$$

where the contours are around the centers of the respective discs[6].

We cut out discs of radius $|\epsilon_{Z_I}|$ around $z = Z_I$ on the z-plane, on the ρ-space it would corresponds to radius r_I, where both are related in the following way:

$$\begin{aligned}|\rho - \rho_I| = \frac{1}{2}|\partial^2\rho(Z_I)||z - Z_I|^2 \Rightarrow \log|z - Z_I| &= \frac{1}{2}\log(2|\rho-\rho_I|) - \frac{1}{2}\log|\partial^2\rho(Z_I)| \\ \Rightarrow \log|\epsilon_{Z_I}| &= \frac{1}{2}\Big(\log(2r_I) - \log|\partial^2\rho(Z_I)|\Big)\end{aligned} \tag{4.116}$$

where we used $|z - Z_I| = |\epsilon_{Z_I}|$ and $|\rho - \rho_I| = r_I$.

Near the interaction point $z = Z_I$ behaves as

$$\begin{aligned}\phi &\approx \log[\partial^2\rho(Z_I)(z - Z_I)] + \log[\bar{\partial}^2\bar{\rho}(\bar{Z}_I)(\bar{z} - \bar{Z}_I)] \\ &= 2\log|\partial^2\rho(Z_I)| + 2\log|\epsilon_{Z_I}| = \log(2r_I) + \log|\partial^2\rho(Z_I)|\end{aligned} \tag{4.117}$$

and $\partial\phi$ behaves as

$$\partial\phi \approx \frac{\partial^2\rho(Z_I)}{(z - Z_I)\,\partial^2\rho(Z_I) + \cdots} = \frac{1}{z - Z_I} + \mathcal{O}(1) \tag{4.118}$$

Using these results,

$$\oint_{z=Z_I} dz\,\phi\,\partial\phi = \Big(\log(2r_I) + \log|\partial^2\rho(Z_I)|\Big)\oint_{z=Z_I}\frac{dz}{z - Z_I} = 2\pi i\Big(\log(2r_I) + \log|\partial^2\rho(Z_I)|\Big) \tag{4.119}$$

[5] This could be derived in the following way: Normal vector at a point on the disc is $\vec{n} = -(\sigma^1\hat{e}_1 + \sigma^2\hat{e}_2) = -\frac{1}{2}(z+\bar{z})\,\hat{e}_1 - \frac{1}{2i}(z-\bar{z})\,\hat{e}_2 \Rightarrow \hat{n} = -\frac{z+\bar{z}}{2|z|}\,\hat{e}_1 - \frac{z-\bar{z}}{2i|z|}\,\hat{e}_2$, $\vec{\nabla} = \hat{e}_1\partial_1 + \hat{e}_2\partial_2 = \hat{e}_1(\partial+\bar{\partial}) + i\hat{e}_2(\partial-\bar{\partial})$. Using these, $\partial_n = \hat{n}\cdot\vec{\nabla} = -\frac{1}{|z|}(z\,\partial + \bar{z}\,\bar{\partial})$. Note that the normal vector $\vec{n}$ has opposite direction than what would be expected from the coordinates on the cut-off discs, this is because it is normal w.r.t the bulk after cutting off the discs

[6] This could be shown easily using $z = re^{i\theta}$, $ds = rd\theta$ so that $\int_\partial ds\left(-\frac{z}{|z|}\right)\phi\partial\phi = i\int_\partial dz\,\phi\partial\phi$ and similarly for the other term.

Similarly,

$$\oint_{\bar{z}=\bar{Z}_I} d\bar{z}\,\phi\,\bar{\partial}\phi = -2\pi i\Big(\log(2r_I) + \log|\partial^2\rho(Z_I)|\Big)$$

and so

$$\int_{\cup_I \partial D_{Z_I}} ds\,\phi\,\partial_n\phi = -4\pi\sum_{I=1}^{n-2}\Big(\log|\partial^2\rho(Z_I)| + \log(2r_I)\Big) \tag{4.120}$$

We also cut-out small discs of radius $|\epsilon_{z_k}|$ around the vertex operator insertions points $z = z_k$. Regularization associated with $|\epsilon_{z_k}|$ corresponds to wave-function renormalization of the vertex operators on the z-plane.

Near a insertion point $z = z_k$, $\phi(z,\bar{z})$ behaves as

$$\rho \sim \alpha_k\,\log(z - z_k) + \bar{\alpha}_k\log(\bar{z} - \bar{z}_k) \Rightarrow \phi \approx 2\log\left(\frac{|\alpha_k|}{|z - z_k|}\right) \tag{4.121}$$

whereas $\partial\phi(z)$ goes as

$$\partial\phi = \frac{\partial^2\rho(z)}{\partial\rho(z)} \approx -\frac{1}{z - z_k} \tag{4.122}$$

And thus

$$\oint_{z=z_k} dz\,\phi\,\partial\phi = -2\log\left(\frac{|\alpha_k|}{|\epsilon_{z_k}|}\right)\oint_{z=z_k}\frac{dz}{z - z_k} = -4\pi i\log\left(\frac{|\alpha_k|}{|\epsilon_{z_k}|}\right) \tag{4.123}$$

which gives (adding contribution from anti-holomorphic part)

$$\int_{\cup_k \partial D_{z_k}} ds\,\phi\,\partial_n\phi = +8\pi\log\left(\prod_{k=1}^{n}\frac{|\alpha_k|}{|\epsilon_{z_k}|}\right) \tag{4.124}$$

Around infinity $(z_\infty, \bar{z}_\infty)$, we have the following behaviour:

$$\begin{aligned} \phi(z,\bar{z}) &\sim \log\left|\sum_{k=1}^{n}\alpha_k z_k\right|^2 - \log|z|^4 + \mathcal{O}\left(\frac{1}{|z|}\right) \\ \partial\phi &\sim -\frac{2}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \end{aligned} \tag{4.125}$$

Using these,

$$\begin{aligned} \oint_{C_{z_\infty}} dz\,\phi\,\partial\phi &= -2\log\left|\sum_{k=1}^{n}\alpha_k z_k\right|^2 \underbrace{\oint_{C_{z_\infty}}\frac{dz}{z}}_{-2\pi i} + 2\underbrace{\oint_{C_{z_\infty}}\frac{dz}{z}\log|z|^4}_{-2\pi i\log|z_\infty|^4} \\ &= 4\pi i\log\left|\sum_{k=1}^{n}\alpha_k z_k\right|^2 - 4\pi i\log|z_\infty|^4 \end{aligned} \tag{4.126}$$

Similarly,

$$\oint_{C_{z_\infty}} d\bar{z}\,\phi\,\bar{\partial}\phi = -4\pi i\log\left|\sum_{k=1}^{n}\alpha_k z_k\right|^2 + 4\pi i\log|z_\infty|^4 \tag{4.127}$$

Adding both, we get

$$\int_{D_{z_\infty}} ds\,\phi\partial_n\phi = -8\pi\log\left[\left|\sum_{k=1}^{n}\alpha_k z_k\right|^2 |z_\infty|^{-4}\right] \tag{4.128}$$

Adding these contributions in (4.113),

$$(\Gamma[\phi])_{\text{kinetic}} = -\log\left[\left|\sum_{k=1}^{n}\alpha_k z_k\right|^{-4\kappa} |z_\infty|^{+8\kappa}\prod_{I=1}^{n-2}\left|\partial^2\rho(Z_I)\right|^{-\kappa}(2r_I)^{-\kappa}\prod_{k=1}^{n}|\alpha_k|^{+2\kappa}|\epsilon_{z_k}|^{-2\kappa}\right] \tag{4.129}$$

Using , we can rewrite it as

$$(\Gamma[\phi])_{\text{kinetic}} = -\kappa\log\left[\left|\sum_{k=1}^{n}\alpha_k z_k\right|^{n-2}\frac{\prod_{k=1}^{n}\prod_{I=1}^{n-2}|z_k-Z_I|^3}{\prod_{j\neq k}^{n}|z_j-z_k|^2\prod_{I\neq J}^{n-2}|Z_I-Z_J|}\frac{|z_\infty|^8}{\prod_{I=1}^{n-2}(2r_I)\prod_{k=1}^{n}|\epsilon_{z_k}|^2}\right] \tag{4.130}$$

4.A.2 Curvature term

Next we have to evaluate for the curvature term of the Liouville action

$$(\Gamma[\phi])_{\text{curvature}} = -\frac{\kappa}{2}\int_{\mathcal{M}}\frac{d^2\sigma}{2\pi}\sqrt{|g|}\,2\hat{R}\,\phi \tag{4.131}$$

Satisfying the relation

$$\int d^2\sigma\sqrt{|g|}\,\hat{R} = 4\pi(2-n) \tag{4.132}$$

we take the following form of the Ricci scalar in $(z,\bar{z})$-coordinates:

$$\left(\sqrt{|g|}\,\hat{R}\right)_{(z,\bar{z})} = 8\pi\,\delta^{(2)}(\tilde{z}-\delta) - 4\pi\sum_{k=1}^{n}\delta^{(2)}(z-z_k) \tag{4.133}$$

where we have a different co-ordinate chart around $\tilde{z}=0$, where $z=-1/\tilde{z}$. With this form,

$$\int d^2z\left(\sqrt{-g}\,\hat{R}\right)_{(z,\bar{z})} = 4\pi(2-n) \tag{4.134}$$

Note that the cut-off δ is related to z_∞ as $\delta=-1/z_\infty$. The composite Liouville field as the covariant form (4.17):

$$\Phi = \log\left[\frac{1}{4}g^{ab}\partial_a\rho\,\partial_b\bar{\rho} + \frac{i}{4}\epsilon^{ab}\partial_a\rho\,\partial_b\bar{\rho}\right] \tag{4.135}$$

To determine Φ around in $(\tilde{z},\bar{\tilde{z}})$ co-ordinate chart, we first determine the change in the metric:

$$(g)_{ab} = \begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix} \Rightarrow (\tilde{g})_{\tilde{a}\tilde{b}} = \frac{1}{|\tilde{z}|^4}\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix} \tag{4.136}$$

where we have used the coordinate transformation $z = -1/\tilde{z}$ and $a, b = z, \bar{z}$; $\tilde{a}, \tilde{b} = \tilde{z}, \bar{\tilde{b}}$. This clearly imply $\tilde{g}^{\tilde{z}\bar{\tilde{z}}} = 2|\tilde{z}|^4$, i.e

$$(\tilde{g})^{\tilde{a}\tilde{b}} = |\tilde{z}|^4 \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{4.137}$$

Similarly ϵ^{ab} changes as

$$(\epsilon^{ab})_{(z,\bar{z})} = \begin{pmatrix} 0 & -2i \\ +2i & 0 \end{pmatrix} \quad \Rightarrow \quad (\epsilon^{\tilde{a}\tilde{b}})_{(\tilde{z},\bar{\tilde{z}})} = |\tilde{z}|^4 \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \tag{4.138}$$

Using these, Φ in $(\tilde{z}, \bar{\tilde{z}})$ co-ordinate chart becomes

$$\phi(\tilde{z}, \bar{\tilde{z}}) = \log\left[|\tilde{z}|^4\, \tilde{\partial}\rho(\tilde{z})\bar{\tilde{\partial}}\bar{\rho}(\bar{\tilde{z}})\right] \tag{4.139}$$

Now

$$\tilde{\partial}\rho\,(z = -1/\tilde{z}) = \frac{\partial z}{\partial \tilde{z}}\frac{\partial \rho}{\partial z}\,[z = -1/\tilde{z}] = \frac{1}{\tilde{z}^2}\frac{\partial \rho}{\partial z}\,[z = -1/\tilde{z}] \tag{4.140}$$

Using the expression of $\partial\rho(z)$:

$$\rho(z) = \sum_{k=1}^{n} \alpha_k \log\,(z - z_k), \quad \partial\rho(z) = \sum_{k=1}^{n} \frac{\alpha_k}{z - z_k} = \left(\sum_{k=1}^{n} \alpha_k z_k\right) \frac{\prod_{I=1}^{n-2}(z - Z_I)}{\prod_{k=1}^{n}(z - z_k)} \tag{4.141}$$

we thus obtain

$$\tilde{\partial}\rho\,(z = -1/\tilde{z}) = \left(\sum_{k=1}^{n} \alpha_k z_k\right) \frac{\prod_{I=1}^{n-2}(1 + Z_I\,\tilde{z})}{\prod_{k=1}^{n}(1 + z_k\,\tilde{z})} \tag{4.142}$$

and so

$$\phi(\tilde{z}, \bar{\tilde{z}}) = \log\left[|\tilde{z}|^4 \left|\sum_{k=1}^{n} \alpha_k z_k\right|^2 \frac{\prod_{I=1}^{n-2}|1 + Z_I\,\tilde{z}|^2}{\prod_{k=1}^{n}|1 + z_k\,\tilde{z}|^2}\right] \tag{4.143}$$

So

$$\begin{aligned}\int_{\mathcal{M}} d^2\sigma\,\sqrt{|g|}\hat{R}\,\phi = 8\pi \log &\left[|\delta|^4 \left|\sum_{k=1}^{n} \alpha_k z_k\right|^2 \frac{\prod_{I=1}^{n-2}|1 + Z_I\,\delta|^2}{\prod_{k=1}^{n}|1 + z_k\,\delta|^2}\right] \\ &+ 4\pi i \log\left[\left(\sum_{k=1}^{n} \alpha_k z_k\right)^n \left(\sum_{k=1}^{n} \bar{\alpha}_k \bar{z}_k\right)^n \frac{\prod_k \prod_I (z_k - Z_I)(\bar{z}_k - \bar{Z}_I)}{\prod_l |\epsilon_{z_l}|^2 \prod_{k\neq j}(z_k - z_j)(\bar{z}_k - \bar{z}_j)}\right]\end{aligned} \tag{4.144}$$

i.e,

$$\int_{\mathcal{M}} d^2\sigma\,\sqrt{|g|}\hat{R}\,\phi = 16\pi \log\left[\left|\sum_{k=1}^{n} \alpha_k z_k\right| \frac{1}{|z_\infty|^2}\right] - 8\pi \log\left[\left|\sum_{k=1}^{n} \alpha_k z_k\right|^n \frac{\prod_{k=1}^{n}\prod_{I=1}^{n-2}|z_k - Z_I|}{\prod_l |\epsilon_{z_l}| \prod_{k\neq j}|z_k - z_j|}\right] \tag{4.145}$$

where we have used $\delta = -1/z_\infty$. Remember that,

$$(\Gamma[\phi])_{\text{curvature}} = -\frac{\kappa}{2\pi}\int_{\mathcal{M}} d^2\sigma\,\sqrt{|g|}\hat{R}\,\phi \tag{4.146}$$

Writing it in a canonical form:

$$\frac{1}{\kappa}\left(\Gamma[\phi]\right)_{\text{curvature}} = -8\log\left[\frac{|\sum_{k=1}^{n}\alpha_k z_k|}{|z_\infty|^2}\right] + 4\log\left[\prod_{k=1}^{n}\frac{|\alpha_k|}{\epsilon_{z_k}}\right] \tag{4.147}$$

Adding (4.129) and (4.147), we obtain

$$\frac{1}{\kappa}\Gamma[\phi] = -\log\left[\left|\sum_{k=1}^{n}\alpha_k z_k\right|^4\prod_{k=1}^{n}|\alpha_k|^{-2}\prod_{I=1}^{n-2}\left|\partial^2\rho\left(Z_I\right)\right|^{-1}\frac{\prod_{k=1}^{n}|\epsilon_{z_k}|^2}{|z_\infty|^8\prod_{I=1}^{n-2}(2r_I)}\right] \tag{4.148}$$

4.B Tensionful term at the Mandelstam Localization

We have the following tensionful term (4.86) in our worldsheet action in the presence of vertex operator insertions:

$$S_0 = \int\frac{d^2z}{2\pi}\,\lambda\pi\left(\partial X^+\bar{\partial}X^- - \bar{\partial}X^+\partial X^-\right) + i\lambda\pi\sum_{k=1}^{n}w_k R\,X^-(z_k) + i\lambda\pi\sum_{k=1}^{n}w_k R\,X^+(\bar{z}_k) \tag{4.149}$$

In the following, we will evaluate

$$S' = \int_{\mathcal{M}} d\rho\wedge d\bar{\rho} = +i\int d^2z\,\left(\partial\rho\,\bar{\partial}\bar{\rho} - \bar{\partial}\rho\,\partial\bar{\rho}\right) \tag{4.150}$$

We can make the integrand as a total derivative and evaluate it

$$S' = \frac{1}{2}\int_{\mathcal{M}} d\left(\rho\,d\bar{\rho} - \bar{\rho}\,d\rho\right) = \frac{1}{2}\int_{\partial\mathcal{M}}\left(\rho\,d\bar{\rho} - \bar{\rho}\,d\rho\right) \tag{4.151}$$

where $(\rho, \bar{\rho})$ are localized at

$$\rho(z) = \sum_{j=1}^{n}\alpha_j\log(z - z_j), \quad \bar{\rho}(\bar{z}) = \sum_{k=1}^{n}\bar{\alpha}_k\log(\bar{z} - \bar{z}_k) \tag{4.152}$$

Along $a_1 + a_1^{-1}$:

$$\begin{aligned}\int_{a_1+a_1^{-1}}\rho\,d\bar{\rho} &= \int_{a_1^{-1}}\left(\oint_{C_{z_1}}d\rho\right)d\bar{\rho} = \oint_{C_{z_1}}d\rho\int_{(z_1,\bar{z}_1)}^{(z_\infty,\bar{z}_\infty)}d\bar{\rho} = 2\pi i\alpha_1\sum_{k=1}^{n}\bar{\alpha}_k\int_{(z_1,\bar{z}_1)}^{(z_\infty,\bar{z}_\infty)}\frac{d\bar{z}}{\bar{z}-\bar{z}_k}\\ &= 2\pi i\alpha_1\bar{\alpha}_1\log\left[\frac{\bar{z}_\infty - \bar{z}_1}{\bar{\epsilon}_{z_1}}\right] + 2\pi i\alpha_1\sum_{k(\neq)1}^{n}\bar{\alpha}_k\log\left[\frac{\bar{z}_\infty - \bar{z}_k}{\bar{z}_1 - \bar{z}_k}\right]\end{aligned} \tag{4.153}$$

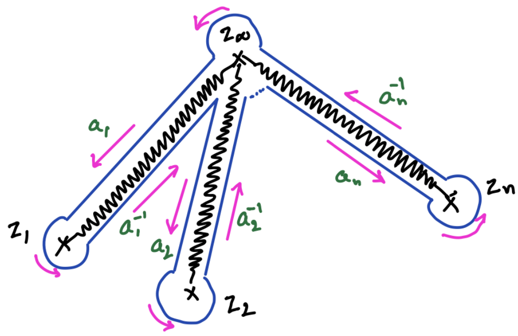

Figure 4.B.1: Octopus diagram to compute the tensionful term in the worldsheet action

where we used $\oint_{C_{z_1}} d\rho = \oint_{C_{z_1}} \frac{dz}{z-z_1} = +2\pi i\alpha_1$. The first equality above follows from

$$\int_{a_1} \rho(a_1)\, d\bar\rho + \int_{a_1^{-1}} \rho(a_1^{-1})\, d\bar\rho = -\int_{a_1^{-1}} \rho(a_1)\, d\bar\rho + \int_{a_1^{-1}} \rho(a_1^{-1})\, d\bar\rho = \int_{a_1^{-1}} \left[\rho(a_1^{-1}) - \rho(a_1)\right]\, d\bar\rho$$
$$\text{and} \quad \delta\rho = \left[\rho(a_1^{-1}) - \rho(a_1)\right] = \oint_{C_{z_1}} d\rho. \tag{4.154}$$

Remember that the countour integral around z_1 in the last step is anti-clockwise. Summing it over all edges of the contour diagram:

$$\sum_{j=1}^{n} \int_{a_j + a_j^{-1}} \rho\, d\bar\rho = 2\pi i \sum_j \alpha_j \bar\alpha_j \log\left[\frac{\bar z_\infty - \bar z_j}{\bar\epsilon_{z_j}}\right] + 2\pi i \sum_{j\neq k} \alpha_j \bar\alpha_k \log\left[\frac{\bar z_\infty - \bar z_k}{\bar z_j - \bar z_k}\right] \tag{4.155}$$

Similarly (using $\oint_{C_{z_k}} d\bar\rho = -2\pi i\bar\alpha_k$),

$$\sum_{j=1}^{n} \int_{a_j + a_j^{-1}} \bar\rho\, d\rho = -2\pi i \sum_j \bar\alpha_j \alpha_j \log\left[\frac{z_\infty - z_j}{\epsilon_{z_j}}\right] - 2\pi i \sum_{j\neq k} \bar\alpha_j \alpha_k \log\left[\frac{z_\infty - z_k}{z_j - z_k}\right] \tag{4.156}$$

So

$$\sum_{j=1}^{n} \int_{a_j + a_j^{-1}} (\rho\, d\bar\rho - \bar\rho\, d\rho) = 2\pi i \sum_{j=1}^{n} |\alpha_j|^2 \log\left[\frac{|z_\infty - z_j|^2}{|\epsilon_{z_j}|^2}\right] + 2\pi i \sum_{j\neq k} \alpha_j \bar\alpha_k \log\left[\frac{(\bar z_\infty - \bar z_k)(z_\infty - z_j)}{(\bar z_j - \bar z_k)(z_k - z_j)}\right] \tag{4.157}$$

Using the fact that $\sum_{j=1}^{n} \alpha_j = 0$, the leading contribution in z_∞ of the above expression vanishes, and thus

$$\boxed{\sum_{j=1}^{n} \int_{a_j + a_j^{-1}} (\rho\, d\bar\rho - \bar\rho\, d\rho) = -2\pi i \sum_{j=1}^{n} |\alpha_j|^2 \log|\epsilon_{z_j}|^2 - 2\pi i \sum_{j\neq k} \alpha_j \bar\alpha_k \log[-|z_j - z_k|^2]} \tag{4.158}$$

Next we need to compute the integrals around $\{z_k\}$:

$$\sum_{l=1}^{n} \oint_{C_{z_l}} \rho \, d\bar{\rho} = \sum_l \sum_k \alpha_k \sum_j \bar{\alpha}_j \oint_{C_{z_l}} d\bar{z} \, \frac{\log(z - z_k)}{\bar{z} - \bar{z}_j} \tag{4.159}$$

For $j \neq l$ and $k \neq l$ the above clearly vanishes.

For $j = l$ and $k \neq l$:

$$\sum_{k \neq l} \alpha_k \bar{\alpha}_l \oint_{C_{z_l}} d\bar{z} \, \frac{\log(z - z_k)}{\bar{z} - \bar{z}_l} = -2\pi i \sum_{k \neq l} \alpha_k \bar{\alpha}_l \, \log(z_l - z_k) \tag{4.160}$$

For $j \neq l$ and $k = l$:

$$\sum_{j \neq l} \alpha_l \bar{\alpha}_j \oint_{C_{z_l}} \frac{\log(z - z_l)}{\bar{z} - \bar{z}_j} = \sum_{j \neq l} \frac{\alpha_l \bar{\alpha}_j}{\bar{z}_l - \bar{z}_j} \oint_{C_{z_l}} d\bar{z} \, \log(z - z_l) = 0 \tag{4.161}$$

For $j = k = l$:

$$\begin{aligned} \sum_{l=1}^{n} \alpha_l \bar{\alpha}_l \oint_{C_{z_l}} d\bar{z} \, \frac{\log(z - z_l)}{\bar{z} - \bar{z}_l} &= \sum_{l=1}^{n} \alpha_l \bar{\alpha}_l \int_0^{2\pi} -i d\theta e^{-i\theta} \epsilon_{z_l} \frac{\log(\epsilon_{z_l}) + i\theta}{\epsilon_{z_l} e^{-i\theta}} \\ &= -2\pi i \sum_{l=1}^{n} |\alpha_l|^2 \log(\epsilon_{z_l}) + 2\pi^2 \sum_l |\alpha_l|^2 \end{aligned} \tag{4.162}$$

So

$$\sum_{l=1}^{n} \oint_{C_{z_l}} \rho \, d\bar{\rho} = -2\pi i \sum_{k \neq l} \alpha_k \bar{\alpha}_l \, \log(z_l - z_k) - 2\pi i \sum_{l=1}^{n} |\alpha_l|^2 \log(\epsilon_{z_l}) + 2\pi^2 \sum_l |\alpha_l|^2 \tag{4.163}$$

Similarly

$$\sum_{l=1}^{n} \oint_{C_{z_l}} \bar{\rho} \, d\rho = +2\pi i \sum_{k \neq l} \bar{\alpha}_k \alpha_l \, \log(\bar{z}_l - \bar{z}_k) + 2\pi i \sum_{l=1}^{n} |\alpha_l|^2 \log(\epsilon_{z_l}) + 2\pi^2 \sum_l |\alpha_l|^2 \tag{4.164}$$

Adding them

$$\boxed{\sum_{l=1}^{n} \oint_{C_{z_l}} (\rho \, d\bar{\rho} - \bar{\rho} \, d\rho) = -2\pi i \sum_{k \neq l} \alpha_k \bar{\alpha}_l \log\left[-|z_k - z_l|^2\right] - 2\pi i \sum_{l=1}^{n} |\alpha_l|^2 \, \log\Big(\epsilon_{z_l}^2\Big)} \tag{4.165}$$

To compute $\oint_{C_{z_\infty}} (\rho \, d\bar{\rho} - \bar{\rho} \, d\rho)$, we first note that (using $\sum_{k=1}^{n} \alpha_k = 0$):

$$\begin{aligned} \rho(z) &= \sum_{k=1}^{n} \alpha_k \left[\log z + \log\left(1 - \frac{z_k}{z}\right)\right] \sim -\frac{\sum_{k=1}^{n} \alpha_k z_k}{z} \\ d\bar{\rho} &= \frac{\sum_{k=1}^{n} \bar{\alpha}_k \bar{z}_k}{\bar{z}^2} \, d\bar{z} \end{aligned} \tag{4.166}$$

so

$$\oint_{C_{z_\infty}} \rho\, d\bar{\rho} = -\left|\sum_{k=1}^{n} \alpha_k z_k\right|^2 \oint_{C_{z_\infty}} \frac{d\bar{z}}{|z|^2\,\bar{z}} = -2\pi i\,\frac{\left|\sum_{k=1}^{n} \alpha_k z_k\right|^2}{|z_\infty|^2} \tag{4.167}$$

where we've used $\oint_{C_{z_\infty}} \frac{d\bar{z}}{\bar{z}} = +2\pi i$. Similarly

$$\oint_{C_{z_\infty}} \bar{\rho}\, d\rho = +2\pi i\frac{\left|\sum_{k=1}^{n} \alpha_k z_k\right|^2}{|z_\infty|^2} \tag{4.168}$$

Since $|z_\infty| \to \infty$, these contributions vanish (Note that the integrals have been computed in the chart that exclude the point z_∞). Therefore the net contribution is:

$$\boxed{\begin{aligned} S_0 &= -2\pi i \sum_{j=1}^{n} |\alpha_j|^2 \log|\epsilon_{z_j}|^2 - 2\pi i \sum_{j\neq k} \alpha_j \bar{\alpha}_k \log[-|z_j - z_k|^2] \\ &= -2\pi i \sum_{j=1}^{n} \alpha_j\, \bar{\rho}(z_j) - 2\pi i \sum_{k=1}^{n} \bar{\alpha}_k\, \rho(z_k) \end{aligned}} \tag{4.169}$$

Therefore, on the Mandelstam maps,

$$\lambda\pi \int \frac{d^2 z}{2\pi} \left(\partial X^+ \bar{\partial} X^- - \bar{\partial} X^+ \partial X^-\right) = -i\lambda\pi \sum_{j=1}^{n} w_j\, X^-(z_j, \bar{z}_j) - i\lambda\pi \sum_{k=1}^{n} \bar{\alpha}_k\, X^+(z_k, \bar{z}_k) \tag{4.170}$$

It's then clear that the tensionful term (4.149) vanishes on the Mandelstam maps:

$$S_0|_{X^+=\rho, X^-=\bar{\rho}} = 0 \tag{4.171}$$

Chapter 5

On the Positivity of Veneziano Amplitude in $D = 4$

5.1 Introduction

Bootstrapping theories with massive particles of arbitrary spin is an old theme of theoretical physics before the advent of quantum chromodynamics. It was motivated by the proliferation of massive higher spin resonances of strongly interacting particles in hadronic physics during 1960s. Arguing for a good high energy behaviour of the tree-level scattering amplitude $A(s,t)$ naturally leads to an infinite sum with an infinite number of exchanged resonances in the following Gegenbauer expansion of $A(s,t)$:

$$A(s,t) = \sum_{n,l} a_{n,l} \frac{C_l^{(\alpha)}\left(1 + \frac{2t}{m_{n,l}^2 - 4m_0^2}\right)}{s - m_{n,l}^2}, \tag{5.1}$$

where $C_l^{(\alpha)}$ is the Gegenbauer polynomial and $\alpha = (D-3)/2$. It was no longer clear if we would need to add contributions from t-channel poles separately to the full amplitude. In fact, [137] advocated such an approximate equality between s- and t-channels with the help of experimental data. Next in 1968, Veneziano proposed [136] an explicit form of the scattering amplitude

$$\mathcal{M}(s,t) = \frac{\Gamma(-\alpha' s - \alpha_0)\Gamma(-\alpha' t - \alpha_0)}{\Gamma(-\alpha' s - \alpha' t - 2\alpha_0)}, \tag{5.2}$$

as a solution to the bootstrap axioms, where α' and α_0 are known as the Regge slope and Regge intercept respectively. But unitarity, which requires $a_{n,l} \geq 0$, is far from being obvious [138, 145], for oscillating nature of the Gegenbauer polynomial in the physical scattering regime. Subsequent spectacular developments unraveled the origin of Veneziano amplitude in strings and unitarity was then implied *indirectly* from the no-ghost theorem in string theory. Thus we expect for (5.2)

$$a_{n,l} \geq 0 \quad \text{for} \quad D \leq 26. \tag{5.3}$$

In this chapter, we will find an explicit form [see eq. (5.36)] for this coefficient on the leading Regge trajectory i.e for $a_{n,n+1}$, in $D = 4$ space-time dimensions, which falls off exponentially for asymptotic values of the spin but always remains positive. Further all other coefficients on the subleading trajectories are expected (though we couldn't prove it rigorously here) to be larger than the leading one for large spins and thus our result indicates positivity of the full Veneziano amplitude in $D = 4$.

Later the Veneziano amplitude was abandoned as a candidate for the strong interactions due to its soft exponential fall-off in the hard scattering regime compared to the power law fall off observed for strong interactions understood in parton model terms and supported by experimental data. In fact, the huge degeneracy of spins at a resonance of given mass in (5.2) is absent for QCD. Hence, in the light of revival of S-matrix bootstrap program [141–143], it might well be important to explicate (5.3) which can help to constructively modify the Veneziano amplitude in $D = 4$ and remove the unwanted degeneracy to come closer to the QCD amplitude. This was the original motivation [135] for this work and we expect that the techniques of this chapter will be helpful in the future, in this regard.

5.2 Bootstrap axioms and Gegenbauer expansion

We consider a $2 \to 2$ scattering of identical lightest scalar glueballs of mass m_0. From simple large N counting, such an amplitude goes as $\frac{1}{N^2}$ indicating that the particles almost fly past each other in the large N limit. We can define the usual Mandlestam invariants

$$s = (k_1 + k_2)^2, \quad t = (k_1 - k_3)^2, \quad u = (k_1 - k_4)^2 \tag{5.4}$$

The amplitude will generally involve contributions from all three channels, where s and u (but not t) channel diagrams have poles in s at the locations of the resonances. For the sake of simplicity we can forget the u-channel resonances, we can always arrange such a process with no contribution from the u-channel with the correct quantum numbers by judiciously choosing a scattering of non-identical particles.

Before the developments of QCD, people tried to extract the physics of hadrons and glueballs by imposing a set of natural postulates on the scattering amplitudes $A(s, t)$, outlined below:

1. **Crossing symmetry:**

$$A(s, t) = A(t, s). \tag{5.5}$$

This is manifest for scattering of identical particles in the description in terms of Feynman diagrams, see [144] for recent developments on an interesting physical interpretation behind it.

2. **Analyticity:** $A(s,t)$ is a meromorphic function in s with poles only on the real axis $s = m_{n,l}^2$, corresponding to the spectrum of the theory (See [139] for some fascinating computation of glueball masses for the first few spins from lattice calculations)

$$\mathrm{Im}[A(s,t)] = \sum_{n,l} f_{n,l}^2\, \delta(s - m_{n,l}^2)\, C_l^{(\alpha)} \left(1 + \frac{2t}{m_{n,l}^2 - 4m_0^2}\right). \tag{5.6}$$

where $C_l^{(\alpha)}(x)$ is the Gegenbauer polynomial which can be expressed in terms of Gaussian hypergeometric (finite) series:

$$C_l^{(\alpha)}(x) = \frac{(2\alpha)_l}{l!}\, {}_2F_1\left(-l, 2\alpha + l; \alpha + \frac{1}{2}; \frac{1-x}{2}\right) \tag{5.7}$$

and

$$\alpha = \frac{D-3}{2}. \tag{5.8}$$

In particular for $D = 4$,

$$C_l^{\frac{1}{2}}(z) = P_l(z) \tag{5.9}$$

where $P_l(z)$ is the standard Legendre polynomial. This analyticity behaviour (5.6) is a consequence of casuality, which states that two regions with a space-like separation don't influence each other.

3. **Unitarity:** In some sense, unitarity is automatically satisfied, since the amplitude goes as $\mathcal{O}\left(\frac{1}{N^2}\right)$ and hence always remains much less than 1 in the large N limit. Still it imposes the following non-trivial constraints since couplings $\{f_{n,l}\}$ at both junctions (see figure 5.2.1) are the same for any exchanged resonance in the scattering of identical particles:

$$f_{n,l}^2 > 0.$$

so that the sum in (5.6) is positive-definite.

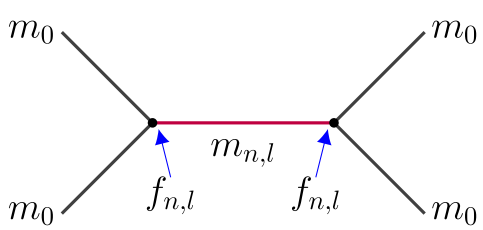

Figure 5.2.1: Scattering process with exchanged resonance of mass $m_{n,l}$.

4. **Boundedness:** There exists a t_0 and a particle of spin L in the spectrum such that

$$\lim_{|s|\to\infty} s^{-L} A(s, t_0) = 0. \tag{5.10}$$

From a modern perspective [140], it can be motivated as an implication of causality for a theory with exchange of a massive higher spin particle of spin $L > 2$. In the hard scattering regime, scattering amplitude of strong interaction falls off according to a power law and so (5.10) is naturally satisfied for some large negative t in that case.

We will now explore some immediate consequences of these four axioms on $A(s,t)$: Given the boundedness condition for some region of t for $L = 0$ [axiom 4], we can exploit the Cauchy integral formula to write

$$A(s,t) = \oint_{C_s} ds' \, \frac{A(s',t)}{s'-s} \tag{5.11}$$

where the contour C_s doesn't enclose any of the poles of $A(s,t)$ as shown in the figure 5.2.2 (a). We can always deform the contour as in fig 5.2.2 (b) to rewrite it as

$$A(s,t) = \int_{m_s^2-|\epsilon|}^{\infty} ds' \, \frac{\mathrm{Im}[A(s',t)]}{s'-s}. \tag{5.12}$$

which is known as the unsubtracted dispersion relation. Plugging (5.6) [axiom 2] into the above relation (5.12), we immediately get

$$A(s,t) = \sum_{n,l} f_{n,l}^2 \, \frac{C_l^{(\alpha)}\left(1 + \frac{2t}{m_{n,l}^2 - 4m_0^2}\right)}{s - m_{n,l}^2}. \tag{5.13}$$

Unitarity [axiom 3] says $f_{n,l}^2 \geq 0$. Crossing symmetry [axiom 1] implies $A(s,t)$ in (5.13) must have poles in t at the same positions $\{m_{n,l}^2\}$ as for the poles in s. It automatically suggests [141] that the sum in (5.13) is an infinite sum, i.e with infinitely many resonances. In fact, there shouldn't be any upper limit on the spin l of exchanged resonance, since otherwise if there exists such an $l_{\max}$, then $\left(\frac{\partial}{\partial t}\right)^{l_{\max}+1} A(s,t) = 0$, contradicting with (5.5).

5.3 Coefficient of the Veneziano amplitude

It is a difficult mathematical problem to find an explicit solution to those four bootstrap axioms. The only known (upto tiny dressings for Heterotic strings) solution proposed by Veneziano [136] is the famous Veneziano amplitude

$$\mathcal{M}(s,t) = \frac{\Gamma(-s-1)\Gamma(-t-1)}{\Gamma(-s-t-2)}. \tag{5.14}$$

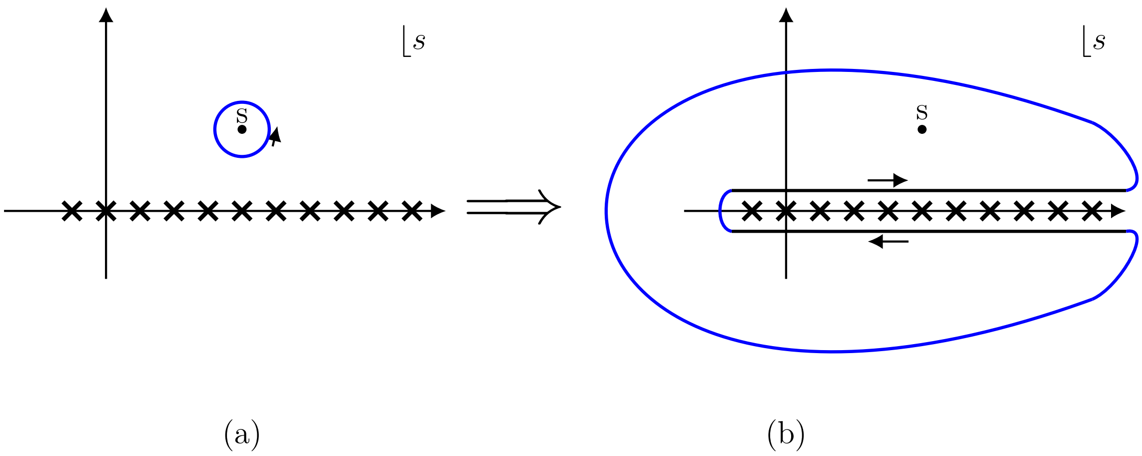

Figure 5.2.2: We can deform the contour in (a) into (b) for computing the scattering amplitude in (5.11). The poles have been drawn in accordance with the Veneziano amplitude (5.14).

It describes the scattering amplitude of four identical tachyons of mass $m_0^2 = -1$.

Clearly the amplitude (5.14) is crossing symmetric with poles in s (and t) at $s = n$ for $n = -1, 0, 1, 2, \ldots$ Boundedness follows from the asymptotic behaviour of Gamma functions for large s and fixed t

$$\mathcal{M}(s,t) \sim \Gamma(-t-1)s^{t+1}. \tag{5.15}$$

Thus for $t < -1$, (5.10) is satisfied. Now to check unitarity, we first calculate the residue at the poles in s from (5.14)

$$\mathrm{Res}_{s=n}\mathcal{M}(s,t) = \frac{(-1)^n}{(n+1)!}\frac{\Gamma(-t-1)}{\Gamma(-t-2-n)} = -\frac{1}{(n+1)!}(t+2)_{n+1}. \tag{5.16}$$

From the Gegenbauer expansion in general dimensions,

$$-\mathrm{Res}_{s=n}\mathcal{M}(s,t) = \sum_{l=0}^{n+1} a_{n,l} C_l^{(\alpha)}\left(1 + \frac{2t}{n+4}\right). \tag{5.17}$$

We note that Veneziano amplitude has huge degeneracy in the spin with particles of spin $l = 0$ to $l = (n+1)$ for the same mass $m_{n,l}^2 = n$. This degeneracy is drastically different from what is observed for QCD. Equating two expressions in (5.16) and (5.17), we get the following

$$\sum_{l=0}^{n+1} a_{n,l} C_l^{(\alpha)}\left(1 + \frac{2t}{n+4}\right) = \frac{1}{(n+1)!}(t+2)_{n+1}. \tag{5.18}$$

We can now use the orthogonality relation of Gegenbauer polynomials

$$\int_{-1}^{+1} dx\, C_l^{(\alpha)}(x)\, C_{l'}^{(\alpha)}(x)\, (1-x^2)^{\alpha-\frac{1}{2}} = 2\,\underbrace{\frac{\pi\Gamma(l+2\alpha)}{2^{2\alpha} l!(l+\alpha)[\Gamma(\alpha)]^2}}_{\Lambda(l,\alpha)}\,\delta_{ll'}, \tag{5.19}$$

to extract out $a_{n,l}$ as

$$a_{n,l} = \frac{1}{\Lambda(l,\alpha)} \frac{1}{(n+4)(n+1)!} \left[\frac{4}{(n+4)^2}\right]^{\alpha-\frac{1}{2}} \int_0^{n+4} dt\, C_l^{(\alpha)}\left(1 - \frac{2t}{n+4}\right)(-t+2)_{n+1}[t(n+4-t)]^{\alpha-\frac{1}{2}}. \tag{5.20}$$

Note that within the integration range, Legendre polynomial changes sign (see figure 5.3.1) and a priori it is not obvious that $a_{n,l} \geq 0$. In fact, we can show that (nearly) half of such coefficients vanish due to the following result:

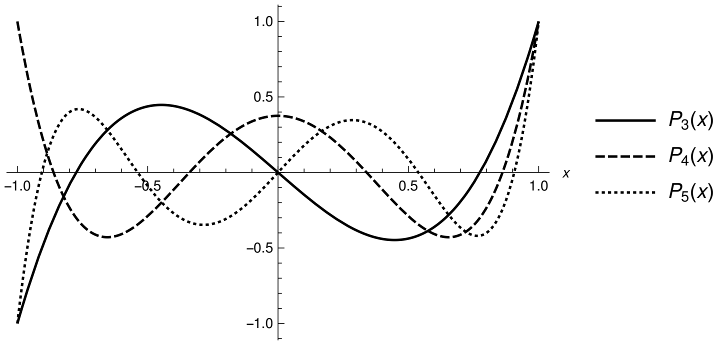

Figure 5.3.1: Few Legendre polynomials and their oscillating behaviours

Claim: $a_{n,l} = 0$ when $(n+l)$ is even.

Proof: Substituting $t = n + 4 - t'$ in (5.20), the integral can be written as,

$$\int_0^{n+4} dt\, C_l^{(\alpha)}\left(1 - \frac{2t}{n+4}\right)(-t+2)_{n+1}[t(n+4-t)]^{\alpha-\frac{1}{2}}$$
$$= (-)^{n+l+1} \int_0^{n+4} dt' C_l^{(\alpha)}\left(1 - \frac{2t'}{n+4}\right)(-t'+2)_{n+1}[t'(n+4-t')]^{\alpha-\frac{1}{2}}, \tag{5.21}$$

where we used $(-x)_m = (-1)^m(x-m+1)_m$ and $C_l^{(\alpha)}(-x) = (-1)^l C_l^{(\alpha)}(x)$. Thus,

$$[1 + (-1)^{n+l}]a_{n,l} = 0. \tag{5.22}$$

This establishes the claim [QED]. Note that the coefficients $a_{n,n+1}$ on the leading Regge trajectories are always non-zero. We have plotted $a_{n,n+1}$ in figure 5.3.2 for first few values of n from (5.20) using Mathematica. Clearly they decrease rapidly with n, but seem to remain positive as far as we can check numerically.

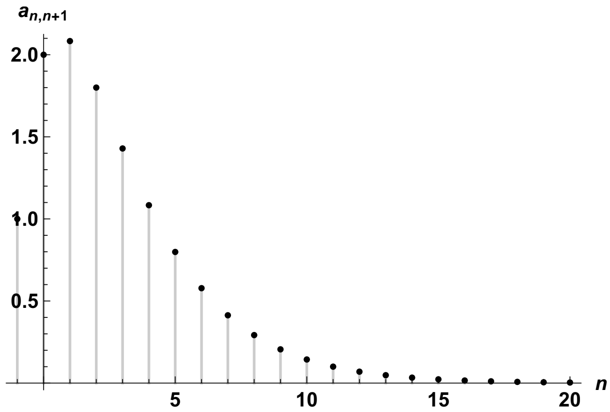

Figure 5.3.2: Coefficients on the leading Regge trajectory in $D = 4$.

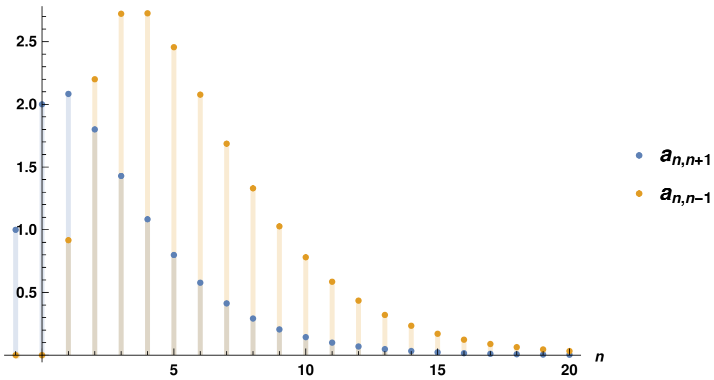

Figure 5.4.1: Coefficients on the leading and sub-leading Regge trajectories in $D = 4$.

5.4 Coefficient on the leading Regge trajectory in $D = 4$

In this chapter we will only be able to calculate an explicit form of the coefficient on the leading Regge trajectory in $D = 4$. In fact we can check (though not proved yet) that all other coefficients on the subleading trajectories are larger than the leading one for asymptotic values of the spin (see figure 5.4.1 for coefficients $a_{n,n+1}$ and $a_{n,n-1}$, remember $a_{n,n} = 0$). It is thus expected that positivity of $a_{n,n+1}$ would imply positivity for all others.

The expression of $a_{n,l}$ in (5.20) simplifies for $D = 4$ as

$$a_{n,l} = \frac{2l+1}{(n+4)(n+1)!} \int_0^{n+4} dt\, P_l\left(1 - \frac{2t}{n+4}\right) (-t+2)_{n+1}. \tag{5.23}$$

First we construct a set of generating functions[1] of $a_{n,l}$ for each value of n as

$$G_n(h) = \sum_{j=0}^{\infty} \frac{1}{2j+1} \frac{a_{n,j}}{h^{j+1}} = \frac{1}{(n+4)(n+1)!} \int_0^{n+4} dt\, \frac{(-t+2)_{n+1}}{[(h-1)^2 + \frac{4ht}{n+4}]^{1/2}}, \tag{5.24}$$

[1] Here h has been analytically continued to any positive real number (both > 1 and < 1).

where we have exploited the standard result

$$\sum_{l=0}^{\infty} P_l(x)\, t^{-l-1} = \frac{1}{\sqrt{1-2xt+t^2}}. \tag{5.25}$$

We can then use the following representation for the Pochhammer symbol,

$$(-t+2)_{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k}\, s(n+1,k)\, (-t+2)^k, \tag{5.26}$$

where $s(m,p)$ is the Stirling number of the first kind, which is defined as $(-1)^{m-p}$ times the number of permutations of $\{1,2,...,m\}$ with exactly p cycles. Thus,

$$G_n(h) = \frac{1}{(n+4)(n+1)!} \sum_{k=0}^{n+1} (-1)^{n+1-k}\, s(n+1,k)\, g_{n,k}(h), \tag{5.27}$$

with the ‘pseudo-generating function’ $g_{n,k}(h)$ defined by

$$g_{n,k}(h) = \int_0^{n+4} dt\, \frac{(-t+2)^k}{[(h-1)^2 + \frac{4ht}{n+4}]^{1/2}}, \tag{5.28}$$

which can be explicitly evaluated as,

$$\begin{aligned} g_{n,k}(h) = \frac{1}{2^{2k+1}} \frac{(n+4)^{k+1}}{h^{k+1}} \left[(h-1)^2 + \frac{8h}{n+4}\right]^k \Bigg[(h+1)\, {}_2F_1\left(\frac{1}{2}, -k; \frac{3}{2}; \frac{(h+1)^2}{(h-1)^2 + \frac{8h}{n+4}}\right) \\ - (h-1)\, {}_2F_1\left(\frac{1}{2}, -k; \frac{3}{2}; \frac{(h-1)^2}{(h-1)^2 + \frac{8h}{n+4}}\right) \Bigg]. \end{aligned} \tag{5.29}$$

More compactly, we can represent it in terms of Appell hypergeometric function of two variales $\mathrm{Appell}F_1(a; b_1, b_2; c; x, y)$:

$$g_{n,k}(h) = 2^k \frac{n+4}{h-1}\, \mathrm{Appell}F_1\left(1; -k, \frac{1}{2}; 2; \frac{n+4}{2}, -\frac{4h}{(h-1)^2}\right), \tag{5.30}$$

whose precise form can be found in (5.44).

To analyse (6.27) further, we can simplify it using the definition of ${}_2F_1$

$${}_2F_1(a,b;c;z) = \sum_{p=0}^{\infty} \frac{(a)_p (b)_p}{(c)_p\, p!} z^p, \tag{5.31}$$

to get

$$g_{n,k}(h) = \frac{1}{2^{2k+1}} \frac{(n+4)^{k+1}}{h^{k+1}} \sum_{p=0}^{k} \frac{(-k)_p}{(2p+1)p!} \left[(h-1)^2 + \frac{8h}{n+4}\right]^{k-p} \sum_{m=0}^{2p+1} \binom{2p+1}{m} h^m (1+(-1)^m). \tag{5.32}$$

Note that the p-sum in (5.31) terminates in the above expression (6.35) of $g_{n,k}(h)$ since one of the arguments of ${}_2F_1$ is a negative integer: $b = -k$ and $(-k)_p = 0$ for $p > k$.

Now $g_{n,k}(h)$ in (6.35) has $(k+1)$ terms with $h^{-1} \cdots h^{-k-1}$. We can find the coefficient of h^{-k-1} from that expansion as

$$d_{n,k} = \frac{(n+4)^{k+1}}{2^{2k+1}} \frac{\sqrt{\pi} k!}{\Gamma(k+3/2)}. \tag{5.33}$$

To find the coefficient on the leading Regge trajectory, we need $a_{n,n+1}$, i.e the term with h^{-n-2} in the expression of $G_n(h)$ in (5.24). From the above analysis of $g_{n,k}(h)$, we can rewrite h-expansion of $G_n(h)$ as

$$\underbrace{\sum_{j=0}^{n+1} [\ldots]\, h^{-j-1}}_{G_n(h)} = \sum_{k=0}^{n+1} [\ldots] \underbrace{\sum_{q=0}^{k} [\ldots]\, h^{-q-1}}_{g_{n,k}(h)}. \tag{5.34}$$

From this expression (6.29), h^{-n-2} term on the R.H.S corresponds to taking the unique term with h^{-n-2} in $g_{n,n+1}(h)$. Note that for subleading trajectories, we would require h^{-k-1} with $k < n$ and there should be contributions from several terms in the R.H.S of (6.29), as discussed thorougly in appendix 5.A.

Hence equating coefficients of h^{-n-2} from both sides of (6.29), we get

$$\frac{1}{2n+3} a_{n,n+1} = \frac{1}{(n+4)(n+1)!} s(n+1, n+1)\, d_{n,n+1}. \tag{5.35}$$

This gives the explicit form of $a_{n,n+1}$

$$a_{n,n+1} = \frac{\sqrt{\pi}}{2^{2n+2}} \frac{(n+4)^{n+1}}{\Gamma(n+\frac{3}{2})}, \ \forall\, n \geq -1. \tag{5.36}$$

It reproduces the plot in figure 5.3.2. Note that for asymptotic values of n, it decays exponentially:

$$a_{n,n+1} \sim \exp[-n \log(4/e)], \tag{5.37}$$

but obviously remains positive.

5.5 Discussion

In this chapter, we found an explicit form for the coefficients in the Gegenbauer expansion of the Veneziano amplitude for the leading Regge trajectory in $D = 4$. They decay exponentially for large spin but remain positive. We also mentioned that the coefficients on the subleading trajectories are expected to be larger than those on the leading ones for large spin and thus our result is a strong indication of the positivity of the full Veneziano amplitude in $D = 4$.

As mentioned in the introduction, one long-term motivation of this work was to understand the consistent deformations of the Veneziano amplitude satisfying the Bootstrap axioms to bring it closer to have features of 4d QCD. In fact, some attempts to generalize the Veneziano amplitude have already been advocated in [146, 147], suggesting the 'Coon amplitude' [141]. It has the explicit form [147]

$$A(s,t) \sim \prod_{r=0}^{\infty} \frac{[(\sigma-1)(s-m^2)+1][(\sigma-1)(t-m^2)+1]-\sigma^r}{[(\sigma-1)(s-m^2)+1-\sigma^r][(\sigma-1)(t-m^2)+1-\sigma^r]}, \tag{5.38}$$

where at $\sigma = 1$ it reduces to the Veneziano amplitude. This generalized amplitude appears to be unitary [147] for $0 \leq \sigma \leq 1$ from numerical verification of the positivity of some set of coefficients $\{a_{n,l}\}$. It would be very gratifying if our analytical strategy can be utilized to show this explicitly.

It is also interesting to note that the exponential decay of the coefficient in (5.37) is somewhat similar to the decay of the partial wave amplitudes with angular momentum quantum number in simple quantum mechanical scattering processes [148]. More explicitly, for spherically symmetric scattering potential, the "scattering amplitude" $f(\vec{k}', \vec{k})$ in the wavefuntion

$$\langle \vec{x} | \Psi \rangle \to A \left[e^{i\vec{k}\cdot\vec{x}} + f(\vec{k}', \vec{k}) \frac{e^{ikr}}{r} \right] \quad r >> 1, \tag{5.39}$$

has the familiar partial wave expansion

$$f(\vec{k}', \vec{k}) = f(\vec{k}, \theta) = \sum_{l=0}^{\infty} \underbrace{(2l+1) \left[\frac{e^{2i\delta_l} - 1}{2ik} \right]}_{a_l} P_l(\cos\theta). \tag{5.40}$$

For small δ_l, $a_l \sim (2l+1)\frac{\delta_l}{k}$. As a concrete example, consider hard sphere scattering with the potential

$$V(r) = \begin{cases} \infty & r < R, \\ 0 & r > R. \end{cases} \tag{5.41}$$

Then in the low-energy limit with $kR << 1$ and large l

$$a_l \sim \frac{1}{k} \exp\left[-2l \log\left(\frac{2l}{kRe}\right)\right]. \tag{5.42}$$

In our context of the Veneziano amplitude, two incoming strings do feel some “force" between them though that should not be modelled by hard sphere potential (5.41). In any case, the connection seems worth exploring.

Appendix

5.A General coefficient in $D=4$

In this section, we will express the coefficients on general Regge trajectories in relatively simpler ways than (5.23). To get the general term with h^{-l-1} from both sides of (6.29), we can proceed with the binomial expansion of $g_{n,k}(h)$ in (6.27) to extract the coefficient of h^{-l-1} and then finally use (5.27) to write the generic coefficient $a_{n,l}$ as

$$\begin{aligned} a_{n,l} = \frac{2l+1}{(n+1)!} & \sum_{k=0}^{n+1} (-)^{n+1-k} s(n+1,k) \frac{1}{2^{2k+1}} (n+4)^k \\ & \times \sum_{p=0}^{k} \sum_{m=0}^{2p+1} \sum_{r=0}^{k-p} \binom{2p+1}{m} \binom{k-p}{r} \binom{2r}{p+r-l-m} \frac{(-k)_p}{(2p+1)p!} \\ & \times \left(\frac{8}{n+4}\right)^{k-p-r} (-1)^{p+r-l-m}(1+(-1)^m). \end{aligned} \tag{5.43}$$

Though looks clumsy, the above Eq. appears algorithmically much simpler than the original expression (5.20).

We can also use another representation (5.30) of $g_{n,k}(h)$,

$$\begin{aligned} g_{n,k}(h) &= 2^k \frac{n+4}{h-1} \mathrm{Appell} F_1 \left(1; -k, \frac{1}{2}; 2; \frac{n+4}{2}, -\frac{4h}{(h-1)^2}\right) \\ &= 2^k \frac{n+4}{h-1} \sum_{p=0}^{k} \sum_{q=0}^{\infty} \frac{(-k)_p (\frac{1}{2})_q}{p! q! (1+p+q)} \left(\frac{n+4}{2}\right)^p \left(-\frac{4h}{(h-1)^2}\right)^q. \end{aligned} \tag{5.44}$$

We can actually perform the p-sum to finally get

$$g_{n,k}(h) = 2^k \frac{n+4}{h-1} \sum_{q=0}^{k} \frac{(\frac{1}{2})_q}{(q+1)!} {}_2F_1\left(-k, 1+q; q+2; \frac{n+4}{2}\right) \left(-\frac{4h}{(h-1)^2}\right)^q. \tag{5.45}$$

We can similarly extract the coefficient of h^{-l-1} from (5.45) to write the general coefficient

$$a_{n,l} = \frac{2l+1}{(n+4)(n+1)!}\sum_{k=0}^{n+1}(-1)^{n+1-k}s(n+1,k)2^k(n+4)\sum_{q=0}^{l}\frac{(\frac{1}{2})_q}{(q+1)!}{}_2F_1\left(-k,1+q;q+2;\frac{n+4}{2}\right) \times(-4)^q\frac{(2q+1)_{l-q}}{(l-q)!}. \tag{5.46}$$

This has relatively simpler form with only two sums (instead of 4-sums in the last expression). Also the $l = 0$ case, $a_{n,0}$ has only a single sum:

$$a_{n,0} = \frac{1}{(n+4)(n+1)!}\sum_{k=0}^{n+1}(-1)^{n-k+1}s(n+1,k)2^k\left(\frac{2^{k+1}}{k+1}+(-1)^k\frac{(n+2)^{k+1}}{k+1}\right). \tag{5.47}$$

Note that the second term in the above expression,

$$\frac{1}{(n+4)(n+1)!}\sum_{k=0}^{n+1}(-1)^{n+1}s(n+1,k)2^k\frac{(n+2)^{k+1}}{k+1} \tag{5.48}$$

is always manifestly positive, since for non-zero $a_{n,0}$, n must be odd. It remains to show the positivity of the first term to ensure $a_{n,0} > 0$.

5.B In general dimensions

In this section, we will write the pseudo-generating function $g^{\alpha}_{n,k}(h)$ in general dimensions (remember that $\alpha = (D-3)/2$, see (5.8)). Introducing the generating functions for $a_{n,l}$ from (5.20)

$$G^{\alpha}_n(h) = \sum_{j=0}^{\infty}\Lambda(j,\alpha)\frac{a_{n,j}}{h^{j+1}} = \frac{1}{(n+4)(n+1)!}\left[\frac{4}{(n+4)^2}\right]^{\alpha-\frac{1}{2}}\int_0^{n+4}dt\frac{(-t+2)_{n+1}[t(n+4-t)]^{\alpha-\frac{1}{2}}}{[(h-1)^2+\frac{4ht}{n+4}]^{\alpha}}, \tag{5.49}$$

where we have used the standard generating function for $C^{(\alpha)}_j$:

$$\sum_{j=0}^{\infty}C^{(\alpha)}_j(x)t^j = [1-2xt+t^2]^{-\alpha}. \tag{5.50}$$

We can now use (5.26) and the following series expansions

$$(n+4-t)^{\alpha-\frac{1}{2}} = \sum_{p=0}^{\infty}\binom{\alpha-\frac{1}{2}}{p}(-1)^p(n+4)^{\alpha-\frac{1}{2}-p}t^p, \tag{5.51}$$

to rewrite the generating function (5.49) as

$$G_n^{\alpha}(h) = \frac{1}{(n+4)(n+1)!}\left[\frac{4}{(n+4)^2}\right]^{\alpha-\frac{1}{2}}\sum_{k=0}^{n+1}(-1)^{n+1-k}s(n+1,k)\sum_{p=0}^{\infty}\binom{\alpha-\frac{1}{2}}{p}(-1)^p(n+4)^{\alpha-\frac{1}{2}-p}g_{n,k,p}^{\alpha}(h) \tag{5.52}$$

with the 'pseudo-generating function' $g_{n,k,p}^{\alpha}(h)$ defined as

$$g_{n,k,p}^{\alpha}(h) = \int_0^{n+4} dt\,\frac{(-t+2)^k\,t^{p+\alpha-\frac{1}{2}}}{[(h-1)^2+\frac{4ht}{n+4}]^{\alpha}}. \tag{5.53}$$

We can evaluate it, as before, in terms of Appell hypergeometric function of two variables

$$g_{n,k,p}^{\alpha}(h) = \frac{2^{k+1}}{2\alpha+2p+1}\frac{(n+4)^{p+\alpha+\frac{1}{2}}}{(h-1)^{2\alpha}}\mathrm{Appell}F_1\left(p+\alpha+\frac{1}{2};-k,\alpha;p+\alpha+\frac{3}{2};\frac{n+4}{2},-\frac{4h}{(h-1)^2}\right). \tag{5.54}$$

Note the extra p-sum in (5.52) compared to $\alpha = 1/2$ case, which complicates the calculation in general dimensions. We might have to find the coefficient with minimum power of h in the expansion of the above expression (5.54) to hope for getting $a_{n,n+1}$. For that, a representation of this Appell function (5.54) like (6.27) would be highly desirable.

In fact we can calculate $a_{n,l}$ numerically for the first few levels and check that the positivity condition breaks down for $D > 26$ [141] i.e, above the critical dimension for bosonic string theory!

Chapter 6

On the Positivity of Coon amplitude in $D = 4$

6.1 Introduction

The Coon amplitude is a $2 \to 2$ scattering amplitude describing an infinite number of higher spin exchanges consistent with the duality constraints [146, 150, 151]. In particular, this mysterious amplitude is the unique solution to duality constraints having an infinity of exchanges with logarithmic trajectories. Recently, it has been brought forward as an interesting case study on its own [141, 152–155] while also serving as a useful reference to the study of accumulation point amplitudes with similar infinite towers of states [141, 145, 156–158].

An intriguing feature of the amplitude is that it interpolates between the stringy and the worldline descriptions, i.e., between the Veneziano amplitude and a scalar amplitude. The expression for the amplitude is as follows:

$$A_q(s,t) = (q-1)\, q^{\frac{\log\sigma \log\tau}{(\log q)^2}} \prod_{n=0}^{\infty} \frac{(\sigma\tau - q^n)(1-q^{n+1})}{(\sigma - q^n)(\tau - q^n)} \tag{6.1}$$

where q is a parameter ranging from $q \in (0,1)$. As it may be apparent, the Coon amplitude can be thought of as a q-deformation of the Veneziano amplitude [159, 160]. Here the parameters σ and τ can be expressed in terms of the Mandelstam variables s and t, respectively:

$$\sigma = 1 + (s-m^2)(q-1), \quad \tau = 1 + (t-m^2)(q-1). \tag{6.2}$$

As we approach the limiting values of q, the amplitude reduces to scalar particle and Veneziano amplitudes, respectively (see Appendix 6.C for further details regarding the limiting cases):

$$
\begin{aligned}
\lim_{q\to 0} A_q(s,t) &= \frac{1}{s-m^2} + \frac{1}{t-m^2} - 1 \\
\lim_{q\to 1} A_q(s,t) &= -\frac{\Gamma(-s+m^2)\Gamma(-t+m^2)}{\Gamma(-s-t+2m^2)}
\end{aligned}
\tag{6.3}
$$

The Regge trajectories corresponding to the amplitude for a generic value of q are logarithmic. The amplitude (6.1) has real s poles at the locations:

$$
\sigma = q^n \;\leftrightarrow\; s = m^2 + \frac{1-q^n}{1-q}. \tag{6.4}
$$

Equation (6.4) indicates the existence of an accumulation point, i.e., a finite energy scale which the Regge poles asymptote to upon taking the limit $n \to \infty$ of (6.4).

$$
s_* = m^2 + \frac{1}{1-q} \tag{6.5}
$$

Apart from certain aspects of asymptotic behaviour, not much about this amplitude is well understood. In particular, details regarding the underlying physical interpretation of the amplitude are lacking, although a proposal was recently put forward regarding the same [155]. Additionally, from previous works [147, 161], it is not clear whether the amplitude (6.1) obeys unitarity constraints, though a recent numerical analysis indicates that it is so [153].[1]

The main aim of our work is to develop an understanding of the unitarity of the amplitude (6.1). One of the important implications of unitarity is the positivity of the coefficients of the partial wave expansion. In our work, we compute the partial wave coefficients corresponding to the leading Regge trajectory of the Coon amplitude in $D = 4$ (see equation (6.31)). We find that the coefficients are always positive, and their magnitude gradually decreases with spin. In addition, we also numerically find that the partial wave coefficients on the subleading trajectories are larger than the coefficients corresponding to the leading Regge trajectory above sufficiently large spin values (see Appendix 6.B). This indicates the consistency of the amplitude with tree-level unitarity in $D = 4$. In Appendix 6.A, we determine the general form of the partial wave coefficient corresponding to the subleading trajectories (see equation (6.36)).

[1] Historically, the conclusions of [161] were instrumental in abandoning the further study of the Coon amplitude, since their numerical work indicated the presence of ghosts, i.e., negative norm states. Our analytical results on positivity indicate the absence of such states at tree-level in $D = 4$, confirming the analysis of [153].

6.2 Positivity of the Coon amplitude

Prior to the computation, we simplify the notational baggage to make our expressions visually less demanding.

6.2.1 Notation

Firstly let us define the q-Pochhammer symbol as follows

$$(a;q)_N = \prod_{n=0}^{N-1}(1-aq^n). \tag{6.6}$$

Next we define the variables α_s and α_t which are functions of the Mandelstam variables s and t as given below

$$\alpha_s = \frac{\log\sigma}{\log q}, \quad \alpha_s = \frac{\log\tau}{\log q}. \tag{6.7}$$

We will also introduce $[n]_q$ as a q-deformed integer with the following expression:

$$[n]_q = \frac{1-q^n}{1-q}. \tag{6.8}$$

Note that as $q \to 1$, the q-deformed integer reduces to ordinary numbers n. Using equations (6.6), (6.7) and (6.8), we can conveniently express the Coon amplitude (6.1) in a compact form

$$A_q(s,t) = (q-1)\, q^{\alpha_s\alpha_t}\frac{(q^{-\alpha_s-\alpha_t};q)_\infty(q;q)_\infty}{(q^{-\alpha_s};q)_\infty(q^{-\alpha_t};q)_\infty} \tag{6.9}$$

The real s poles given in (6.4) now take the following form:

$$s = m_n^2 \equiv m^2 + [n]_q \;\leftrightarrow\; \sigma = q^n. \tag{6.10}$$

6.2.2 Partial wave decomposition

The residue of (6.1) at the pole $s = m_k^2$ from (6.10) is

$$\mathrm{Res}_{s=m_k^2}A_q(s,t) = \tau^k\left[\prod_{\substack{n=0\\n\neq k}}^{\infty}\frac{(q^k\tau-q^n)(1-q^{n+1})}{(q^k-q^n)(\tau-q^n)}\right]\times\frac{q^k(\tau-1)}{\tau-q^k}(1-q^{k+1}). \tag{6.11}$$

Alternatively, using the q-Pochhammer symbol, the expression for the residue takes the following form:

$$\begin{aligned} R_k(\tau, q) \equiv \mathrm{Res}_{s=m_k^2} A_q(s,t) &= \tau^k \frac{(1/(q^k\tau); q)_\infty (q;q)_\infty}{(1/\tau; q)_\infty} \times \lim_{\sigma \to q^k} \frac{(\sigma - q^k)}{(1/\sigma; q)_\infty} \\ &= \frac{(q;q)_\infty q^k}{\prod_{\substack{n=0 \\ n\neq k}}^{\infty} (1-q^{n-k})} \times \left[\tau^k \frac{\left(\frac{1}{\tau}\frac{1}{q^k}; q\right)_\infty}{\left(\frac{1}{\tau}; q\right)_\infty} \right]. \end{aligned} \tag{6.12}$$

We can decompose the product in the denominator in the following useful fashion:

$$\prod_{\substack{n=0 \\ n\neq k}}^{\infty} (1-q^{n-k}) = \prod_{n=0}^{k-1} (1-q^{n-k}) \prod_{n=0}^{\infty} (1-q^{n+1}) = (q;q)_\infty \prod_{n=0}^{k-1} (1-q^{n-k}). \tag{6.13}$$

Using equation (6.13), the residue (6.12) simplifies to give the expression:

$$R_k(\tau, q) = \frac{q^k}{\prod_{n=0}^{k-1}(1-q^{n-k})} \times \left[\tau^k \frac{\left(\frac{1}{\tau}\frac{1}{q^k}; q\right)_\infty}{\left(\frac{1}{\tau}; q\right)_\infty} \right] \tag{6.14}$$

To get the partial wave coefficients, we need to expand the residue obtained in (6.14) in terms of a partial wave decomposition. We decompose the residue in the following form in $D = 4$:

$$R_k(\tau, q) = \sum_{l=0}^{k} a_{k,l}^q \, P_l \left(1 + \frac{2t}{[k]_q - 3m^2} \right) \tag{6.15}$$

Note that the Legendre polynomials $P_l(x)$ entering the partial wave expansion satisfy the following orthogonality relation:

$$\int_{-1}^{+1} dx \, P_l(x) \, P_{l'}(x) = \frac{2}{2l+1} \, \delta_{l,l'} \tag{6.16}$$

We can use the orthogonality relation given in (6.16) to read off the partial wave amplitudes as

$$a_{k,l}^q = \frac{2l+1}{[k]_q - 3m^2} \int_{t=3m^2-[k]_q}^{t=0} dt \, R_k(\tau, q) \, P_l \left(1 + \frac{2t}{[k]_q - 3m^2} \right) \tag{6.17}$$

We have plotted $a_{k,k}^q$ for first few values of k in figures 6.2.1 and 6.2.2. They decrease with spin k but remain positive.

6.2.3 Calculation of residues

Our method for analytically determining the residues obtained in (6.17) involves utilizing the generating function of the Legendre polynomials $P_l(x)$:

$$\sum_{l=0}^{\infty} P_l(x) \, h^{-l-1} = (1 - 2xh + h^2)^{-1/2}. \tag{6.18}$$

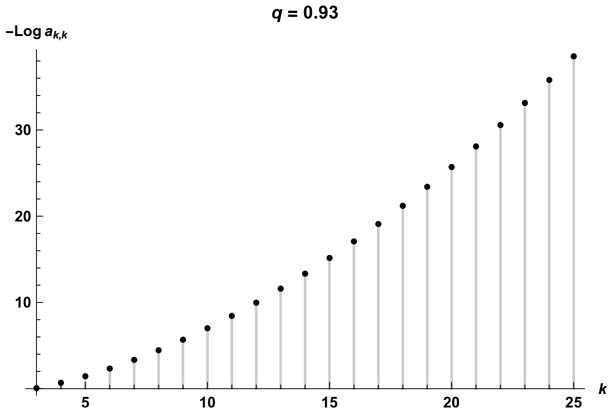

Figure 6.2.1: Decreasing magnitude of leading Regge coefficients $a^q_{k,k}$ with spin k for $q = 0.93$ and $m^2 = -1$ (the y axis here denotes negative values).

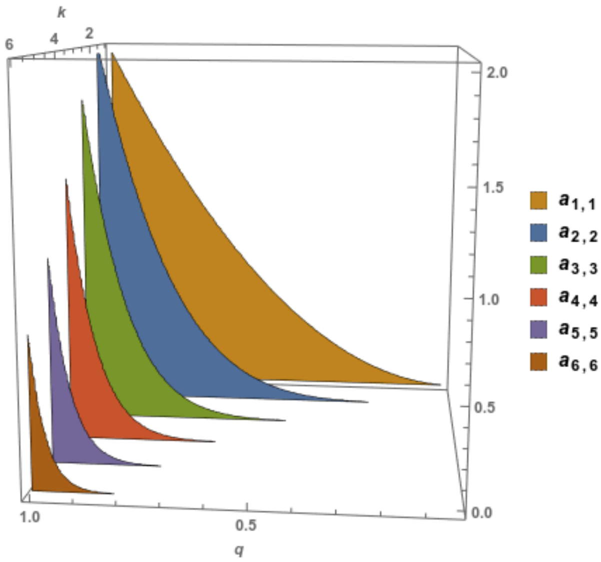

Figure 6.2.2: Plot displaying behaviour of leading Regge coefficients with q and spin (for $m^2 = -1$).

Using the generating function of the polynomials in (6.18), we can construct a set of generating functions for the partial wave coefficients $a_{n,j}$ by dividing with $(2j+1)h^{j+1}$ on both sides and then taking the sum over all spins j:

$$G_k(h) = \sum_{j=0}^{\infty} \frac{1}{2j+1} \frac{a^q_{k,j}}{h^{j+1}} = \frac{1}{[k]_q - 3m^2} \int_{t=3m^2-[k]_q}^{t=0} dt \, \frac{R_k(\tau, q)}{\left[(h-1)^2 - \frac{4ht}{[k]_q - 3m^2}\right]^{1/2}} \tag{6.19}$$

In order to compute (6.19), we need to expand the original expression for the residue given in (6.12) in terms of the variable τ. This operation necessitates the usage of the following identity

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \tag{6.20}$$

where we have used the notation introduced in (6.6). Then, the identity (6.20) helps us in evaluating a part of the residue from (6.12) as given below:

$$\frac{\left(\frac{1}{\tau}\frac{1}{q^k};q\right)_\infty}{\left(\frac{1}{\tau};q\right)_\infty} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{q^k};q\right)_n}{(q;q)_n}\frac{1}{\tau^n} = \sum_{n=0}^{k}\frac{\left(\frac{1}{q^k};q\right)_n}{(q;q)_n}\frac{1}{\tau^n} \tag{6.21}$$

where in the second equality, we have restricted the upper limit of the product from ∞ to k using the fact that the numerator vanishes for $n > k$, i.e.

$$\left(\frac{1}{q^k};q\right)_n = 0 \text{ for } n > k \tag{6.22}$$

With the simplification introduced in (6.21) the residue given in (6.12) takes the following form:

$$R_k(\tau, q) = \frac{q^k}{\prod_{n=0}^{k-1}(1-q^{n-k})}\sum_{n=0}^{k}\frac{\left(\frac{1}{q^k};q\right)_n}{(q;q)_n}\tau^{k-n} \tag{6.23}$$

Bound on spin

As clear from the expression (6.23), the residue at a given pole $s = m_k^2$ is a polynomial in t of degree k, corresponding to particle exchanges of spin $j = 0, \ldots, k$. Note that the poles of our amplitude $s = m^2 + [k]_q$ start from $k = 0$, as opposed to [20] where the poles $(s = k)$ start from $k = -1$. Thus we need to set $m^2 = -1$ and shift $k \to (k+1)$ in the expressions of $a^q_{k,j}$ to compare our results to [20]. In our convention, the leading Regge trajectory is associated with coefficients of the form $a^q_{k,k}$.

6.2.4 Reading off the coefficients for leading Regge trajectories

Using the simplified residues from (6.23), the expression for the generating function $G_k(h)$ given in (6.19) takes the following form.

$$G_k(h) = \frac{1}{[k]_q - 3m^2}\frac{q^k}{\prod_{n=0}^{k-1}(1-q^{n-k})}\sum_{n=0}^{k}\frac{\left(\frac{1}{q^k};q\right)_n}{(q;q)_n}\int_{t=3m^2-[k]_q}^{t=0} dt\,\frac{\tau^{k-n}}{\left[(h-1)^2 - \frac{4ht}{[k]_q-3m^2}\right]^{1/2}} \tag{6.24}$$

As can be seen from (6.19), in order to obtain the expression from $a^q_{k,j}$, we have to read off the coefficient of h^{-j-1} from the expression. To extract this coefficient, we can define a *pseudo-generating function* $g_{k,r}(h)$ in the subsequent fashion:

$$g_{k,r}(h) = \int_{t=3m^2-[k]_q}^{t=0} dt\,\frac{\tau^r}{\left[(h-1)^2 - \frac{4ht}{[k]_q-3m^2}\right]^{1/2}}. \tag{6.25}$$

Using the definition introduced in (6.25), the expression for the original generating function $G_k(h)$ from (6.24) has the expression

$$G_k(h) = \frac{1}{[k]_q - 3m^2} \frac{q^k}{\prod_{n=0}^{k-1}(1-q^{n-k})} \sum_{r=0}^{k} \frac{\left(\frac{1}{q^k};q\right)_{k-r}}{(q;q)_{k-r}} g_{k,r}(h) \tag{6.26}$$

The pseudo generating function introduced in (6.25) can be explicitly evaluated and can be expressed as a sum of two ${}_2F_1$ hypergeometric functions:

$$\begin{aligned} g_{k,r}(h) = &\frac{(q-1)^r}{2^{2r+1}} \frac{([k_q]-3m^2)^{r+1}}{h^{r+1}} \left[(h-1)^2 + \frac{4hs_*}{3m^2-[k]_q}\right]^r \times \\ &\left[(h+1)_2F_1\left(\frac{1}{2},-r;\frac{3}{2};\frac{(h+1)^2}{(h-1)^2+\frac{4hs_*}{3m^2-[k]_q}}\right) - (h-1)_2F_1\left(\frac{1}{2},-r;\frac{3}{2};\frac{(h-1)^2}{(h-1)^2+\frac{4hs_*}{3m^2-[k]_q}}\right)\right]. \end{aligned} \tag{6.27}$$

where s_* is the accumulation point calculated in (6.5).

Reading off the leading Regge trajectory

The pseudo generating function $g_{k,r}(h)$ comprises of $(r+1)$ terms accompanying powers of h, i.e. terms of the form $h^{-1}, h^{-2}, \cdots, h^{-r-1}$. We can easily extract the coefficient of h^{-r-1} from the expansion (6.27), which gives us the following expression.

$$d_{k,r} = \frac{\sqrt{\pi}}{2^{2r+1}} \frac{r!}{\Gamma(r+3/2)} (q-1)^r([k_q]-3m^2)^{r+1} \tag{6.28}$$

In order to read off the partial wave coefficients, we will use a similar argument to the one used while extracting the leading Regge coefficients for the Veneziano amplitude [20]. Firstly, we note that the h-expansion of $G_k(h)$ can be represented in the following form using equations (6.19), (6.26) and the expansion of $g_{k,r}(h)$ in h:

$$\underbrace{\sum_{j=0}^{k} [\ldots]\, h^{-j-1}}_{G_k(h)} = \sum_{r=0}^{k} [\ldots] \underbrace{\sum_{p=0}^{r} [\ldots]\, h^{-p-1}}_{g_{k,r}(h)}. \tag{6.29}$$

In general, it is difficult to extract the coefficients $a^q_{k,l}$ from (6.29). However, we can determine the expression for the leading Regge trajectory, i.e., $a^q_{k,k}$. In order to determine $a^q_{k,k}$ from the coefficient of h^{-k-1} in the RHS of (6.29), we need to consider the term with h^{-k-1} in $g_{k,k}(h)$, which is a unique term. Using this term, while equating both sides of (6.29), we obtain the following expression for $a^q_{k,k}$:

$$\frac{1}{2k+1} a^q_{k,k} = \frac{1}{[k]_q - 3m^2} \frac{q^k}{\prod_{n=0}^{k-1}(1-q^{n-k})} \frac{\left(\frac{1}{q^k};q\right)_0}{(q;q)_0} \times d_{k,k} \tag{6.30}$$

Substituting the result of $d_{k,k}$ from (6.28) into (6.30), we obtain the our result for the partial wave coefficients $a^q_{k,k}$

$$a^q_{k,k} = \frac{\sqrt{\pi}}{2^{2k}} \frac{k!}{\Gamma(k+1/2)} \frac{([k]_q - 3m^2)^k q^k (1-q)^k}{\prod_{n=0}^{k-1}(q^{n-k}-1)}. \tag{6.31}$$

Our analytic result in (6.31) matches with numerical results obtained from directly integrating the rhs of (6.17). For instance, we are able to reproduce the plot in figure 6.2.2 using our formula. Our result in (6.31) is also manifestly positive for $m^2 < \frac{1}{3}\min\left([k]_q\right) = \frac{1}{3}$ [2] and $0 < q < 1$. This upper bound on m^2 is consistent with the bound obtained in [153].

As another check, we take the $q \to 1$ limit of equation (6.31), where we expect to recover the coefficients for the Veneziano amplitude. In the limit $q \to 1$, we have the following reduction:

$$\prod_{n=0}^{k-1}(q^{n-k}-1) \to (1-q)^k k!. \tag{6.32}$$

Taking the limit $q \to 1$ in (6.31) and substituting (6.32) in the same, we obtain

$$\lim_{q\to 1} a^q_{k,k} = \frac{\sqrt{\pi}}{2^{2k}} \frac{(k-3m^2)^k}{\Gamma(k+1/2)} \tag{6.33}$$

which precisely matches (accounting for the shifts in 6.2.3) with the result for Veneziano amplitude in [20, 162].

6.3 Conclusion

In our work, we have calculated the partial wave coefficients that govern the leading Regge behaviour in $D = 4$. We have shown that the associated coefficients are positive. Since the coefficients governing the subsequent subleading Regge trajectories have increasingly larger values (see Appendix 6.B for details regarding the same), our work indicates the positivity of the Coon amplitude for all partial wave coefficients in $D = 4$. An interesting direction would be to investigate the same for general D, possibly along the lines of [162].

In general, negative partial wave coefficients imply that exchanges involving negative norm states take place. Our present work indicates the absence of such states at the tree level. However, it may still be possible that the underlying physical description suffers from problems resulting from negative norm states when loops are taken into account, and hence the physical description is inconsistent in the first place. In order to check such issues, one needs to understand the physics behind the amplitude, which may be similar to the prescription suggested by [155].

The accumulation point in (6.5) leads to the existence of a branch cut, and another

[2] Note that for $k = 0$, $a^q_{0,0} = 1 > 0$.

interesting question is to understand the physical structure behind the same.

Appendix

6.A Expression for the general Regge coefficient $a^q_{k,j}$

Using the following expansion of the hypergeometric function

$$ {}_2F_1\left(\frac{1}{2}, -r; \frac{3}{2}; z\right) = \sum_{p=0}^{r} \frac{(-r)_p}{(2p+1)\,p!}\, z^p, \tag{6.34}$$

we can expand the expression for the pseudo-generating function $g_{k,r}(h)$ given in (6.27) as follows:

$$\begin{aligned} g_{k,r}(h) = \frac{(q-1)^r}{2^{2r+1}} \frac{([k]_q - 3m^2)^{r+1}}{h^{r+1}} \sum_{p=0}^{r} \frac{(-r)_p}{(2p+1)p!} \left[(h-1)^2 + \frac{4hs_*}{3m^2 - [k]_q}\right]^{r-p} \\ \times \sum_{d=0}^{2p+1} \binom{2p+1}{d} h^d (1 + (-1)^d). \end{aligned} \tag{6.35}$$

Now we can make binomial expansion of $\left[(h-1)^2 + \frac{4hs_*}{3m^2-[k]_q}\right]^{r-p}$ in the above expression and compute the coefficient of h^{-j-1} from r.h.s of (6.26) to find the generic coefficient on any Regge trajectory as

$$\begin{aligned} a^q_{k,j} = \frac{(2j+1)\,q^k}{\prod_{n=0}^{k-1}(1-q^{n-k})} \sum_{r=0}^{k} \frac{1}{2^{2r+1}} \frac{\left(\frac{1}{q^k}; q\right)_{k-r}}{(q;q)_{k-r}} (q-1)^r ([k]_q - 3m^2)^r \\ \times \sum_{p=0}^{r} \sum_{d=0}^{2p+1} \sum_{i=0}^{r-p} \binom{r-p}{i} \binom{2p+1}{d} \binom{2m}{p+i-j-d} \frac{(-r)_p}{(2p+1)\,p!} \\ \times \left(\frac{4s_*}{3m^2 - [k]_q}\right)^{r-p-i} (-1)^{p+i-j-d} \left[1 + (-1)^d\right] \end{aligned} \tag{6.36}$$

Algorithmically (6.36) is much simpler than the original expression (6.17) as an integral.

6.B Leading versus subleading coefficients

In this appendix, we compare the partial wave coefficients on the leading Regge trajectory ($a^q_{k,k}$) versus the coefficients on the subleading Regge trajectory ($a^q_{k,j}$, $j < k$).

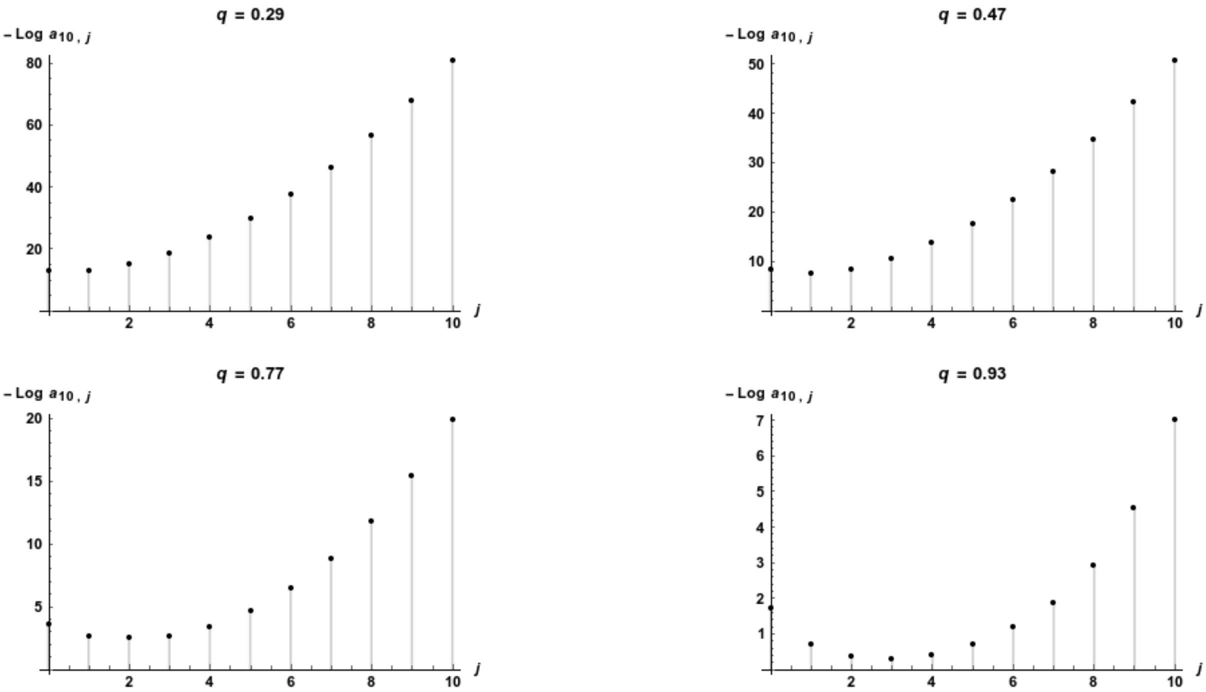

Figure 6.B.1: Coefficients $a^q_{10,j}$ for $j = 0, \ldots, 10$ at different values of q for $m^2 = -1$.

In figure 6.B.1, we perform the comparison between leading and subleading coefficients by fixing $k = 10$, and plotting the negative logarithm of all coefficients j with $j \leq 10$, for $m^2 = -1$. The increase in the negative logarithm of $a^q_{10,j}$ with j implies that the magnitude of the coefficients $a^q_{10,j}$ decreases with j.

We also plot the coefficients corresponding to the leading ($a^q_{k,k}$) and the subleading ($a^q_{k,k-1}$) trajectories at different values of q in figure 6.B.2. Here again, the increase in the negative logarithm of $a^q_{k,j}$ with j implies that the magnitude of the coefficients $a^q_{k,j}$ decreases with j.

6.C Asymptotic q-limits of the Coon Amplitude

In this appendix, we look at the asymptotic q-limits of the Coon amplitude.

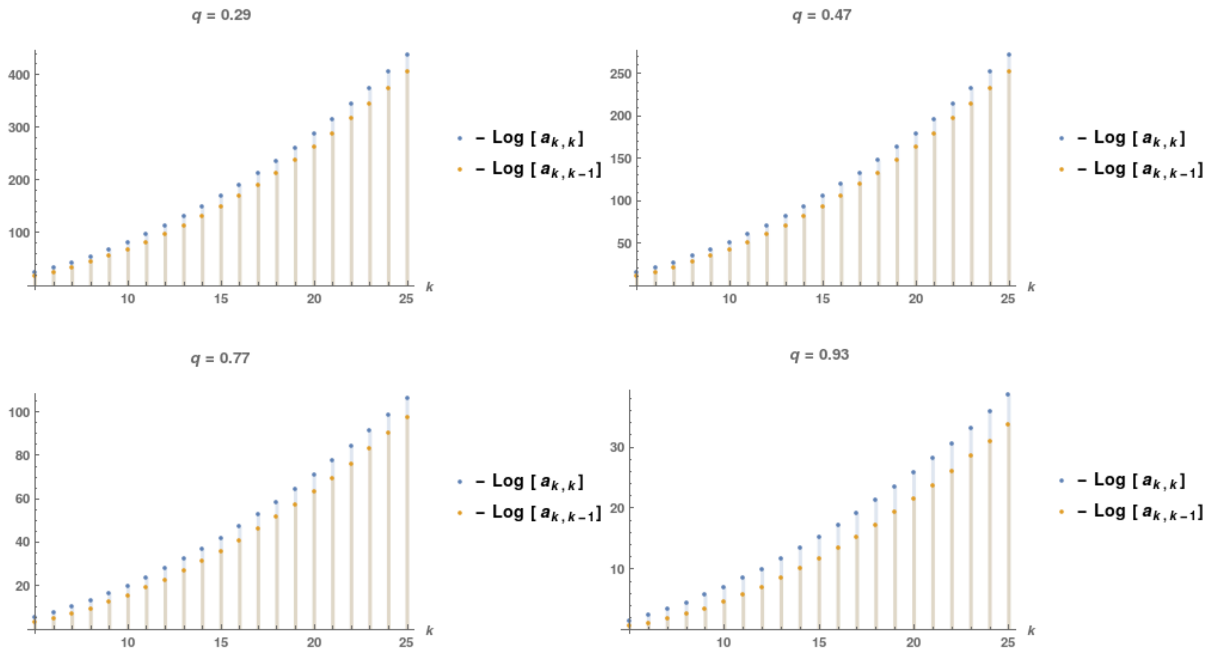

Figure 6.B.2: Coefficients on the leading ($a^q_{k,k}$) and subleading ($a^q_{k,k-1}$) trajectories at different values of q for $m^2 = -1$.

The $q \to 1$ limit

To recover the Veneziano amplitude as $q \to 1$, we rewrite the product form of the Coon amplitude (6.1) as

$$\prod_{n=0}^{\infty} \frac{(\sigma\tau - q^n)(1-q^{n+1})}{(\sigma - q^n)(\tau - q^n)} = \frac{(-s-t+2m^2) + (1-q)(-s+m^2)(-t+m^2)}{(-s+m^2)(-t+m^2)(1-q)} \times \prod_{n=1}^{\infty}(1-q^n)$$
$$\times \prod_{n=1}^{\infty} \frac{(-s-t+2m^2) + (1-q)\,(-s+m^2)(-t+m^2) + [n]_q}{(1-q)[(-s+m^2)+[n]_q][(-t+m^2)+[n]_q]} \tag{6.37}$$

where we have used equation (6.2). Next, we note that

$$\prod_{n=1}^{\infty} \frac{1-q^n}{1-q} = \prod_{n=1}^{\infty} [n]_q,$$

using which we take the limit $q \to 1$ such that (6.37) takes the following form:

$$\frac{1}{1-q}\,\frac{x+y}{xy} \prod_{n=1}^{\infty} \frac{n(x+y+n)}{(x+n)(y+n)} = \frac{1}{1-q}\,\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{6.38}$$

Here $x = (-s+m^2)$ and $y = (-t+m^2)$, and we have used an identity of beta function in the second equality above. The other factors in (6.1) have the limit

$$(q-1)\, q^{\frac{\log\sigma\,\log\tau}{(\log q)^2}} \to (q-1) \tag{6.39}$$

Combining this with (6.38), we get the familiar Veneziano amplitude in (6.3) as $q \to 1$.

The $q \to 0$ limit

In the limit $q \to 0$, only the $n = 0$ factor contributes to equation (6.37), which takes the form

$$\begin{aligned} \prod_{n=0}^{\infty} \frac{(\sigma\tau - q^n)(1 - q^{n+1})}{(\sigma - q^n)(\tau - q^n)} &\to \frac{(-s - t + 2m^2) + (-s + m^2)(-t + m^2)}{(-s + m^2)(-t + m^2)} \\ &= \frac{1}{s - m^2} + \frac{1}{t - m^2} - 1 \end{aligned} \tag{6.40}$$

In the same limit, the pre-factor becomes minus one, and we thus recover the familiar point particle (scalar) result in (6.3) as $q \to 0$.

Bibliography

[1] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. B **72** (1974) 461.

[2] J.M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38** (1999) 1113 [Adv. Theor. Math. Phys. **2** (1998) 231] [hep-th/9711200].

[3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428** (1998) 105 [hep-th/9802109].

[4] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150].

[5] J. M. Maldacena and H. Ooguri, "Strings in AdS(3) and SL(2,R) WZW model 1.: The Spectrum," J. Math. Phys. **42**, 2929-2960 (2001) [arXiv:hep-th/0001053 [hep-th]].

[6] J. M. Maldacena, H. Ooguri and J. Son, "Strings in AdS(3) and the SL(2,R) WZW model. Part 2. Euclidean black hole," J. Math. Phys. **42**, 2961-2977 (2001) [arXiv:hep-th/0005183 [hep-th]].

[7] J. M. Maldacena and H. Ooguri, "Strings in AdS(3) and the SL(2,R) WZW model. Part 3. Correlation functions," Phys. Rev. D **65**, 106006 (2002) [arXiv:hep-th/0111180 [hep-th]].

[8] M. R. Gaberdiel and R. Gopakumar, "Tensionless string spectra on AdS_3," JHEP **05**, 085 (2018) [arXiv:1803.04423 [hep-th]].

[9] L. Eberhardt, M. R. Gaberdiel and R. Gopakumar, "The Worldsheet Dual of the Symmetric Product CFT," JHEP **04** (2019), 103 [arXiv:1812.01007 [hep-th]].

[10] L. Eberhardt, M. R. Gaberdiel and R. Gopakumar, "Deriving the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence," JHEP **02** (2020), 136 [arXiv:1911.00378 [hep-th]].

[11] A. Dei, M. R. Gaberdiel, R. Gopakumar and B. Knighton, "Free field world-sheet correlators for AdS_3," JHEP **02**, 081 (2021) [arXiv:2009.11306 [hep-th]].

[12] M. R. Gaberdiel and R. Gopakumar, "String Dual to Free N=4 Supersymmetric Yang-Mills Theory," Phys. Rev. Lett. **127** (2021) no.13, 131601 [arXiv:2104.08263 [hep-th]].

[13] R. Gopakumar, "From free fields to AdS: III," Phys. Rev. D **72** (2005), 066008 [arXiv:hep-th/0504229 [hep-th]].

[14] S. S. Razamat, "On a worldsheet dual of the Gaussian matrix model," JHEP **07**, 026 (2008) [arXiv:0803.2681 [hep-th]].

[15] M. R. Gaberdiel, R. Gopakumar, B. Knighton and P. Maity, "From symmetric product CFTs to AdS_3," JHEP **05**, 073 (2021) [arXiv:2011.10038 [hep-th]].

[16] F. Bhat, R. Gopakumar, P. Maity and B. Radhakrishnan, "Twistor coverings and Feynman diagrams," JHEP **05**, 150 (2022) [arXiv:2112.05115 [hep-th]].

[17] D. J. Gross and W. Taylor, "Two-dimensional QCD is a string theory," Nucl. Phys. B **400** (1993), 181-208 [arXiv:hep-th/9301068 [hep-th]].

[18] S. Komatsu, P. Maity, "String Dual of Two-dimensional Yang-Mills and Symmetric Product Orbifolds," [*to appear*].

[19] P. Goddard and C. B. Thorn, "Compatibility of the Dual Pomeron with Unitarity and the Absence of Ghosts in the Dual Resonance Model," Phys. Lett. B **40**, 235-238 (1972)

[20] P. Maity, "Positivity of the Veneziano amplitude in D = 4," JHEP **04**, 064 (2022) [arXiv:2110.01578 [hep-th]].

[21] J. Chakravarty, P. Maity and A. Mishra, "On the positivity of Coon amplitude in D = 4," JHEP **10**, 043 (2022) [arXiv:2208.02735 [hep-th]].

[22] R. Gopakumar, "From free fields to AdS space. I," Phys. Rev. D **70** (2004) 025009 [hep-th/0308184].

[23] R. Gopakumar, "From free fields to AdS space. II," Phys. Rev. D **70** (2004) 025010 [hep-th/0402063].

[24] O. Aharony, Z. Komargodski and S.S. Razamat, "On the worldsheet theories of strings dual to free large N gauge theories," JHEP **0605** (2006) 016 [hep-th/0602226].

[25] S.S. Razamat, "From Matrices to Strings and Back," JHEP **1003** (2010) 049 [arXiv:0911.0658 [hep-th]].

[26] R. Gopakumar, "What is the Simplest Gauge-String Duality?," arXiv:1104.2386 [hep-th].

[27] R. Gopakumar and R. Pius, "Correlators in the Simplest Gauge-String Duality," JHEP **1303** (2013) 175 [arXiv:1212.1236 [hep-th]].

[28] R. de Mello Koch and L. Nkumane, "Topological String Correlators from Matrix Models," JHEP **1503** (2015) 004 [arXiv:1411.5226 [hep-th]].

[29] I. Yaakov, "Open and closed string worldsheets from free large N gauge theories with adjoint and fundamental matter," JHEP **0611** (2006) 065 [hep-th/0607244].

[30] K. Furuuchi, "From free fields to AdS: Thermal case," Phys. Rev. D **72** (2005) 066009 [hep-th/0505148].

[31] J.R. David and R. Gopakumar, "From spacetime to worldsheet: Four point correlators," JHEP **0701** (2007) 063 [hep-th/0606078].

[32] O. Aharony, J.R. David, R. Gopakumar, Z. Komargodski and S.S. Razamat, "Comments on worldsheet theories dual to free large N gauge theories," Phys. Rev. D **75** (2007) 106006 [hep-th/0703141].

[33] J.R. David, R. Gopakumar and A. Mukhopadhyay, "Worldsheet Properties of Extremal Correlators in AdS/CFT," JHEP **0810** (2008) 029 [arXiv:0807.5027 [hep-th]].

[34] S. Lal, "On Gluing CFT Correlators to Higher-Spin Amplitudes in AdS," arXiv:2009.10110 [hep-th].

[35] A. Giveon, D. Kutasov, E. Rabinovici and A. Sever, "Phases of quantum gravity in AdS_3 and linear dilaton backgrounds," Nucl. Phys. B **719** (2005) 3 [hep-th/0503121].

[36] M.R. Gaberdiel, R. Gopakumar and C. Hull, "Stringy AdS_3 from the worldsheet," JHEP **07** (2017), 090 [arXiv:1704.08665 [hep-th]].

[37] K. Ferreira, M.R. Gaberdiel and J.I. Jottar, "Higher spins on AdS_3 from the worldsheet," JHEP **07** (2017), 131 [arXiv:1704.08667 [hep-th]].

[38] G. Giribet, C. Hull, M. Kleban, M. Porrati and E. Rabinovici, "Superstrings on AdS_3 at $k = 1$," JHEP **1808** (2018) 204 [arXiv:1803.04420 [hep-th]].

[39] O. Lunin and S.D. Mathur, "Correlation functions for M^N/S_N orbifolds," Commun. Math. Phys. **219** (2001) 399 [hep-th/0006196].

[40] O. Lunin and S.D. Mathur, "Three point functions for M^N/S_N orbifolds with $\mathcal{N}=4$ supersymmetry," Commun. Math. Phys. **227** (2002) 385 [hep-th/0103169].

[41] L. Eberhardt, "$\mathrm{AdS}_3/\mathrm{CFT}_2$ at higher genus," JHEP **2005** (2020) 150 [arXiv:2002.11729 [hep-th]].

[42] L. Eberhardt, "Partition functions of the tensionless string," arXiv:2008.07533 [hep-th].

[43] A. Dei, M.R. Gaberdiel, R. Gopakumar and B. Knighton, "Free field world-sheet correlators for AdS_3," arXiv:2009.11306 [hep-th].

[44] R. de Mello Koch, A. Jevicki, K. Suzuki and J. Yoon, "AdS Maps and Diagrams of Bi-local Holography," JHEP **1903** (2019) 133 [arXiv:1810.02332 [hep-th]].

[45] O. Aharony, S.M. Chester and E.Y. Urbach, "A Derivation of AdS/CFT for Vector Models," arXiv:2011.06328 [hep-th].

[46] D.J. Gross and P.F. Mende, "String Theory Beyond the Planck Scale," Nucl. Phys. B **303** (1988) 407.

[47] J. Maldacena, D. Stanford and Z. Yang, "Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space," PTEP **2016** (2016) 12C104 [arXiv:1606.01857 [hep-th]].

[48] A.M. Polyakov, "Fine Structure of Strings," Nucl. Phys. B **268** (1986) 406.

[49] H. Kleinert, "The Membrane Properties of Condensing Strings," Phys. Lett. B **174** (1986) 335.

[50] P. Horava, "Topological strings and QCD in two-dimensions," hep-th/9311156.

[51] A. Dei and L. Eberhardt, "Correlators of the symmetric product orbifold," JHEP **2001** (2020) 108 [arXiv:1911.08485 [hep-th]].

[52] F. Liu and B. Osserman, "The irreducibility of certain pure-cycle Hurwitz spaces," American Journal of Mathematics **130** (2008) 1687.

[53] F. Aprile and P. Vieira, "Large p explorations. From SUGRA to big STRINGS in Mellin space," arXiv:2007.09176 [hep-th].

[54] K. Roumpedakis, "Comments on the S_N orbifold CFT in the large N-limit," JHEP **1807** (2018) 038 [arXiv:1804.03207 [hep-th]].

[55] R. Dijkgraaf and C. Vafa, "Toda Theories, Matrix Models, Topological Strings, and N=2 Gauge Systems," arXiv:0909.2453 [hep-th].

[56] H. Itoyama, K. Maruyoshi and T. Oota, “The Quiver Matrix Model and 2d-4d Conformal Connection,” Prog. Theor. Phys. **123** (2010) 957 [arXiv:0911.4244 [hep-th]].

[57] T. Eguchi and K. Maruyoshi, “Penner Type Matrix Model and Seiberg-Witten Theory,” JHEP **1002** (2010) 022 [arXiv:0911.4797 [hep-th]].

[58] R. Schiappa and N. Wyllard, “An A(r) threesome: Matrix models, 2d CFTs and 4d N=2 gauge theories,” J. Math. Phys. **51** (2010) 082304 [arXiv:0911.5337 [hep-th]].

[59] A. Mironov, A. Morozov and S. Shakirov, “Matrix Model Conjecture for Exact BS Periods and Nekrasov Functions,” JHEP **1002** (2010) 030 [arXiv:0911.5721 [hep-th]].

[60] S.R. Wadia, “Dyson-Schwinger Equations Approach to the Large-N Limit: Model Systems and String Representation of Yang-Mills Theory,” Phys. Rev. D **24** (1981) 970.

[61] M. Mariño, “Les Houches lectures on matrix models and topological strings,” hep-th/0410165.

[62] V. Ovsienko and S. Tabachnikov, “What is the Schwarzian derivative?", Notices of the AMS **56(1)** (2009) 34.

[63] S. Mukhi, “Topological matrix models, Liouville matrix model and c = 1 string theory,” hep-th/0310287.

[64] S.K. Ashok, F. Cachazo and E. Dell’Aquila, “Strebel differentials with integral lengths and Argyres-Douglas singularities,” hep-th/0610080.

[65] R. de Mello Koch and S. Ramgoolam, “From Matrix Models and quantum fields to Hurwitz space and the absolute Galois group," arXiv:1002.1634 [hep-th].

[66] C. Itzykson and J.B. Zuber, “Matrix integration and combinatorics of modular groups," Commun. Math. Phys. **134** (1990) 197.

[67] P. Di Francesco and C. Itzykson, “A Generating Function for Fatgraphs,” Annales Poincare Phys. Theor. **59** (1993) 117 [hep-th/9212108].

[68] M. Mulase and M. Penkava, “Ribbon Graphs, Quadratic Differentials on Riemann Surfaces, and Algebraic Curves Defined over $\bar{Q}$,” math-ph/9811024.

[69] G. Mandal, P. Nayak and S.R. Wadia, “Coadjoint orbit action of Virasoro group and two-dimensional quantum gravity dual to SYK/tensor models,” JHEP **1711** (2017) 046 [arXiv:1702.04266 [hep-th]].

[70] S. Sachdev and J. Ye, "Gapless spin fluid ground state in a random, quantum Heisenberg magnet," Phys. Rev. Lett. **70** (1993) 3339 [cond-mat/9212030].

[71] A. Kitaev and S.J. Suh, "The soft mode in the Sachdev-Ye-Kitaev model and its gravity dual," JHEP **1805** (2018) 183 [arXiv:1711.08467 [hep-th]].

[72] A. Kitaev, "A simple model of quantum holography", talks at KITP, Santa Barbara, 7 April and 27 May 2015.

[73] J. Maldacena and D. Stanford, "Remarks on the Sachdev-Ye-Kitaev model," Phys. Rev. D **94** (2016) 106002 [arXiv:1604.07818 [hep-th]].

[74] Y.H. He and J. Read, "Dessins d'enfants in $\mathcal{N} = 2$ generalised quiver theories," JHEP **1508** (2015) 085 [arXiv:1503.06418 [hep-th]].

[75] R. Gopakumar, "Open-Closed-Open String Duality", talk at the Johannesburg workshop: 'Correlation Functions and the AdS/CFT Correspondence', April 27, 2010. http://neo.phys.wits.ac.za/workshop_2/pdfs/rajesh.pdf

[76] T. Bargheer, F. Coronado and P. Vieira, "Octagons I: Combinatorics and Non-Planar Resummations," JHEP **2019** (2020) 162 [arXiv:1904.00965 [hep-th]].

[77] Y. Jiang, S. Komatsu and E. Vescovi, "Structure constants in $\mathcal{N} = 4$ SYM at finite coupling as worldsheet g-function," JHEP **2007** (2020) 037 [arXiv:1906.07733 [hep-th]].

[78] O. Aharony and Z. Komargodski, "The Space-time operator product expansion in string theory duals of field theories," JHEP **0801** (2008) 064 [arXiv:0711.1174 [hep-th]].

[79] S. Ghosh, S. Sarkar and M. Verma, "Implications of the AdS/CFT Correspondence on Spacetime and Worldsheet OPE Coefficients," arXiv:1703.06132 [hep-th].

[80] R. Gopakumar, presentation at "Bootstrapping String Theory," Bootstrap Collaboration Meeting, Caltech, May 18th, 2020.

[81] D.J. Gross, "Two-dimensional QCD as a string theory," Nucl. Phys. B **400** (1993) 161 [hep-th/9212149].

[82] S. Cordes, G.W. Moore and S. Ramgoolam, "Large N 2D Yang-Mills theory and topological string theory," Commun. Math. Phys. **185** (1997) 543 [hep-th/9402107].

[83] T. Bargheer, J. Caetano, T. Fleury, S. Komatsu and P. Vieira, "Handling handles. Part II. Stratification and data analysis," JHEP **1811** (2018) 095 [arXiv:1809.09145 [hep-th]].

[84] M. R. Gaberdiel and R. Gopakumar, "String Dual to Free N=4 Supersymmetric Yang-Mills Theory," Phys. Rev. Lett. **127** (2021) no.13, 131601 [arXiv:2104.08263 [hep-th]].

[85] M. R. Gaberdiel and R. Gopakumar, "The Worldsheet Dual of Free Super Yang-Mills in 4D," JHEP **11**, 129 (2021) [arXiv:2105.10496 [hep-th]].

[86] N. Berkovits, "A New Limit of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ Sigma Model," JHEP **0708** (2007) 011 [hep-th/0703282].

[87] N. Berkovits and C. Vafa, "Towards a Worldsheet Derivation of the Maldacena Conjecture," JHEP **0803** (2008) 031 [AIP Conf. Proc. **1031** (2008) 21] [arXiv:0711.1799 [hep-th]].

[88] N. Berkovits, "Perturbative Super-Yang-Mills from the Topological $\mathrm{AdS}_5 \times \mathrm{S}^5$ Sigma Model," JHEP **0809** (2008) 088 [arXiv:0806.1960 [hep-th]].

[89] N. Berkovits, "Sketching a Proof of the Maldacena Conjecture at Small Radius," JHEP **1906** (2019) 111 [arXiv:1903.08264 [hep-th]].

[90] L. Eberhardt, "$\mathrm{AdS}_3/\mathrm{CFT}_2$ at higher genus," JHEP **2005** (2020) 150 [arXiv:2002.11729 [hep-th]].

[91] A. Dei, M. R. Gaberdiel, R. Gopakumar and B. Knighton, "Free field world-sheet correlators for AdS_3," JHEP **02** (2021), 081 [arXiv:2009.11306 [hep-th]].

[92] B. Knighton, "Higher genus correlators for tensionless AdS_3 strings," JHEP **04**, 211 (2021) [arXiv:2012.01445 [hep-th]].

[93] M. R. Gaberdiel, B. Knighton and J. Vošmera, "D-branes in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ at $k = 1$ and their holographic duals," [arXiv:2110.05509 [hep-th]].

[94] M. R. Gaberdiel, R. Gopakumar and C. Hull, "Stringy AdS_3 from the worldsheet," JHEP **07**, 090 (2017) [arXiv:1704.08665 [hep-th]].

[95] K. Ferreira, M. R. Gaberdiel and J. I. Jottar, "Higher spins on AdS_3 from the worldsheet," JHEP **07**, 131 (2017) [arXiv:1704.08667 [hep-th]].

[96] G. Giribet, C. Hull, M. Kleban, M. Porrati and E. Rabinovici, "Superstrings on AdS_3 at $k = 1$," JHEP **08**, 204 (2018) [arXiv:1803.04420 [hep-th]].

[97] A. Giveon, D. Kutasov, E. Rabinovici and A. Sever, "Phases of quantum gravity in AdS(3) and linear dilaton backgrounds," Nucl. Phys. B **719**, 3-34 (2005) [arXiv:hep-th/0503121 [hep-th]].

[98] B. Balthazar, A. Giveon, D. Kutasov and E. J. Martinec, "Asymptotically Free AdS3/CFT2," [arXiv:2109.00065 [hep-th]].

[99] E. J. Martinec, "AdS3's with and without BTZ's," [arXiv:2109.11716 [hep-th]].

[100] L. Eberhardt, "A perturbative CFT dual for pure NS-NS AdS_3 strings," [arXiv:2110.07535 [hep-th]].

[101] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, "Strings in flat space and pp waves from N=4 superYang-Mills," JHEP **04** (2002), 013 [arXiv:hep-th/0202021 [hep-th]].

[102] N. Berkovits, "An Alternative string theory in twistor space for N=4 superYang-Mills," Phys. Rev. Lett. **93** (2004), 011601 [arXiv:hep-th/0402045 [hep-th]].

[103] L. Mason and D. Skinner, "Ambitwistor strings and the scattering equations," JHEP **1407** (2014) 048 [arXiv:1311.2564 [hep-th]].

[104] E. Witten, "Perturbative gauge theory as a string theory in twistor space," Commun. Math. Phys. **252** (2004), 189-258 [arXiv:hep-th/0312171 [hep-th]].

[105] C. Ahn, "Worldsheet Free Fields, Higher Spin Symmetry and Free $\mathcal{N} = 4$ Super Yang-Mills," [arXiv:2109.09260 [hep-th]].

[106] O. Lunin and S.D. Mathur, "Correlation functions for M^N/S_N orbifolds," Commun. Math. Phys. **219** (2001) 399 [arXiv:hep-th/0006196].

[107] A. Pakman, L. Rastelli, and S. Razamat, "Diagrams for Symmetric Product Orbifolds," JEHP **0910** (2009) 034 doi:10.1088/1126-6708/2009/10/034 [arXiv:0905.3448 [hep-th]].

[108] K. Strebel, "Quadratic Differentials," Springer (1980).

[109] M. Mulase, M. Penkava, "Ribbon Graphs, Quadratic Differentials on Riemann Surfaces, and Algebraic Curves Defined over $\bar{\mathbb{Q}}$," Asian J. Math. **02** (1998), 875. [arXiv:math-ph/9811024].

[110] T. Adamo, D. Skinner and J. Williams, "Twistor methods for AdS_5," JHEP **08** (2016), 167 [arXiv:1607.03763 [hep-th]].

[111] T. Adamo, "Lectures on twistor theory," PoS **Modave2017**, 003 (2018) [arXiv:1712.02196 [hep-th]].

[112] M. R. Gaberdiel, R. Gopakumar, B. Knighton and P. Maity, (work in progress).

[113] B. Eynard, "Lectures on Compact Riemann Surfaces" [arXiv:1805.06405 [math-ph]]

[114] K. Roumpedakis, "Comments on the S_N orbifold CFT in the large N-limit," JHEP **1807** (2018) 038 [arXiv:1804.03207 [hep-th]].

[115] G. 't Hooft, "A Two-Dimensional Model for Mesons," Nucl. Phys. B **75**, 461-470 (1974)

[116] M. R. Douglas, "Conformal field theory techniques in large N Yang-Mills theory," [arXiv:hep-th/9311130 [hep-th]].

[117] N. Callebaut, J. Kruthoff and H. Verlinde, "$T\overline{T}$ deformed CFT as a non-critical string," JHEP **04** (2020), 084 [arXiv:1910.13578 [hep-th]].

[118] J. Gomis and H. Ooguri, "Nonrelativistic closed string theory," J. Math. Phys. **42** (2001), 3127-3151 [arXiv:hep-th/0009181 [hep-th]].

[119] U. H. Danielsson, A. Guijosa and M. Kruczenski, "Newtonian gravitons and d-brane collective coordinates in wound string theory," JHEP **03** (2001), 041 [arXiv:hep-th/0012183 [hep-th]].

[120] J. Polchinski and A. Strominger, "Effective string theory," Phys. Rev. Lett. **67** (1991), 1681-1684

[121] S. Hellerman, S. Maeda, J. Maltz and I. Swanson, "Effective String Theory Simplified," JHEP **09** (2014), 183 [arXiv:1405.6197 [hep-th]].

[122] J. Gomis, Z. Yan and M. Yu, "KLT factorization of winding string amplitudes," JHEP **06** (2021), 057 [arXiv:2103.05013 [hep-th]].

[123] A. Hashimoto and D. Kutasov, "$T\overline{T}, J\overline{T}, T\overline{J}$ partition sums from string theory," JHEP **02** (2020), 080 [arXiv:1907.07221 [hep-th]].

[124] A. Hashimoto and D. Kutasov, "Strings, symmetric products, $T\bar{T}$ deformations and Hecke operators," Phys. Lett. B **806** (2020), 135479 [arXiv:1909.11118 [hep-th]].

[125] J. Polchinski, "Evaluation of the One Loop String Path Integral," Commun. Math. Phys. **104** (1986), 37

[126] Y. Baba, N. Ishibashi and K. Murakami, "Light-Cone Gauge String Field Theory in Noncritical Dimensions," JHEP **12**, 010 (2009) [arXiv:0909.4675 [hep-th]].

[127] R. Dijkgraaf, G. W. Moore, E. P. Verlinde and H. L. Verlinde, "Elliptic genera of symmetric products and second quantized strings," Commun. Math. Phys. **185** (1997), 197-209 [arXiv:hep-th/9608096 [hep-th]].

[128] K. Okuyama and K. Sakai, "Holomorphic anomaly of 2d Yang-Mills theory on a torus revisited," JHEP **08** (2019), 025 [arXiv:1903.10510 [hep-th]].

[129] C. Vafa, "Two dimensional Yang-Mills, black holes and topological strings," [arXiv:hep-th/0406058 [hep-th]].

[130] R. Dijkgraaf, "Les Houches lectures on fields, strings and duality," [arXiv:hep-th/9703136 [hep-th]].

[131] J. A. Minahan and A. P. Polychronakos, "Equivalence of two-dimensional QCD and the C = 1 matrix model," Phys. Lett. B **312** (1993), 155-165 [arXiv:hep-th/9303153 [hep-th]].

[132] C. A. Keller, "Phase transitions in symmetric orbifold CFTs and universality," JHEP **03** (2011), 114 [arXiv:1101.4937 [hep-th]].

[133] Y. Baba, N. Ishibashi and K. Murakami, "Light-Cone Gauge Superstring Field Theory and Dimensional Regularization," JHEP **10** (2009), 035 [arXiv:0906.3577 [hep-th]].
[134]

[134] W. Donnelly and G. Wong, "Entanglement branes in a two-dimensional string theory," JHEP **09**, 097 (2017) [arXiv:1610.01719 [hep-th]].

[135] Z. Komargodski, "The S-matrix Bootstrap program revived", Lecture on YouTube.

[136] G. Veneziano, "Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories," Nuovo Cim. A **57** (1968), 190-197.

[137] R. Dolen, D. Horn and C. Schmid, "Finite energy sum rules and their application to pi N charge exchange," Phys. Rev. **166** (1968), 1768-1781.

[138] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 1: Introduction".

[139] H. B. Meyer and M. J. Teper, "Glueball Regge trajectories and the pomeron: A Lattice study," Phys. Lett. B **605** (2005), 344-354 [arXiv:hep-ph/0409183 [hep-ph]].

[140] X. O. Camanho, J. D. Edelstein, J. Maldacena and A. Zhiboedov, "Causality Constraints on Corrections to the Graviton Three-Point Coupling," JHEP **02** (2016), 020 [arXiv:1407.5597 [hep-th]].

[141] S. Caron-Huot, Z. Komargodski, A. Sever and A. Zhiboedov, "Strings from Massive Higher Spins: The Asymptotic Uniqueness of the Veneziano Amplitude," JHEP **10** (2017), 026 [arXiv:1607.04253 [hep-th]].

[142] M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees and P. Vieira, "The S-matrix bootstrap. Part I: QFT in AdS," JHEP **11** (2017), 133 [arXiv:1607.06109 [hep-th]].

[143] M. Correia, A. Sever and A. Zhiboedov, "An Analytical Toolkit for the S-matrix Bootstrap," [arXiv:2006.08221 [hep-th]].

[144] S. Mizera, "Crossing symmetry in the planar limit," Phys. Rev. D **104** (2021) no.4, 045003 [arXiv:2104.12776 [hep-th]].

[145] N. Arkani-Hamed, T. C. Huang and Y. T. Huang, "The EFT-Hedron," [arXiv:2012.15849 [hep-th]].

[146] D. D. Coon, "Uniqueness of the veneziano representation," Phys. Lett. B **29** (1969), 669-672.

[147] D. B. Fairlie and J. Nuyts, "A fresh look at generalized Veneziano amplitudes," Nucl. Phys. B **433** (1995), 26-40 [arXiv:hep-th/9406043 [hep-th]].

[148] Sakurai, J., Napolitano, J., Modern Quantum Mechanics (2nd ed.). Cambridge: Cambridge University Press, 2017.

[149] N. Arkani-Hamed, L. Eberhardt, Y. t. Huang and S. Mizera, "On Unitarity of Tree-Level String Amplitudes," [arXiv:2201.11575 [hep-th]].

[150] M. Baker and D. D. Coon, "Dual resonance theory with nonlinear trajectories," Phys. Rev. D **2**, 2349-2358 (1970)

[151] D. D. Coon, U. P. Sukhatme and J. Tran Thanh Van, "DUALITY AND PROTON PROTON SCATTERING AT ALL ANGLES," Phys. Lett. B **45**, 287-291 (1973)

[152] I. Ridkokasha, "Confronting Dual Models of the Strong Interaction," Mod. Phys. Lett. A **36**, no.05, 2150031 (2021) [arXiv:2008.11418 [hep-ph]].

[153] F. Figueroa and P. Tourkine, "Unitarity and Low Energy Expansion of the Coon Amplitude," Phys. Rev. Lett. **129**, no.12, 121602 (2022) [arXiv:2201.12331 [hep-th]].

[154] N. Geiser and L. W. Lindwasser, "Properties of infinite product amplitudes: Veneziano, Virasoro, and Coon," JHEP **12**, 112 (2022) [arXiv:2207.08855 [hep-th]].

[155] J. Maldacena and G. N. Remmen, "Accumulation-point amplitudes in string theory," JHEP **08**, 152 (2022) [arXiv:2207.06426 [hep-th]].

[156] Z. Bern, D. Kosmopoulos and A. Zhiboedov, "Gravitational effective field theory islands, low-spin dominance, and the four-graviton amplitude," J. Phys. A **54**, no.34, 344002 (2021) [arXiv:2103.12728 [hep-th]].

[157] Y. t. Huang and G. N. Remmen, "UV-complete gravity amplitudes and the triple product," Phys. Rev. D **106**, no.2, L021902 (2022) [arXiv:2203.00696 [hep-th]].

[158] L. Y. Chiang, Y. t. Huang, W. Li, L. Rodina and H. C. Weng, "(Non)-projective bounds on gravitational EFT," [arXiv:2201.07177 [hep-th]].

[159] M. Chaichian, J. F. Gomes and P. Kulish, "Operator formalism of q deformed dual string model," Phys. Lett. B **311**, 93-97 (1993) [arXiv:hep-th/9211029 [hep-th]].

[160] L. L. Jenkovszky, M. Kibler and A. Mishchenko, "Two parametric quantum deformed dual amplitude," Mod. Phys. Lett. A **10**, 51-60 (1995) [arXiv:hep-th/9407071 [hep-th]].

[161] D. D. Coon and S. Yu, "Dual Four Point Functions with No Negative Residues," Phys. Rev. D **10**, 3780 (1974)

[162] N. Arkani-Hamed, L. Eberhardt, Y. t. Huang and S. Mizera, "On unitarity of tree-level string amplitudes," JHEP **02**, 197 (2022) [arXiv:2201.11575 [hep-th]].

List of publications relevant to the thesis

- M. R. Gaberdiel, R. Gopakumar, B. Knighton and P. Maity, "From symmetric product CFTs to AdS_3," JHEP 05 (2021) 073 [arXiv:2011.10038 [hep-th]].
- F. Bhat, R. Gopakumar, P. Maity and B. Radhakrishnan, "Twistor coverings and Feynman diagrams," JHEP 05 (2022) 150 [arXiv:2112.05115 [hep-th]].
- S. Komatsu, P. Maity, "String Dual of Two-dimensional Yang-Mills and Symmetric Product Orbifold" (*to appear*).
- P. Maity, "Positivity of the Veneziano amplitude in D = 4," JHEP 04 (2022) 064 [arXiv:2110.01578 [hep-th]].
- J. Chakravarty, P. Maity and A. Mishra, "On the positivity of Coon amplitude in D = 4," JHEP 10 (2022) 043 [arXiv: 2208.02735 [hep-th]].