On some aspects of Dijkgraaf-Witten theory for finite 2-groups

A thesis submitted to the
Tata Institute of Fundamental Research
for the degree of Doctor of Philosophy
in Mathematics
by

Srikanth Pai
International Centre for Theoretical Sciences
Tata Institute of Fundamental Research
Bengaluru 560 089, India

November, 2023

Final version submitted in November, 2023

DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Pranav Pandit at the International Centre for Theoretical Sciences, at Bengaluru, of the Tata Institute of Fundamental Research, Mumbai.

30/11/23

Srikanth Pai B

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Pranav Pandit

Date: November 30, 2023

Abstract

Dijkgraaf-Witten's construction of a topological quantum field theory (DW TQFT) for finite gauge groups gives us connections between the representation theory of finite groups and manifold invariants. Just like groups arise as symmetries of sets, 2-groups arise as symmetries of categories. Topologically, finite 2-groups can be identified with connected homotopy 2-type whose homotopy groups are finite.

A d-dimensional extended TQFT Z^d is a symmetric monoidal 2-functor from a symmetric monoidal bicategory of d-dimensional bordisms. It is well known that symmetric monoidal bicategories are challenging to construct directly because of the definition's many coherences and coherence laws. An indirect approach, following Shulman, is to work with symmetric monoidal double categories. This thesis proves that a 2-category of isofibrant pseudo double categories is biequivalent to a bicategory of isofibrant Segal categories. An analogous statement for symmetric monoidal double categories is conjectured. The main idea is that working with Segal conditions reduces the number of coherence checks.

Given a CW complex X, a d-extended DW TQFT Z_X^d is defined by factoring through a symmetric monoidal bicategory of bispans, i.e. $Z_X^d = Q_X \circ \mathcal{F}_X^d$. In physics parlance, the TQFT is obtained by quantizing (Q_X) a classical field theory ($\mathcal{F}_X^d$). The central result of this thesis is the characterization of the 2-functors out of the bicategory of bispans (in a given category $\mathcal{C}$) by a universal property. A stronger statement involving a symmetric monoidal structure will construct a DW TQFT. Roughly, the universal property is that a pair of functors out of $\mathcal{C}$ satisfying a higher dimensional version of the well-known Beck-Chevalley (BC) conditions induce a 2-functor out of the bicategory of bispans in $\mathcal{C}$. The thesis also discusses an example of a pair of functors satisfying BC conditions, which induces a 2-functor $\tilde{Q}$, from the bicategory of bispans of groupoids to the bicategory of linear categories, previously considered by Morton.

A finite 2-group is a connected 2-type X with finite homotopy groups. The functor $\mathcal{F}_X^d = Maps(\_, X)$, on k-manifolds for $d-2 \leq k \leq d$, induces a d-extended DW TQFT. It is shown that the path groupoid of $\mathcal{F}_X^3(S^2)$ is the action groupoid for the standard action of $\pi_1(X,*)$ on $\pi_2(X,*)$. The path groupoid of $\mathcal{F}_X^d(S^1)$ is also described for $d \in \{1,2\}$. Morton's 2-functor $\tilde{Q}$ is evaluated on these groupoids, and the evaluations can be interpreted as the assignment of an extended DW TQFT on the spheres.

Acknowledgements

First and foremost, I am deeply grateful to my advisor, Pranav Pandit. My present understanding of modern mathematics, including all the ideas in this thesis, is entirely due to him. His ideas and perspectives have rejuvenated my mathematical spirit, and I have a newfound admiration for research and abstraction. His support, candour and care have helped me stay strong throughout my PhD.

I am grateful to the thesis committee members, Siddhartha Gadgil, Rukmini Dey, Joseph Samuel, Carlos Simpson, Samik Basu and Vishal Vasan. I thank all of them for giving me their valuable time, helpful comments and encouragement. Carlos Simpson's encouraging review of my thesis and insightful questions during my viva have had a very motivating effect on me.

I thank Abhishodh Prakash for nurturing my interests in TQFTs and being an enthusiastic mentor. I thank Lakshya Bhardwaj, Naveen Prabhakar, and Aswin Balasubramanian for enlightening discussions. I also thank the ICTS faculty, Abhishek Dhar and Loganayagam, for their outstanding teaching, moral support, and patience.

I deeply owe my colleague Bhanu Kiran S for incredibly influencing me. His hard work and discipline have inspired my work ethic, and his caring persona and helping nature have inspired me to be a better person. He is a kindred soul, a sounding board, a humble intellectual and, above all, a fantastic friend. I thank Kohinoor Ghosh and Varun Dubey, who kept me ecstatic by discussing excellent mathematics in the initial days at ICTS.

I cannot imagine ICTS without the sweet, intelligent and caring batchmates Sugan, Aswin, Pronobesh, Arnab, Pinak, Bhawik, Monica, Shashank, Prashant, Vijay, and Soummyadip. Apart from my batchmates, my life at ICTS was memorable due to Akhil, Junaid, Joydeep, Manisha, Sumith and Santhosh. I thank my colleagues Mayank, Chetan, Adittya, Prokash, Arideep, Poorna, Anantadulal, Priyadarshini, Himanshu, and Viswanathan for the fun discussions and support.

The unwavering faith and unending love of Amma and Aanu are the source of my energy and confidence. I am deeply indebted to them for their support and encouragement. I am grateful to my love, Darlu, for patiently listening to my troubles, teaching me to focus and manage time, being mindful of my health, and being the most fantastic sweetheart.

I am also very grateful to the incredible support and warmth from Jeeva, Arshitha, Veena, Nidhi, Ramya, Jenny, Basavaraj, Mahendra, Suresh and Anuradha.

Dedicated to Amma, Aanu and Darlu.

List of Figures

1.1 A figure depicting the relation between spans. 11
1.2 A figure depicting a pullback diagram used for composing spans. 11
1.3 Classical Field theory on bordisms 13
1.4 Manifold with corners whose boundaries are red and blue manifolds with boundaries. The green manifolds are corners of the original manifolds. . 18
1.5 The manifolds with corners cut in a cubical way can be glued in two independent directions. 19
1.6 A two-cell: α is a 2-morphism, M, N are horizontal morphisms, f, g are vertical morphisms and the rest are objects. 20
1.7 Two step TQFT: Extended classical field theory $\mathcal{F}$ followed by quantization Q. 28

2.1 A fundamental morphism of a pseudo double category. 62
2.2 The lifting criteria diagram associated with the (iso)fibration (S, T). . . 63
2.3 A fundamental morphism of the pseudo double category span(C) is a commuting diagram. 65
2.4 A lifting diagram for the Morita category Mor. 66
2.5 A lift of a morphism showing that source and target are fibrations for the pseudo double category HB. 66
2.6 A map of comparisons. 87
2.7 A simplified map of comparisons. 87

3.1 2-morphisms in a span bicategory and a bispan bicategory. 90

3.2 An object of $P_s([2])$ from Construction 3.2.12. 107

3.3 The boundaries of F from Figure 3.2 108

3.4 A morphism of $P_s([2])$ between black colored F and blue colored G is shown in green. 108

3.5 A 2-cell in the double category of bispans from Construction 3.2.14. 109

4.1 The columns are the Beck-Chevalley composable morphism, the computation in the category of bispan and a simplification. 124

4.2 Computation of the long path of HBC commutative diagram 129

4.3 Vertical 2-functoriality constraint 131

4.4 Horizontal 2-functoriality constraint 131

4.5 Adjunction moves 135

4.6 Coherence isomorphism of a 2-functor 135

4.7 Two conditions denoting invertibility of functorial coherences 136

4.8 Functorial coherence laws 136

4.9 Breaking pushforward and pullback of identity morphism 137

4.10 Beck-Chevalley morphism $BC[\sigma_*]$ 138

4.11 The partly special ambijunction condition gives a move where the circle on left hand side can be deleted. 138

4.12 Vertical BC condition 138

4.13 Horizontal BC condition 139

4.14 String diagram for g_*f^* 140

4.15 String diagram for 2-morphism assigned to F 141

4.16 Composition coherence of F 141

4.17 Well definedness of the 2-functor 142

4.18 Proof of F preserves identity 2-morphisms. 143

4.19 Vertical composition of 2-cells 145

4.20 Final step of the proof of Proposition 4.3.4 146

4.21 Proposition 4.3.6 147

4.22 Proof of Proposition 4.3.6 147

4.23 Naturality of the composition coherence 149

4.24 Naturality check: Step 1. 149

4.25 Naturality check: Step 2. 150

4.26 Naturality check: Step 3. 151

4.27 Naturality check: Step 4. 152

4.28 Functorial coherence axioms checked in Propositions 4.3.9 and 4.3.8 . . . 153

4.29 Step 1 of the composition coherence 2-morphism on RHS of the first equality in Figure 4.28. 154

4.30 Step 2 of the composition coherence 2-morphism on RHS of the first equality in Figure 4.28. 155

4.31 Step 3 of the composition coherence 2-morphism on RHS of the first equality in Figure 4.28. 156

4.32 Step 4 of the composition coherence 2-morphism on RHS of the first equality in Figure 4.28. 157

4.33 Step 1 of identity axiom check 158

4.34 Step 2 of the identity axiom check 158

4.35 Identity Coherence Axiom. 159

5.1 Three equivalent descriptions of 2-groups. 162

5.2 Composition for G_2 in Construction 5.1.18 170

5.3 Action of G_2 on G_1 in Construction 5.1.18 171

5.4 LES of pairs. The red arrow is part of the data of a Whitehead crossed module. 172

Contents

Abstract v

1 Introduction 1

1.1 Preface 1

1.1.1 On the categorical formalism for quantum field theory 2

1.1.2 Quantization of a classical field theory 9

1.1.3 Once extended field theories 15

1.1.4 Grothendieck's Homotopy Hypothesis 23

1.1.5 Classical 2-group gauge theory 25

1.1.6 Quantization 27

1.1.7 Dijkgraaf-Witten theory 29

1.2 Objectives 31

1.3 Results 34

1.3.1 Results of Chapter 2 34

1.3.2 Results of Chapter 3 35

1.3.3 Results of Chapter 4 36

1.3.4 Results of Chapter 5 37

1.4 Related Works 38

List of Symbols 43

1.5 Notations 43

1.6 Conventions 44

1.7 Prerequisites and Warnings 45

2 Segal Fibrations and Double Categories **47**

2.1 Preliminaries 50

2.1.1 The categories $\triangle$, Γ 51

2.1.2 Bicategories 52

2.1.3 Fibrations 55

2.1.4 Nerve of a category 58

2.1.5 Double categories 61

2.1.6 Symmetric Monoidal Double categories 67

2.1.7 Nerve of a strict double category 68

2.2 Segal Categories 70

2.2.1 Biequivalence of Segal categories and pseudo double categories . 71

2.3 Segal Fibrations over $\triangle$ 74

2.3.1 Pinched Segal Fibrations 75

2.4 Segal fibration over $\triangle \times \Gamma$ 80

2.4.1 Strict symmetric monoidal bicategories 82

2.4.2 Construction of symmetric monoidal bicategories from Segal fibrations 86

3 The bicategory of bispans **89**

3.1 Preliminaries 90

3.2 Spans and Bispans as Segal Fibrations 93

3.2.1 Segal fibration over $\triangle$ of spans in a category 94

3.2.2 Segal fibration of spans of groupoids 96

3.2.3 Fibration of bispans of groupoids 102

3.2.4 Double categories of spans and bispans 105

3.3 Miscellaneous properties of bispans 111

4 Double Beck Chevalley condition 113

4.1 Hierarchy of Spans 116

4.2 Basic moves 134

4.2.1 String diagram notations 134

4.2.2 Ambijunction and Beck-Chevalley moves 137

4.3 Proof of the main theorem 139

5 Applications to gauge theories 161

5.1 Preliminaries 162

5.1.1 Model categories 163

5.1.2 Homotopy theory of 2-groupoids and 2-types 165

5.1.3 Crossed Modules 168

5.1.4 Mapping space of crossed modules 171

5.1.5 An example of a DBC functor data 174

5.2 Computations 181

5.2.1 Groupoid presentations of certain mapping spaces 181

5.2.2 Values of the tentative DW theory 185

Bibliography 189

Chapter 1

Introduction

1.1 Preface

Given a finite set S, let $\#S$ denote the number of elements in S and given a connected topological space Σ, let $\pi_1\Sigma$ denote the fundamental group of Σ at some base point. It is well known that for a compact manifold Σ, the fundamental group $\pi_1(\Sigma)$ is finitely generated. Thus, given a finite group G, the set of group homomorphisms $\mathrm{Hom}(\pi_1(\Sigma), G)$ is finite.

In 1906, Frobenius and Schur computed a formula for the number of elements in $\mathrm{Hom}(\pi_1(\Sigma), G)$ when Σ is a non-orientable surface. In 1978, Mednykh computed a formula for the number of elements in this set when Σ is an orientable surface (in [46]). Elements of the representation theory of finite groups feature in both formulae. The Mednykh formula is stated in the following theorem.

Theorem 1. *Given a finite group G, let $Irrep(G)$ denote the set of irreducible representations of G. Let a closed 2-dimensional oriented manifold Σ of genus g be given. Then*

$$\frac{\#Hom(\pi_1\Sigma, G)}{\#G} = \sum_{V \in Irreps(G)} \left(\frac{dim\, V}{\#G}\right)^{2-2g},$$

where $dim(V)$ represents the dimension of the representation V.

Mednykh's proof of this formula used classical Riemann surface theory. This formula is a relation between representation theory and manifold invariants. Interesting facts about representation theory can be deduced by plugging in various specific manifolds. Choosing the Riemann surface as the sphere and the torus gives well-known facts about the dimension of the regular representation and the number of irreducible representations of G, respectively. New proofs of special cases of the Mednykh formula were found by computations of partition functions in quantum field theory [14][19]. A discussion of the formulae in non-orientable and orientable cases, due to Frobenius-Schur and Mednykh, respectively, is given in [61].

Generalization of the Mednykh formula and its interaction with topological quantum field theories is one of the primary motivations of this thesis. The results of this thesis are generalizations of a TQFT due to Dijkgraaf and Witten and computations presented by Freed, Freed-Quinn. The subsections of Section 1.1 introduce all the actors required to understand the objectives of this thesis (found in Section 1.2).

In Subsection 1.1.1, the definition of a TQFT, due to Atiyah-Segal-Kontsevich, is motivated from their physical origins. In this definition, categories are used to model glueing of manifolds with boundaries. Similarly, double categories model glueing of manifolds with (cubical) corners and the notion of a once extended field theory is discussed in Subsection 1.1.3. In Subsection 1.1.4, three presentations of groupoids are discussed. In Subsection 1.1.5 and Subsection 1.1.6, proposals of 2-group gauge theories and quantization of 2-group gauge theories are sketched. In Subsection 1.1.7, Dijkgraaf-Witten TQFT is discussed and its value on a circle are derived.

The discussions in Subsections 1.1.3, 1.1.5 and 1.1.6 are imprecise and tentative in nature since it is the subject matter of this thesis and only the motivations are discussed in this section. The precise contributions of this thesis are discussed in Section 1.3. The materials in the rest of the subsections are well-known.

1.1.1 On the categorical formalism for quantum field theory

The path integral formalism dates back to Feynman's ideas surrounding the 'sum over histories method'. In this method for constructing TQFTs, a phase (i.e. an element of $U(1)$) associated with the functional S (in units of $\hbar$), written as e^{iS} is associated with every trajectory of a particle moving on a given manifold. The probability amplitude of a transition from an initial state to a final state is given by integrating the phase over all possible trajectories with given initial and final states. A quantum field theory in the path integral formalism is similarly constructed by choosing a space (of fields on a given manifold), a functional on the space (the action) and performing a "path integral" on the exponentiated action over the space of fields given the boundary conditions. The path integral computed on a given manifold is a complex number, and it is called the "partition function" of the theory. A rigorous theory for path integral formalism is not available in general. Still, it has infused fresh ideas into geometry and topology. We will review the case that is the focus of this thesis: Topological quantum field theories.

In the '90s, Witten explained that the Jones polynomial for knots arises from computations in quantum gauge theory constructed from a compact semisimple Lie group (the famous Chern-Simons theory, see [70]). Since this field theory did not depend on the metric and depended, only[1] on the underlying smooth structure of the given manifold, the quantum field theory was called "topological". In [1], Atiyah noted that various new developments (due to Donaldson, Floer, Gromov, Witten and others) in the study of manifold invariants were motivated by examples of topological quantum field theories. So, he proposed a definition for topological quantum field theories (TQFT) (inspired by Segal's definition of a conformal field theory, see [56]).

Let us see how to arrive at categorical definition starting from the path integral formalism. Let M denote the d-dimensional closed manifold of interest and $\mathcal{F}(M)$ denote the space of relevant classical fields on M. For the purposes of this subsection, the space of classical fields $\mathcal{F}$ is defined as follows:

Definition 1.1.1. Pre-classical fields is a (pseudo)functor $\mathcal{F} : \mathrm{Man}^{op} \to \mathcal{C}$, where Man denotes the category whose objects are smooth manifolds with boundaries and corners, morphisms are smooth maps, and $\mathcal{C}$ is a (higher) category.

Example 1.1.2. Consider the example of "gauge fields" $\mathcal{F}_G$ for a finite group G. Here $\mathcal{C}$ is the bicategory of groupoids. For a manifold M, $\mathcal{F}_G(M)$ is the groupoid of principal G-bundles on M.

Example 1.1.3. Fix a CW complex X and let $Maps$ denote the homotopy type of the mapping space between CW complexes. Define $\mathcal{F}_X(M) = Maps(M, X)$. Here $\mathcal{C}$ is a a category of spaces. We could also get a groupoid by defining $\mathcal{F}_X(M) = \pi_{\leq 1}(Maps(M, X))$.

The action S is a functional on $\mathcal{F}(M)$ i.e. $S : \mathcal{F}(M) \to \mathbb{R}$. The path integral Z can be written schematically as

$$Z(M) = \int_{\phi \in \mathcal{F}(M)} e^{iS(\phi)} \text{``}d\phi\text{''}.$$

The integral is schematic because $d\phi$ may not be a measure. The exponentiated action is an element of the group $U(1)$. One embeds the group $U(1) \hookrightarrow \mathbb{C}$ and then sums up the complex numbers. For a closed manifold M,

$$Z(M) \in \mathbb{C}.$$

---

[1] up to a choice of the framing of the manifold.

The quantum field theory we will consider is topological[2] and hence Z only depends on the underlying smooth structure.

One of the fundamental principles of quantum field theories is *locality*. Roughly speaking, locality in QFTs is the idea that physical phenomena in a region can be explained by fields supported on a region's neighbourhood. We follow the exposition in [19] where locality is implemented for TQFTs by demanding that the (action) functional S satisfy a criteria we currently discuss. Given manifolds M_1 and M_2, since $\mathcal{F}$ is a functor, we have the canonical map

$$\mathcal{F}(M_1 \amalg M_2) \to \mathcal{F}(M_1) \times \mathcal{F}(M_2) \tag{1.1.1}$$

obtained by evaluating the functor $\mathcal{F}$ on the structure maps of the coproduct $M_1 \amalg M_2$ (in the category Man). Let us write the map as $\phi \mapsto (\phi_1, \phi_2)$. The functional S is *local* if it is of the form $S(\phi) = S_1(\phi_1) + S_2(\phi_2)$. Tentatively, in a "local" classical field theory determined by a pair $(S, \mathcal{F})$, the functional S is local and the map in Equation 1.1.1 is an isomorphism/equivalence. In fact, we impose that $\mathcal{F}$ is a symmetric monoidal functor with the cartesian monoidal structure for Man^{op} and a symmetric monoidal category $\mathcal{C}$ as a rigorous axiom for locality. So we have the following definition:

Definition 1.1.4. The classical fields functor $\mathcal{F} : (\mathrm{Man}^{op}, \amalg) \to (\mathcal{C}, \otimes)$ is a symmetric monoidal functor.

In the case of TQFTs, this will mean the path integral satisfies

$$Z(M_1 \amalg M_2) = Z(M_1)Z(M_2).$$

In this thesis, we will *only* consider the following two-step method for constructing TQFTs: In the first step, we consider the space of classical fields on a given manifold where $\mathcal{F}$ is a symmetric monoidal functor. In the second step, we quantize the theory by computing the path integral over the space of fields. We revisit this two-step process precisely at the end of this subsection.

So in the simplest form, a TQFT Z assigns a complex number to each isomorphism class of manifolds. This definition is quite common in the literature, and state sum models often produce the manifold invariant $Z(M)$ for each manifold M.

Now we discuss the glueing properties of a TQFT, which leads to the Atiyah-Segal-Kontsevich definition of a functorial form of a TQFT. Classical fields $\mathcal{F}$ on a manifold

---

[2] A topological quantum field theory is a theory that does not depend on the metric or conformal structure on the manifold.

M with boundary ∂M is the data of a tuple $(M, \partial M, |_{\partial M} : \mathcal{F}(M) \to \mathcal{F}(\partial M))$. The elements of $\mathcal{F}(M)$ will be called "bulk fields", the elements of $\mathcal{F}(\partial M)$ will be called "boundary fields" and the map $|_{\partial M} : \mathcal{F}(M) \to \mathcal{F}(\partial M)$ will be called a "restriction". The boundary fields are fixed to compute the path integral, and the path integral is performed over all the bulk fields restricted to the given boundary fields. For a given boundary field $\phi \in \mathcal{F}(\partial M)$, the space of bulk fields with boundary field ϕ is given by

$$\mathcal{F}(M, \phi) = \{\psi \in \mathcal{F}(M) | \psi|_{\partial M} = \phi\}.$$

The path integral, in this case, is

$$Z(M, \phi) = \int_{\psi \in \mathcal{F}(M,\phi)} e^{iS(\psi)} \text{``}d\psi\text{''}.$$

The assignment $Z(M) := Z(M, -)$ can be viewed as a functional on the space $\mathcal{F}(N)$ where $N \simeq \partial M$. This space of functionals $\mathbb{C}^{\mathcal{F}(N)}$ forms a vector space which we denote by $Z(N)$. This vector space is called the *space of states* on N. Thus, for every manifold M with boundary, $Z(M) \in Z(\partial M)$.

Given two manifolds M_1, M_2 with a boundary diffeomorphic to N, we have two states $Z(M_1)$ and $Z(M_2)$ in $Z(N)$. Now, suppose we can glue the manifolds along the boundary to get a closed manifold $M = M_1 \coprod_N M_2$. The path integral $Z(M)$ is computed by "summing over the histories":

$$Z(M) = \int_{\psi \in \mathcal{F}(N)} \text{``}d\psi\text{''}\, Z(M_1, \psi) Z(M_2, \psi) = \langle Z(M_1), Z(M_2) \rangle$$

Thus, we get a bilinear form on $Z(N)$, which can be an inner product. Further, the above argument also illustrates that this bilinear form can be used to compute $Z(M)$ in a different way, i.e. by cutting M into M_1 and M_2, computing Z on the pieces and then pairing the vectors using the bilinear form.

We have seen that the path integral assigns a Hilbert space

$$Z(M) = \mathbb{C}^{\mathcal{F}(M)}$$

to a closed $(d-1)$-dimensional manifolds M. Given a manifold N with in-boundary M_1 and out-boundary M_2, and boundary fields ψ_1, ψ_2, the path integral assignment

$$Z(N, \psi_1, \psi_2) = \int\limits_{\psi \in \mathcal{F}(N, \psi_1, \psi_2)} e^{iS(\psi)} \text{``}d\psi\text{''}$$

can be considered a matrix element indexed by fields on M_1 and M_2. Here $\mathcal{F}(N, \psi_1, \psi_2)$ is the space of all fields on N that restrict to ψ_1 on M_1 and restrict to ψ_2 on M_2. In other words, we have a linear transformation $Z(N) : Z(M_1) \to Z(M_2)$ given by

$$(Z(N)(f))(\phi_2) = \int_{\psi_1 \in \mathcal{F}(M_1)} \text{``}d\psi_1\text{''} f(\psi_1) \int_{\psi \in \mathcal{F}(N,\psi_1,\psi_2)} e^{iS(\psi)} \text{``}d\psi\text{''}$$

A manifold N with in-boundary M_1 and out-boundary M_2 is called a "bordism" from M_1 to M_2. Further suppose M_1 and M_2 are a pair of $(d-1)$-dimensional manifolds then, from the discussion on locality, the space of fields on $M_1 \coprod M_2$ is $\mathcal{F}(M_1) \times \mathcal{F}(M_2)$. The space of functionals is tentatively

$$Hom(\mathcal{F}(M_1) \times \mathcal{F}(M_2), \mathbb{C}) \simeq \mathbb{C}^{\mathcal{F}(M_1)} \otimes \mathbb{C}^{\mathcal{F}(M_2)} \simeq Z(M_1) \otimes Z(M_2).$$

This is a schematic argument since we have not specified the nature of the "space of fields" throughout the discussion.

Remark 1.1.5. Let us summarise the previous four paragraphs. A d-dimensional topological quantum field theory (TQFT) Z provides the following assignments.

1. Given a closed oriented d-manifold M, $Z(M)$ is a complex number.
2. Given a closed oriented $(d-1)$-manifold N, $Z(N)$ is a vector space.
3. Given a compact oriented manifold N with boundary $\partial N \simeq M$, $Z(N)$ is a vector in $Z(\partial N)$.
4. For a closed oriented $(d-1)$-manifold N, the vector space $Z(N)$ has an inner product.
5. Given two closed oriented $(d-1)$-manifolds N_1 and N_2 and bordism $M : N_1 \to N_2$, there is a linear transformation $Z(M) : Z(N_1) \to Z(N_2)$.
6. Given a disjoint union of oriented manifolds $N_1 \amalg N_2$,

$$Z(N_1 \amalg N_2) = Z(N_1) \otimes Z(N_2).$$

Table 1.1 presents this list using pictures.

The observations in Remark 1.1.5 follows from a precise definition of a TQFT. Before the definition can be stated, some preparations are needed. A diagram in the category

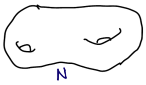
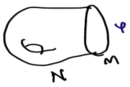
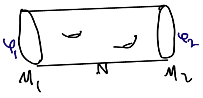
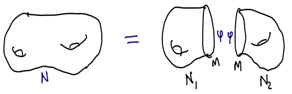
Picture	Fields	Path integral
	$\mathcal{F}(N)$	$Z(N) = \int_{\phi \in \mathcal{F}(N)} e^{iS(\phi)}$ “$d\phi$”
	$\mathcal{F}(N,\phi) \subseteq \mathcal{F}(N)$	$Z(N,\phi) = \int_{\psi \in \mathcal{F}(N,\phi)} e^{iS(\psi)}$ “$d\psi$”
	$\mathcal{F}(N,\phi_1,\phi_2) \subseteq \mathcal{F}(N)$	$Z(N,\phi_1,\phi_2) = \int_{\psi \in \mathcal{F}(N,\phi_1,\phi_2)} e^{iS(\psi)}$ “$d\psi$”
	$\mathcal{F}(M)$	$\langle Z(N_1), Z(N_2)\rangle = \int_{\phi \in \mathcal{F}(M)}$ “$d\phi''Z(N_1,\phi)Z(N_2,\phi)$

TABLE 1.1

of manifolds

$$\begin{tikzcd} & \Sigma & \\ M_1 \arrow[ur, "i_1"] & & M_2 \arrow[ul, "i_2"'] \end{tikzcd}$$

is a bordism if Σ is a manifold with boundaries and the maps i_1, i_2 provide a decomposition of the boundary, i.e. we have a diffeomorphism

$$\partial\Sigma \simeq M_1 \amalg M_2.$$

First we discuss the category $(Bord^{d,d-1}, \amalg)$. The objects of this category are closed, oriented $(d-1)$-manifolds. If N is an oriented manifold, then let $\overline{N}$ denote the same smooth manifold but equipped with the opposite orientation.

Definition 1.1.6. Let M be a d-dimensional compact oriented manifold with boundary with a specified diffeomorphism

$$\partial M \simeq N_1 \amalg \overline{N_2},$$

then M is a *bordism* from N_1 to N_2.

We will denote this situation by $M : N_1 \to N_2$. The diffeomorphism classes of bordisms from N_1 to N_2 (relative to the boundaries) are the morphisms from N_1 to N_2 in $(Bord^{d,d-1}, \amalg)$.

Given bordisms $M_1 : N_1 \to N_2, M_2 : N_2 \to N_3$, we can glue the manifolds along the common boundary N_2. Topologically it is the pushout $X := M_1 \amalg_{N_2} M_2$. To induce a manifold structure, we have to be able to give coordinates at all points. On all points of the pushout that are not in the image of N_2 (under the canonical map of the pushout) come from $X \backslash N_2 := (M_1 \backslash \partial M_1) \amalg (M_2 \backslash \partial M_2)$. Since $M \backslash \partial M$ is an open subset of M, there is a natural choice of coordinates around every point of M, and thus we have coordinates on points of $X \setminus N_2$. Now it remains to describe the coordinates in neighbourhoods of points in the image of N_2 If we choose a "collar" around the boundaries (i.e. an open neighbourhood U of M_2 in N_1 that is diffeomorphism to $M_2 \times [0, 1)$), then clearly there is a way to assign coordinates to the points of M_2 in $N_1 \amalg_{M_2} N_2$. The existence of collars is guaranteed by Brown's collaring theorem (see [8]), and it turns out that different choices give rise to the same diffeomorphism class of manifolds. The composition in the category $(Bord^{d,d-1}, \amalg)$ is defined by glueing bordism along the common boundary.

The disjoint union of manifolds $\amalg$ is the monoidal structure, and the empty manifold is the monoidal unit.

The codomain category is the category of complex vector spaces and linear transformations. The monoidal structure is the tensor product of vector spaces, and the monoidal unit is vector space $\mathbb{C}$.

We can summarise the situation in Table 1.1 succinctly using symmetric monoidal categories[3] following Atiyah-Segal-Kontsevich's definition [1].

Definition 1.1.7. A TQFT Z is a symmetric monoidal functor

$$Z : (Bord^{d,d-1}, \amalg) \to (Vect_{\mathbb{C}}, \otimes)$$

Now note that the TQFT Z has upgraded to a functor (from a function that was a manifold invariant). The functoriality captures the compatibility of the partition function with gluing manifolds. This gives us greater flexibility in computing the manifold invariant. The functoriality of TQFT corresponds to the locality of the theory in physics.

The definition of a symmetric monoidal functor recovers the list in Remark 1.1.5:

1. The TQFT Z is a functor and thus assigns objects to objects. So a vector space $Z(N)$ (of states) is assigned to a $(d-1)$-dimensional closed, oriented manifold N.

---

[3] Symmetric monoidal categories are categories equipped with a commutative multiplication. It's like an Abelian monoid in categories. A symmetric monoidal functor is like a homomorphism of abelian monoids. All the monoids have units, and monoid homomorphism should map units to units.

2. The TQFT Z is a functor and thus assigns morphisms to morphisms. So a linear transformation $Z(M) : Z(N_1) \to Z(N_2)$ is assigned to a bordism $M : N_1 \to N_2$.

3. If M is a manifold with boundary N, then we can view it as a bordism $M : e \to N$ where e is the empty manifold. Since $Z(e) \simeq \mathbb{C}$, we have a linear transformation $Z(M) : \mathbb{C} \to Z(N)$, which is determined by $Z(M)(1)$. Thus $Z(M)$ is a vector in $Z(N)$.

4. The monoidal part of the monoidal functor gives $Z(N_1 \amalg N_2) \simeq Z(N_1) \otimes Z(N_2)$, and $Z(\phi) = \mathbb{C}$.

5. A closed d-manifold M is a bordism from the empty manifold to itself (up to diffeomorphism). By the monoidal nature of the functor Z, the assignment $Z(M)$ is a morphism from $\mathbb{C}$ to itself and hence a complex number. So, each closed d-manifold (up to diffeomorphism) is assigned a complex number.

6. It turns out that $Z(\overline{N}) = Z(N)^\vee$.[4] If M_1, M_2 are manifolds with N as boundary (but opposite orientations), then we can compose $\overline{M_1} : e \to N$, $M_2 : N \to e$ in the bordism category. Since Z is a functor, it maps composites to composites. So we get a complex number by composing: $\langle Z(M_1), Z(M_2) \rangle$. This pairing is bilinear since composition in the category of vector spaces is bilinear.

The TQFT was constructed in a two-step process. Roughly speaking, first, we consider the space of classical fields on manifolds, then we quantize the space of fields. Can the categorical TQFT (i.e. Definition 1.1.7) also be factored into a product of two functors? If this is the case, what is the target of the Bordism category? We will answer these questions in the next subsection.

1.1.2 Quantization of a classical field theory

In this subsection, we define classical field theories and discuss the construction of TQFTs via quantizing a classical field theory. The notion of a span is the crucial ingredient used to define a classical field theory.

---

[4] The proof of this fact uses the symmetric monoidal structure on the categories and functor. More generally, it arises from the fact that tensor duals are mapped to tensor duals.

Definition 1.1.8. Let $\mathcal{C}$ be a category. A pair of morphisms (f, g) in $\mathcal{C}$ of the form $f : a \to b$ and $g : a \to c$ is called *a span* in $\mathcal{C}$. It is usually depicted in the following way:

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[dr, "g"] & \\ b & & c \end{tikzcd}$$

Spans are ubiquitous in mathematics. A relation R on a set X is a subset $R \subset X \times X$, and it can be viewed as a span $X \xleftarrow{a} R \xrightarrow{b} X$ where a, b are projections. Thus, a span is also called a *generalized relation* or a *correspondence* in the literature. For a category $\mathcal{C}$ with pullbacks, Benabou constructed a bicategory whose 1-morphisms where spans in $\mathcal{C}$ [6, §2.6]. We present a category whose morphisms are spans and are obtained by identifying isomorphic 1-morphisms in Benabou's construction.

Construction 1.1.9. Let $\mathcal{C}$ be a category with pullbacks. Define a symmetric monoidal category $(\mathrm{Span}(\mathcal{C}), \times)$ as follows:

- The set of objects of $\mathrm{Span}(\mathcal{C})$ is the set of objects of $\mathcal{C}$.
- Given two objects A, B in $\mathrm{Span}(\mathcal{C})$, a 1-morphism is a set of spans $(f : X \to A, g : X \to b)$ in $\mathcal{C}$ modulo the following relation: Spans $(f : X \to A, g : X \to b)$ and $(f' : X' \to A, g' : X' \to b)$ are related if there is an isomorphism $k : X \to X'$ such that $f'k = f$ and $g'k = g$. In other words, in Figure 1.1, we say the outer spans commute if a dashed isomorphism makes both triangles commute. We denote the morphism by $[(f, X, g)] : A \to B$ or $[(f, X, g)]$ for short.
- The identity morphism for an object A is the equivalence class of the span $[(id_A, A, id_A)]$. The composition of $[(f, X, g)] : A \to B$ and $[(h, Y, k)] : B \to C$ is defined as $[(\pi_1, X \times_B Y, \pi_2)]$ where the span is a part of a pullback diagram depicted in Figure 1.2. Even though a pullback is not unique, the composition is well-defined since a pullback is unique up to a unique isomorphism.
- The symmetric monoidal structure is given by product on objects. On morphisms, define the monoidal product of $[(f, X, g)] : A \to B$ and $[(h, Y, k)] : C \to D$ as $[(f \times h, X \times Y, g \times k)] : (A \times C) \to (B \times D)$.

Matrix multiplication can be motivated as a functor out of $\mathrm{Span}(FinSets)$ where $FinSets$ is the category of finite sets and maps between them.

Construction 1.1.10. Consider a span in finite sets $X \xleftarrow{a} R \xrightarrow{b} Y$ along with a function (called a *kernel*) $t : R \to k$ for some field k, R is a finite set and k^Z denotes the vector

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dashed, "k"] & \\
A & & B \\
& X' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

FIGURE 1.1: A figure depicting the relation between spans.

$$\begin{tikzcd}
& X \times_B Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\
& B &
\end{tikzcd}$$

FIGURE 1.2: A figure depicting a pullback diagram used for composing spans.

space of k-valued functions on Z. We can construct a linear transformation $T : k^X \to k^Y$ using the span and the function t in the following manner:

$$T(f)(y) = \sum_{r:b(r)=y} t(r)f(a(r))$$

Note that $R = X \times Y$ recovers matrix multiplication in the usual form.

We can recognize three steps in Construction 1.1.10: Start with a function $f \in k^X$

1. In the first step, we compose $f : X \to k$ with $a : R \to X$ to get

$$a^*f = f \circ a.$$

   This step, i.e. the map $a^* : k^X \to k^R$ is generally called "pullback".

2. The function $g : R \to k$ can be "pushed along b" to obtain a function $b_*g : Y \to k$. We do this by "integrating along the fibres". Given a $y \in Y$, the *fiber* of b over y is simply the preimage of y:

$$b^{-1}(y) = \{r \in R | b(r) = y\}$$

Define

$$b_*g(y) = \sum_{r\in b^{-1}(y)} t(r)g(r)$$

Note that the sum is well-defined since R is a finite set. The function t is called a *kernel*.

3. So we can briefly write

$$T(f) = b_*a^*f.$$

In summary, given a pair of functors $()^*, ()_* : \mathrm{FinSets} \to \mathrm{Vect}_k$ which agree on objects and the former is contravariant, we can define a "pull-push" operation on a span of finite sets as follows:

$$(X \xleftarrow{a} R \xrightarrow{b} Y) \mapsto (b_*a^* : k^X \to k^Y)$$

The above procedure is called "pull-push". The pushforward required a process of summation. The ubiquity of spans is tied to the ubiquity of push-pull operations in mathematics. Fourier transforms, Fourier-Mukai transform, Mackey functors are different examples of push-pull operations in mathematics [5].

In physics, pull-push arises as a way to quantize classical field theories. Typically, a quantum field theory is constructed by first constructing the classical field theory and then quantizing the classical fields.

First of all, what is a classical field theory?

Definition 1.1.11. Let $\mathcal{C}$ be a category with finite limits and let $(\mathrm{Span}(\mathcal{C}), \times)$ denote the symmetric monoidal category constructed in Construction 1.1.9. A $\mathcal{C}$-valued classical field theory C is a symmetric monoidal functor

$$C : (Bord^{d,d-1}, \amalg) \to (\mathrm{Span}(\mathcal{C}), \times).$$

So, let us discuss the example of a classical field theory that arises in physics. Let M be a manifold with boundary $\partial M \simeq N_1 \amalg N_2$ as shown on the left of Figure 1.3.

Let $\mathcal{F}$ denote a classical fields functor (see Definition 1.1.4). By restricting the fields on M to the boundary components, we get a span of spaces on the right of Figure 1.3. In fact, this induces a classical field theory $C_{\mathcal{F}}$ that assigns a manifold M, the space $\mathcal{F}(M)$. Next, we ask what is quantization?

---

[5] For a leisurely discussion, see Morton's n-category cafe entry: `https://golem.ph.utexas.edu/category/2010/11/integral_transforms_and_pullpu.html`

$$\begin{tikzcd} & M & \\ N_1 \arrow[ur, hook] & & N_2 \arrow[ul, hook'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{F}(M) \arrow[dl] \arrow[dr] & \\ \mathcal{F}(N_1) & & \mathcal{F}(N_2) \end{tikzcd}$$

Figure 1.3: Classical Field theory on bordisms

Definition 1.1.12. Let $\mathcal{C}$ be a category with finite limits and $(\mathrm{Span}(\mathcal{C}), \times)$ denote the symmetric monoidal category constructed in Construction 1.1.9. Let $(Vect_{\mathbb{C}}, \otimes)$ denote the symmetric monoidal category of complex vector spaces with the usual tensor product. A quantization functor Q is a symmetric monoidal functor

$$Q : (\mathrm{Span}(\mathcal{C}), \times) \to (Vect_{\mathbb{C}}, \otimes).$$

A functor $\tilde{Q} : \mathrm{Span}(\mathcal{C}) \to Vect_{\mathbb{C}}$ is called a *prequantization* functor.

The matrix multiplication discussion in Construction 1.1.10 gives an example of a prequantization functor for $\mathcal{C} = FinSets$.

Now we assume that the category $\mathcal{C}$, in the definition of classical fields functor, is the category of spaces (or equipped with a functor to a category of spaces of some kind). We can quantize a theory by assigning the vector space of complex-valued functions on the space of fields to the boundaries

$$Z(N) = \mathbb{C}^{\mathcal{F}(N)}.$$

Let $S : \mathcal{F}(M) \to \mathbb{R}$ be a function (the "action" functional in physics) on the space of fields. The complex-valued function e^{iS} will serve as the kernel (see Construction 1.1.10) for pull-push.

Let us specify the pushforward in this case. Let $f \in \mathbb{C}^{\mathcal{F}(M)}$ and $b : \mathcal{F}(M) \to \mathcal{F}(N)$ be the restriction of the fields on a compact manifold M to a component N of the boundary of M. Then define the pushforward of f along b as:

$$(b_* f)(p) = \sum_{\{q \in \pi_0(\mathcal{F}(\mathcal{M})) | b(q) \simeq p\}} e^{iS(q)} f(a(q)) \mu(q)$$

where

$$\mu(q) = \frac{\#\Omega_p \mathcal{F}(N)}{\#\Omega_q \mathcal{F}(M)}.$$

Here $\#Z$ of a space Z is the homotopy cardinality[6]. The choice of the measure μ determines the pushforward operation $(\_)_*$.

Spans can be composed by using pullbacks (see Construction 1.1.9).Let $Q(X) = \mathbb{C}^X$ and let Q send the span of finite groupoids to a linear transformation of vector spaces via pull-push (as described in the previous paragraph). We pose the following question (note the absence of the monoidal structure):

Find conditions under which the operation $Q : Span(Gpds^f) \to Vect_{\mathbb{C}}$ *is functorial.*

The composition of solid lines on the right should match the dotted lines in the figure below:

$$\begin{tikzcd}
A & X \arrow[l, "l"'] \arrow[d, "m"'] & Z \arrow[l, dotted, "a"'] \arrow[d, dotted, "b"] \arrow[dl, phantom, very near start, "\llcorner"] \\
 & B & Y \arrow[l, "n"] \arrow[d, "p"'] \\
 & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
Q(A) \arrow[dr, "m_*l^*"] \arrow[ddrr, dotted, bend right, "(pb)_*(la)^*"'] & & \\
 & Q(B) \arrow[dr, "p_*n^*"] & \\
 & & Q(C)
\end{tikzcd}$$

The compositions match $p_*b_*a^*l^* = p_*n^*m_*l^*$ iff

$$b_*a^* = n^*m_*$$

This condition is a simple version of the famous "Beck-Chevalley conditions". Let us formalize the functoriality condition into a proposition.

Definition 1.1.13. Let $\mathcal{C}$ be a category with finite limits. A pair of functors

$$()^*, ()_* : \mathcal{C} \to \mathcal{D}$$

is called a *Beck-Chevalley pair* if the former functor is contravariant, the functors agree on objects and for any pullback square in $\mathcal{C}$:

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \bullet \arrow[d, "m"] \\
\bullet \arrow[r, "n"'] & \bullet
\end{tikzcd}$$

the following condition holds:

$$b_*a^* = n^*m_*.$$

---

[6] See Subsection 1.1.6 for a precise definition due to Kontsevich.

Let $\mathcal{C}$ be a category with finite limits. Let $Span(\mathcal{C})$ denote the category constructed in Construction 1.1.9. In the next proposition, we summarize the discussion on the prequantization functor using a universal property.

Theorem 1.1.14. *Let $\mathcal{C}$ be a category with finite limits. Let $\mathcal{D}$ be a category. The pair of functors i^*, i_* are universal Beck-Chevalley functors. In other words, given any pair of Beck-Chevalley functors $()^*, ()_*$, it uniquely factors through i^*, i_* as shown in the diagram below.*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "{i^*,i_*}"] \arrow[dr, "{()^*,()_*}"'] & Span(\mathcal{C}) \arrow[d, dotted, "Q"] \\
 & \mathcal{D}
\end{tikzcd}$$

When constructing a quantum field theory, classical field theories are quantized by choosing a measure to do the path integral. The above proposition imposes a condition on the measure that allows prequantization.

Now we define the phrase "quantizing a classical field theory".

Definition 1.1.15. A TQFT Z is obtained by *quantizing a classical field theory* if there exists a category $\mathcal{C}$ with finite limits such that Z factors as follows

$$\begin{tikzcd}
 & (\mathrm{Span}(\mathcal{C}), \times) \arrow[dr, "Q"] & \\
(Bord^{d,d-1}, \amalg) \arrow[ur, "C"] \arrow[rr, "Z"] & & (Vect_{\mathbb{C}}, \otimes)
\end{tikzcd}$$

A classical fields functor $\mathcal{F} : (Man^{op}, \amalg) \to (\mathcal{C}, \otimes)$ naturally induces a classical field theory $C_{\mathcal{F}} : (Bord^{d,d-1}, \amalg) \to (\mathrm{Span}(\mathcal{C}), \times)$ such that $C_{\mathcal{F}}(M) = \mathcal{F}(M)$. Recall the discussion on locality leading up to Definition 1.1.4 where it was explained that monoidal structure on the functors corresponds to the locality of the theory. Theorem 1.1.14 gives us the conditions for constructing the prequantization functor without imposing locality. Thus, having the monoidal version of Beck-Chevalley conditions (see Definition 1.1.13), which produces a monoidal version of Theorem 1.1.14 gives us a way to construct a quantization functor Q. Thus, we obtain a TQFT $Z = QC_{\mathcal{F}}$ by quantizing a classical field theory.

1.1.3 Once extended field theories

In this subsection, the definition of an once extended cubical field theory is motivated.. But we first discuss the history of extended field theories. In the '90s, following ideas of Freed in [19] and Baez-Dolan [2], Atiyah-Segal-Kontsevich's definition of TQFT was generalized. This definition of a d-dimensional TQFT can be paraphrased as assigning a complex number to a closed-oriented d-manifold and a vector space to a codimension 1 submanifold of a d-manifold. Freed suggested that we should be able to extend this definition to higher codimensions. He proposed that a category can be assigned to a codimension 2-manifold (a 2-category can be assigned to a codimension 3-manifold and so on). Freed illustrated this idea by working out parts of a once-extended 3-dimensional Dijkgraaf-Witten theory for finite groups. He argued that the circle (which is codimension 2 for a 3-dimensional theory) should be assigned the representation category of the Drinfeld double of the group. Baez and Dolan proposed the notion of a fully extended TQFT (assuming we had a good model for symmetric monoidal higher categories).

Before we discuss extended TQFTs, a clarification about the type of higher categories is overdue. There are two types of theories that are typically called *higher category theories*. A higher category can be defined by specifying the objects, morphisms, composition and associated data along with a collection of rules (called "coherences") that interrelate the data. For instance, this is how categories are described in [43] and bicategories are described in [6, 43, 33] (Also see Definition 2.1.3 for the definition of a bicategory). But working with higher categories beyond bicategories (i.e. tricategories [25] and tetracategories) becomes very cumbersome. On the other hand, higher categories can also be studied within the framework of ∞-category theory. In [42], Lurie contrasts these two different approaches and then formulates the definition of a fully extended TQFT using the framework of ∞-category theory. The methods employed in this thesis is inspired by the methods used in ∞-category theory (that suppresses complicated coherences) but no ∞-category theory is required to understand the any result or proof in this thesis. So when we say higher categories we mean a theory of the first type but using the methods of the second type.

Now we discuss the notion of a fully extended TQFT informally. Given a category $\mathcal{C}$ and two objects x, y in $\mathcal{C}$, let $\mathcal{C}(x, y)$ denote the set of morphisms from x to y. We follow the exposition of higher categories from [42, Section 1]. An inductive definition of strict higher categories is the following: A strict n-category is a category enriched in the category of $(n-1)$-categories. A higher category is a weakened version of strict higher category where the associative and unital laws are weakened to hold up to isomorphisms. If we unravel the definition of a higher category, we see that a higher category $\mathcal{C}$ is specified by

- A collection of objects $x, y, \cdots$.
- For every pair of objects, a collection of $1-$*morphisms* $f, g, \cdots$.

- For every pair of 1−morphisms, a collection of 2−*morphisms* $A, B, \cdots$.
- For every pair of 2−morphisms, a collection of 3−*morphisms* $\alpha, \beta, \cdots$.
- $\cdots$

The notion of a bicategory is a precise example of a n-category for the case $n = 2$. See Definition 2.1.3.

Back to the discussion on extended field theories: Roughly speaking, the d-category $Bord^d$ of d-dimensional bordisms has i-dimensional bordisms as i-morphisms for $i > 0$ and 0-dimensional manifolds as the objects of the d-category. The coproduct of manifolds induces the symmetric monoidal structure on $Bord^d$.

Definition 1.1.16 (Sketch)**.** Let $\mathcal{C}$ be a symmetric monoidal higher category. A fully extended $\mathcal{C}$-valued TQFT Z is a symmetric monoidal functor

$$Z : Bord^d \to \mathcal{C}.$$

Baez and Dolan proposed the "cobordism hypothesis" that fully extended TQFTs are characterized by the TQFT's value on a point. Lurie sketched a construction of a symmetric monoidal d-category of bordisms in [42] and it was rigorously constructed by Calaque and Scheimbauer in [10]. We remark that the constructions are in the framework of ∞-categories, but the distinction is not very important for our informal discussion here. Lurie also wrote down the precise statement of the cobordism hypothesis/theorem and sketched a proof of the theorem [42, Section 2.4]. Roughly speaking, the claim is: A fully extended TQFT is characterized by its value on a point and the value on the point is given by a "fully dualizable object" in the codomain d-category of the TQFT. The notion of full dualizability is internal to the symmetric monoidal category and is a generalization of the notion of dualizability in monoidal categories. We will not talk about dualizability in this thesis, so we will not discuss it further. The notable point is that all known fully dualizable objects have a rich structure. The interested reader should refer to [42, Section 2.3].

The focus of this thesis is on generalizations of TQFTs proposed by Dijkgraaf and Witten (discussed in Subsection 1.1.7). It is natural to wonder if there exists a fully extended version of this theory. While it is believed that TQFTs arising in physics are fully extendable, a rigorous construction of such a fully extended TQFT is not available presently. The Atiyah-Segal-Kontsevich TQFT in Definition 1.1.7 can be constructed using the tools of category theory which is well studied and classical. On the other hand, Definition 1.1.16 requires methods from higher category theory which is relatively

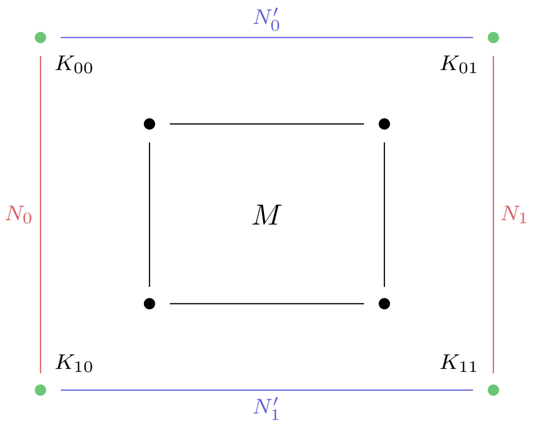


Figure 1.4: Manifold with corners whose boundaries are red and blue manifolds with boundaries. The green manifolds are corners of the original manifolds.

new and the definitions are long and complex (For instance, look at the definition of a tetracategory due to Trimble in [32].) At the present moment, bicategories (Definition 2.1.3) can be considered as a sweet spot for the trade off between complexity and applicability. Historically, bicategories defined by Benabou [6] are precise models for weak 2-categories. Double categories are mild generalizations of bicategories and discussed in this subsection. In this thesis, the construction of an once extended TQFT is considered which is double categorical in nature.

So now we discuss the notion of an once extended cubical TQFT. Suppose M is a manifold with corners as shown below in Figure 1.4. Let $\psi \in \mathcal{F}(K)$, then by the reasoning for (2) in Remark 1.1.5, we get $Z(N_i, \psi) = \mathbb{C}^{\mathcal{F}(N_i)}$.

At this point, there are various ways to chop the manifold into smaller parts. We will take a cubical point of view. The simplest example is shown in Figure 1.4, where a manifold M with corners is drawn. It has four boundary components N_0, N_1, N_0', N_1'. Each of these manifolds has boundaries denoted in green and labelled by K.

We can assemble these cubical bordisms into some kind of category. The gluing of manifolds along diffeomorphic boundary pieces will be the composition of this category. The manifold with corners can be glued in two independent directions as shown in Figure 1.5 and thus the partition function on these manifolds can be composed in two ways.

We note that if fields are fixed on the boundary manifolds and on the corners in a compatible way, then the path integral can be computed. In other words, if $\psi = (\psi_i)$ are the fields on boundaries labelled by N, N' and $\theta = (\theta_i)$ are the fields on the corners

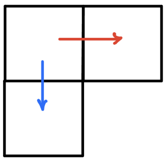


Figure 1.5: The manifolds with corners cut in a cubical way can be glued in two independent directions.

labelled by K as shown in Figure 1.4, then by integrating over the all the fields on M that are compatible with the boundary conditions, we can obtain the partition function of the field theory:

$$Z(M, \psi, \theta) \in \mathbb{C}$$

Now we see that $Z(M,, \theta)$ is a linear functional:

$$Z(M,, \theta) \in \mathbb{C}^{\mathcal{F}_{\partial M}(\theta)}.$$

Let us denote $\mathbb{C}^{\mathcal{F}_{\partial M}(\theta)}$ by $Z(\partial M, \theta)$. Recall that composition of morphisms was related to a pairing of vectors. Now we will have two independent directions to pair vectors.

Further, we can cut a big square into 4 smaller squares by drawing vertical and horizontal mid-lines. The four smaller squares can be glued horizontally and then vertically or the other way around. Either way, we get the big square as the composite. The partition function Z on the full square given the boundary condition can be computed directly or by composing the pieces in different ways, the answer should remain the same.

The situation described in the paragraphs above with a pair of compositions is captured by working with a double category instead of a category. The disjoint union of manifolds continues to be the monoidal structure.

Double categories were introduced by Ehresmann in [15]. We present a definition of a strict double category here. The reader can look at Definition 2.1.39 in Chapter 2 for a precise definition of a weaker notion that will be used in the thesis.

Definition 1.1.17. A double category $\mathbb{D}$ is given by

Data:

1. A pair of categories $(\mathbb{D}_0, \mathbb{D}_1)$ called object category and arrow category respectively.

2. A collection of functors $S : \mathbb{D}_1 \to \mathbb{D}_0$, $T : \mathbb{D}_1 \to \mathbb{D}_0$ and $U : \mathbb{D}_0 \to \mathbb{D}_1$ called source, target and unit functors respectively.
3. A *composition* functor $\odot : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ where the pullback is over the diagram: $\mathbb{D}_1 \xrightarrow{T} \mathbb{D}_0 \xleftarrow{S} \mathbb{D}_1$.

satisfying the following **conditions**

1. The source, target and unit functors satisfy $SU = id$ and $TU = id$.
2. The source, target and composition satisfy $S\odot = S\pi_2$ and $T\odot = T\pi_1$.
3. The composition is associative and unital.

A double functor between double categories is the data of functors between object and arrow categories that commute with all the structure maps (i.e. $S, T, U, \odot$). Double categories and double functors form a category.

If we spell out the definition above, then a double category $\mathbb{D}$ has a set of objects $\mathbb{D}_{00}$, a set of vertical morphisms $\mathbb{D}_{01}$, a set of horizontal morphisms $\mathbb{D}_{10}$ and a set of 2-morphisms $\mathbb{D}_{11}$ equipped with source and target maps,

$$\begin{tikzcd}
\mathbb{D}_{11} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathbb{D}_{10} \arrow[d, shift left] \arrow[d, shift right] \\
\mathbb{D}_{01} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{D}_{00}
\end{tikzcd}$$

A sketch of a two-cell of a double category is shown below. In Figure 1.6 the 2-morphism α is a morphism between morphisms in two ways: $\alpha : M \to N$ and simultaneously $\alpha : f \to g$. The 2-morphisms can be composed horizontally by pasting horizontal squares and the composition will be denoted by $\odot$. The 2-morphisms can be composed vertically by pasting vertical squares and the composition will be denoted by $\cdot$.

$$\begin{tikzcd}
A \arrow[r, "M", "|" marking] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r, "N"', "|" marking] & D
\arrow[from=1-1, to=2-2, phantom, "\alpha"]
\end{tikzcd}$$

FIGURE 1.6: A two-cell: α is a 2-morphism, M, N are horizontal morphisms, f, g are vertical morphisms and the rest are objects.

For a collection of 2-morphisms $\alpha, \beta, \gamma, \delta$ in the following shape,

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\gamma"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\delta"] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

the functoriality of composition in Definition 1.1.17 corresponds to the "middle four exchange" formula

$$(\alpha \odot \beta) \cdot (\gamma \odot \delta) = (\alpha \cdot \gamma) \odot (\beta \cdot \delta).$$

The category formed by objects and horizontal morphisms is called the horizontal category of $\mathbb{D}$. The category formed by objects and vertical morphisms is called the vertical category of $\mathbb{D}$.

Recall that a 2-category has objects, 1-morphisms between objects and 2-morphisms between 1-morphisms with the same source and target objects. The composition of 1-morphisms is associative and unital. For a precise definition, look at Benabou [6].

Remark 1.1.18. Certain two-cells in a double category look like 2-categories.

$$\begin{tikzcd}
A \arrow[r, "M", "|" marking] \arrow[dd, equal, "id"'] \arrow[ddr, phantom, "\alpha"] & B \arrow[dd, equal, "id"] & & & \\
& & \overset{\mathcal{H}}{\mapsto} & A \arrow[r, bend left, "M", ""{name=U, below}] \arrow[r, bend right, "N"', ""{name=D}] & B \\
A \arrow[r, "N"', "|" marking] & B & & &
\arrow[Rightarrow, from=U, to=D, "\alpha"]
\end{tikzcd}$$

Given a double category $\mathbb{D}$, if we discard all the 2-morphisms with non-identity vertical morphisms, we get the bicategory (Definition 2.1.3) of horizontal morphisms $\mathcal{H}(\mathbb{D})$. This construction is functorial (See [59, Page 4]).

A strict symmetric monoidal double category can be defined as well.

Definition 1.1.19. A strict symmetric monoidal double category $(\mathbb{D}, \otimes, I)$ is a double category $\mathbb{D}$ equipped with functors $\otimes : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ and $I : * \to \mathbb{D}$ such that

$$\begin{aligned} \otimes \circ (\mathrm{Id} \times \otimes) &= \otimes \circ (\otimes \times \mathrm{Id}) \\ \otimes \circ (\mathrm{Id} \times I) &= \mathrm{Id} \\ \otimes \circ (I \times \mathrm{Id}) &= \mathrm{Id} \\ \otimes \circ \tau &= \otimes \\ \tau^2 &= \mathrm{Id} \end{aligned}$$

where Id denotes identity functor, $\tau : \mathbb{D}^2 \to \mathbb{D}^2$ is the braiding double functor that swaps coordinates and $*$ denotes the terminal double category.

A symmetric monoidal double functor is a double functor that respects the monoidal structure and the braiding. For a weak version, see [59, Definition 2.9].

In this thesis, we discuss double categories with weakly associative composition in the horizontal direction (called a pseudo double category) and consider the double categorical analogue of a TQFT in the spirit of Atiyah. Let DBord^d denote the double category of bordisms cut in a cubical manner as discussed at the start of this section. This pseudo double category has a symmetric monoidal structure given by the disjoint union of manifolds. A construction is outlined in Schommer-Pries' thesis [54, §3.1.4].

Definition 1.1.20. A once extended cubical TQFT Z is a symmetric monoidal double functor

$$Z : (\mathrm{DBord}^d, \amalg, \phi) \to (\mathbb{D}, \otimes, I)$$

The computation of the partition function by cutting manifolds in two independent directions naturally leads to a cubical extended TQFT.

Remark 1.1.21. The choice of cutting the manifolds in a cubical manner naturally suggests the use of the pseudo double category of bordisms. It is plausible that other ways of cutting manifolds may model the extended TQFT differently, but the author hopes that the data of the extended TQFT will be model independent.

The composition in the pseudo double category will be weakly associative since the gluing process is not canonical.

It should be clear from this section, that an extended TQFT is used to compute the partition function on a closed manifold. In simple words, we are cutting the manifold only to glue them later. A key insight in Lurie's construction of extended TQFTs [42] is that we can choose to remember where to cut without really cutting the manifold. The results in this thesis were motivated by the problem of giving a rigorous construction of once extended cubical TQFTs.

1.1.4 Grothendieck's Homotopy Hypothesis

We will use the term "space" to mean a topological space that is homotopy equivalent to a CW complex.

Definition 1.1.22. A space X such that the homotopy groups $\pi_k(X, *) = 0$ for $k > n$ is called a n-type.

If X is an n-type and it has finite homotopy groups, X is called a finite n-type.

If X is a finite n-type for some n, then X is called a *homotopy finite space.*

In this subsection, the well known dictionary between groups, one object groupoids and connected 1-types is presented. The corresponding dictionary between crossed modules, one object 2-groupoids and connected 2-types is discussed in Subsection 5.1.2. The subsection ends with the statement of Grothendieck homotopy hypothesis.

Given a CW complex X, we can construct the *fundamental path groupoid* $\pi_{\leq 1}(X)$ with points of X as the objects and the homotopy classes of paths between points as morphisms. A CW complex $|\mathcal{X}|$, called the geometric realization, can be constructed from a groupoid $\mathcal{X}$. Recall that a map of CW complexes $f : X \to Y$ is a quasiisomorphism if $\pi_0(f)$ and $\pi_1(f)$ are isomorphisms. Let Ho(1types) denote the category of 1-types localized at the set of quasi-isomorphisms and Ho(Gpds) denote the category of groupoids localized at equivalences of groupoids. The pair $(\pi_{\leq 1}, |(-)|)$ induce an equivalence of categories:

$$\begin{tikzcd} \mathrm{Ho}(1-\mathrm{types}) \arrow[r, bend left, "\pi_{\leq 1}"] & \mathrm{Ho}(\mathrm{Gpds}) \arrow[l, bend left, "|(-)|"] \end{tikzcd}$$

Under the equivalence above, the fundamental path groupoid of a connected 1-type is a one object groupoid.

Construction 1.1.23. So now we have three equivalent descriptions of a group:

- *Algebraic description:* Category with groups as objects and group homomorphisms as morphisms. Denote this category by Grps. We can choose weak equivalences as isomorphisms and denote this data as the pair (Grps, Iso).

- *Categorical description:* Category of one-object groupoids. The morphisms are functors between groupoids. Denote this category by Gpds_*. Choose weak equivalences as equivalences of groupoids and denote this data as the pair $(\mathrm{Gpds}_*, \mathrm{Equiv})$.
- *Topological description:* Category of connected 1-types. The morphisms are continuous maps between spaces. Denote this category by $\mathrm{Conn1-types}$. The weak equivalences are quasi-isomorphisms and denote this data as the pair $(\mathrm{Conn1-types}, \mathrm{QIso})$.

Given an object G in Grps, denote $*//G$ as the groupoid with one object $*$ and the morphisms $(*//G)(*, *) = G$ with composition of morphisms defined by the multiplication in G. It turns out we have a functor

$$*//(-) : \mathrm{Grps} \to \mathrm{Gpds}_*$$

which maps isomorphisms of groups to isomorphisms of groupoids. Given a one object groupoid $\mathcal{X}$ with $*$ as the object, the set of morphisms $G(\mathcal{X}) = \mathcal{X}(*, *)$ along with the composition data of the groupoid gives $G(\mathcal{X})$ a group structure. The map G can be upgraded to a functor which maps equivalences of groupoids to isomorphism of groups.

Proposition 1.1.24. *The functors $*//(-)$ and $|(-)|$ described in Construction 1.1.23 induce equivalences of homotopy categories.*

One imagines that a n-groupoid will have n levels of invertible morphisms. However, writing a precise definition of a weak n-groupoid will involve a prohibitively high amount of coherence isomorphisms and very complicated coherence conditions (look at the definition of a weak 4-category in [32, Section 3.2] due to Todd Trimble). So how can we define weak n-groupoids? Grothendieck envisioned a remarkably simple proposal to this conundrum [29] which can be motivated by Proposition 1.1.24 (and the analogous Theorem 5.1.11 for 2-types in Chapter 5).

***Grothendieck's homotopy hypothesis*: A CW complex X such that the homotopy groups $\pi_k(X, *) = 0$ for $k > n$ is a model for n-groupoids.**

Equivalence of n-groupoids is given by weak homotopy equivalence of CW complexes.

Essentially, the homotopy hypothesis suggests that we set the categorical description *to be equal* to the topological description. Proposition 1.1.24 notes that the homotopy category of connected 1-types is equivalent to the category of groups. This proposition along with Grothendieck's homotopy hypothesis suggests that a homotopy theory of connected n-types could reasonably be called the theory of "n-groups" (generally called *higher groups* when n is not specified). The homotopy hypothesis for the case $n = 2$ has been studied in the literature [68, 44, 47] and the reader is directed to Theorem 5.1.11 due to Moerdijk and Svensson. Figure 5.1 gives a map of the discussions, in Chapter 5, that relate 3 distinct descriptions of 2-groups.

1.1.5 Classical 2-group gauge theory

In this subsection, we discuss Dijkgraaf-Witten theory for higher groups. A Chern-Simons theory is specified by a Lie group G, a manifold M, and a principal G-bundle on M equipped with a connection . The space of classical fields in this case is the space of G-bundles equipped with a flat connection. The focus of Dijkgraaf-Witten theory is on a finite group G and in this case, there is a unique flat connection. Thus the space of classical fields in this case is simply the space of principal G-bundles. Since G is discrete, any Eilenberg- MacLane space $K(G, 1)$ is a classifying space BG for G-bundles. Note that BG is a connected 1-type with a finite fundamental group. This motivates the definition of classical field theories for finite n-groups (see Willerton [69] for a similar discussion).

Let X be a finite n-type and M be a manifold. The mapping space $\mathrm{Maps}(M, X)$ has a physical interpretation of classical gauge fields on M. In this case, from Definition 1.1.1, classical gauge field functor can be considered as a contravariant functor $\mathcal{F}_X$ into the category of spaces. In the informal discussion below, when we say space of classical fields on M and denote it by $\mathcal{F}_X(M)$, a mapping space $\mathrm{Maps}(M, X)$ is a good example to keep in mind. We remark that in the physics literature this model for fields is called a sigma model.

We motivate the bicategory of bispans of spaces as a natural home for a 2-functor of classical fields. Given a manifold M with corners, the classical fields on M restrict to classical fields on submanifolds of all codimensions. Schematically, the space of classical fields $\mathcal{F}_X$ can assemble into functor that sends bordisms $M : N_1 \to N_2$ to spans (or correspondences) of spaces.

$$\begin{tikzcd}
 & M & & & & \mathcal{F}_X(M) \arrow[dl] \arrow[dr] & \\
N_1 \arrow[ur, hook] & & N_2 \arrow[ul, hook'] & {} \arrow[r, squiggly, "\mathcal{F}"'] & \mathcal{F}_X(N_1) & & \mathcal{F}_X(M)
\end{tikzcd}$$

The classical fields double functor on the manifold shown in Figure 1.4 looks like this:

$$\begin{tikzcd}
\mathcal{F}_X(K_{00}) & \mathcal{F}_X(N_0') \arrow[l] \arrow[r] & \mathcal{F}_X(K_{01}) \\
\mathcal{F}_X(N_0) \arrow[u] \arrow[d] & \mathcal{F}_X(M) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathcal{F}_X(N_1) \arrow[u] \arrow[d] \\
\mathcal{F}_X(K_{10}) & \mathcal{F}_X(N_1') \arrow[l] \arrow[r] & \mathcal{F}_X(K_{11})
\end{tikzcd}$$

All the arrows in the above diagram are restriction functors and all squares commute. This image should be interpreted as a 2-morphism of a pseudo double category whose horizontal and vertical morphisms are spans of spaces and 2-morphisms are spans of spaces in both directions (called bispans of groupoids). The space of classical fields $\mathcal{F}_X$ can be enhanced to a 2-functor, if the composition in the pseudo double category of bordisms corresponds to the composition in the tentative pseudo double category of bispans of groupoids. This motivates the composition of spans of spaces: Suppose a manifold M is cut along a codimension 1 submanifold N into two manifolds (with boundary) M_1 and M_2. Lets denote the gluing of M_1, M_2 along N by $M_1 \amalg_N M_2 \simeq M$. If the restriction map

$$\mathrm{Maps}(M, X) \to \mathrm{Maps}(M_1, X) \times^h_{\mathrm{Maps}(N,X)} \mathrm{Maps}(M_2, X)$$

is an equivalence of spaces, which means we can reconstruct classical fields on M from its parts. This should follow from showing that gluing along boundaries produces homotopy colimits of spaces. Thus the composition of the spans of spaces is given by a homotopy pullback of spaces.

In the model of double categories we consider in this thesis, the composition in the vertical direction has to be strict. So we shall identify the vertical spans up to equivalences. The compositions of 2-morphisms in two independent directions are computed using homotopy limit diagrams which we won't spell out here. Also, by definition of coproduct,

$$\mathcal{F}_X(M_1 \amalg M_2) \simeq \mathcal{F}_X(M_1) \times \mathcal{F}_X(M_2)$$

The above discussion can be summarized as saying that there is a symmetric monoidal 2-functor $\mathcal{F}$ induced from the classical fields functor $\mathcal{F}$ on manifolds. Motivated by [20, Section 3], higher category of spans are used to model classical field theory. While only the bicategory of bispans is considered in this thesis, the (∞, n)-category of n-fold spans has been constructed in [30].

Definition 1.1.25. A $\mathcal{C}$-valued once extended cubical classical field theory is a symmetric monoidal double functor

$$\mathcal{F} : (DBord^d, \amalg) \to (D\text{Bispan}(\mathcal{C}), \times).$$

Remark 1.1.26. In the physics literature, the pair of space of fields $\mathcal{F}_X(M)$ on M and the action functional S constitutes a classical field theory. So the definition 1.1.25 may seem restrictive. However, by suitably modifying the target category of classical fields functor, the action data can be incorporated into the above definition. Definition 1.1.25 is differs slightly from the one in [20, Section 3]. The objects of the bordism category in *loc.cit* is a pair (M, ϕ) where $\phi \in \mathcal{F}(M)$ and the objects of $\mathcal{C}$ are spaces equipped with a local system.

We can also discuss a bicategorical version of classical field theory.

Definition 1.1.27. A $\mathcal{C}$-valued once extended classical field theory is a symmetric monoidal 2-functor

$$\mathcal{F} : (\mathbf{Bord}^d, \amalg) \to (\mathbf{Bispan}(\mathcal{C}), \times).$$

A preclassical field theory is a 2-functor

$$\mathcal{F} : \mathbf{Bord}^d \to \mathbf{Bispan}(\mathcal{C}).$$

A tentative example of a preclassical field theory arises from the assignment $\mathcal{F}_X(M) = Maps(M, X)$ where $\mathcal{C}$ is a category of topological spaces. It is expected that the assignment $M \mapsto Maps(M, X)$ for manifolds M induces a pre-classical field theory $\tilde{\mathcal{F}}_X$.

1.1.6 Quantization

The path integral quantization can be carried out once we have an integration measure for the space of classical fields and a real functional (action) on the space of classical fields. However as the discussion in Section 1.1.1 showed, we demand that the partition function of a manifold should be computable by composing the value of the partition function on the pieces. In other words, we demand functoriality of the quantization functor.

In Subsection 1.1.2, we discussed quantization of a classical field theory. This discussion naturally leads to the problem of characterizing the category of spans by a universal property. Recall that the universal property was in terms of the Beck-Chevalley conditions (see Theorem 1.1.14). It is natural to wonder if Bispan$(\mathcal{C})$ discussed in the previous section has an analogous universal property. A universal property aids in the construction of a 2-functor $Q : \mathrm{Bispan}(\mathcal{C}) \to \mathcal{B}$. If we compose the classical field theory $\mathcal{F}$ in Definition 1.1.25 with a functor Q described in this section, then we can obtain a TQFT (Definition 1.1.20) if Q is a symmetric monoidal 2-functor. Thus we have

Definition 1.1.28. Let $(\mathcal{B}, \otimes)$ be a symmetric monoidal pseudo double category and let $\mathcal{C}$ be a category with finite limits. A symmetric monoidal pseudo double functor

$$Q : (\mathbf{Bispan}(\mathcal{C}), \times) \to (\mathcal{B}, \otimes)$$

is called an extended *quantization* double functor.

Note that the two-step formalism to compute the partition function can be upgraded to extended cubical TQFTs (see Figure 1.7):

$$\begin{tikzcd}
(\mathrm{DBord}, \amalg) \arrow[r, "\mathcal{F}"] \arrow[dr, "Z"'] & (\mathrm{Bispan}(\mathcal{C}), \times) \arrow[d, "Q"] \\
 & (\mathcal{D}, \otimes)
\end{tikzcd}$$

FIGURE 1.7: Two step TQFT: Extended classical field theory $\mathcal{F}$ followed by quantization Q.

Usually, in the literature, an extended TQFT is defined as a symmetric monoidal 2-functor of symmetric monoidal bicategories. Freed, Hopkins, Lurie and Teleman outlined the construction of extended TQFTs using a two-step formalism [20]. Following that paper, there was a lot of activity in constructing extended DW theory using two-step formalism for finite groups [69, 49, 50, 48]. More details are discussed at the start of Section 1.4. In this case, we can analogously define a extended quantization 2-functor and classical field theory 2-functor. We also define a prequantization 2-functor (cf. Definition 1.1.12). Let $(\mathbf{Bispan}(\mathcal{C}), \times)$ denote a symmetric monoidal bicategory of spans where the product of objects and spans provides the monoidal structure [30],

Definition 1.1.29. Let $(\mathcal{B}, \otimes)$ be a symmetric monoidal bicategory and let $\mathcal{C}$ be a category with limits. A symmetric monoidal 2-functor $Q : (\mathbf{Bispan}(\mathcal{C}), \times) \to (\mathcal{B}, \otimes)$ is called an extended *quantization* 2-functor.

An *extended $\mathcal{B}$-valued prequantization 2-functor* $\tilde{Q}$ is a 2-functor of the form

$$\tilde{Q} : \mathbf{Bispan}(\mathcal{C}) \to \mathcal{B},$$

for some category $\mathcal{C}$ with pullbacks. We may simply refer to it as a prequantization 2-functor.

Once the universal property of the symmetric monoidal bicategory of bispans is available, it can be used to determine which integration theories (read path integral measures) produce plausible candidates for quantization.

1.1.7 Dijkgraaf-Witten theory

We now specify the Dijkgraaf-Witten theory for a finite group. Let X be a finite n-group, then $\mathcal{F}_X(M) = \mathrm{Maps}(M, X)$ is also a finite n-groupoid. A d-dimensional TQFT is specified if two ingredients are given: a functional on the space of fields (action) and a tentative path integral measure.

The first ingredient is the action functional. Dijkgraaf and Witten argued that for a finite group (a connected 1-type) X, the action functional for a 3-dimensional theory is given by a 3-cocycle $\omega \in H^3(X, U(1))$ [14, Section 3]. Let M be an oriented 3-manifold and $f \in \mathcal{F}_X(M)$ is a classical field, then the action is the pairing $\langle f^*\omega, [M]\rangle$ where $[M]$ is the fundamental class of M. We can generalize this to oriented d-manifolds and finite n-groups X. We can consider the action functional as a d-cocycle in the cohomology of X with $U(1)$ coefficients (See Willerton's introduction [69]).

The other ingredient is the path integral measure over finite n-groupoids. Kontsevich constructed a topological quantum field theory with a specific measure (he calls it *homotopy Euler characteristic*).

Definition 1.1.30. [37, Section 7] Let F be a finite n-groupoid for some n (i.e. a finite n-type). Given a function $f : F \to \mathbb{C}$, the integral $\int_F f d\mu$ is defined as

$$\int_{\mathcal{G}} f d\mu = \sum_{[x]\in\pi_0(F)} f(x) \prod_{i\in\mathbb{Z}_{>0}} (-1)^i \# \pi_i(F, x).$$

The symbol $\pi_i(F, x)$ represents the ith homotopy group of F based at x. The volume or the "homotopy cardinality" of the space F is the integral of the constant function 1. We will denote it by $\#F$.

Note that the sum is finite and well-defined since all the homotopy groups are trivial after a certain stage. We list some nice properties of homotopy cardinality:

1. The homotopy cardinality is clearly a homotopy invariant.

2. If we have a Serre fibration $F \to E \to X$ for a connected space X and all the spaces are homotopy finite spaces, Kontsevich notes that $\#F\#X = \#E$.

3. Let X_1 and X_2 be two homotopy finite spaces, then

$$\#(X_1 \amalg X_2) = \#X_1 + \#X_2.$$

In this thesis, we have set the action functional to zero, and such Dijkgraaf-Witten theories are called *untwisted.* For finite 1-groupoids (in the categorical description), one can show that the basic Beck-Chevalley conditions (Definition 1.1.13) hold with the homotopy cardinality measure and zero action functional[71]. In other words, Kontsevich's measure for finite 1-groupoids gives an integration theory that induces a quantization functor.

Once the quantization functor is computed, we have the Dijkgraaf-Witten theory (DW theory) $Z_G = Q \circ \mathcal{F}$ for a finite group G. A dictionary between topological and algebraic descriptions of groupoids gives a bridge between TQFTs and finite group theory. Let us discuss this in the case of DW theory for finite groups.

First, we need the definition of action groupoid.

Definition 1.1.31. Let a group G act on a set X, then the groupoid $X//G$ has X as the set of objects and $\mathrm{Hom}(x, y)$ is a pair (x, g) such that $y = g \cdot x$. The composition is given by the composition of the group elements.

The following proposition is an example of a dictionary between different descriptions of a groupoid. Let $LX = \mathrm{Maps}(S^1, X)$ denote the free loop space of X. The following lemma is well known[7].

Lemma 1.1.32. *Let X be a connected 1-type and let G be the fundamental group of X at some base point, then the fundamental path groupoid of the free loop space of X, $\pi_{\leq 1}(LX)$, is equivalent to the action groupoid $G//G$ where G acts on itself via conjugation.*

Here is a proof sketch: Use the dictionary between topological, algebraic and categorical descriptions of groupoids mentioned in Subsection 1.1.4. Compute the groupoid

$$[\pi_{\leq 1}(S^1), \pi_{\leq 1}(X)] \simeq [BZ, BG] \simeq G//G,$$

---

[7] It appears as Proposition 4.1 on the webpage https://ncatlab.org/nlab/show/free+loop+space+object. However, the original reference could not be traced.

where $[X, Y]$ denoted the category of functors and natural transformations between X and Y, and BG presents the group G as a one-object groupoid. The fundamental path groupoid $\pi_{\leq 1}$ is described in Subsection 1.1.4.

By computing the isomorphism classes of the groupoid $G//G$, we get

Corollary 1.1.33. *Let X be a connected 1-type and G be the fundamental group of X, then $\pi_0(LX)$ is the set of conjugacy classes in G.*

Let $\mathrm{Cl}(G)$ denote the vector space of complex-valued class functions on G. We recover the classic results from [14], [21].

Corollary 1.1.34. *Let Z_G denote the 2-dimensional DW theory for a finite group G and let S^1 denote the circle. Then*

$$Z_G(S^1) = \mathit{Cl}(G).$$

Proof.

$$Z_G(S^1) = Q(\mathcal{F}_{|BG|}(S^1)) = Q(L|BG|) = \mathbb{C}^{\pi_0(L|BG|)} = \mathrm{Cl(G)}.$$

The last equality follows from the definition of class functions and Corollary 1.1.33. □

In Freed's seminal work [19], he computed some aspects of a once extended 3-dimensional DW TQFT Z_G^{ex} for a finite group G. He showed that Z_G^{ex} assigns a tensor category of representations of a nice Hopf algebra (called the Drinfeld double of the gauge group) to the circle. Proposition 1.1.32 directly gives this linear category of representations (See [69] for a discussion on the twisted version of this story. Recall that twisted theories are DW theories where the action functional may not be zero).

Generalizations of Corollary 1.1.34 to finite 2-groups are considered in this thesis.

1.2 Objectives

The primary objective of the thesis is the construction of a once-extended DW theory for finite 2-groups. As discussed in Subsection 1.1.3, there is an intuitive way of thinking of a once-extended TQFT as a symmetric monoidal double functor. Now there are two aspects of such constructions that pose difficulties:

1. The first is that we have to construct weak symmetric monoidal double categories and symmetric monoidal double functors, which have a modest number of coherence isomorphisms and coherence axioms. See below Definition 2.9 in [59]. It shows that there are 10 conditions to check.

2. The second issue is that extended TQFTs are defined as symmetric monoidal 2-functors out of a symmetric monoidal bicategory of bordisms ([2], [19]). The construction of a symmetric monoidal bicategory has even more coherences, and the coherence axioms are equalities of polyhedral pasting diagrams. We do not discuss this complication here since it is discussed thoroughly in other places in the literature (See the appendix of [54] for the definition, and the introduction in [59] for pointers to the literature).

Apart from the tedious/complicated nature of such constructions, any construction of an extended TQFT in the language of symmetric monoidal bicategories will be long and will generally omit details. So we have a new objective.

Objective 1. Construction of extended TQFTs using a different model of symmetric monoidal pseudo double category where we have to check very little.

Now, more details about the model are discussed. Let $\triangle$ denote the category of totally ordered finite sets. A functor $X : \triangle^{op} \to \mathcal{C}$ is called a *simplicial object* in $\mathcal{C}$. A simplicial object in the category of sets is called a simplicial set, and a simplicial object in the 2-category of categories is called a simplicial category. In the literature, the term simplicial category is also used to refer to categories enriched in simplicial sets. In this thesis, the term will always refer to a simplicial object in the category of small categories.

Let Γ denote the opposite category of finite pointed sets. Symmetric monoidal categories can be defined as fibrations over Segal's Γ category [56] [45]. For now, a fibration of categories can be loosely thought of as a directed version of fibrations in topology, i.e. it is a functor that has an arrow-lifting property.

Definition 1.2.1. [64],[60, Section 5.3] A Segal fibration $p : X \to \triangle \times \Gamma$ is a fibration of categories satisfying two "Segal conditions". If the fibre of p over $([0], S)$, where $[0]$ is the initial object in $\triangle$ and S is any finite pointed set, is a category with no non-identity morphisms, then Segal fibration is called pinched. (For a precise version, see Definition 2.4.4.)

The key point to note is that everything in sight is a category; thus, there are no coherences. One of our beliefs is that Segal fibrations are equivalent to (weak) symmetric monoidal double categories, and pinched Segal fibrations are symmetric monoidal bicategories.

Assuming the conjecture is true, we have a precise form of the primary objective:

Objective 2. Build a once-extended DW TQFT for 2-groups using Segal fibrations.

Now suppose we have constructed a tentative symmetric monoidal bicategory of bordisms. In Subsection 1.1.5, a sketch of the bicategory of bispan of spaces is outlined. This bicategory is the natural target for the classical field theory. As we discussed in Subsection 1.1.6, a quantization 2-functor can be constructed if the bicategory of bispans has a universal property. This leads to another objective.

Objective 3. Let $\mathcal{C}$ be a category with finite limits. Characterize the symmetric monoidal bicategory of bispans in $(\mathcal{C}, \times)$ by a universal property.. A simpler objective is to characterize the bicategory of bispans in $\mathcal{C}$ by a universal property.

Note that a bicategory of bispans in $\mathcal{C}$ can be constructed for any category $\mathcal{C}$ with pullbacks [53]. Finally, we discussed in Subsection 1.1.7, that the DW theory for 1-group assigned vector space of class functions to the circle. In that subsection, the computation of the value of TQFTs in the language of group theory was possible by using the equivalences between topological, algebraic and categorical descriptions of groups (Proposition 1.1.24). In this thesis, we are interested in working with 2-groups.

Given a finite 2-group X, if we consider the groupoid $\pi_{\leq 1}(Maps(M, X))$ as a model for the space of classical fields on a manifold M, then we can consider the associated preclassical field theory 2-functor $\tilde{\mathcal{F}}_X$ (see discussion below Definition 1.1.27). Define $\tilde{Z}_X^{DW} = \tilde{Q} \circ \tilde{\mathcal{F}}_X$ where $\tilde{Q}$ denotes Morton's 2-functor. Since we are interested in 2-groups, we have the following objective.

Objective 4. Compute the partition function of the low dimensional DW TQFTs for finite 2-groups on the circle and orientable surfaces.

Here, low dimensional means dimensions two and three. More precisely, the aim is to compute the values of $\tilde{Z}_X^{DW}(M)$ for oriented surfaces M and the circle S^1 for a finite 2-group X.

Let us summarize the objectives.

1. Construction of a different model for symmetric monoidal double categories (and symmetric monoidal bicategories) and functors in which there are few conditions

to check.

2. Construction of DW TQFT for finite 2-groups using the simpler model.
3. Let $\mathcal{C}$ be a category with finite limits. Characterize the symmetric monoidal bicategory of bispans in $(\mathcal{C}, \times)$ by a universal property. *Basic version:* Let $\mathcal{C}$ be a category with pullbacks. Characterize the bicategory of bispans in $\mathcal{C}$ by a universal property.
4. Let a d-dimensional extended DW theory for a finite 2-group be denoted by Z. The computation of the value of Z on a circle and orientable surfaces in terms of crossed modules.

1.3 Results

The main results of this thesis are presented in this section, and it will be in line with the objectives in the previous section. The discussion of the results can also be used as a guide to navigate the thesis.

1.3.1 Results of Chapter 2

First, we discuss the results relating to Objective 1, which is the focus of Chapter 2. Following Tamsamani and Simpson [64],[60], we define Segal categories as simplicial categories[8] satisfying Segal conditions (Definition 2.2.1). In Proposition 2.2.6, it is proved that the nerve of an isofibrant pseudo double category (see Definition 2.1.41) is an isofibrant Segal category (Definition 2.2.1). In Theorem 2.2.8, the nerve is shown to fully faithfully map onto the 2-category of isofibrant Segal categories.

Theorem 1.3.1. *The horizontal nerve 2-functor N from the 2-category of isofibrant pseudo double categories to the 2-category of isofibrant Segal categories is an equivalence.*

The strategy of the proof is as follows: First, we discuss essential surjectivity (see Definition 2.1.6). In [18], the authors show that their (strict) nerve functor (see Definition 2.1.55) N_s that sends a strict double category (see Definition 2.1.39) to a simplicial category has a left adjoint c_h. We show that the counit of the adjunction is an equivalence when computed on a Segal category (Proposition 2.2.7). The fully faithfulness on the level of Hom categories is formal. It follows from 2-categorical Yoneda embedding (See, for instance [33]).

---

[8] Recall that a simplicial category is a simplicial object in a category.

The Grothendieck construction allows us to recast Segal categories as fibrations over $\triangle$ satisfying Segal conditions (see Definition 2.3.1). Denote the bicategory of Segal fibrations over $\triangle$ by $\mathbf{SegFib}(\triangle)$. We discuss pinched Segal fibration over $\triangle$ (see Definition 2.3.3) and denote the bicategory of pinched Segal fibrations by $\mathbf{PSegFib}(\triangle)$. In Proposition 2.3.9, the inclusion of pinchable Segal fibrations over $\triangle$ into Segal fibrations over $\triangle$ is shown to have a right adjoint P. We show that $\mathbf{PSegFib}(\triangle)$ is equivalent to Lack's bicategory of bicategories, $\mathbf{Bicat}_2$ [38]. A precise statement can be found in Theorem 2.1.10.

Let Γ be the opposite category of finite pointed sets (see Definition 2.1.2). It is well known that Γ objects in a category, satisfying some conditions due to Segal, are models for Abelian monoid objects in that category [55], [45], [57]. Toen has already considered Γ objects in pinched Segal categories[9] (satisfying Segal conditions) as a model for symmetric monoidal bicategories in [65, §2]. Similarly, we work with fibrations and define *Segal fibrations over* $\triangle \times \Gamma$ as fibrations over $\triangle \times \Gamma$ that satisfy Segal conditions along $\triangle$ as well as Γ (see Definition 2.4.4). Then we conjecture that Segal fibrations over $\triangle \times \Gamma$ are equivalent to Shulman's symmetric monoidal double categories [59, Definition 2.9] in Conjecture 2.4.6.

1.3.2 Results of Chapter 3

Now we discuss the results relating to Objective 2 and the results of Chapter 3. In Chapter 2, a strict double category is obtained after applying Fiore-Paoli-Pronk's horizontal categorification functor c_h to the associated pseudofunctor of the Segal fibration over $\triangle$. We will say that this is the "associated double category" with the Segal fibration.

Given a category $\mathcal{C}$ with pullbacks, Benabou has a construction of bicategory of spans in $\mathcal{C}$ [6, §2.6], Grandis and Pare have constructed a pseudo double category of spans in $\mathcal{C}$ [26, §3.2], Barwick [5], Haugseng [30] have constructed categories of spans in the framework of ∞-categories. We closely follow Haugseng (but work entirely within category theory) and construct a Segal fibration $p : Sp\mathcal{C} \to \triangle$ in Subsection 3.2.1. In Construction 3.2.12, it is shown that the associated double category is a strict double category of spans in $\mathcal{C}$.

We discuss the construction of a Segal fibration of spans of groupoids. This is a variation on Grandis-Pare's pseudo double category of bispans since instead of isomorphisms of groupoids and pullbacks of groupoids, we work with equivalences of groupoids

---

[9] Segal categories with a discrete category of 0-simplices are called "pinched Segal categories". A discrete category has only identity morphisms.

and homotopy pullback of groupoids (see Construction 2.1.26). In Subsection 3.2.2, we construct a Segal fibration over $\triangle \times \Gamma$ and Construction 3.2.13 shows that the associated double category (in the $\triangle$ direction) is a strict double category of spans of groupoids. The fibration in $\triangle$ direction is a special case of Haugseng's construction [30, Section 5] and it can also be found in [5, Section 2].

The novelty of this chapter is the proposal of a fibration over $\triangle \times \Gamma$ whose associated double category should be the bicategory of bispans of groupoids. The construction of a fibration of bispans in groupoids is Construction 3.2.10 in Section 3.2.3. Checking the Segal condition remains open. However, Construction 3.2.14 shows that the fibration has the right morphisms for a double category of bispans of groupoids.

1.3.3 Results of Chapter 4

The bicategory of bispans of groupoids was constructed in the previous chapter. We prove a universal property of the bicategory of bispans in Chapter 4 along the lines of the Objective 3. However, we do not have a statement for the universal property of the symmetric monoidal bicategory of bispans.

We investigate the pair of inclusion functors $i^*, i_* : \mathcal{C} \to \mathbf{Bispan}(\mathcal{C})$ that is defined as identity on objects and on morphisms

$$i_*(f : x \to y) \quad = \quad \begin{tikzcd} & x \arrow[dl, "id_x"'] \arrow[dr, "f"] & \\ x & & y \end{tikzcd}$$

and

$$i^*(f : x \to y) \quad = \quad \begin{tikzcd} & x \arrow[dl, "f"'] \arrow[dr, "id_x"] & \\ y & & x \end{tikzcd}$$

It is shown that the pair of functors i_*, i^* satisfy 5 interesting axioms (see propositions 4.1.8, 4.1.3, 4.1.13 and 4.1.14). Any functor out of $\mathcal{C}$ into a bicategory $\mathcal{B}$ that satisfies these axioms (called "Double Beck-Chevalley conditions") is called a "Double Beck-Chevalley functor" (see Definition 4.1.15). The main highlight of the chapter is Theorem

4.3.10, which proves a universal property of the 2-category of bispans. The main theorem that we will prove is

Theorem 1.3.2. *Let $\mathcal{C}$ be a category with finite limits and $\mathcal{D}$ be a bicategory. Given a pair of 2-functors F_*, F^* that satisfy the double Beck-Chevalley conditions, it induces a unique 2-functor*

$$F : Bisp\mathcal{C} \to \mathcal{D}.$$

The proof uses the technique of string diagrams and occupies the rest of the chapter. Suppose the Theorem 1.3.2 is generalized to include the symmetric monoidal structure. In that case, it can be used to construct an extended quantization 2-functor. However, we can only construct an extended prequantization 2-functor.

Note that the collection of groupoids forms a 2-category with invertible 2-morphisms, and the 2-category of bispans of groupoids constructed in the previous chapter uses homotopy pullbacks (instead of ordinary pullbacks of groupoids). However, as stated, our theorem is proved for a category $\mathcal{C}$, so it looks like we should not be able to apply it to construct a universal 2-functor. However, since we have used universal properties and string diagrams for all the proofs in this chapter, the proofs will work for groupoids.

Alternatively, Morton [49] deals with such a situation by working with only skeletal groupoids. One can follow Morton and apply the above theorem to a category of skeletal groupoids.

1.3.4 Results of Chapter 5

Finally the Objective 4 is achieved in Chapter 5.

Let X be a finite 2-group and let $\pi_{\leq 1}$ denote the fundamental path groupoid of space (see Construction 5.1.10). This section has three parts:

1. Compute $\pi_{\leq 1}(Maps(S^n, X))$ for $n = 1, 2$.
2. Using the universal property proved in Chapter 4 to construct an example of a prequantization 2-functor $\tilde{Q}$ (see Definition 1.1.29).
3. Composing the above two steps and computing $\tilde{Q}(\pi_{\leq 1}(Maps(S^n, X)))$ for $n = 1, 2$.

Using Noohi's results on derived mapping 2-groupoids (see Subsection 5.1.4) we find 2-groupoid presentations of a few mapping spaces (in Subsection 5.2.1).

Definition 1.3.3. A linear category is a category enriched in vector spaces. The collection of linear categories, linear functors and linear natural transformations form a 2-category called **LinCat**.

Let X be a finite groupoid. We will denote the functor category $\underline{Fun}(X, \mathrm{Vect})$ by Vect^X to stress the analogy with the vector space of functions $\mathbb{C}^X$. Let $f : Y \to X$ be a map of finite groupoids. Then the pullback functor $f^* : \mathrm{Vect}^X \to \mathrm{Vect}^Y$ given by

$$f^*F = F \circ f.$$

For $F \in \mathrm{Vect}^Y$ define the pushforward f_*F as the left Kan extension $\mathrm{Lan}_f F$. So we have a pair of functors out of the category of finite groupoids

$$()_*, ()^* : \mathrm{Gpds}^f \to \mathrm{LinCat}$$

.

Construction 5.1.34 recalls the Beck-Chevalley conditions satisfied by the pair of functors $()_*, ()^*$ constructed by Morton in [50]. In Theorem 5.1.35, we show that $()_*, ()^*$ satisfy all Double Beck-Chevalley conditions. Thus it induces the 2-functor $\tilde{Q}$ from Theorem 1.3.2. The proof follows mostly from details in Morton's paper. Note that this 2-functor is a prequantization 2-functor (see Definition 1.1.29).

Finally we compute $\tilde{Q}(\pi_{\leq 1}(Maps(S^n, X)))$ for $n = 1, 2$ by evaluating Morton's 2-functor $\tilde{Q}$ on the groupoid $\pi_{\leq 1}(Maps(S^n, X))$ computed from Noohi's results. These values can be interpreted as the partition function of the tentative Dijkgraaf-Witten TQFT for a finite 2-group X on the spheres.

1.4 Related Works

Freed, Hopkins, Lurie and Teleman (FHLT), in 2010, sketched partial constructions of extended field theories using the two-step procedure of classical field theory and quantization [20]. They describe possible values of the TQFT on a point and circle for dimensions 1,2,3,4. Although the fully rigorous construction of the extended DW theory was not available, Willerton computed the value assigned to tori in [69] following Freed. From 2010 to 2015, Morton published a series of papers that formally constructed the classical field theory and the quantization of a once extended DW TQFT for finite groups, using an algebraic (read explicit) description of symmetric monoidal bicategories

[48], [50], [49]. This was in the spirit of the 2 step formalism outlined by FHLT. All these constructions are for finite groups.

Symmetric monoidal (∞, n)-categories have been defined as contravariant functors out of the category Γ into the $(\infty, 1)$ category of (∞, n)-categories satisfying Segal conditions [65, Section 2.2](for an English exposition, see [10, Section 1.3]). By the Grothendieck construction, these constructions can be viewed as fibrations satisfying Segal conditions. The definition of a symmetric monoidal higher category using fibration suppresses the coherence information neatly. It is worthwhile to consider 1-categorical analogues of this definition to model symmetric monoidal bicategories, which have a huge number of coherences. Kaledin's paper has studied adjunctions in a 2-category from this point of view in [35]. In Kaledin's paper [35, Section 2.5], strict 2-categories are defined using fibrations. This fibrational definition is the same as the Definition 2.3.3, where the author of this thesis calls it a "pinchable simplicial fibration". In Chapter 2 of this thesis, Theorem 1.3.1 (and using Grothendieck construction) shows that the nerve functor from pseudo double categories to special simplicial fibrations is an equivalence of 2-categories. When this nerve functor is restricted to the category of strict 2-categories, we obtain Kaledin's 'simplicial replacement' on strict 2-categories [35, Section 2.2]. The equivalence of the nerve functor restricts to an equivalence between bicategories and pinchable simplicial fibrations. Therefore, a contribution of this thesis is proof of the equivalence between bicategories (Benabou's explicit definition [6]) and Kaledin's bicategories (pinchable simplicial fibrations). Secondly, Kaledin defines symmetric monoidal category structure as fibrations over Segal's Γ category that satisfy certain Segal conditions [35, Section 4.1]. He defines fibrations over the category $\triangle$ (Kaledin's strict 2-category) and fibrations over the category Γ (Kaledin's symmetric monoidal category). However, in this thesis, the fibrations over $\triangle \times \Gamma$ (called 'Segal fibrations', see Definition 2.4.4) are discussed that conjecturally correspond to symmetric monoidal *bicategories*.

Further, the thesis proposes the equivalence between symmetric monoidal double categories and Segal fibrations (Definition 2.4.4). Morton's constructions of symmetric monoidal bicategories and 2-functors, [48],[50],[49], are explicit and many coherence conditions and axioms are suppressed from the presentation. It is the most detailed presentation of the once-extended DW theory for finite groups available now. The work in this thesis is motivated by the idea that the language of Segal fibrations may allow for a transparent construction on DW TQFTs

The universal property of the span bicategory has a long history, starting in the 60s in Chevalley's seminar. A universal property of the bicategory of spans was published by Hermida (See in [31, Theorem A.2]). A universal property of a double category of spans is discussed by Dawson, Pare, and Pronk [13].

The bicategory of bispans in a category (with pullbacks) is constructed explicitly in Rebro's preprint [53]. This thesis discusses the universal property of Rebro's bicategory of bispans. Recall that the universal property in Chapter 4 is described by functors that satisfy 5 conditions called the "Double Beck Chevalley conditions". In 2018, Balmer and Dell'Ambrogio described the universal property of a bicategory of certain restricted bispans [4]. In their bicategory, the one-morphisms (spans) are assumed to have a faithful morphism as one of the legs. In our version, there are no such restrictions. While the universal property described in this thesis has 5 conditions, their version does not have the last two conditions (i.e. Vertical and Horizontal Beck Chevalley conditions). It will be interesting to check whether the additional restriction on the spans is making the last two conditions automatic since the last two conditions are very complicated to check. Both works use string diagrams for proofs, and the string diagram computations look similar.

In 2020, Stefanich discovered a universal property of the (∞, n)-category of n-fold spans presented iteratively [62]. He uses Haugseng's iterative definition of span categories [30]. The bispan bicategory considered in this thesis is the homotopy bicategory of the $(\infty, 3)$ category of spans in Stefanich's work. Our result is not iterative and directly presents the universal property of the homotopy bicategory of tricategory of spans. On a cursory glance, the conditions look simple and different. For instance, there is no ambidexterity condition. It will be very interesting to prove that Stefanich's universal property implies the one in this thesis. It may be the case that the last two complicated conditions in the Double Beck Chevalley conditions are a shadow of simpler higher categorical conditions.

The DW theory for finite 2-groups is expected to relate manifold invariants with the representation theory of 2-groups. Elgueta [16] classified the representations of a finite 2-group in semisimple linear categories. The fully extended 3-dimensional DW theory for 2-groups is expected to assign this bicategory of representations of the finite 2-group to the point.

Ganter and Kapranov considered an action of a group on categories by pseudofunctors. This corresponds to a 2-group action (when a group is considered a 2-group) [23]. They propose a definition of a 'categorical trace' for a 1-endomorphism in a bicategory. The trace is valued in sets (or vector spaces). This naturally leads to a 'categorical character' of the 2-group action of the group on a linear category. In this thesis, Proposition 5.2.10 computes a linear category of functors that is assigned to a circle by the

3-dimensional DW theory. The categorical characters of Ganter-Kapranov are objects of this linear category for the case when the group is considered a 2-group.

In a recent preprint [17], Martins and Porter construct once extended TQFTs using a two-step formalism similar to the classical field theory-quantization factorization (See Figure 1.7) considered in this thesis and they compute the value of the partition function on some manifolds. First, we discuss the construction of TQFTs: The construction in [17] uses explicit descriptions of symmetric monoidal bicategories instead of the approach using fibrations considered in this thesis. A symmetric monoidal structure on the bicategory of $d-$bordisms is given the most detailed attention. A symmetric monoidal structure on the bicategory **Prof** with linear categories as objects, Vect-enriched profunctors as 1-morphisms and natural transformations as 2-morphisms is discussed in Section 2.8 of *loc.cit.* In Section 2.8.4 of *loc.cit*, it is claimed that symmetric monoidal structure on this bicategory follows from [12, Section 5.2]. As usual, the verification of many coherence axioms are left to the reader. Secondly, Martins and Porter use Brown-Higgins crossed complexes ([9]) to model spaces as opposed to Moerdijk-Svensson-Noohi's 2-groupoid models (see Subsection 5.1.2) which is used in this thesis. The models used to compute the partition function in this thesis (in Chapter 5) are restricted to 2-types. In contrast, Martins and Porter's models are general.. The values of the 3-dimensional and 2-dimensional TQFTs on the circle and sphere worked out in Chapter 5, i.e. Propositions 5.2.8 and 5.2.10 in this thesis, match with Theorems 276 and 283 in *loc.cit* respectively.

Notations, Prerequisites and Warnings

1.5 Notations

DW	Dijkgraaf-Witten
$\mathbb{C}^X$	The vector space of complex valued functions on the set $\pi_0(X)$.
$Fun(A, B)$	The set of functors from category A to category B.
$\underline{Fun}(A, B)$	The category of functors from category A to category B.
$\mathbf{Fun}(A, B)$	The bicategory of 2-functors from bicategory A to bicategory B.
$F \dashv G$	F is left adjoint to G for 1-morphisms in a given 2-category.
$(f_*, f^*, \eta, \epsilon, \eta^!, \epsilon^!)$	An ambidextrous adjunction (4.1.7).
$\triangle$	The skeletal category of finite ordinals and order-preserving maps.
Γ	The opposite category of finite pointed sets.
n_+	The set $\{*, 1, 2, 3, \cdots, n\}$ where $*$ is a basepoint.
DS	The discrete category associated to the set S.
$\mathcal{C}, \mathcal{D}$	Symbols used to represent categories.
$\mathbb{D}, \mathbb{E}$	Pseudo double categories (See Definition 2.1.39).
$\mathcal{B}$	Symbol used to represent a bicategory.
Man	Category of smooth manifolds and smooth maps
Cat	The category of small categories and functors.
$Vect$	The category of complex vector spaces and linear transformations.
$sSets$	The category of simplicial objects in sets (or $[\triangle^{op}, \text{Sets}]$).
$CrsMod$	The category of crossed modules and homomorphisms (5.1.14)
$2 - Gpds/\mathbf{2} - \mathbf{Gpds}$	The category/bicategory of 2-groupoids.
$2 - Gpds_*/\mathbf{2} - \mathbf{Gpds}_*$	The category/bicategory of pointed 2-groupoids.
$\mathbf{Cat}$	The 2-category of small categories.
$\mathbf{Dbl}$	The 2-category of small pseudo double categories.
$s\mathbf{Cat}$	The 2-category of simplicial objects in categories.
$\mathbf{Gpds}^f$	The 2-category of finite groupoids.
$\mathbf{LinCat}$	The 2-category of linear categories.
$\mathbf{KV2Vect}$	The 2-category of Kapranov-Voevodsky two-vector spaces (5.1.28)
$\mathbf{Bispan}(\mathcal{C})$	The bicategory of bispans in $\mathcal{C}$ (3.2.15).
$X \times^h_Y Z$	Homotopy pullback of spaces/groupoids

$\mathrm{Maps}(M, X)$	The homotopy type of a space of continuous maps from M to X.
$X//G$	The action groupoid for the action of a group G on a set X.
$\#X$	Homotopy cardinality of a homotopy finite space X.
$\pi_i(X, *)$	The ith homotopy group of the based space $(X, *)$.
$\pi_{\leq 1}(X, *)$	The fundamental path groupoid of a space X (5.1.10).
$WX.$	Whitehead 2-groupoid of a CW complex X (5.1.9)
$\mathcal{W}$	The Whitehead crossed module (5.1.20).
$N(\_)$	Nerve of a category, double category, 2-groupoid.
$\mathrm{Aut}(x)$	The group of automorphisms of the object x.
$\mathbf{Ho}(\mathcal{C})$	The homotopy category of a model category $\mathcal{C}$.

1.6 Conventions

1. Bicategories are denoted by a boldface string. For instance **Cat** denotes the bicategory of categories as opposed to Cat which represents the category of categories.

2. Let $\mathcal{C}$ be either $\triangle$, Γ or $\triangle \times \Gamma$, the symbol $\mathbf{SegFib}(\mathcal{C})$ denotes the bicategory of Segal fibrations over $\mathcal{C}$. The appropriate Segal conditions can be assumed, for instance look at Definition 2.3 and Definition 2.4.

3. $Maps(M, X)$ for simplicial sets M, X is the derived internal hom object in the simplicial model category of simplicial sets. Note that we use the same notation for spaces but the meaning should be clear from context.

4. $\mathcal{C}$ is used for a category and $\mathcal{B}$ is used for a bicategory.

5. When dealing with an adjunction we always use η for unit and ϵ for counit.

6. A commutative square of the following type with a marking on the top represents a pullback square.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

7. In case of classical field theory and quantization, we denote the lack of monoidal structure by a tilde on the top. For instance, $\tilde{Q}$ represents a prequantization functor.

Leinster	**Thesis**
bicategories	bicategories
2-categories	strict bicategory
homomorphism	2-functors
strict homomorphism	strict 2-functor
strong transformation	pseudonatural transformation
modification	modification
biequivalence	biequivalence

Table 1.2: Comparison of terminology with Leinster's article [40].

1.7 Prerequisites and Warnings

Prerequisites: We assume the reader is comfortable with category theory (see [43]), bicategory theory [33] and string diagrams, fibrations and Grothendieck construction [66]. We also assume acquaintance with model categories [24, Chapter II].

Warnings:

1. The terminology in this thesis regarding bicategories might be non-standard and the literature does not have a standard reference. So here is a comparison with Tom Leinster's article [40] (See Table 1.2).

2. In this thesis, the term simplicial category refers to a simplicial object in categories. This should not be confused with simplicially enriched categories.

3. Quantization, classical field theory and TQFT have various adjectives in this thesis and let us denote these three constructions commonly as X.

    - The symbol X, without the adjective "extended", is a functor and "extended X" is a 2-functor.
    - The lack of adjective 'pre-' before X implies that the (2-)functor X is symmetric monoidal. Otherwise it is simply a (2-)functor.
    - The adjective 'cubical' before X implies that the 2-functor is pseudo double functor of double categories. Otherwise it is simply a (2-)functor.

    For instance, a *cubical extended preclassical field theory* is a pseudo double functor from a bordism double category to a double category of spans.

Chapter 2

Segal Fibrations and Double Categories

The nerve construction is ubiquitous in mathematics, and is used to convert a categorical gadget into a simplicial gadget. Grothendieck constructed the nerve of a category in [29, §4.1] and characterized it as a simplicial set, i.e. a functor $X : \triangle^{op} \to Sets$, such that the (Segal) morphisms $X_{n+m} \to X_m \times_{X_0} X_n$ are isomorphisms (called *Segal conditions*; see 2.1.30 for details). Similarly, Simpson and Tamsamani constructed a nerve of a bicategory and characterized it as a simplicial object in categories, i.e. a 2-functor $\mathcal{C} : \triangle^{op} \to \mathbf{Cat}$ that satisfies Segal conditions and $\mathcal{C}_0$ has only identity morphisms [60],[64]. They replaced Segal isomorphisms by equivalences of categories and called it a *Segal category.*

Lack notes, in [38], that a bicategory of bicategories,$\mathbf{Bicat}_2$, can be constructed by choosing special 2-morphisms called 'icons' (see Example 2.1.51). Let $\mathbf{Fun}(\triangle^{op}, \mathbf{Cat})$ denote the bicategory of pseudofunctors, pseudonatural transformations and modifications. Lack and Paoli construct a "2-nerve", i.e. 2-functor $N_2 : \mathbf{Bicat}_2 \to \mathbf{Fun}(\triangle^{op}, \mathbf{Cat})$ and characterise bicategories as a Segal category with discrete category[1] of objects and two other technical conditions.

Double categories are generalizations of strict bicategories that contain an additional class of 1-morphisms (See 2.1.5 for an introduction). In [18], a strict nerve functor is discussed for double categories and to the best of my knowledge, the strict nerve of a double category has not been characterised. We consider a weakened version of double categories called *pseudo double categories*, due to Grandis and Pare [26], where the composition of a class of 1-morphisms is not strictly associative or unital. In Section 2.2 of this chapter, a nerve of pseudo double category is constructed, and the nerve

---

[1] Recall that a discrete category has only identity morphisms.

of a pseudo double category, satisfying a mild condition[2], is characterized as a Segal category. Let **Dbl** denote the bicategory of isofibrant pseudo double categories (2.1.45), s**Dbl** denote the bicategory of double categories. As Lack notes in [38, §6.1], we have an embedding $j : \mathbf{Bicat}_2 \to \mathbf{Dbl}$ (discussed in 2.1.51). It is unclear how to construct an isofibrant pseudo double category from a strict double category. So we have the following situation.

$$\begin{tikzcd}
\mathbf{Bicat}_2 \arrow[r, "j"] \arrow[dr, "Lack{,}Paoli"'] & \mathbf{Dbl} \arrow[d, "Thesis"'] & s\mathbf{Dbl} \arrow[dl, "Fiore{,}Paoli{,}Pronk"] \\
 & s\mathbf{Cat} &
\end{tikzcd}$$

On the other hand, we can also construct a bicategory $\mathcal{H}\mathbb{D}$ from a double category. Shulman, in [59], defined symmetric monoidal pseudo double categories and symmetric monoidal double functors (defined at 2.1.52, 2.1.53). Then he showed that the bicategory $\mathcal{H}\mathbb{D}$ inherits a symmetric monoidal structure if $\mathbb{D}$ is symmetric monoidal and it satisfies a mild condition (See Theorem 2.1.54).

From the work of Grothendieck on fibrations, it is known that fibrations over a category are equivalent to pseudofunctors out of the category [28]. Further, Grothendieck argued that fibrations are considerably easier to construct and work with since only categories are used (instead of bicategories). The key insight is that fibrations suppress much of the coherence data in the definition of an equivalent pseudofunctor. Now recall that Segal categories are pseudofunctors $X : \triangle^{op} \to \mathbf{Cat}$. Using the Grothendieck construction (see 2.1.3), we get a biequivalence between the bicategory **Dbl** and fibrations over $\triangle$ satisfying Segal conditions (see 2.3.2). We call these fibrations as *Segal fibrations over* $\triangle$ (see 2.3.1). Following Grothendieck's argument, it looks more practical to work with fibrations over $\triangle$ than Segal categories since it suppresses coherence information. This view is hardly new as it is ubiquitous in the ∞-category literature (see, for instance, [30]). In terms of category theory, Kaledin, in [35], defines bicategories as fibrations over $\triangle$ satisfying Segal conditions with a discrete fibre over object $[0]$ of the category $\triangle$. Segal fibrations over $\triangle$ are Kaledin's 2-categories without the discrete fibre condition.

So nerves of isofibrant pseudo double categories (See Definition 2.1.39) induce a biequivalence with the bicategory of Segal fibrations over $\triangle$ (See Theorem 2.3.2). Then we construct the analogue of the functor $\mathcal{H}$ for Segal fibrations over $\triangle$ which we call the *pinching construction* P (see 2.3.9).

---

[2] the condition is isofibrancy, see Definition 2.1.41. We say it is mild because many standard examples satisfy this condition (see 2.1.5).

We wish to describe symmetric monoidal double categories/bicategories using the theory of Grothendieck fibrations of categories. It is well known that symmetric monoidal categories can be described as "special Γ-categories". Special Γ categories were introduced in [56, §2]. Mandell shows the equivalence of homotopy theories for various flavours of strict and non-strict symmetric monoidal categories [45, Theorem 3.9]. The homotopy theories of symmetric monoidal categories and special Γ-categories are known to be equivalent [57, Theorem 6.18]. By applying the Grothendieck construction to a Γ-category, we can obtain a fibration over Γ and thus, a symmetric monoidal category can be considered as a fibration over Γ satisfying Segal conditions. This view is considered in Kaledin [35, §4.1]. Toen defines a symmetric monoidal bicategory as a special Γ object in the category of Segal categories [65, §2]. Applying Grothendieck construction to Toen's definition, we are led to the notion of *Segal fibrations over* $\triangle \times \Gamma$ which are fibrations over $\triangle \times \Gamma$ that satisfy Segal conditions along $\triangle$ as well as Γ. We conjecture that the bicategory of symmetric monoidal double categories is biequivalent to the bicategory of Segal fibrations over $\triangle \times \Gamma$ (2.4.6). Here is a schematic of the various actors in this chapter and their relations:

$$\begin{tikzcd}
& \mathbf{SegFib}(\triangle \times \Gamma) \arrow[rrr, "P"] \arrow[dl] \arrow[dd, squiggly, no head] & & & \mathbf{PSegFib}(\triangle \times \Gamma) \arrow[dl] \arrow[dd, dotted] \\
\mathbf{SegFib}(\triangle) \arrow[rrr, "P"] \arrow[dd, dashed, red, no head, "N"'] & & & \mathbf{PSegFib}(\triangle) \arrow[dd, no head, "N"] & \\
& \mathbf{SMDbl} \arrow[rrr] \arrow[dl] & & & \mathbf{SMBicat} \arrow[dl] \\
\mathbf{Dbl} \arrow[rrr, "\mathcal{H}"] & & & \mathbf{Bicat}_2 &
\end{tikzcd}$$

The red dashed arrow is proven to be a biequivalence in this chapter. A part of the conjecture is the existence of a bicategory structure on **SMBicat** and **SMDbl**. The conjecture is the biequivalence of the dotted arrow equipped with a compatible pinching construction (i.e. commutativity of the squares of the cube).

Suppose Segal morphisms of a Segal fibration over $\triangle \times \Gamma$ are isomorphisms in the Γ direction. In this case, a symmetric monoidal pseudo double category can be constructed (see 2.4.11) and then applying $\mathcal{H}$, a symmetric monoidal bicategory is obtained (see Theorem 2.4.13).

Organisation

In Section 2.1, we introduce the preliminaries required to understand the rest of the chapter. It contains quick descriptions of fairly standard material on fibrations in Subsection 2.1.3, bicategories in Subsection 2.1.2 and nerves in Subsection 2.1.4. It also contains an introduction to pseudo double categories in Subsection 2.1.5, Shulman's symmetric monoidal structures on pseudo double categories in Subsection 2.1.6 and Fiore-Paoli-Pronk's results about strict nerves in Subsection 2.1.7.

In Section 2.2, we discuss Segal categories and prove the biequivalence of the 2-category of isofibrant pseudo double categories with the 2-category of Segal categories via the nerve construction (Theorem 2.2.8).

Section 2.3 discusses the definition of Segal fibration over $\triangle$. In Subsection 2.3.1, we discuss pinched Segal fibration corresponding to bicategories.

In Section 2.4, Segal fibrations over $\triangle \times \Gamma$ are discussed. The main thrust of this section is the conjecture about the biequivalence of Segal fibrations over $\triangle \times \Gamma$ and symmetric monoidal double categories (see Conjecture 2.4.6). In Subsection 2.4.1, the construction of a symmetric monoidal double category from strict Segal fibration over $\triangle \times \Gamma$ is presented. This is the well-known construction of a commutative monoid object in a cartesian monoidal category **Dbl**. In Subsection 2.4.2, a construction of a symmetric monoidal bicategory from a strict Segal fibration over $\triangle \times \Gamma$ is presented.

2.1 Preliminaries

In this section, we collect the preliminary materials required for stating and proving the results in the subsequent sections. There are no new results in this section. The content of Subsections 2.1.1,2.1.2, 2.1.3 is fairly standard. The rest of the subsections are on double categories, and some of it is not standard. In Subsection 2.1.5, we discuss pseudo double categories, and in Subsection 2.1.6, we define Shulman's symmetric monoidal double categories. Subsection 2.1.7 discusses the preliminaries of a double categorical generalization of discussion in Subsection 2.1.4. The content of this subsection is from Fiore, Paoli and Pronk [18].

2.1.1 The categories $\triangle$, Γ

All the materials in this section are standard, and the purpose of this subsection is to collect notations in a single place.

Definition 2.1.1. Let $\triangle$ denote the category whose

1. objects are total ordered sets of the form $[n] = \{0 < 1 < 2 < \cdots < n\}$ for a natural number n. We will interchangeably think of $[n]$ as a category whose object set is $\{1, 2, 3, \cdots, n\}$ and a unique morphism from i to j for every i, j such that $0 \leq i < j \leq n$.
2. morphisms are nondecreasing maps. When the objects are considered as categories, the morphisms will be considered as functors.

Throughout this thesis, the following notations will be used:

1. For $0 \leq l \leq n - k$, define the morphism $i_l : [k] \to [n]$ by the assignment
$$i_l(j) = j + l.$$
Clearly it is an order preserving injection.
2. The morphism $\partial_i : [n-1] \to [n]$ is the unique order preserving injection that does not have i in the range. The morphism $s_0 : [1] \to [0]$ is the unique morphism from $[1]$ to $[0]$ in $\triangle$.

Note that $Fun(\triangle^{op}, Cat)$ represents the *set* of functors, $\underline{Fun}(\triangle^{op}, Cat)$ represents the *category* of functors and natural transformations and $\mathbf{Fun}(\triangle^{op}, \mathbf{Cat})$ represents the bicategory of pseudofunctors, pseudonatural transformations and modifications.

Next, we define the category Γ:

Definition 2.1.2. Let Fin_* be the skeletal category of finite pointed sets and pointed maps with objects
$$n_+ := \{*, 1, 2, 3, \cdots, n\}$$
for every natural number n. The opposite category Fin_*^{op} is usually denoted by Γ.

Throughout this thesis, the following notations will be used:

1. For $1 \leq i \leq n$, let $i^n_k : n_+ \to 1_+$ be defined by $i^n_k(j) = 1$ iff $j = k$. Maps injective on the domain minus the base point are called *inert maps*. The collection of maps i^n_k constitutes all the inert maps from n_+ to 1_+.
2. Maps, where only the base point goes to the base point, are called active maps. We will also use the following notation for *active maps*: for a map $n_+ \to m_+$, if the preimage of k is I_k, then we will denote the map by $u_{I_1,I_2,...}$. We will write the set I_k as a string in increasing order.

2.1.2 Bicategories

Benabou introduced bicategories [6]. While the theory of bicategories is a prerequisite for reading this thesis, we collect some definitions and note some conventions so that we may use them in the future. The reader unfamiliar with bicategories is urged to refer to [33] whenever needed. A quick primer due to Leinster is also recommended [40].

First, we record the definition of

Definition 2.1.3. A *bicategory* $\mathcal{B}$ consists of

- a collection of *objects* denoted by $a, b, c, \cdots$
- for any two objects a, b, a category $\mathcal{B}(a, b)$ of *morphisms*. An object f of this category is called a 1-*morphism* from a to b and denoted by $f : a \to b$. A morphism $\alpha : f \to g$ of $\mathcal{B}(a, b)$ is called a 2-*morphism* from f to g. The composition of 2-morphisms in a category of morphisms is called *vertical composition*.
- for any three objects a, b, c, there is a *horizontal composition* functor

$$\circ : \mathcal{B}(a, b) \times \mathcal{B}(b, c) \to \mathcal{B}(a, c)$$

such that for any three 1-morphisms of the type $f : a \to b, g : b \to c, h : c \to d$, there are natural 2-isomorphisms $h \circ (g \circ f) \simeq_\alpha (h \circ g) \circ f$ (called *an associator isomorphism*) and $f \circ 1_a \simeq_\rho f \simeq_\lambda 1_b \circ f$ (called *right* and *left unitor isomorphisms* respectively) such that: 1) Pentagon axiom: The diagram

$$\begin{tikzcd}
k \circ (h \circ (g \circ f)) \arrow[r] \arrow[dd] & k \circ ((h \circ g) \circ f) \arrow[dr] & \\
& & (k \circ (h \circ c)) \circ f \arrow[dl] \\
(k \circ h) \circ (g \circ f) \arrow[r] & ((k \circ h) \circ g) \circ f &
\end{tikzcd}$$

where all the arrows are associator isomorphisms commutes.

2) Triangle axiom: The following diagram

$$\begin{tikzcd}
(g \circ 1_b) \circ f \arrow[dr, "\rho"'] & & g \circ (1_b \circ f) \arrow[ll, "\alpha"'] \arrow[dl, "\lambda"] \\
& g \circ f &
\end{tikzcd}$$

commutes.

We will only consider the homomorphisms in [40, Variant 1.1], calling them *2-functors*. We also consider strong transformations in [40, Variants 1.2], calling them *pseudonatural transformations*. Pseudonatural transformations that are invertible up to a modification are *pseudonatural equivalences*.

Definition 2.1.4. A category is *discrete* if the set of morphisms of the category is only identity morphisms. A bicategory is *discrete* if the set of 2-morphisms of the bicategory are only identity 2-morphisms.

Definition 2.1.5. Let *Bicat* denote the category of bicategories and 2-functors. Given a category $\mathcal{C}$, let $D\mathcal{C}$ denote the discrete bicategory with the same objects and 1-morphisms as the category $\mathcal{C}$. It is easy to see that the construction $D(\_)$ is functorial.

We will need the following terms:

Definition 2.1.6. A collection of definitions about various kinds of 2-functors.

1. Given a 2-functor $F : \mathcal{B} \to \mathcal{A}$, we say F is a *bequivalence* if there is a 2-functor $G : \mathcal{A} \to \mathcal{B}$ and there are pseudonatural equivalences $FG \to 1_{\mathcal{A}}$, $GF \to 1_{\mathcal{B}}$.
2. Given a 2-functor $F : \mathcal{B} \to \mathcal{A}$, we say F is *fully faithful* if F induces an equivalence on Hom categories.

3. Given a 2-functor $F : \mathcal{B} \to \mathcal{A}$, we say F is an *embedding* if F is fully faithful and F is an injection on the set of objects.

4. Given a bicategory $\mathcal{B}$, let B_0 denote a subset of objects of $\mathcal{B}$. The full subbicategory with B_0 as objects is a bicategory $\tilde{B_0}$ with an embedding $\tilde{B_0} \to \mathcal{B}$.

5. Given a 2-functor $F : \mathcal{B} \to \mathcal{A}$, the *essential image* of F is the set of all objects x in $\mathcal{A}$ such that there is an object b in $\mathcal{B}$ and an equivalence $F(b) \to x$. Sometimes it is also used to denote the full subbbicategory spanned by these objects. But we will never use it in this sense.

6. A 2-functor $F : \mathcal{B} \to \mathcal{A}$ is a *strict left 2-adjoint* to $G : \mathcal{A} \to \mathcal{B}$ if for every pair of objects x, y in $\mathcal{B}$ and $\mathcal{A}$ respectively there is a natural isomorphism of strict 2-categories
$$\mathcal{B}(F(x), y) \simeq \mathcal{A}(x, G(y))$$
which is a strict pseudonatural transformation in x, y.

Some examples of bicategories that will appear in the thesis:

Definition 2.1.7. The bicategory **Cat** is the strict bicategory of categories, functors and natural transformations.

Definition 2.1.8. Let $\mathcal{D}$ be a category. Then the over bicategory $\mathbf{Cat}_{/\mathcal{D}}$ is the strict bicategory of functors $F : \mathcal{C} \to \mathcal{D}$ with 1-morphisms $\alpha : F \to F'$ as a functor $G : \mathcal{C} \to \mathcal{C}'$ such that $F' \circ G = F$. Let $G.G' : F \to F'$, then a 2-morphism $\theta : G \to G'$ is a natural transformation such that $F'(\theta_x) = id_{F(x)}$ for every object x in $\mathcal{C}$.

Definition 2.1.9. The 2-category $s\mathbf{Cat} := \mathbf{Fun}(\triangle^{op}, \mathbf{Cat})$ is defined as the 2-category of simplicial categories, simplicial functors and simplicial natural transformations.

Bicategories, 2-functors and pseudonatural transformations do not form a bicategory (discussed in [38, §3]). In order to fix this issue, Lack defined the notion of an 'icon' (stands for *i*dentity *c*omponent *o*plax *n*atural transformation). We do not define icons here but look at Example 2.1.51 for a double-categorical way of thinking about them. We record the theorem due to Lack.

Theorem 2.1.10. *[38, Theorem 3.2] There is a strict bicategory* $\mathbf{Bicat}_2$ *of bicategories, 2-functors and icons.*

Remark 2.1.11. In [38], Lack proves the existence of a 2-category of bicategories, 'lax 2-functors' and icons. Since we never consider lax 2-functors in this thesis, we will not discuss them further.

Now we discuss the definition of bicategory PBCat (appears in Subsection 3.2.1). In [7, Theorem 3], it is shown that the category of categories with pullbacks and pullback preserving functors is cartesian closed. The following definition is just before [11, Lemma 2.8]

Definition 2.1.12 (Carboni-Johnstone)**.** A *cartesian natural transformation* is a natural transformation if its naturality squares are pullback squares. More precisely,a natural transformation $\eta : F \to G$ (for $F, G : \mathcal{C} \to \mathcal{D}$) is a *cartesian natural transformation* iff for any $f : c \to c'$ the naturality square

$$\begin{tikzcd}
Fc \arrow[r, "Ff"] \arrow[d, "\eta_c"'] & Fc' \arrow[d, "\eta_{c'}"] \\
Gc \arrow[r, "Gf"'] & Gc'
\end{tikzcd}$$

is a pullback square.

It can easily be checked that the composition of pullback preserving functors is pullback preserving. The vertical and horizontal composition of cartesian natural transformations are cartesian natural transformations. This follows from the pasting law for pullback squares. So, in effect, we have a bicategory:

Definition 2.1.13. Let PBCat denote the 2-category of categories with pullback, pullback preserving functors and cartesian natural transformations. We will treat PBCat as a subbicategory of Cat.

2.1.3 Fibrations

Analogous to path lifting criteria in algebraic topology for Serre/Hurewicz fibrations, one can define morphism lifting criteria and fibrations for categories. All the ideas in this section are due to Grothendieck [28] and are fairly standard material in category theory. The interested reader should look at [66, Chapter 3] for a quick exposition and [33, Chapter 9,10] for a detailed and rigorous exposition. We note that the definition of fibration given here is slightly different from the above texts (for instance, it can be found in [35, Definition 1.5]).

Definition 2.1.14. Let $p : \mathcal{C} \to \mathcal{D}$ be a functor. A morphism $f : x \to x'$ is *weak p-cartesian*, if given any morphism $g : x'' \to x'$ with $p(f) = p(g)$, there exists a unique morphism $h : x'' \to x$ such that $p(h) = id$ and $g = f \circ h$.

Note that isomorphisms of $\mathcal{C}$ are weak cartesian morphisms.

Using weak cartesian morphisms, we can define (Grothendieck) fibrations:

Definition 2.1.15. Let $p : \mathcal{C} \to \mathcal{D}$ be a functor. The functor p is a *fibration* if given any object $x \in \mathcal{C}$ and any morphism $\alpha : c \to p(x)$, there exists a weak cartesian morphism $f : \alpha^* x \to x$ in $\mathcal{C}$ such that $p(f) = \alpha$ and composition of weak cartesian morphisms is weak cartesian. Weak cartesian morphisms of a fibration are called *cartesian morphisms*, and a morphism of the form $f : \alpha^* x \to x$ is called a *cartesian lift* of (α, x).

If we demand only isomorphisms to be lifted, then we call functors as *isofibrations.*

Definition 2.1.16. Let $p : \mathcal{C} \to \mathcal{D}$ be a fibration. A collection l of cartesian lifts $l_{f,a} : f^* a \to a$ for every 1-morphism $f : x \to y$ in $\mathcal{D}$ and an object a in $\mathcal{C}$ with $p(a) = y$ is called a *cleavage.* We say it is a *splitting cleavage* if $l_{gf,a} = l_{f,g^*a} l_{g,a}$.

Note that every fibration has a cleavage.

Definition 2.1.17. Let $p : \mathcal{C} \to \mathcal{D}$ be a fibration. Let x be an object of $\mathcal{D}$. The pullback $x \times_{\mathcal{D}} \mathcal{C}$ is called the *fibre* of p at x and denoted by $\mathcal{C}_x$.

A *fibration in groupoids* is a fibration whose fibres are all groupoids.

Remark 2.1.18. Isofibrations are also called “functors with i.p.l (invertible path lifting) property” in [34].

The following is the motivating example of fibrations and explains why the term ‘cartesian’ was coined.

Example 2.1.19. Let $\mathcal{C}$ be a category with pullbacks and let $\partial_1 : [0] \to [1]$ denote the map $\partial_1(0) = 1$. Consider the induced functor

$$\partial_1^* : \underline{Fun}([1], \mathcal{C}) \to \underline{Fun}([0], \mathcal{C}).$$

The morphisms in $\underline{Fun}([1], \mathcal{C})$ are commutative squares in $\mathcal{C}$. It can be checked that cartesian morphisms are cartesian squares (i.e. pullback squares). The existence of cartesian lifts corresponds to the existence of pullbacks in $\mathcal{C}$.

Now, we discuss the morphism of fibrations.

Definition 2.1.20. Let $p_1 : \mathcal{C}_1 \to \mathcal{D}$ and $p_2 : \mathcal{C}_2 \to \mathcal{D}$. A morphism of fibration over $\mathcal{D}$ is a functor $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ such that $p_2 \phi = p_1$ and ϕ maps cartesian morphisms to cartesian morphisms.

Definition 2.1.21. Given a category $\mathcal{D}$, let $\mathbf{Cat}_{/\mathcal{D}}$ denote the bicategory of categories over $\mathcal{D}$ (Definition 2.1.8). The bicategory $\mathrm{Fib}(\mathcal{D})$ is the subbicategory of $\mathbf{Cat}_{/\mathcal{D}}$ whose objects are fibrations over $\mathcal{D}$, 1-morphisms are morphisms of fibrations and 2-morphisms are 2-morphisms of $\mathbf{Cat}_{/\mathcal{D}}$.

Next, we discuss the Grothendieck construction (See [66, Chapter 3] for an exposition).

Definition 2.1.22. Let $\mathcal{D}$ be a category that is viewed as a bicategory with only identity 2-morphisms and let $\mathbf{Fun}(\mathcal{D}^{op}, \mathbf{Cat})$ denote the bicategory of 2-functors, pseudonatural transformations and modifications (See [40, Section 1.3] for modifications). An object of the bicategory $\mathbf{Fun}(\mathcal{D}^{op}, \mathbf{Cat})$ is called a *pseudofunctor*.

Construction 2.1.23. From [66, Proposition 3.11], a fibration $p : \mathcal{C} \to \mathcal{D}$ with a cleavage l(See Definition 2.1.16) induces a pseudofunctor $P_l : \mathcal{D}^{op} \to \mathbf{Cat}$ such that $P_l(x) = \mathcal{C}_x$ and on morphisms $f : x \to y$, $P_l(f) = f^*$ (specified by the cleavage). If $f : x \to y, g : y \to z$ are morphisms in $\mathcal{D}$ and a is an object of $P_l(z)$, then $f^*g^*a \simeq (g \circ f)^*a$ is the canonical isomorphism arising from the cartesian lift $(g \circ f)^*a \to a$. The assignment

$$\Phi : \mathrm{Fib}(\mathcal{D}) \to \mathbf{Fun}(\mathcal{D}^{op}, \mathbf{Cat})$$

that maps a fibration $p : \mathcal{C} \to \mathcal{D}$ to the pseudofunctor P_l for the choice of some cleavage l, is a 2-functor. If the cleavage is splitting, then P_l is a functor (instead of *pseudo*functor).

The following theorem and its proof can be found in [33, Theorem 10.6.16].

Theorem 2.1.24 (Grothendieck)**.** *The 2-functor*

$$\Phi : \mathit{Fib}(\mathcal{D}) \to \mathbf{Fun}(\mathcal{D}^{op}, \mathbf{Cat})$$

is a biequivalence of bicategories.

Construction 2.1.25. An inverse 2-functor to Φ in Theorem 2.1.24, denoted by $\int$, is described in [33, Proposition 10.3.22]. This 2-functor is called the *Grothendieck construction.*

Now we describe the Grothendieck construction. Given a pseudofunctor $P : \mathcal{D}^{op} \to \mathbf{Cat}$, the category $\int P$ (called the *category of elements of* P) has

1. the collection of pairs (a, x) as objects where a is an object of $\mathcal{D}$ and x is an object of the category $P(x)$.
2. the collection of pairs (f, g) as morphisms. Suppose $(f, g) : (a, x) \to (b, y)$, then $f : a \to b$ is a morphism in $\mathcal{D}$ and $g : x \to Pf(y)$ is a morphism in $P(a)$.

The functor $p : \int P \to \mathcal{D}$ defined by $p(x, a) = x$ is a fibration, and the assignment $P \mapsto \int P$ is the inverse to Φ.

Now, we will see an interesting application of isofibrations from [34].

Construction 2.1.26. Given a diagram of categories

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "G"'] \\ \mathcal{E} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

Let P denote the pullback in the category of categories. There is a notion of a *homotopy pullback* (called *pseudopullback* in [34]) H whose objects are (a, x, b) where a, b are objects from $\mathcal{C}$ and $\mathcal{E}$ respectively and $x : G(a) \to F(b)$ is an isomorphism. A morphism $u : (a, x, b) \to (c, y, d)$ is a pair of maps $(u_1 : a \to c, u_2 : b \to d)$ so that $F(u_2)x = yG(u_1)$. The composition is a componentwise composition of morphisms.

The comparison functor $\phi : P \to H$, which sends (a, b) to (a, id, b) is fully faithful.

So when do homotopy pullbacks coincide with pullbacks? The answer is when one of the morphisms is an isofibration as Joyal and Street [34, Theorem 1] show:

Proposition 2.1.27 (Joyal-Street)**.** *The comparison functor ϕ (from Construction 2.1.26) is an equivalence for all functors F iff G is an isofibration.*

2.1.4 Nerve of a category

Definition 2.1.28. The category of functors $\underline{Fun}(\triangle^{op}, Sets)$ is the category of simplicial sets (denoted by $sSets$).

Example 2.1.29. For a category $\mathcal{C}$, the assignment

$$NC([n]) := \mathrm{Fun}([n], \mathcal{C})$$

gives a simplicial set NC associated to a category $\mathcal{C}$. The construction $N : Cat \to \mathrm{sSets}$ is the *nerve* of the category $\mathcal{C}$.

Now consider the following diagram in $\triangle$:

$$\begin{tikzcd}
{[0]} \arrow[r, "i_0"] \arrow[d, "i_n"'] & {[m]} \arrow[d, "i_n"] \\
{[n]} \arrow[r, "i_0"'] & {[m+n]}
\end{tikzcd}$$

where $i_k(j) = j + k$.

The following definition of a Segal set (not named as such) and the characterization that follows first appeared in Grothendieck's FGA [27, §4.1].

Definition 2.1.30. A *Segal set* X is a simplicial set such that for every pair of integers $m, n \geq 0$

$$\begin{tikzcd}
X_{n+m} \arrow[r, "i_n^*"] \arrow[d, "i_0^*"'] & X_m \arrow[d, "i_0^*"] \\
X_n \arrow[r, "i_n^*"'] & X_0
\end{tikzcd}$$

is a pullback square in *Sets*. The pullback square condition is called a *Segal condition*.

Proposition 2.1.31 (Grothendieck [27])**.** *The functor*

$$N : Cat \to sSets$$

is fully faithful, and the essential image is the set of Segal sets.

In Gabriel and Zisman's book [22, §4.1], a left adjoint c is constructed to the nerve N.

We need some preparations to describe it. We follow Fiore,Paoli,Pronk [18].

Definition 2.1.32. A tuple $X = (X_1, X_0, s, t : X_1 \to X_0, u : X_0 \to X_1)$ is called a *reflexive directed graph* where X_0, X_1 are sets and s, t, u are functions such that compositions tu and su are identities.

Elements of X_1 are called *edges*, and elements of X_0 are called *vertices*. If f is an edge with $s(f) = a$ and $t(f) = b$, we denote the edge by $f : a \to b$ or $a \xrightarrow{f} b$. We will denote $u(b) = 1_b$ for a vertex b.

The collection of reflexive directed graphs forms a category *RefGraphs*.

Construction 2.1.33. Given a category $\mathcal{C}$, forgetting the composition gives a reflexive graph $U\mathcal{C}$. This construction upgrades to a functor $U : Cat \to RefGraphs$, which has a left adjoint F. Given a reflexive graph X, FX is a category with X_0 as objects. Elements of X_1 freely generate the morphisms of FX. See [63, §1] for a detailed description of FX.

Construction 2.1.34. Let X be a simplicial set and let $\tilde{X}$ denote the reflexive graph given by the data: $\partial_0^*, \partial_1^* : X_1 \to X_0$ and $s_0^* : X_0 \to X_1$.

Now we describe the construction of the left adjoint to the nerve given in Gabriel-Zisman's book (but we follow [18]). We have to quotient a category in the sequel, so we need the notion of a congruence (See [18, Definition 3.9]):

Definition 2.1.35. A *congruence* on a category $\mathcal{C}$ is a collection of equivalence relations $\sim$ on the morphism sets $\mathcal{C}(a, b)$ for every pair $a, b \in \mathcal{C}$ such that if f, g are composable, $f \sim f'$ and $g \sim g'$ then $fg \sim f'g'$.

Example 2.1.36. Let $F : \mathcal{C} \to \mathcal{D}$, define $f \sim f'$ iff $F(f) = F(f')$. Then $\sim$ is a congruence by the functoriality of F.

The following definition is from [18, Definition 3.14]

Definition 2.1.37. Let $\mathcal{C}$ be a category and $\sim$ be a congruence on $\mathcal{C}$. Then the *quotient category* $\mathcal{C}/\sim$ has the same objects as $\mathcal{C}$. Define

$$(\mathcal{C}/\sim)(a, b) = \mathcal{C}(a, b)/\sim .$$

The composition in $\mathcal{C}/\sim$ is induced from $\mathcal{C}$. We say $\mathcal{C}/\sim$ is the category "$\mathcal{C}$ modulo the congruence $\sim$".

Note that there is a canonical functor $\pi : \mathcal{C} \to \mathcal{C}/\sim$ with $\pi(f) = \pi(f')$ if $f \sim f'$ and this functor is universal for this property.

Construction 2.1.38. Let X be a simplicial set and let $\tilde{X}$ denote the reflexive graph described in Construction 2.1.34. The *categorification* of X is the free category (Construction 2.1.33) on $\tilde{X}$ modulo the smallest congruence $\sim$ such that for any $t \in X_2$ with $\partial_2^*(t) = f, \partial_0^*(t) = g$ and $\partial_1^*(t) = h$ we have $g \circ f \sim h$. In [18, Proposition 6.1], it is proved that c is left adjoint to N.

2.1.5 Double categories

In this subsection, we quickly recall the basics of double category theory. Double categories are originally due to Ehresmann [15], who proposed a strict version of the definition. We will follow Grandis and Pare [26] and adopt a weaker notion called *pseudo double categories*. We follow [59] for the exposition. A technical notion of isofibrant (and fibrant) pseudo double category is discussed, and we verify isofibrancy in standard examples of pseudo double categories.

Definition 2.1.39. (Ehresmann-Grandis-Pare, [59, Definition 2.1]) The following data gives a pseudo double category $\mathbb{D}$:

1. A pair of categories $(\mathbb{D}_0, \mathbb{D}_1)$ equipped with functors S, T, U

$$\begin{tikzcd} \mathbb{D}_1 \arrow[r, shift left=2, "S"] \arrow[r, shift right=2, "T"'] & \mathbb{D}_0 \arrow[l, "U" description] \end{tikzcd}$$

2. A 'loose composition' functor $\odot : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$

3. Natural isomorphisms $(\mathfrak{a}, \mathfrak{r}, \mathfrak{l})$ given by

$$\mathfrak{a} : (M \odot N) \odot P \xrightarrow{\cong} M \odot (N \odot P) \qquad \text{associator} \tag{2.1.1}$$

$$\mathfrak{r} : M \odot U(A) \xrightarrow{\cong} M \tag{2.1.2}$$

$$\text{right unitor}\mathfrak{l} : U(B) \odot M \qquad \xrightarrow{\cong} M\text{left unitor} \tag{2.1.3}$$

satisfying the properties:

$$SU(A) = A \tag{2.1.4}$$

$$TU(A) = A \tag{2.1.5}$$

$$S(M \odot N) = SN \tag{2.1.6}$$

$$T(M \odot N) = TM \tag{2.1.7}$$

The S and T functors map the components of associators and unitors to identity, and these natural isomorphisms satisfy the pentagon and triangle equations.

If the associator and unitors are identities, then we say that the pseudo double category is *strict*.

Now, we discuss the terminologies used in double category theory.

$$\begin{tikzcd}
A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
C \arrow[r, "N"', "\shortmid"{marking}] & D
\end{tikzcd}$$

FIGURE 2.1: A fundamental morphism of a pseudo double category.

Definition 2.1.40. Given a pseudo double category $\mathbb{D}$,

1. the categories $\mathbb{D}_0$ and $\mathbb{D}_1$ are called *category of objects* (or 'object category of $\mathbb{D}$') and *category of arrows* (or 'arrow category of $\mathbb{D}$') respectively.
2. the objects and morphisms of $\mathbb{D}_0$ are called *objects* and *tight 1-morphisms* of $\mathbb{D}$.
3. the objects and morphisms of $\mathbb{D}_1$ are called *loose 1-morphisms* and *2-morphisms* of the pseudo double category.
4. a morphism $\mathfrak{b}$ in $\mathbb{D}_1$ is *globular* if $S(\mathfrak{b}) = T(\mathfrak{b}) = id$.

Note that the associators and unitors are globular according to Definition 2.1.39.

Next, we discuss diagrams in a pseudo double category: The objects of the category $\mathbb{D}_1$ represent horizontal 1-morphisms with a dash and the morphisms of $\mathbb{D}_1$ by 2-morphisms. A fundamental 2-morphism is shown in Figure 2.1.

Note that an arrow with a dash on it denotes a loose 1-morphism. The composition of loose 1-morphisms is not associative on the nose, unlike tight 1-morphisms. In the figure, $S(M) = A, T(M) = B, S(N) = C, T(N) = D$, $S(\alpha) = f, T(\alpha) = g$.

The composition of tight 1-morphisms is obtained from composition in $\mathbb{D}_0$. The composition of loose 1-morphisms is obtained from the functor $\odot$ acting on objects. The 2-morphisms can be composed along the tight morphisms using the composition in $\mathbb{D}_1$ and along the loose morphisms using the action of $\odot$ on morphisms. We will line up composable loose 1-morphisms along the horizontal direction and composable tight 1-morphisms along the vertical direction. The functoriality of $\odot$ states that horizontal and vertical composition commute.

Now we define fibrancy conditions on pseudo double categories from [59, Remark 3.22].

Definition 2.1.41. A pseudo double category $\mathbb{D}$ is

$$\begin{tikzcd} A \arrow[r, dashed, "M"] \arrow[d, "f"'] \arrow[d, phantom, shift left=6, "\Downarrow \alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "|" marking] & D \end{tikzcd}$$

FIGURE 2.2: The lifting criteria diagram associated with the (iso)fibration (S, T).

1. *isofibrant* if the functor

   $$\mathbb{D}_1 \xrightarrow{S,T} \mathbb{D}_0 \times \mathbb{D}_0$$

   is an isofibration (Definition 2.1.15).

2. *fibrant* if the functor

   $$\mathbb{D}_1 \xrightarrow{S,T} \mathbb{D}_0 \times \mathbb{D}_0$$

   is a fibration (Definition 2.1.15).

In [59], the definition of fibrant pseudo double categories looks different. The definitions are equivalent, which is proved in [58, Theorem 4.1].

Remark 2.1.42 (Source and Target fibrations). As per Definition 2.1.41 and Definition 2.1.15, in a *isofibrant* pseudo double category, if given a horizontal morphism $C \xrightarrow{N} D$ and vertical isomorphisms $A \xrightarrow{f} C, B \xrightarrow{g} D$, there exists an horizontal morphism $A \xrightarrow{M} B$ and a 2-morphism $M \xrightarrow{\alpha} N$ such that $S(\alpha) = f, T(\alpha) = g$. If the above property holds for all vertical morphisms f, g (i.e. not merely isomorphisms), then the pseudo double category is *fibrant*.

This situation is depicted in Figure 2.2 where the solid arrows are given and dashed arrows are the lifts.

As we will see in the examples below, the source-target isofibration condition is satisfied by natural examples. This condition guarantees that a homotopy pullback of the diagram

$$\begin{tikzcd} & \mathbb{D}_1 \arrow[d, "S"] \\ \mathbb{D}_1 \arrow[r, "T"] & \mathbb{D}_0 \end{tikzcd}$$

coincide with strict pullbacks (Proposition 2.1.27). It will also be useful in proving Proposition 2.2.6. Henceforth, while proving that (S, T) is a (iso)fibration, we will use the right-hand side of the lifting criteria diagram shown in Figure 2.2.

Definition 2.1.43. Let $\mathbb{D}$ and $\mathbb{E}$ be two pseudo double categories. A (horizontal) pseudo double functor $F : \mathbb{D} \to \mathbb{E}$ is given by the following:

1. A pair of functors $F_0 : \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1 : \mathbb{D}_1 \to \mathbb{E}_1$ that commute with source and target maps of the pseudo double categories.
2. A natural globular isomorphism, called the "composition coherence data", $F^{\odot} : F_1 \odot F_1 \to F_1 \circ \odot$ where the functors are between $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ and $\mathbb{E}_1$. The natural isomorphism satisfies the compatibility with the associators of each pseudo double category for composing three horizontal morphisms.
3. A natural globular isomorphism, called "unit coherence data", $F^U : U \circ F_0 \to F_1 \circ U$ that is compatible with the left unitor, right unitor and associators of both pseudo double categories for composing horizontal morphisms on either side of horizontal units.

If the globular isomorphisms are identities, then we say that the pseudo double functor is *strict*.

Definition 2.1.44. Let $\mathbb{D}$ and $\mathbb{E}$ be two pseudo double categories. Let $F, G : \mathbb{D} \to \mathbb{E}$ be pseudo double functors. A vertical (natural) transformation θ is given by the following:

1. A pair of natural transformations $\theta^0 : F_0 \to G_0$ and $\theta^1 : F_1 \to G_1$ such that $S \circ \theta^1 = \theta^0 \circ S$, $T \circ \theta^1 = \theta^0 \circ T$.
2. The natural transformations satisfy the following coherence laws: Given any two composable horizontal morphisms M, N and any object A in $\mathbb{D}$:

$$\theta^1_{M \odot N} \circ F^{\odot}_{M,N} = G^{\odot}_{M,N} \circ (\theta^1_M \odot \theta^1_N)$$
$$\theta^1_{UA} \circ F^U_A = G^U_A \circ U_{\theta^0_A}$$

The composition of the components defines the composition of pseudo double functors and vertical transformations. Note that composition of pseudo double functors is strictly associative.

Definition 2.1.45. The symbol **Dbl** denotes the 2-category of isofibrant pseudo double categories, pseudo double functors and vertical transformations.

We will see that isofibrancy conditions are satisfied in some standard examples.

A theorem due to Grandis and Paré in [26, Theorem 7.5] states that every pseudo double category is equivalent to a strict double category, and every pseudo double functor is equivalent to a strict double functor. We state this theorem below so that we can use it later.

$$\begin{tikzcd}
c_1 \arrow[d] & c_2 \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] & c_3 \arrow[d] \\
c_1' & c_2' \arrow[l] \arrow[r] & c_3'
\end{tikzcd}$$

FIGURE 2.3: A fundamental morphism of the pseudo double category span(C) is a commuting diagram.

Theorem 2.1.46 (Grandis-Paré)**.** *For every pseudo double category $\mathbb{D}$ there exists a strict double category $\mathbb{E}$ and a pair of pseudo double functors $F : \mathbb{D} \to \mathbb{E}$ and $G : \mathbb{E} \to \mathbb{D}$ such that $GF = id$ and $FG \simeq id$. Further, every pseudo double functor $F : \mathbb{D} \to \mathbb{E}$ can be replaced by an equivalent strict functor $F' : \mathbb{D}' \to \mathbb{E}'$.*

We now define a bicategory associated with a pseudo double category, which is important in constructing symmetric monoidal bicategories (Theorem 2.1.54).

Definition 2.1.47. Given a pseudo double category $\mathbb{D}$, the *horizontal bicategory* $\mathcal{H}(\mathbb{D})$ is a bicategory consisting of the objects, 1-morphisms, and globular 2-morphisms of the pseudo double category $\mathbb{D}$.

The examples below will show bicategories arising as horizontal bicategories of pseudo double categories. The following examples follow Shulman's exposition [59] except for the discussion on isofibrancy in each example.

Example 2.1.48. Given a category C with finite limits, define the pseudo double category, following Grandis and Pare [26, §3.2], span(C) with C as the object category and the objects of the arrow category are spans in C, and arrows are morphisms of spans. A fundamental morphism of span(C) is shown in Figure 2.3.

The horizontal composition of horizontal 1-morphisms $c_1 \xleftarrow{f} c_2 \xrightarrow{g} c_3$ and $c_3 \xleftarrow{h} c_4 \xrightarrow{k} c_5$ is defined using a pullback:

$$\begin{tikzcd}
 & & c_2 \times_{c_3} c_4 \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & c_2 \arrow[dl, "f"'] \arrow[dr, "g"] & & c_4 \arrow[dl, "h"'] \arrow[dr, "k"] & \\
c_1 & & c_3 & & c_5
\end{tikzcd}$$

The horizontal composition is a choice of a pullback $c_1 \leftarrow c_2 \times_{c_3} c_4 \to c_5$. Associators and unitors are canonical morphisms between limit diagrams. The pentagonal and triangle

$$\begin{tikzcd}
A \arrow[rr, dashed, "{_A(C \underset{C}{\otimes} N \underset{D}{\otimes} D)_B}"] \arrow[d, "f"'] & {} \arrow[d, Rightarrow, dashed] & B \arrow[d, "g"] \\
C \arrow[rr, "{_CN_D}"'] & {} & D
\end{tikzcd}$$

Figure 2.4: A lifting diagram for the Morita category Mor.

$$\begin{tikzcd}
A \arrow[rr, dashed, "f"] \arrow[d, equal] & {} \arrow[d, Rightarrow, dashed, "id_f"] & B \arrow[d, equal] \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}$$

Figure 2.5: A lift of a morphism showing that source and target are fibrations for the pseudo double category HB.

laws follow from the uniqueness of canonical morphisms between limit diagrams. The horizontal bicategory $\mathcal{H}(\text{span}(C))$ is the bicategory of spans considered by Benabou [6, §2.6].

Example 2.1.49. A Morita pseudo double category consists of rings as objects, ring homomorphisms as tight 1-morphisms, bimodules as loose 1-morphisms and equivariant maps of bimodules as 2-morphisms. A loose 1-morphism $M : A \to B$ is a $A{-}B$ bimodule M. The tensor product of bimodules gives the horizontal composition.

The associator, the unitors constructed form the universal properties of the tensor product. The pentagonal and triangle laws follow from the canonicalness of universal arrows.

The source and target functors are fibrations, as seen in Figure 2.4. The horizontal lift is essentially a restriction of bimodules. Note that, in this case, the source and target functors are bifibrations.

In this case, the horizontal bicategory is the well-known bicategory of rings, bimodules and equivariant maps of bimodules (often called the Morita bicategory).

The last example relates to bicategories and pseudo double categories.

Definition 2.1.50. A *pinched pseudo double category* is a pseudo double category with only globular 2-morphisms.

Example 2.1.51. Given a bicategory $\mathcal{B}$, we can construct a pinched pseudo double category $j\mathcal{B}$ with 1-morphisms of $\mathcal{B}$ as horizontal 1-morphisms of $j\mathcal{B}$ and all vertical 1-morphisms are identity. This construction is 2-functorial, i.e. $j : \mathbf{Bicat}_2 \to \mathbf{Dbl}$ is a 2-functor where $\mathbf{Bicat}_2$ is from Theorem 2.1.10. It turns out that j is an embedding (as discussed in [38, §6.1]), and the essential image consists of pinched pseudo double categories. Thus, if we consider bicategories as pinched pseudo double categories, then 'icons' are vertical natural transformations between pinched pseudo double categories.

In this case, the source and target functors are fibrations (hence isofibrations). See Figure 2.5 where the lift is shown in dashed lines. The horizontal bicategory construction $\mathcal{H}$ is right adjoint to j. Also, note that $\mathcal{H}(j\mathcal{B})$ is $\mathcal{B}$ itself (i.e. the unit of the adjunction is identity).

2.1.6 Symmetric Monoidal Double categories

A notion of symmetric monoidal structure on pseudo double categories is introduced in Shulman [59]. We will follow his notations and exposition . Theorem 2.1.54 is the main tool used to construct symmetric monoidal bicategories out of symmetric monoidal pseudo double categories.

Definition 2.1.52. (Shulman [59, Definition 2.9]) A symmetric monoidal pseudo double category is a pseudo double category $\mathbb{D}$ equipped with functors $\otimes : \mathbb{D}\times\mathbb{D} \to \mathbb{D}$, $I : * \to \mathbb{D}$ and tight natural equivalences

1. $\otimes(\otimes \times 1) \simeq_\alpha \otimes(1 \times \otimes)$.
2. $\otimes(I \times 1) \simeq_\rho id, \otimes(1 \times I) \simeq_\lambda id.$
3. $\otimes\tau \simeq_\beta \otimes$ where $\tau(x, y) = (y, x)$ is the twist functor.

satisfying the pentagon law, triangle law and the hexagon laws.

The symmetric monoidal structure is *strict* if the natural equivalences are identities.

Definition 2.1.53. [67, Definition 2.14] A symmetric monoidal pseudo double functor between symmetric monoidal pseudo double categories $F : \mathbb{D} \to \mathbb{E}$ is a pseudo double functor along with tight natural equivalences

1. $\mathbb{F}\otimes \simeq \otimes(F \times F)$.
2. $FI \simeq I$.

3. $\otimes(F \times F)\tau \simeq \otimes(F \times F)$.

satisfying usual axioms of a symmetric monoidal functor. If the natural equivalences are identities, then we say the symmetric monoidal functor is *strict*.

Given a pseudo double category $\mathbb{D}$, we can extract a bicategory $\mathcal{H}(\mathbb{D})$ (Definition 2.1.47). Recall from Definition 2.1.41, that a pseudo double category $\mathbb{D}$ is *fibrant* if the functor $\mathbb{D}_1 \xrightarrow{S \times T} \mathbb{D}_0^2$ is a fibration of categories.

Theorem 2.1.54 ([59] Theorem 5.1)**.** *If a symmetric monoidal pseudo double category* $\mathbb{D}$ *is fibrant, then* $\mathcal{H}(\mathbb{D})$ *is a symmetric monoidal bicategory.*

This theorem is very useful to construct symmetric monoidal bicategories and was used by Schommer-Pries [54, §3.1.4] to sketch the construction of a symmetric monoidal bicategory of bordisms. The example of symmetric monoidal bicategory of vector space valued profunctors is discussed in Shulman and used in Martins and Porter's paper [17, §6.6.1].

2.1.7 Nerve of a strict double category

The 2-category of pseudo double categories contain a subbicategory of strict double categories and strict double functors which we will denote by $s\mathbf{Dbl}$. In this subsection, we recall results about the strict horizontal nerve (2-functor) $N_s : s\mathbf{Dbl} \to s\mathbf{Cat}$ from Fiore, Paoli and Pronk's paper [18, §5]. The key results are that N_s is fully faithful, N_s has a (2-)left adjoint (see Definition 2.1.6) denoted by c_h (called *horizontal categorification*). These results can be viewed as analogues of results about nerves of categories.

Given a category $[n]$, consider it as a bicategory with only identity 2-morphisms. Then Example 2.1.51 constructs the pseudo double category $j[n]$. A quick reminder that $\mathbf{Cat}$ is the 2-category of categories and *Cat* is the category of categories. The symbol $s\mathbf{Cat}$ stands for the bicategory $\mathbf{Fun}(\triangle^{op}, \mathbf{Cat})$.

The following definition is from [18, Definition 5.1].

Definition 2.1.55. Let $\mathbb{D}$ be a strict double category. The functor

$$Dbl(j(\_), \mathbb{D}) : \triangle^{op} \to \mathbf{Cat}$$

is called the strict nerve $N_s\mathbb{D}$ of $\mathbb{D}$. The 2-functor

$$N_s : s\mathbf{Dbl} \to s\mathbf{Cat}$$

is the strict nerve 2-functor.

Recall the definition of a fully faithful 2-functor from Definition 2.1.6. In [18, Proposition 5.9], they show

Proposition 2.1.56. *The strict nerve 2-functor N_s is a fully faithful 2-functor.*

Recall the definition of strict left 2-adjoint of a 2-functor from Definition 2.1.6. In [18, Theorem 5.6], the following result is proved:

Theorem 2.1.57. *The strict nerve 2-functor*

$$N_s : s\mathbf{Dbl} \to s\mathbf{Cat}$$

has a strict left 2-adjoint c_h (called horizontal categorification).

Construction 2.1.58. Given a simplicial category X, we describe the strict double category c_hX briefly. For full details, refer to [18, Definition 6.3]. Double analogues of congruence (Definition 2.1.35), quotient (Definition 2.1.37) and free category (Construction 2.1.33) are constructed in *loc.cit.*

Let $Obj : Cat \to Sets$ denote the functor that sends a category to its set of objects. Clearly this induces a functor $Obj_* : sCat \to sSets$. Let category of objects of c_hX be X_0. The loose 1-morphisms are morphisms of the categorification $c(Obj_*(X))$ (See Definition 2.1.38). The horizontal categorification c_hX is the double category freely generated by morphisms of X_1 modulo the smallest congruence that satisfies (we will denote the free composite as a string of symbols):

1. If α, β are composable in morphisms of X_1, then the vertical composite $\alpha\beta$ in the free product is related to the composite in X_1.
2. If t is a morphism of X_2 so that $\partial_2^*(t) = f, \partial_0^*(t) = g$ and $\partial_1^*(t) = h$, then we have $gf \sim h$.
3. For any morphism j in X_0, the vertical identity id_j satisfies $id_j \sim s_0^*(j)$ where s_0 is the degeneracy map.
4. For any object f in X_1, the horizontal identity Uf satisfies $Uf \sim \mathrm{id}_f$ where id_f is the identity map for object f in X_1.

Remark 2.1.59. In [18, §3], the notion a free double category and a quotient double category is constructed. The quotient double category generalizes in an analogous way to the construction of a quotient category (Definition 2.1.37).

However some care is needed to define a free double category. If we represent a 2-morphism of a strict double category by a rectangle and represent composable 2-morphisms by pasting rectangles, then a typical composite will look like a rectangle equipped with a subdivision into smaller rectangles. The key issue is that there are rectangles with a subdivision that do not arise as pastings of 2-morphisms in a strict double category. For an example, look at page 1871 of *loc.cit.*

2.2 Segal Categories

In this section, we state the definition of Segal categories. This is categorification of the story discussed in Subsection 2.1.4. Instead of simplicial sets satisfying Segal conditions, we consider simplicial categories satisfying Segal conditions.

Let $\triangle$ denote the skeletal category of finite totally ordered sets (See Subsection 2.1.1). Consider the following diagram in $\triangle$:

$$\begin{tikzcd}
{[0]} \arrow[r, "i_0"] \arrow[d, "i_n"'] & {[m]} \arrow[d, "i_n"] \\
{[n]} \arrow[r, "i_0"'] & {[m+n]}
\end{tikzcd}$$

where $i_k(j) = j + k$.

A simplicial category is a functor $\mathcal{C} : \triangle^{op} \to \mathbf{Cat}$. The notion of a simplicial category $\mathcal{C}$ satisfying Segal conditions was considered by Tamsamani [64] and Simpson (see [60, Section 5.3]).

Definition 2.2.1 (Tamsamani-Simpson)**.** A Segal category X is a simplicial category such that for every pair of integers $m, n \geq 0$

$$\begin{tikzcd}
X_{n+m} \arrow[r, "i_n^*"] \arrow[d, "i_0^*"] & X_m \arrow[d, "i_0^*"] \\
X_n \arrow[r, "i_n^*"] & X_0
\end{tikzcd}$$

is a homotopy pullback square[3] in Cat.

---

[3] See Construction 2.1.26

A Segal category can be constructed from any category.

Example 2.2.2. Let C be a category. The nerve of C, denoted by NC, is a simplicial set and hence a simplicial category (since every set can be considered as discrete category). First we note that NC_n can be characterized as the set of n-composable arrows. Now we check the Segal conditions: If x_n, x_m are n-composable arrows and m-composable arrows respectively, then the condition $x_n|_0 = x_m|_1$ means that x_n and x_m are composable. So we can define the $(n+m)$-composable arrow (x_n, x_m) which restricts to x_n and x_m.

Definition 2.2.3. **SegCat** is defined as the full subbicategory of simplicial categories with Segal categories as objects.

2.2.1 Biequivalence of Segal categories and pseudo double categories

In this section, we prove the biequivalence between the 2-category of Segal categories and 2-category of pseudo double categories (Theorem 2.2.8).

Since the category of sets embed into the category of categories, the 2-category **Cat** embeds into **Dbl**. We call the pseudo double category in the essential image as discrete double categories.

Definition 2.2.4. A pseudo double category whose category of objects and category of arrows have only identity morphisms is called a *discrete double category.*

Let i denote the composite $\triangle \hookrightarrow \mathbf{Cat} \hookrightarrow \mathbf{Dbl}$ which induces

$$i^* : \mathbf{Fun}(\mathbf{Dbl}^{op}, \mathbf{Cat}) \hookrightarrow \mathbf{Fun}(\triangle^{op}, \mathbf{Cat}).$$

Let $Y : \mathbf{Dbl} \hookrightarrow \mathbf{Fun}(\mathbf{Dbl}^{op}, \mathbf{Cat})$ be the 2-Yoneda functor (see [33, Section 8.2] for the definition and facts relating to 2-Yoneda functor), then we have the following definition:

Definition 2.2.5. The nerve functor $N : \mathbf{Dbl} \to \mathbf{Fun}(\triangle^{op}, \mathbf{Cat})$ is defined as

$$N := i^* \circ Y.$$

The main theorem of this section is the biequivalence of **Dbl** and **SegCat** (Theorem 2.2.8).We briefly sketch the strategy. In Proposition 2.2.6, it is shown that the nerve of a isofibrant pseudo double category satisfies the Segal condition. This is useful in showing that the nerve construction of an isofibrant pseudo double category is an object of **SegCat**. This proposition also proves that commutative squares in $\triangle$ that appear in definition of Segal conditions are pushout squares.In Construction 2.1.58, a pseudo double category is built from the data of a Segal category. Along with Proposition 2.2.7, the construction will be used to show the essential surjectivity (onto the essential image) in the main theorem. Finally, the above theorem is proved in Theorem 2.2.8 using 2-Yoneda lemma and assembling all the bits above.

Proposition 2.2.6. *Let $\mathbb{D}$ be an isofibrant double category and let N denote the nerve from Definition 2.2.5 with $N\mathbb{D}_n := N\mathbb{D}([n])$. Then the induced map $\phi : N\mathbb{D}_{m+n} \to N\mathbb{D}_m \times^h_{N\mathbb{D}_0} N\mathbb{D}_n$ is an equivalence of categories.*

Proof. We will first show that ϕ is essentially surjective. Given an object (G, H, η) in the target of ϕ, we construct an object F in the source of ϕ with $\phi(F) \cong (G, H, \eta)$. Define F on objects as

$$F(i) = \begin{cases} G(i) & 0 \leq i \leq m \\ H(i-m) & m < i \leq m+n \end{cases}$$

Before we define F on arrows, we need some preparations. Since $\eta : Gi_n \to Hi_0$ is a pseudonatural isomorphism on $[0]$, we have a vertical isomorphism $\eta_0 : G(m) \to H(0)$. Using the fact that $(S, T) : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is an isofibration, we can lift the isomorphism $(\eta_0, id_{H(1)}) : (G(m), H(1)) \to (H(0), H(1))$:

$$\begin{tikzcd} G(m) \arrow[r, dashed, "F(m{,}m+1)"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\overset{\alpha_m}{\Downarrow}"] & H(1) \arrow[d, "id_{H(1)}"] \\ H(0) \arrow[r, "H(0{,}1)"'] & H(1) \end{tikzcd}$$

The dashed arrows indicate the lifts and the 2-morphism α_m is invertible.

Define F on arrows as

$$F(ij) = \begin{cases} G(ij) & 0 \leq i \leq j \leq m \\ F(m, m+1) \odot G(i, m) & 0 \leq i < m, j = m+1 \\ H(1, j-m) \odot F(m, m+1) \odot G(im) & 0 \leq i < m, m+1 < j \leq m+n \\ F(m, m+1) & i = m, j = m+1 \\ H(1, j-m) \odot F(m, m+1) & i = m, m+1 < j \leq m+n \\ H(i-m, j-m) & m < i \leq j \leq m+n \end{cases}$$

Define the coherence isomorphisms associated to composition $f_{i,j,k} : F(j,k) \odot F(i,j) \to F(i,k)$ using the coherence isomorphisms of the pseudofunctors G, H and identities for distinct i,j,k. The coherences associated to units $f_i : UF_i \to F_{ii}$ is directly inherited from the unit coherences of G and H. For the case when some two indices are equal, i.e. $f_{i,j,j} : F_{j,j} \odot F_{i,j} \to F_{i,j}$, the unital law $(f_j \odot F_{ij})f_{i,j,j} = id$ fixes the value (we have used whiskering notation). Since the isomorphisms are either coherence isomorphisms of pseudofunctors or identities, the the pseudofunctorial coherence conditions will hold.

Therefore we have constructed a pseudofunctor $F : [n+m] \to \mathbb{D}$. Next we show that $\phi(F) \cong (G, H, \eta)$ in the target of ϕ. We construct $\alpha : (Fi_m, Fi_n, id) \to (G, H, \eta)$ as the pair of pseudonatural isomorphisms (α_g, α_h). By construction $Fi_m = G$, and we choose $\alpha_g : Fi_m \to G$ as identity pseudonatural isomorphism. The pseudonatural isomorphism $\alpha_h : H \to Fi_n$ on objects (i.e. $\mathbb{D}_0$) is given by $\alpha_{hi} = id$ for $i \neq 0$ and $\alpha_{h0} = \eta_0^{-1}$. On the arrow category $\mathbb{D}_1$, $\alpha_{hij} = id$ for $i,j > 0$ and $\alpha_{h00} = U\eta_0$ and $\alpha_{h01} = \alpha_m$. The pseudonatural coherences boil down to the relation $\alpha_m \odot U\eta = \alpha_m$ which holds because our pseudo double category $\mathbb{D}$ is strict. For α to be a morphism in the fiber product category, we must have

$$\begin{tikzcd}
Gt \arrow[r, "\eta"] \arrow[d, "\alpha_g t"'] & Hs \arrow[d, "\alpha_h s"] \\
Fi_m t \arrow[r, equal] & Fi_n s
\end{tikzcd}$$

This square commutes in $N\mathbb{D}_0$ since it suffices to evaluate it on the object 0, which gives the commuting square

$$\begin{tikzcd}
Gm \arrow[r, "\eta_0"] \arrow[d, equal] & H0 \arrow[d, "\eta_0^{-1}"] \\
Gm \arrow[r, equal] & Gm
\end{tikzcd}$$

Now it remains to show that ϕ is fully faithful. Given a morphism $\beta : F \to F'$ in $N\mathbb{D}_{m+n}$, $\phi(\beta) = (\beta i_m, \beta i_n)$. If $\phi(\beta) = \phi(\gamma)$, then $\beta i_m = \gamma i_m$ and $\beta i_n = \gamma i_n$. Therefore, β and γ are equal (since they agree pointwise on objects and morphisms) and therefore ϕ is faithful. Given a morphism (γ, δ) between $\phi(F)$ and $\phi(F')$, i.e. $\gamma : Fi_m \to F'i_m$ and $\delta : Fi_n \to F'i_n$ such that $\gamma t = \delta s$, we construct a pseudonatural transformation $\beta : F \to F'$. On objects, define β via the restrictions γ and δ. Since $\gamma t = \delta s$, the isomorphism $\beta_m = \gamma t(0) = \delta s(0)$ is unambiguously defined. On morphisms, $\beta_{i,i+1}$ is given by $\gamma_{i,i+1}$ or $\delta_{i,i+1}$ and the remaining β_{ij} are fixed by the coherences of a pseudonatural transformation. By construction, $\beta i_m = \gamma$ and $\beta i_n = \delta$, therefore $\phi(\beta) = (\gamma, \delta)$ and we have established that ϕ is full. □

Next, we use horizontal categorical functor c_h stated in Theorem 2.1.57 to obtain a pseudo double category c_hX. For the proof of the next proposition, we have to use the description of c_hX given in Construction 2.1.58.

Proposition 2.2.7. *If X is a Segal category, then the unit map $\eta : X \to Nc_hX$ is an equivalence of categories.*

Proof. From the proof of [18, Theorem 6.7], the unit map $\eta : X \to Nc_hX$ is obtained once we have a 2-truncated morphism $\eta_{\leq 2} : \tau_{\leq 2}X \to \tau_{\leq 2}Nc_hX$. This morphism can be seen to be identity on the category of 0-simplices and 1-simplices. For 2-simplices $\eta_2 : X_2 \to X_1 \times_{X_0} X_1$ is the Segal morphism. This implies that the functor $\eta_{\leq 2}$ is an equivalence. Since $N\mathbb{D}$ is 2-coskeletal, this implies the induced map η is an equivalence. □

Theorem 2.2.8. *The nerve construction*

$$N : \mathbf{Dbl} \to \mathbf{Fun}(\triangle^{op}, \mathbf{Cat})$$

is fully faithful and the essential image[4] *is the set of Segal categories.*

Proof. Since the nerve is the restriction of the 2-Yoneda embedding, it is 2-fully faithful (See [33, Lemma 8.3.12]) [5]. By Proposition 2.2.6, the nerve of a pseudo double category is a Segal category. Given a Segal category X, Construction 2.1.58 gives a pseudo double category $c_h(X)$. By Proposition 2.2.7 it follows that the nerve construction is essentially surjective onto its image. □

2.3 Segal Fibrations over $\triangle$

In Theorem 2.2.8, we showed that every pseudo double category arises from a pseudofunctor $D : \triangle^{op} \to \mathbf{Cat}$ satisfying Segal conditions. Applying the Grothendieck construction (Construction 2.1.25) to the pseudofunctor D we obtain fibration over $\triangle$. This motivates the definition of Segal fibrations.

---

[4] The terms essential image and fully faithful have been defined for bicategories in Definition 2.1.6.

[5] The [33, Corollary 8.3.13] states that for strict bicategories, we have an *isomorphism* of Hom categories! In our case this applies since **Dbl** is a strict bicategory.

Definition 2.3.1. A *Segal fibration over* $\triangle$ is a fibration $p : \mathcal{C} \to \triangle$ such that

$$\begin{tikzcd}
\mathcal{C}_{n+m} \arrow[r, "i_n^*"] \arrow[d, "i_0^*"] & \mathcal{C}_m \arrow[d, "i_0^*"] \\
\mathcal{C}_n \arrow[r, "i_n^*"] & \mathcal{C}_0
\end{tikzcd}$$

is a homotopy pullback square [6] in **Cat**. Here $\mathcal{C}_m$ denotes the fibre of p over $[m]$.

Let **SegFib**$(\triangle)$ denote the full subbicategory of fibrations **Fib**$(\triangle)$ with Segal fibrations as objects.

From Theorem 2.1.24, we know that the Grothendieck construction is a biequivalence. This establishes a biequivalence between the bicategories **SegFib**$(\triangle)$ and **SegCat**. And thus a Segal fibration is a model for a pseudo double category. We record this as a theorem.

Theorem 2.3.2. *The composite 2-functor*

$$\mathbf{Dbl} \xrightarrow{N} \mathbf{SegCat} \xrightarrow{\int} \mathbf{SegFib}(\triangle)$$

is a biequivalence. Here N represents the nerve construction (Definition 2.2.5) and $\int$ corresponds to the Grothendieck construction (Construction 2.1.25).

Thus the bicategory **Dbl** *is equivalent to the bicategory* **SegFib**$(\triangle)$*.*

The proof follows from the biequivalence of N (Theorem 2.2.8) and the biequivalence of $\int$ (Theorem 2.1.24).

2.3.1 Pinched Segal Fibrations

Every bicategory can be considered as a pseudo double category (with only globular 2-morphisms) via the 2-functor j (See Example 2.1.51). From Theorem 2.3.2 we know that Segal fibrations over $\triangle$ are equivalent to pseudo double categories via the biequivalence $\int N$.

So we have the following situation:

$$\mathbf{Bicat}_2 \xrightarrow{j} \mathbf{Dbl} \xrightarrow{\int N} \mathbf{SegFib}(\triangle)$$

---

[6] See Construction 2.1.26

We characterize the essential image as the set of pinched Segal fibrations over $\triangle$ of the composite in this subsection.

In Proposition 2.3.9, we construct a pinched Segal fibration out of a given Segal fibration. The pinching construction is the analogue of the horizontal bicategory $\mathcal{H}$. The definition of a pinched Segal fibration in essence goes back to Tamsamani [64]. The main ideas and propositions in this section are already contained in [30, §4] (albeit in an ∞-category setting). In particular, Proposition 2.3.9 is [30, Lemma 4.17].

Definition 2.3.3. A pinched Segal fibration over $\triangle$ is a Segal fibration over $\triangle$, $p : X \to \triangle$ such that the fibre over $[0]$ is a discrete category[7]. The full subbicategory of $\mathbf{SegFib}(\triangle)$ with pinched Segal fibrations over $\triangle$ as objects will be denoted by $\mathbf{PSegFib}(\triangle)$.

The following theorem is a consequence of Theorem 2.3.2 and discussion at Example 2.1.51.

Theorem 2.3.4. *The bicategory* $\boldsymbol{Bicat_2}$ *is biequivalent to* $\boldsymbol{PSegFib}(\triangle)$.

In other words, the essential image of $\int Nj : \mathbf{Bicat}_2 \to \mathbf{SegFib}(\triangle)$ the set of pinched Segal fibrations where j is from Example 2.1.51 and $\int$ is the Grothendieck construction from Construction 2.1.25.

Remark 2.3.5. Kaledin defines a strict bicategory explicitly as it is done in this thesis. Whereas he *defines* a bicategory as a pinched Segal fibration over $\triangle$ [35, Definition 2.5]. In contrast, we prove the equivalence via the $\int Nj$ construction.

Remark 2.3.6. Lack and Paoli show that "2-nerve" of a bicategory is a Segal category $\mathcal{C}$ with a discrete $\mathcal{C}_0$ [39, Theorem 7.2]. After applying the Grothendieck construction, we obtain a pinched Segal fibration. The Proposition 2.3.4 recovers Lack and Paoli's claim.

The next lemma follows from [30, Lemma 4.14].

Lemma 2.3.7. *Let* $p : \mathcal{E} \to \mathcal{C}$ *be a cartesian fibration and let* $j : \tilde{\mathcal{C}} \to \mathcal{C}$ *be a functor with a right adjoint. Let*

$$\begin{tikzcd}
\tilde{\mathcal{E}} \arrow[r, "J"] \arrow[d, "\tilde{p}"'] & \mathcal{E} \arrow[d, "p"] \\
\tilde{\mathcal{C}} \arrow[r, "j"'] & \mathcal{C}
\end{tikzcd}$$

be a pullback square. Then the functor J *has a right adjoint.*

---

[7] See Definition 2.1.4

Using the above lemma, we have a way to construct a pinched Segal fibration from a given Segal fibration. This is also discussed in Kaledin [35, §2.2]. The strategy is to show the inclusion $i : PSegFib(\triangle) \to SegFib(\triangle)$ has a right adjoint by applying Lemma 2.3.7.

We need some preparations. The propositions and proofs are adapted from the ideas presented in the lemmas of [30, §4]. Any simplicial category can be projected to the category of vertices. In other words, the projection $p_0 : sCat \to Cat$ sends $X \mapsto X_0$. The following lemma is the analogue of [30, Lemma 4.15].

Lemma 2.3.8. *The projection $p_0 : SegCat \to Cat$ is a fibration of categories.*

Proof. First we note that p_0 has a right adjoint given by the following right Kan extension.

$$\begin{tikzcd}
{[0]} \arrow[r, "\mathcal{C}"'] \arrow[d, hook, "i"'] & \mathbf{Cat} \\
\triangle^{op} \arrow[ur, dotted, "V=Ran_i(\mathcal{C})"'] &
\end{tikzcd}$$

Note that $\mathcal{C}$ denotes both the functor and its image.

We can compute the Kan extension to be

$$V(\mathcal{C}) = Ran_i(\mathcal{C}) = \lim_{[0]/[n]} \mathcal{C} = \mathcal{C}^{n+1}.$$

Also note that the counit of the adjunction is identity, i.e. $\epsilon_{\mathcal{C}} : p_0V(\mathcal{C}) \to \mathcal{C}$ and the components of the unit

$$\eta_{X,n} : X_n \to Vp_0(X)_n = X_0^{n+1}$$

is induced by all the maps from $[0]$ to $[n]$.

In order to show that p_0 is a fibration, we will show it on the level of simplicial categories itself. Let X be a Segal category, $\mathcal{C}$ be a category with a given functor $\gamma : \mathcal{C} \to p_0(X)$, we have to construct a Cartesian lift.

Consider the following pullback square in sCat:

$$\begin{tikzcd}
X \times_{Vp_0X} V\mathcal{C} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta_X"] \\
V\mathcal{C} \arrow[r, "V\gamma"'] & Vp_0X
\end{tikzcd}$$

We claim that π_1 is the cartesian lift of γ. Let us denote the pullback $X \times_{Vp_0X} V\mathcal{C}$ by $\gamma^*_{\mathcal{C}}(X)$.

First we show that $p_0(\pi_1) = \gamma$. Note that

$$p_0(\pi_2) : p_0(\gamma^*_{\mathcal{C}}(X)) \to p_0V(\mathcal{C})$$

is identity. Indeed $p_0V(\mathcal{C}) = \mathcal{C}$,

$$p_0(\gamma^*_{\mathcal{C}}(X)) = p_0(X) \times_{p_0VX_0} p_0V(\mathcal{C}) = X_0 \times_{X_0} \mathcal{C} = \mathcal{C}$$

and the projection map onto the second coordinate is sent to identity under the above identifications.

Applying p_0 to the pullback diagram above and using the naturality of the counit ϵ, we can paste two commutative squares together to get the following commutative rectangle

$$\begin{tikzcd}
p_0(X \times_{Vp_0X} V\mathcal{C}) \arrow[r, "p_0(\pi_1)"] \arrow[d, "p_0(\pi_2)"] \arrow[dr, phantom, "\lrcorner", very near start] & p_0(X) \arrow[d, "p_0(\eta_X)"] \\
p_0V\mathcal{C} \arrow[r, "p_0V\gamma"'] \arrow[d, "\epsilon_{\mathcal{C}}"] & p_0Vp_0X \arrow[d, "\epsilon_{p_0(X)}"] \\
\mathcal{C} \arrow[r, "\gamma"'] & p_0X
\end{tikzcd}$$

From the outer rectangle, we get

$$\epsilon_{p_0(X)}p_0(\eta_X)p_0(\pi_1) = \gamma\epsilon_{\mathcal{C}}p_0(\pi_2).$$

But since $p_0(\pi_2), \epsilon$ are identities and using one of the zig-zag identities of the adjunction $p_0 \dashv V$, we get $p_0(\pi_1) = \gamma$. Thus π_1 is indeed a lift of γ.

Next we show that π_1 is a cartesian morphism. Let $a : Y \to X$ be a map of simplicial categories and let $\beta : p_0Y \to \mathcal{C}$ be such that

$$p_0(a) = \gamma\beta.$$

We have to show there is a unique functor

$$F : Y \to \gamma_{\mathcal{C}}^{*}(X)$$

such that $F\pi_1 = a$ and $p_0(F) = \beta$.

Composing $V\beta$ with η_Y, we get the map $V(\beta)\eta_Y : Y \to V(\mathcal{C})$. Now consider the following square:

$$\begin{tikzcd}
Y \arrow[r, "a"] \arrow[d, "\eta_Y"'] \arrow[dr, dotted] & X \arrow[dr, "\eta_X"] & \\
Vp_0Y \arrow[r, "V\beta"'] \arrow[rr, bend right, "Vp_0(a)"'] & V\mathcal{C} \arrow[r, "V\gamma"'] & Vp_0X
\end{tikzcd}$$

The outer rectangle commutes by naturality of η. The inner commuting parallelogram gives the required lift of β:

$$F : Y \to \gamma_{\mathcal{C}}^{*}X.$$

This map clearly satisfies $F\pi_1 = a$. The condition $p_0(F) = \beta$ follows from the fact that counit of $p_0 \dashv V$ is identity and that $p_0V(\beta) = \beta$.

Finally note that if X is a Segal category, then $\gamma_{\mathcal{C}}^{*}(X)$ is also a Segal category. Hence the lift is contained in $SegCat$. □

Using Theorem 2.3.2, we identify Segal fibrations over △ with Segal categories.

Proposition 2.3.9. *The inclusion i : PSegFib(△) $\to$ SegFib(△) has a right adjoint P (will be called the "pinching functor").*

Proof. Since by Proposition 2.3.8, the map p_0 is a fibration. We apply Lemma 2.3.7 to the following pullback square

$$\begin{tikzcd}
\text{PSegFib}(\triangle) \arrow[r, "i"'] \arrow[d] & \text{SegFib}(\triangle) \arrow[d, "p_0"] \\
\text{Sets} \arrow[r, "D"'] & Cat
\end{tikzcd}$$

using the fact that D has a right adjoint. □

Remark 2.3.10. Recall the functor $\mathcal{H} : Dbl \to Bicat$ (See Definition 2.1.47). The pinching functor P is the analogue of functor $\mathcal{H}$. The precise statement is that the following diagram commutes:

$$\begin{tikzcd}
Dbl \arrow[r, "\int N"] \arrow[d, "\mathcal{H}"'] & SegFib(\triangle) \arrow[d, "P"'] \\
Bicat \arrow[r, "\int N"'] & PSegFib(\triangle)
\end{tikzcd}$$

2.4 Segal fibration over $\triangle \times \Gamma$

It has already been established that Segal fibrations over $\triangle$ correspond to pseudo double categories (Theorem 2.3.2) and pinched Segal fibrations over $\triangle$ correspond to bicategories (Theorem 2.3.4). In this subsection, we want to propose possible analogues of symmetric monoidal pseudo double category (Definition 2.1.52) and symmetric monoidal bicategories.

The definition of Γ and the notations can be found in Subsection 2.1.1.

Definition 2.4.1. A Gamma object in pseudo double categories is a pseudofunctor $\mathbb{D} : \Gamma^{op} \to \mathbf{Dbl}$ such that $\mathbb{D}(*)$ is the terminal pseudo double category. A strict Gamma object is a Gamma object $\mathbb{D}$ that is a functor.

Definition 2.4.2. A special Gamma double category is a Gamma object in double categories $\mathbb{D}$ if the induced map

$$\mathbb{D}(n_+) \xrightarrow{i_{1*,2*,\ldots,n*}} \mathbb{D}(1_+)^n$$

is an equivalence of categories (where $i_{1*,2*,\ldots,n*} = i_{1*} \times i_{2*} \times \ldots i_{n*}$).

The conditions of equivalences will be termed *Segal conditions.* We say that special Gamma double category is *strict* if the equivalences are isomorphisms.

Again using Grothendieck construction, a tentative definition of a symmetric monoidal double category can be stated entirely in in the language of fibrations of categories. Instead of the skeletal category of finite pointed sets, if we consider Γ as the category of finite pointed sets, then we may describe the Segal conditions of a special Γ objects in a compact way. We follow [35, §4] for the definition. Let S, T be finite pointed sets and let $S \vee T$ denote the coproduct of finite pointed sets. Consider the map $S \vee T \to S$ that projects all points of S and send all the points of T to the base point. Similarly, we can define $S \vee T \to T$.

Definition 2.4.3. ([35, §4.1]) A functor $p : X \to \Gamma$ is a *Segal fibration over* Γ, if it is a fibration with $X(*) = *$ and for $S, T \in \Gamma$ the induced functor

$$X(S \vee T) \to X(S) \times X(T)$$

is an equivalence of categories.

Now we come to the main definition of this section.

Definition 2.4.4. A functor $p : X \to \triangle \times \Gamma$ is a *Segal fibration over* $\triangle \times \Gamma$ if

1. $X_{[n],\_}$ is a special Segal Γ fibration for all $[n] \in \triangle$.
2. $X_{\_,S}$ is a Segal fibration for all $S \in \Gamma$.

The Segal fibration over $\triangle \times \Gamma$ is *pinched* if $X_{\_,S}$ is a pinched Segal fibration over $\triangle$ for all $S \in \Gamma$.

If the map induced by ∂_0, ∂_1,

$$X_{[1],S} \to X_{[0],S} \times X_{[0],S}$$

is a fibration, then we say that the Segal fibration is “fibrant”.

Definition 2.4.5. Let **SegFib**$(\triangle \times \Gamma)$ denote the full subbicategory of **Fib**$(\triangle \times \Gamma)$ whose objects are Segal fibrations over $\triangle \times \Gamma$. Similarly let **PSegFib**$(\triangle \times \Gamma)$ denote the full subbicategory of **Fib**$(\triangle \times \Gamma)$ containing the pinched Segal fibrations.

A bicategory enriched in the cartesian monoidal 2-category **Dbl** is called a locally cubical bicategory. Hansen and Shulman show that the 2-functor $\mathcal{H}$: **Dbl** $\to$ **Bicat**$_2$ can be upgraded to locally cubical functor of locally cubical bicategories [67, §4]. Let $SymMonDbl$ denote the category of symmetric monoidal pseudo double categories. Hoping that there is a way to choose 2-morphisms so that we have bicategory **SMBicat** of symmetric monoidal bicategories and a bicategory of symmetric monoidal pseudo double categories **SMDbl**, we conjecture that

Conjecture 2.4.6. *The bicategory* **SegFib**$(\triangle \times \Gamma)$ *(and* **PSegFib**$(\triangle \times \Gamma)$ *resp.) are equivalent to* **SMDbl** *(and* **SMBicat** *resp.) such that the faces of the cube given below commutes.*

The following cube demonstrates the dictionary between the fibration picture and the pseudo double category picture.

$$\begin{tikzcd}
& & \mathbf{SegFib}(\triangle \times \Gamma) \arrow[rr, "P"] \arrow[dll, "U"'] \arrow[dd, squiggly, dash] & & \mathbf{PSegFib}(\triangle \times \Gamma) \arrow[dl, "U"] \arrow[dd, dotted] \\
\mathbf{SegFib}(\triangle) \arrow[rrr, "P"] \arrow[dd, dashed, red, dash, "N"'] & & & \mathbf{PSegFib}(\triangle) \arrow[dd, dash, "N"] & \\
& & \mathbf{SMDbl} \arrow[rr] \arrow[dll, "U"'] & & \mathbf{SMBicat} \arrow[dl, "U"] \\
\mathbf{Dbl} \arrow[rrr, "\mathcal{H}"] & & & \mathbf{Bicat}_2 &
\end{tikzcd}$$

I believe the dotted arrow can be constructed and it can be shown to be an equivalence. The crooked line represents the equivalence described in Conjecture 2.4.6. The red dashed line is the equivalence described in Theorem 2.2.8. The front square with red dashed line is the content of the pinching adjunction in Proposition 2.3.9. Shulman has shown that the bottom square commutes for a fibrant pseudo double category [59].

The Definition 2.4.4 is important for the thesis. In the next chapter, we construct examples of Segal fibrations over $\triangle \times \Gamma$. In fact, the construction of titular extended DW TQFT in the language of Segal fibrations is one of our aims.

Remark 2.4.7. Given a pinched Segal fibration $p : X \to \triangle \times \Gamma$, we can tentatively construct some aspects of the equivalent bicategory (assuming the conjecture). The Theorem 2.2.8 and Definition 2.4.3, the fibre over $(0, 1_+)$ is the set of objects of the bicategory. The fibre over $(1, 1_+)$ is a category whose objects are 1-morphisms of the bicategory and the morphisms are the 2-morphisms of the bicategory. The fibre over $(2, 1_+)$ parametrises 2-composable 1-morphisms and 2-morphisms of the bicategory along with the compositions.

2.4.1 Strict symmetric monoidal bicategories

In this section, we will first sketch a construction of symmetric monoidal bicategories from strict Segal fibrations. We show that a strict symmetric monoidal pseudo double category can be constructed from a strict Segal fibration. And thus using Theorem 2.1.54, Theorem 2.3.2 and Proposition 2.3.9, we can construct strict symmetric monoidal bicategory from a strict Segal fibration.

Definition 2.4.8. The category of *strict-special Gamma pseudo double categories* is denoted by $s\Gamma Dbl$ which is a subcategory of $\underline{Fun}(\Gamma^{op}, Dbl)$ of special Γ objects for which the Segal morphisms are isomorphisms of pseudo double functors.

Analgously *strict Segal fibrations over* $\triangle \times \Gamma$ are Segal fibrations for which the Segal morphisms in the Γ directions are isomorphisms. The subcategory of $SegFib(\triangle \times \Gamma)$ consisting of strict Segal fibrations will be denoted by $sSegFib(\triangle \times \Gamma)$.

The following construction extracts a commutative monoid object in $(\mathbf{Dbl}, \times)$ from a strict-special Gamma pseudo double category.

Construction 2.4.9. Given a strict-special Gamma pseudo double category X, let us define a strict-symmetric monoidal structure on the pseudo double category $\mathbb{D}(X) := X(1_+)$ (we will abbreviate it as $\mathbb{D}$). Let $I : X_0 \to \mathbb{D}$ be identified with the image of the unique object in $\mathbb{D}$. The object 'I' will serve as a monoidal unit.

We will denote $X(n_+)$ as $\mathbb{D}_n$ in this construction. Using the unique active map from 2_+ to 1_+, $u_{12} : 2_+ \to 1_+$, and the special Segal condition for $n = 2$, we define the monoidal structure.

$$\begin{tikzcd}
2_+ \arrow[r, "i_1 \times i_2"] \arrow[d, "u_{12}"'] & 1_+ \times 1_+ \\
1_+ &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{D}_2 \arrow[r, "i_{1*,2*}"'] \arrow[d, "u_{12*}"] & \mathbb{D} \times \mathbb{D} \arrow[l, bend right, "(i_{1*,2*})^{-1}"'] \arrow[dl, "\otimes"] \\
\mathbb{D} &
\end{tikzcd}$$

So we define

$$\otimes = u_{12*}(i_{1*} \times i_{2*})^{-1}.$$

The braiding morphism is defined as the map that swaps the coordinates $\tau : \mathbb{D}^2 \to \mathbb{D}^2$ and is evidently self-inverse. Now we will verify the axioms stated in Definition 2.1.52.

The associator is identity and this arises from the dotted square in the following cube

$$\begin{tikzcd}
\mathbb{D}_3 \arrow[rr, "u_{12,3*}"'] \arrow[dr, "i_{1*,2*,3*}"'] \arrow[dd, "u_{1,23*}"'] & & \mathbb{D}_2 \arrow[dr, "i_{1*,2*}"] \arrow[dd, "u_{12*}"' near end] & \\
& \mathbb{D}^3 \arrow[rr, dotted, "\otimes \times 1"] \arrow[dd, dotted, "1 \times \otimes"' near end] & & \mathbb{D}^2 \arrow[dd, dotted, "\otimes"] \\
\mathbb{D}_2 \arrow[rr, "u_{12*}" near start] \arrow[dr, "i_{1*,2*}"'] & & \mathbb{D} \arrow[dr, "id"] & \\
& \mathbb{D}^2 \arrow[rr, dotted, "\otimes"'] & & \mathbb{D}
\end{tikzcd}$$

Note that the blue arrows are Segal isomorphisms and can be inverted. We have to show that all the squares in the cube commute. The red square commutes because it is the functorial image of a commutative square in Γ^{op}.

The commutativity of the right square and bottom square follows from the definition of $\otimes$.

Note that the commutativity of the top and left square in the cube needs to be proved. The top square is shown below.

$$\begin{tikzcd}
\mathbb{D}_3 \arrow[r, "u_{12,3*}"] \arrow[d, "i_{1*,2*,3*}"] & \mathbb{D}_2 \arrow[d, "i_{1*,2*}"'] \\
\mathbb{D}^3 \arrow[r, dotted, "\otimes\times 1"'] \arrow[u, bend left=60] & \mathbb{D}^2 \arrow[u, bend right=60]
\end{tikzcd}$$

The above square commutes because one can check for instance that

$$\begin{aligned}
(\otimes \times 1)(i_{1*} \times i_{2*} \times i_{3*}) &= \otimes(i_{1*} \times i_{2*}) \times i_{3*} \\
&= (u_{12*}(i_{1*} \times i_{2*})^{-1}(i_{1*} \times i_{2*})) \times i_{3*} \\
&= u_{12*} \times i_{3*} \\
&= (i_{1*} \times i_{2*})u_{12,3*}.
\end{aligned}$$

Similarly one can check that the left square in the cube commutes. Thus we conclude the dotted square commutes and we have strict associativity.

The compatibility with the unit comes from the following diagram:

$$\begin{tikzcd}
& \mathbb{D}_2 \arrow[dd, "u_{12*}"] \arrow[dr, "i_{1*,2*}"] & \\
\mathbb{D} \arrow[ur, "u_{2*}"] \arrow[rr, dotted, "I\times 1"' near end] \arrow[dr, dotted, "id"'] & & \mathbb{D}^2 \arrow[dl, dotted, "\otimes"] \\
& \mathbb{D} &
\end{tikzcd}$$

The right face on the back commute by definition of $\otimes$. The left face commutes because $u_{12}u_1 = id$ on 1_+ and the face with $I \times 1$ commutes because $i_1 u_2$ maps everything to the base point and $i_1 u_2 = id$.

Next we check that $\tau\otimes = \otimes$. For this we use the morphism is defined $\mathrm{sw} = u_{1,2}$ where $\mathrm{sw} : 2_+ \to 2_+$ is the based map that swaps 1 and 2. Look at the following commutative pyramid:

$$\begin{tikzcd}
\mathbb{D}_2 \arrow[rr, "u_{2,1*}"] \arrow[d, "i_{1*,2*}"'] \arrow[rdd, "u_{12*}"] & & \mathbb{D}_2 \arrow[d, "i_{1*,2*}"] \arrow[ldd, "u_{12*}"'] \\
\mathbb{D}^2 \arrow[rr, dotted, "\tau"] \arrow[rd, dotted, "\otimes"'] & & \mathbb{D}^2 \arrow[ld, dotted, "\otimes"] \\
& \mathbb{D} &
\end{tikzcd}$$

The left and right face commute by the definition of $\otimes$. We have to check the commutativity of the behind square:

$$\begin{aligned}
\tau i_{1*,2*} &= i_{2*,1*} \\
&= i_{2*} \times i_{1*} \\
&= i_{1*}u_{2,1*} \times i_{2*}u_{2,1*} \\
&= i_{1*,2*}u_{2,1*}.
\end{aligned}$$

Thus we have a strict symmetric monoidal structure $(\otimes, I)$ on $X(1_+) = \mathbb{D}$.

Construction 2.4.10. Given a natural transformation between two Gamma categories $f : X \to Y$, we will show that $\mathbb{D}(f) := F = f(1_+)$ is a strict symmetric monoidal pseudo double functor. First note that in construction 2.4.9 we have shown that $X(1_+), Y(1_+)$ has a structure of a strict symmetric monoidal pseudo double category. Now we will check the axioms stated in Definition 2.1.53.

First, we prove $F\otimes = \otimes(F \times F)$ which is the dotted red square in the following prism diagram:

$$\begin{tikzcd}
X_2 \arrow[rr, "u_{12*}"] \arrow[rd] \arrow[dd, "f(2_+)"'] & & X_1 \arrow[dd, dotted, "F"] \\
& X_1^2 \arrow[ru, dotted, "\otimes"'] \arrow[dd, dotted, "F\times F"' near start] & \\
Y_2 \arrow[rr, "u_{12*}"] \arrow[rd] & & Y_1 \\
& Y_1^2 \arrow[ru, dotted, "\otimes"'] &
\end{tikzcd}$$

The blue arrows are Segal isomorphisms. The side squares commute because of the naturality of f. The top and bottom triangles commute because of the definition of $\otimes$.

The axiom $FI = I$ again follows from the naturality of f when it is evaluated on the initial/terminal category $*$. The compatibility with braiding is actually automatic.

Thus we have a symmetric monoidal functor.

Lemma 2.4.11. *Given a strict-special Γ pseudo double category X, $\mathbb{D}(X)$ given in Construction 2.1.52 and for a morphism f of special-Γ pseudo double category, double functor $\mathbb{D}(f)$ in Construction 2.1.53 assemble into a functor $\mathbb{D} : s\Gamma Dbl \to SymMonDbl$.*

Proof. The Constructions 2.1.52 and 2.1.53 already show that $\mathbb{D}$ maps objects and morphisms of $s\Gamma Dbl$ to objects of SymMonDbl. The functoriality is clear from Construction 2.1.53 and composition of natural transformations:

$$\mathbb{D}(g \circ f) = (g \circ f)(1_+) = g(1_+)f(1_+) = \mathbb{D}(g)\mathbb{D}(f).$$

Also, note that all the components of identity natural transformations are identity. ☐

Remark 2.4.12. Let $p : X \to \triangle \times \Gamma$ be a Segal fibration. From Construction 2.1.52, the fibre over (n, m) is m-fold monoidal products of n-composable horizontal morphisms. So the guiding principle for constructing Segal fibration will be the collection of all m-fold monoidal products of n-composable horizontal morphisms.

2.4.2 Construction of symmetric monoidal bicategories from Segal fibrations

In this subsection, we will sketch a construction of a symmetric monoidal bicategories from strict Segal fibrations culminating into Theorem 2.4.13. Lemma 2.4.11 that constructs a strict symmetric monoidal pseudo double category out of a strict-special Γ object is crucial for Theorem 2.4.13. This theorem simply translates Shulman's theorem 2.1.54 in the language of pinched Segal fibrations. Thus, in essence, there are no new results in this section.

Let us denote the category of strict Segal fibrations over $\triangle \times \Gamma$ by $sSegFib(\triangle \times \Gamma)$. A strict symmetric monoidal pseudo double category can be constructed from a strict Segal fibration. And thus using with Theorem 2.1.54, we can describe strict symmetric monoidal bicategories using strict Segal fibrations.

Let us record all the results, discussed in this chapter, in a single diagram as shown in Figure 2.6.

The equivalence denoted by a pink $\sim$ in the diagram is via the cartesian closedness of the category of small categories. The other equivalence denoted by a red $\sim$ is via the Grothendieck construction. The top arrow St is the inverse of the Grothendieck

$$\begin{tikzcd}
sSegFib(\triangle \times \Gamma) \arrow[r, "\Phi"] \arrow[ddr, dotted] & \underline{Fun}(\Gamma^{op} \times \triangle^{op}, Cat) \arrow[d, dash, "\sim"] & \\
& \underline{Fun}(\Gamma^{op}, \underline{Fun}(\triangle^{op}, Cat)) & \\
& \underline{Fun}(\Gamma^{op}, SegCat) \arrow[u, hook] \arrow[r, "c_{h*}"'] \arrow[d, dash, "\sim"'] & \underline{Fun}(\Gamma^{op}, Dbl) \\
& \underline{Fun}(\Gamma^{op}, SegFib(\triangle)) \arrow[dd, "P_*"'] & s\Gamma Dbl \arrow[u, hook] \arrow[d, "\mathbb{D}"] \\
& & SymMonDbl \arrow[d, "\mathcal{H}"] \\
& \underline{Fun}(\Gamma^{op}, PSegFib(\triangle)) & SymMonBicat
\end{tikzcd}$$

FIGURE 2.6: A map of comparisons.

construction. The blue N is the horizontal nerve and in Theorem 2.2.8, it shown to be an equivalence. The pinching functor P is from Proposition 2.3.9. The orange map $\mathbb{D}$ is the map described in Lemma 2.4.11 and the green $\mathcal{H}$ is described in Section 2.1.6. The Segal condition along the $\triangle$ direction in the definition of Segal fibration gives us the dotted red lift.

Now we will redraw this diagram after removing the obvious equivalences in Figure 2.7. We discuss the construction of a red dashed arrow at the bottom. Recall that Φ is

$$\begin{tikzcd}
sSegFib(\triangle \times \Gamma) \arrow[r, "c_{h*}\Phi"] \arrow[dr, dotted] \arrow[dd, "B"] & \underline{Fun}(\Gamma^{op}, \mathbf{Dbl}) \\
& s\Gamma Dbl \arrow[u, hook] \arrow[d, "\mathbb{D}"] \\
\underline{Fun}(\Gamma^{op}, SegFib(\triangle)) \arrow[d, shift right, "P_*"'] & SymMonDbl \arrow[d, "\mathcal{H}"] \\
\underline{Fun}(\Gamma^{op}, PSegFib(\triangle)) \arrow[u, shift right, "I_*"'] \arrow[r, dashed] & SymMonBicat
\end{tikzcd}$$

FIGURE 2.7: A simplified map of comparisons.

an inverse to Grothendieck construction (discussed in Construction 2.1.23 and Theorem

2.1.24). Note that the red dotted lift on the top exists since strict Segal fibrations satisfy strict Segal conditions along the Γ direction. Given a pinched strict Segal fibration over $\triangle$, (using I_*) it is also a strict Segal fibration over $\triangle$. Now we can construct a strict ΓDbl using the horizontal categorification (outlined in Construction 2.1.58). Finally composing along $\mathbb{D}, \mathcal{H}$ gives us a symmetric monoidal bicategory.

Recall the notion of a fibrant Segal fibration from Definition 2.4.4. Using Shulman's Theorem 2.1.54, we can construct a symmetric monoidal bicategory from a pinched strict fibrant Segal fibration.

We record the conclusion of the arguments in the last couple of paragraphs formally as a theorem.

Theorem 2.4.13. *Let $\mathcal{H}$ denote horizontal bicategory functor (Definition 2.1.47), Φ denote an inverse to Grothendieck construction (Construction 2.1.23) and $\mathbb{D}(\_)$ denote the construction of a strict symmetric monoidal pseudo double category (from Lemma 2.4.11). Given a strict fibrant pinched Segal fibration $\pi : X \to \Gamma \times \triangle$, the bicategory $\mathcal{H}\mathbb{D}c_{h*}\Phi(\pi)$ is a symmetric monoidal bicategory.*

Chapter 3

The bicategory of bispans

In this chapter, we construct examples of Segal fibrations of spans of different kinds. Given a category $\mathcal{C}$ with pullbacks, Benabou introduced the bicategory of spans in [6, §2.6] and the pseudo double category of spans was introduced in [26, §3.2] (both constructions are defined and discussed in Example 2.1.48). In this chapter, we construct a Segal fibration (over $\triangle$) of spans in $\mathcal{C}$ (see Subsection 3.2.1). The idea of using twisted categories $Tw_{(\_)}$ to construct spans is not new (see [30]). A minor novelty is the explicit construction of a strict double category of spans from the Segal fibration over $\triangle$ (see Construction 3.2.12). The explicit construction involves applying the (Grothendieck) Construction 2.1.23 to the Segal fibration to obtain a Segal category and the horizontal categorification functor (see Construction 2.1.58) is applied.

For the purposes of this thesis, we are interested in the bicategory of spans (and bispans) in groupoids in which we want to compose spans using *homotopy pullback* of groupoids. From Proposition 2.1.27, it is known that homotopy pullback of groupoids is equivalent to pullback if one of the legs of the diagram is an isofibration. So we can construct a bicategory of spans of groupoids by setting $\mathcal{C}$ as the category whose objects are groupoids and morphisms are isofibrations. While this approach may work, naturally occurring examples of functors are not isofibrations. For instance, the canonical map of groupoids $* \hookrightarrow \mathcal{G}$, where $*$ is the terminal category and $\mathcal{G}$ is any groupoid other than the terminal groupoid, is not an isofibration. It is preferable to construct the Segal fibrations of spans in groupoids without demanding isofibrations as one of the legs. See Theorem 3.2.8 for a construction.

In [53, Theorem 8], Rebro constructs the bicategory of bispans in a category. The bicategory of spans and bispans can be contrasted by noting the differences in the shape of the 2-morphisms (in green) in Figure 3.1. Note that the objects (black) and the 1-morphisms (red) have the same form. The bicategory of bispans has spans for 2-morphisms. The horizontal composition of 2-morphisms is obtained by computing limits

$$\begin{tikzcd}
& d_1 \arrow[dl] \arrow[dr] \arrow[dd] & \\
c_1 & & c_2 \\
& d_2 \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& d_1 \arrow[dl] \arrow[dr] & \\
c_1 & e \arrow[u] \arrow[d] & c_2 \\
& d_2 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

FIGURE 3.1: 2-morphisms in a span bicategory and a bispan bicategory.

of diagrams (see [53, Pg. 12]). Haugseng, in [30], constructs the (∞, n) category of n-fold spans using the machinery of Complete Segal Spaces. However, the constructions in this chapter are entirely internal to the bicategory of categories. We construct a fibration over $\triangle \times \Gamma$ that should produce a double category of bispans in groupoids after applying the Grothendieck construction and horizontal categorification. In the discussion at Construction 3.2.14, we argue that the fibration produces a tentative 2-morphism of the expected form. However, we have not been able to verify the Segal conditions, and thus, the construction of the double category of bispans of groupoids remains incomplete.

Organization In Section 3.1 we introduce the shape categories Tw_n, B_n and their properties. In Section 3.2, we discuss the constructions of three different kinds of Segal fibrations of spans/bispans. In Section 3.3, we discuss a few miscellaneous properties of bispans that will be useful in the next chapter.

3.1 Preliminaries

In this section, we define the shape categories Tw_n, B_n and discuss a universal property for Tw_n in Proposition 3.1.5. There are no new results in this section. For instance, all the definitions and the results in this section appear in [30].

The following construction of the twisted arrow category is well known (originally from [41, §1]).

Construction 3.1.1. Let $\mathcal{C}$ be a category. Define a category $Tw(\mathcal{C})$ (called the *twisted arrow category of* $\mathcal{C}$):

1. The objects of $Tw(\mathcal{C})$ are morphisms $f : x \to y$.

2. A morphism $l : f \to f'$ between $f : x \to y, f' : x' \to y'$ is a pair (l_1, l_2) of morphisms $l_1 : x \to x'$ and $l_2 : y' \to y$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
x \arrow[r, "l_1"] \arrow[d, "f"'] & x' \arrow[d, "f'"] \\
y & y' \arrow[l, "l_2"]
\end{tikzcd}$$

commutes in $\mathcal{C}$.

3. The compositions of morphisms in $Tw(\mathcal{C})$ is given by componentwise composition of morphisms in $\mathcal{C}$.

Since functors map commutative squares to commutative squares, the construction $Tw(\_)$ is a functor $Tw : Cat \to Cat$.

Remark 3.1.2. Given a category $\mathcal{C}$, the functor $Hom : \mathcal{C} \times \mathcal{C}^{op} \to Sets$ can be viewed as pseudofunctor valued in a discrete category[1]. It can be checked that the $Tw(\mathcal{C})$ is the category of elements (See Construction 2.1.25) of Hom.

Now we describe the twisted arrow category of $[n]$ explicitly.

Construction 3.1.3. Let n be a nonnegative integer, then $Tw_n := Tw([n])$ denotes the category whose

1. objects are pairs of natural numbers ij with $0 \leq i \leq j \leq n$.
2. morphisms $\mathrm{Hom}_{Tw_n}(ij, mn)$ is a singleton denoted by $ijmn$ if $i \leq m$ and $j \geq n$, otherwise empty. Composition and identity are defined in an obvious manner.

Given a morphism $\alpha : [n] \to [m]$ in $\triangle$. Define

$$Tw(\alpha)(ij) := \alpha(i)\alpha(j),$$

and the morphisms in an obvious way.

The above category is a poset with the property that any pair of elements has a supremum. In category language, it's a category with pullbacks.

---

[1] See Definition 2.1.4.

The categories Tw_n are drawn for $n = 0, 1, 2$ below:

$$\begin{tikzcd}
 & & & & & & 02 \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & & 01 \arrow[dl] \arrow[dr] & & & 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & \\
00 & 00 & & 11 & 00 & & 11 & & 22 \\
Tw_0 & & Tw_1 & & & & Tw_2 & &
\end{tikzcd}$$

Construction 3.1.4. We will construct a subfunctor B via a natural transformation $j : B \to Tw$ Let B_n denote the full subcategory of Tw_n whose objects ij satisfy the condition $0 \leq |i-j| \leq 1$. On morphisms in $\triangle$, B acts via restriction of Tw. The (strict) natural transformation $j_n : B_n \to Tw_n$ is the inclusion functor. (The naturality can be checked by hand.)

The category Tw_n is obtained by filling in all the pullbacks possible in B_n. We formalise it below in Lemma 3.1.5. Recall the definition of PBCat from Definition 2.1.13.

Lemma 3.1.5. *Let $\mathcal{C} \in PBCat$, then the functor induced by composition along j_n*

$$j_n^* : PBCat(Tw_n, \mathcal{C}) \to \mathbf{Cat}(B_n, \mathcal{C})$$

is an equivalence of categories.

Proof. We will prove j_n^* is essentially surjective and fully faithful.

Let $F : B_n \to \mathcal{C}$ be a functor. Define $\tilde{F} : Tw_n \to \mathcal{C}$ so that it agrees on the subcategory B_n. In other words, we have defined $\tilde{F} : Tw_n \to \mathcal{C}$ on ij with $0 \leq j - i \leq 1$. Define

$$\tilde{F}(ij) := \tilde{F}(i(j-1)) \times_{\tilde{F}_{(i+1)(j-1)}} \tilde{F}((i+1)j)$$

iteratively for $j - i \geq 2$. Note that $i+1 \leq j-1$ is equivalent to $j - i \geq 2$. The arrows will be canonically defined from the definition of pullback. Using the fact that a pullback of a pullback is also a pullback, we see that $\tilde{F}$ is a pullback preserving functor. Clearly $j_n^* \tilde{F} = F$.

Next, we show full faithfulness. Let $P, Q : Tw_n \to \mathcal{C}$ be two pullback-preserving functors. Let $j_n^* : \mathrm{PBCat}(P, Q) \to \mathbf{Cat}(j_n^* P, j_n^* Q)$ denote the map $\theta \to \theta j_n$.

Conversely, if $\eta \in \mathbf{Cat}(j_n^* P, j_n^* Q)$, define $\tilde{\eta}_{ij} : P(ij) \to Q(ij)$ as $\tilde{\eta}(ij) = \eta(ij)$ for $0 \leq j - i \leq 1$. Assume $\tilde{\eta}_{ij}$ is defined for $j - i < m$ for some $m \geq 1$. Let $j = i + m$, in the figure below

$$\begin{tikzcd}
 & & Q(ij) \arrow[d] \arrow[dr] & \\
P(ij) \arrow[urr, dashed] \arrow[d] \arrow[dr] & & Q(i(j-1)) \arrow[dr] & Q((i+1)j) \arrow[d] \\
P(i(j-1)) \arrow[urr] \arrow[dr] & P((i+1)j) \arrow[urr] & & P((i+1)(j-1)) \\
 & P((i+1)(j-1)) \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

all the solid arrows are already defined either because they are arrows defined via the functors P, Q (these are the arrows on either side of the diagram) or because of our assumption that $\tilde{\eta}_{ij}$ is defined for $j - i < m$ for some $m \geq 1$ (these are the arrows that are going across). From the universal property of pullbacks, there exists a unique arrow, from $P(ij)$ to $Q(ij)$ that makes the cube commute, which we will define as $\tilde{\eta}_{ij}$. By construction, $j_n^*(\tilde{\eta}_{ij}) = \eta_{ij}$ proving surjectivity. Uniqueness $\tilde{\eta}$ proves injectivity. Thus, j_*^n is fully faithful and, therefore, an equivalence. □

Remark 3.1.6. The summary of the above theorem:

1. Given a functor $F : B_n \to \mathcal{C}$, for a category $\mathcal{C}$ with pullbacks, there is a pullback preserving functor $\tilde{F} : Tw_n \to \mathcal{C}$ such that $\tilde{F} \circ j_n = F$.
2. The functor $\tilde{F}$ satisfying $\tilde{F} \circ j_n = F$ is not unique. But it is unique up to natural isomorphism. In other words, if there is a $\tilde{F'}$ satisfying $\tilde{F'} \circ j_n = F$, then there is a unique natural isomorphism that witnesses $\tilde{F} \cong \tilde{F'}$.

3.2 Spans and Bispans as Segal Fibrations

In this subsection, we construct three examples of Segal fibrations. Given a category $\mathcal{C}$ with pullbacks, a Segal fibration of spans in $\mathcal{C}$ is constructed in Subsection 3.2.1. In Subsection 3.2.2, we construct a Segal fibration (over $\triangle \times \Gamma$) of spans in groupoids. In Subsection 3.2.3, we construct a fibration (over $\triangle \times \Gamma$) of bispans in groupoids that is hopefully a Segal fibration. In Subsection 3.2.4, we discuss the double categories that arise from fibrations discussed in the previous three subsections.

The fact that these fibrations are nerves of a double category of appropriate spans/bispans is justified in Subsection 3.2.4.

The main focus of this subsection is the construction of a Segal fibration over $\triangle$ of spans.

3.2.1 Segal fibration over $\triangle$ of spans in a category

In this section, we will discuss the construction of a Segal fibration over $\triangle$, which we denote by $Sp\mathcal{C}$, for a category $\mathcal{C}$ with pullbacks. This construction is well known and can be found in [30], for instance, for any ∞-category $\mathcal{C}$ with pullbacks (in particular, the construction presented below is subsumed in *loc.cit*). We present the construction using only the tools of category theory. The main benefit of the detailed discussion is that the Construction 3.2.1 presents a natural cleavage (see the first line of the proof of Theorem 3.2.3) that allows us to construct a double category in Construction 3.2.12.

Now, we construct a Segal fibration over $\triangle$ using the Tw_n categories (from Subsection 3.1).

Construction 3.2.1. Let $\mathcal{C}$ be a category with pullbacks. Define a category $Sp\mathcal{C}$ with

1. objects as pairs (n, F) where n, m are nonnegative integers and $F \in \mathrm{PBCat}(Tw_n, \mathcal{C})$,
2. a morphism between $(n, F) \to (n', F')$ as a pair (α, η) where $\alpha : [n] \to [n']$, and a natural transformation $\eta : F \to F' \circ Tw(\alpha)$.
3. identity as (id, id) and composition as

$$(\alpha', \eta') \circ (\alpha, \eta) := (\alpha' \circ \alpha, (\eta' \circ \eta)).$$

Define a functor $p : Sp\mathcal{C} \to \triangle$ that projects the objects and morphisms onto the first component and second component.

Lemma 3.2.2. *The morphisms $(\alpha, \theta) : (n, F) \to (m, G) \in Sp\mathcal{C}$ is cartesian iff θ is an isomorphism.*

Proof. Cartesian morphisms are weak cartesian morphisms that are closed under composition. Suppose $(\alpha, \theta) \in Sp\mathcal{C}$ is weak cartesian. This means that given any morphism $(\alpha, \eta) : (n, H) \to (m, G)$, there is a unique lift over identity $(id, \rho) : (n, H) \to (n, F)$ such that $\theta \circ \rho = \eta$. Put $H = G \circ Tw(\alpha)$ and $\eta = id$, to obtain a unique lift $(id, \bar{(}\theta))$. Note that $\overline{\theta} : G \circ Tw(\alpha) \to F$ and $\theta \circ \overline{\theta} = id$. Now put $H = F$ and $\eta = \theta$. Since both

$\theta \circ (\overline{\theta} \circ \theta) = \theta$ and $\theta \circ id = \theta$, by uniqueness of the lift over identity, $\overline{\theta} \circ \theta = id$. Thus we conclude θ is an isomorphism.

Conversely, if θ is an isomorphism, then for any given morphism $(\alpha, \eta) : (n, H) \to (m, G)$ to factor through (α, θ), ρ has to be

$$\rho := \theta^{-1} \circ \eta.$$

Thus, there is a unique lift over identity, and (α, η) is weak cartesian. Since isomorphisms are closed under composition, (α, η) is cartesian. □

Proposition 3.2.3. *The functor $p : Sp\mathcal{C} \to \triangle$ (from Construction 3.2.1) is a strict Segal fibration over $\triangle$.*

Proof. First, we show that p is a fibration. Given an object $(m, F) \in X$, and a morphism $\alpha : [n] \to [m]$ in $\triangle$, the lift

$$([n], F \circ Tw(\alpha)) \xrightarrow{(\alpha, id)} ([m], F)$$

is cartesian. Indeed, if $(\alpha, \eta) : (n, H) \to (m, F)$ is any morphism over α, then since $\eta : H \to F \circ Tw(\alpha)$, we can us it to construct $(id, \eta) : (n, H) \to (n, F \circ Tw(\alpha))$, a morphism over identity so that (α, η) factors through (id, η).

Now we show it is a Segal fibration over $\triangle$. For this, we use the lifting criteria stated at the start of the section. Given a diagram of cartesian arrows

$$(n, F_n) \xleftarrow{(\partial_n, id)} (0, F_0) \xrightarrow{(\partial_0, id)} (m, F_m),$$

we have

$$F_n \circ Tw(\partial_n) = F_0 = F_m \circ Tw(\partial_0) \Leftrightarrow F_n(nn) = F_0(00) = F_m(00).$$

We obtain the functors $j_n^* F_n : B_n \to \mathcal{C}$, $j_m^* F_m : B_m \to \mathcal{C}$ using j_m^* defined in Lemma 3.1.5. We can define $G' : B_{n+m} \to \mathcal{C}$ by defining $G' \circ B(i_0) = j_n^* F_n$ and $G' \circ B(i_n) = j_m^* F_m$. By unravelling the definitions,

$$G'(ij) = F_n(ij) \text{ for } 0 \leq i, j \leq n \tag{3.2.1}$$

$$G'((i+n)(j+n)) = F_m(ij) \text{ for } 0 \leq i, j \leq m \tag{3.2.2}$$

which is well defined since $F_n(nn) = F_m(00)$. The functor $G' : B_{n+m} \to \mathcal{C}$ can be factorised as $G' = F_{n+m} \circ j_{n+m}$ upto a unique natural isomorphism, following the Remark 3.1.6.

Since

$$\begin{aligned}(F_{n+m} \circ Tw(i_0)) \circ j_n &= F_{n+m} \circ (Tw(i_0) \circ j_n) \\ &= F_{n+m} \circ (j_{m+n} \circ B(i_0)) \\ &= G' \circ B(i_0) \\ &= j_n^* F_n \\ &= F_n \circ j_n,\end{aligned}$$

Remark 3.1.6 suggests that there is a unique natural isomorphism

$$\theta_n : F_n \to F_{n+m} \circ Tw(i_0)$$

such that $\theta_m j_m = id$. Similarly, there is a unique natural isomorphism

$$\theta_m : F_m \to F_{n+m} \circ Tw(i_n).$$

such that $\theta_n j_n = id$. From Lemma 3.2.2, (i_0, θ_n) is a cartesian lift of (i_0, F_{m+n}) and (i_n, θ_m) is a cartesian lift of (i_n, F_{m+n}). It remains to show that the following square commutes

$$\begin{tikzcd}
(0, F_0) \arrow[r, "(\partial_0, id)"'] \arrow[d, "(\partial_n, id)"] & (m, F_m) \arrow[d, "(i_n, \theta_m)"'] \\
(n, F_n) \arrow[r, "(i_0, \theta_n)"] & (m+n, F_{m+n})
\end{tikzcd}$$

This involves checking $\theta_m Tw(\partial_0) = \theta_n Tw(\partial_n)$. We first note that

$$F_{m+n}(nn) = F_{m+n}(j_{m+n}(nn)) = G'(nn) = F_m(00) = F_0(00).$$

The third equality above follows from Equation 3.2.1. Now $\theta_m Tw(\partial_0)(00) = \theta_m(00) = id$ since $\theta_m j_m = id$. and $\theta_n Tw(\partial_n)(00) = \theta_n(nn) = id$ since $\theta_n j_n = id$. □

3.2.2 Segal fibration of spans of groupoids

In this section, the Segal fibration over $\triangle \times \Gamma$ of spans of groupoids will be constructed using fibrations. The guiding principle of constructing Segal fibration over $\triangle \times \Gamma$s (see Remark 2.4.12) is that fibre over n, m parametrizes m-fold monoidal products of n-composable horizontal morphisms.

The data of n-composable spans in finite groupoids look like pullback preserving functors $F : \mathrm{Tw}_n \to \mathrm{Gpds}^f$. We will use Grothendieck construction and present n-composable spans in Gpds^f as a functor $p : A \to \mathrm{Tw}_n^{op}$ that is a fibration in finite groupoids.

Definition 3.2.4. Let $p : X \to \mathcal{C}^{op}$ be a fibration in groupoids and let X_c denote the fibre of p over $c \in \mathcal{C}$. The fibration will be called *special* if given any pullback square in $\mathcal{C}$

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd}$$

the associated square of groupoids

$$\begin{tikzcd} X_a \arrow[r] \arrow[d] & X_b \arrow[d] \\ X_c \arrow[r] & X_d \end{tikzcd}$$

is a homotopy pullback square of groupoids.

Now we construct a Segal fibration over $\triangle \times \Gamma$ of groupoids

Construction 3.2.5. Define a category $\mathcal{T} = \mathrm{Sp}(\mathrm{Gpds})$ as follows:

1. The objects are tuples $(m, S, p : X \to \mathrm{Tw}_m^{op} \times DS)$ where m is a natural number, S is a finite pointed set, DS is the discrete category associated to the pointed set S and the functor p is a special fibration (as defined in Definition 3.2.4). Let p_1, p_2 be the components of the projection of p along Tw_m^{op} and DS respectively. We demand p_1 over the base point of DS is the terminal groupoid.

2. A morphism between two objects $(m, S, p : X \to \mathrm{Tw}_m^{op} \times DS) \to (n, T, q : Y \to \mathrm{Tw}_n^{op} \times DT)$ is a tuple (α, a, f), where $\alpha : [m] \to [n]$ is a morphism in $\triangle$, the arrow $a : T \to S$ is a map of pointed sets and $f : X \to Y$ is a functor such that the following squares commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_2"'] & Y \arrow[d, "q_2"] \\ DS & DT \arrow[l, "Da"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_1"'] & Y \arrow[d, "q_1"] \\ \mathrm{Tw}_m^{op} \arrow[r, "\mathrm{Tw}(\alpha)"'] & \mathrm{Tw}_n^{op} \end{tikzcd}$$

3. The identity morphism is identity map in all coordinates. Composition given by pasting commutative squares.

Clearly, we have a projection map $\pi : \mathcal{T} \to \triangle \times \Gamma$ that projects on to the first two coordinates.

Proposition 3.2.6. *The projection $\pi : \mathcal{T} \to \triangle \times \Gamma$ is a fibration of categories.*

Proof. First we show that the functor π is a fibration. Given an object $(n, T, q : Y \to \mathrm{Tw}_n^{op} \times DT)$, and a morphism (α, a) where $\alpha : [m] \to [n]$ is an order preserving map and $S \overset{a}{\leftarrow} T$ is a map of finite pointed sets. Let $p : \alpha^*Y \to \mathrm{Tw}_m^{op}$ be the pullback of q_1 along $\mathrm{Tw}^{op}(\alpha)$. Then composing the canonical map $g : \alpha^*Y \to Y$ with Daq_2 gives a map $\alpha^*Y \to DS$.

Now, we show that p is special. Since p is obtained as a pullback of fibrations, note that the fibre of p at x is the fibre of q at $[\mathrm{Tw}(\alpha)](x)$ and morphisms are also pulled back from the image of $\mathrm{Tw}(\alpha)$. Thus the square of groupoids induced by a square σ in Tw_m is the square that is induced by the image of σ. Since the functor q is special and all squares in Tw_n are pullback squares, the functor p is special.

Next we will prove that p is a cartesian lift. Given a morphism

$$\begin{tikzcd}
Z \arrow[d, "r_1"'] \arrow[dr, "k"] & & Z \arrow[d, "r_2"'] \arrow[dr, "k"] & \\
\mathrm{Tw}_l^{op} \arrow[dr, "\tilde{\beta}"'] & Y \arrow[d, "q_1"'] & DU & Y \arrow[d, "q_2"] \\
& \mathrm{Tw}_n^{op} & & DT \arrow[ul, "b"]
\end{tikzcd}$$

where $\tilde{\beta} = \mathrm{Tw}^{op}(\beta)$. We will follow this tilde notation for the rest of the proof. Suppose we are also given maps $\gamma : \mathrm{Tw}_l^{op} \to \mathrm{Tw}_m^{op}$ and a map of finite pointed sets $c : S \to U$ such that $\tilde{\alpha}\gamma = \tilde{\beta}$ and $ca = b$. In order to show that the map (α, a, g) is a cartesian morphism, we have to show there is a unique f (shown by a dashed arrow) that makes the diagrams below commute.

$$\begin{tikzcd}
& Z \arrow[d, "r_1"] \arrow[ldd, dashed, "f"'] \arrow[rdd, "k"] & \\
& \mathrm{Tw}_l^{op} \arrow[ldd, "\tilde{\gamma}"'] \arrow[rdd, "\tilde{\beta}"] & \\
\alpha^*Y \arrow[rr, "g"'] \arrow[d, "p_1"'] & & Y \arrow[d, "q_1"] \\
\mathrm{Tw}_m^{op} \arrow[rr, "\tilde{\alpha}"'] & & \mathrm{Tw}_n^{op}
\end{tikzcd}
\qquad
\begin{tikzcd}
& Z \arrow[d, "r_2"] \arrow[ldd, dashed, "f"'] \arrow[rdd, "k"] & \\
& DU & \\
\alpha^*Y \arrow[rr, "g"'] \arrow[d, "p_2"'] & & Y \arrow[d, "q_2"] \\
DS \arrow[ruu, "c"] & & DT \arrow[ll, "a"] \arrow[luu, "b"']
\end{tikzcd}$$

First we focus on the diagram on the left. It shows that f maps into a pullback square. Further the requirement that the top triangle and left square commute forces f to be the unique morphism such that $gf = k$ and $\tilde{\gamma}r_1 = p_2 f$. So the uniqueness of f is clear. We have to simply check the compatibility with the right diagram, so we focus on it. The top triangle is the same as the one on the left, so we check whether the left square commutes:

$$\begin{aligned} cp_2 f &= caq_2 gf \\ &= caq_2 k \\ &= bq_2 k = r_2 \end{aligned}$$

Thus the functor p is a cartesian lift and this establishes $\pi : \mathcal{T} \to \triangle \times \Gamma$ is a fibration. □

Construction 3.2.7. Let $PbFib(\mathrm{Tw}_n^{op})$ denote domain category of pullback preserving fibrations (of groupoids) over Tw_n^{op} and $Fib(B_n^{op})$ denote the codomain category of fibrations over B_n^{op}. Suppose the Lemma 3.1.5 is applied with the target category as category of groupoids and then Grothendieck construction used on domain and codomain. Then the restriction map $r_n : PbFib(\mathrm{Tw}_n^{op}) \to Fib(B_n^{op})$ is an equivalence of categories. Denote the adjoint inverse by e.

Theorem 3.2.8. *The functor $\pi : \mathcal{T} \to \triangle \times \Gamma$ is a fibrant Segal fibration over $\triangle \times \Gamma$.*

Proof. First, we prove the Segal condition in the direction of $\triangle$. Fix a finite pointed set S. Let $\mathcal{T}_{n,S}$ denote the fibre of π over $([n], S)$. We have to show the Segal condition: The canonical functor $K : \mathcal{T}_{m+n,S} \to \mathcal{T}_{m,S} \times_{\mathcal{T}_{0,S}} \mathcal{T}_{n,S}$ is an equivalence of categories.

We construct an inverse functor to the K. Given a pair of fibrations $p : X \to \mathrm{Tw}_m^{op} \times DS$, $q : Y \to \mathrm{Tw}_n^{op} \times DS$ that agree (upto an equivalence), when restricted along the Segal structure maps, with $s : Z \to \mathrm{Tw}_0^{op} \times DS$. Using the restriction functor (see

Construction 3.2.7), we get the following diagram:

$$\begin{tikzcd}
& r_0Z \arrow[dl] \arrow[d] \arrow[drr] & & \\
r_nX \arrow[d] \arrow[drr, dotted] & B_0^{op} \arrow[dl] \arrow[drr] & & r_mY \arrow[d] \arrow[dl, dotted] \\
B_n^{op} \arrow[drr] & & W \arrow[d, dashed] & B_m^{op} \arrow[dl] \\
& & B_{n+m}^{op} &
\end{tikzcd}$$

Note that we have suppressed $\times DS$ on all the base categories. The category W is the homotopy pushout of the span on the top involving r_nX, r_mY, r_0Z. From the universal property of pushout, we obtain the dashed morphism.

Next, we apply the functor e (described in Construction 3.2.7), to the fibration $W \to B_{n+m}^{op}$ to obtain a special fibration $eW \to \mathrm{Tw}_{n+m}^{op} \times DS$. It can be checked that the functor to K sends the tuple $(X, p), (Y, q), (Z, s)$ to the fibration $eW \to \mathrm{Tw}_{n+m}^{op} \times DS$ is an inverse to K.

Next we argue that $\pi : \mathcal{T} \to \triangle \times \Gamma$ satisfies strict Segal conditions in the Γ direction. Fix $[m] \in \triangle$ and let n_+ denote the finite pointed set $\{*, 1, 2, 3, \cdots, n\}$. Let the fibre of π over $([m], n_+)$ be denoted by $\mathcal{T}_{m,n_+}$. Let $i_j : n_+ \to 1_+$ denote the maps that sends j to 1 and rest to the base point. These are the structure maps of Segal in the Γ direction. The map induced by the maps i_j for $j = 1, 2, ..., n$

$$I : \mathcal{T}_{m,n_+} \to \mathcal{T}^n_{m,1_+}$$

is given by

$$(m, p : Y \to \mathrm{Tw}_n^{op} \times Dn_+) \mapsto (m, i_jp : Y \to \mathrm{Tw}_n^{op} \times D1_+)_{j=1,2...,n}.$$

Conversely, given a collection of maps $Y_i \xrightarrow{\beta_i} \mathrm{Tw}_m^{op} \times D1_+$, consider the coproduct (or wedge sum) of these maps

$$Y \to \mathrm{Tw}_m^{op} \times (\bigvee_j D1_+),$$

consider the maps $a_j : 1_+ \to n_+$ which sends 1 to j. The coproduct of pointed sets

$$(\bigvee_j D1_+) \to Dn_+$$

is an isomorphism and thus we get a fibration $Y \to \mathrm{Tw}_m^{op} \times Dn_+$. It can be checked that this assignment is an inverse isomorphism to I.

Thus $\pi : \mathcal{T} \to \triangle \times \Gamma$ is a Segal fibration over $\triangle \times \Gamma$. Finally, we check that the Segal fibration over $\triangle \times \Gamma$ is fibrant. So we have to prove that

$$\mathcal{T}_{1,1_+} \xrightarrow{(\partial_1^*,\partial_0^*)} \mathcal{T}_{0,1_+} \times \mathcal{T}_{0,1_+}$$

is a fibration[2]. For this proof, we switch to the pseudofunctorial view. We have to prove equivalence of

$$Fun(\mathrm{Tw}_1 \times D1, \mathrm{Gpds}) \xrightarrow{(\partial_1^*,\partial_0^*)} Fun(\mathrm{Tw}_0 \times D1, \mathrm{Gpds})^2$$

Note that $D1$ is not pointed. Another way to write this using the closed cartesian monoidal structure of Cat is

$$Fun(\mathrm{Tw}_1, \mathrm{Gpds}) \xrightarrow{(\partial_1^*,\partial_0^*)} Fun(\mathrm{Tw}_0, \mathrm{Gpds})^2$$

The lifting criteria corresponds to supplying the groupoid L and the dashed arrows along with the two-cells in the following diagram:

$$\begin{tikzcd}
M \arrow[d] & L \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] & N \arrow[d] \\
B & A \arrow[l] \arrow[r] & C
\end{tikzcd}$$

Note all the capital letters represent groupoids and the commutative squares commute up to a natural isomorphism of groupoids. The homotopy limit of the non-dashed diagram gives L and the dashed arrows (along with canonical two-cells). The cartesian property of L simply follows from the universal property of the limit. Thus $(\partial_1^*, \partial_0^*)$ is a fibration and $\pi : \mathcal{T} \to \triangle \times \Gamma$ is a fibrant Segal fibration over $\triangle \times \Gamma$. □

Remark 3.2.9. The notion of fibrations and Grothendieck constructions can be generalized to bicategories [3]. All the constructions in this section pertaining to groupoids only use the homotopy pullbacks and properties of pasting diagrams of such pullbacks. Since these arguments extend to 2-groupoids, I suspect that the construction of Segal fibration over $\triangle \times \Gamma$ of spans in 2-groupoids will be the same as above.

---

[2] This proof is similar to the proof in Example 2.1.48.

3.2.3 Fibration of bispans of groupoids

In this section, we will construct a fibration that is tentatively the Segal fibration over $\triangle \times \Gamma$ of bispans in groupoids. The Segal condition verification for this fibration remains open.

Construction 3.2.10. Define a category $\mathcal{S} = \text{Bisp(Gpds)}$ as follows:

1. The objects are tuples $(m, S, p : X \to \text{Tw}_m^{op} \times DS)$ where m is a natural number, S is a finite pointed set, DS is the discrete category associated to the pointed set S and the functor p is a pullback preserving fibration (as defined in Definition 3.2.4). Let p_1, p_2 be the components of the projection of p along Tw_m^{op} and DS respectively. We demand p_1 over the base point of DS is the terminal groupoid.

2. Given two objects $(m, S, p : X \to \text{Tw}_m^{op} \times DS) \to (n, T, q : Y \to \text{Tw}_n^{op} \times DT)$, consider a set of tuples (α, a, f, g, E), where $\alpha : [m] \to [n]$ is a morphism in $\triangle$, the arrow $a : T \to S$ is a map of pointed sets and $f : X \to Y$ is a functor such that the following squares commute:

$$\begin{tikzcd}
& E \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X \arrow[d, "p_1"'] & & Y \arrow[d, "q_1"] \\
DS & & DT \arrow[ll, "Da"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& E \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X \arrow[d, "p_2"'] & & Y \arrow[d, "q_2"] \\
\text{Tw}_m^{op} \arrow[rr, "\text{Tw}(\alpha)"'] & & \text{Tw}_n^{op}
\end{tikzcd}$$

Define a relation $\sim$: $(\alpha_1, a_1, f_1, g_1, E_1) \sim (\alpha_2, a_2, f_2, g_2, E_2)$ if $\alpha_1 = \alpha_2$, $a_1 = a_2$ and there exists a an equivalence $\phi : E_1 \to E_2$ such that the following diagram commutes up to natural isomorphisms.

$$\begin{tikzcd}
& E_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[dd, "\phi"] & \\
X & & Y \\
& E_2 \arrow[ul, "f_2"] \arrow[ur, "g_2"'] &
\end{tikzcd}$$

3. The identity morphism at $(m, S, p : X \to \text{Tw}_m^{op} \times DS)$ is the equivalence class of the tuple (id, id, id, id, X). Let

$$[(\alpha, a, f, g, E)] : (l, S, p : X \to \text{Tw}_l^{op} \times DS) \to (m, T, q : Y \to \text{Tw}_m^{op} \times DT)$$

and

$$[(\beta, b, h, k, F)] : (m, T, q : Y \to \mathrm{Tw}_m^{op} \times DT) \to (n, U, r : Z \to \mathrm{Tw}_n^{op} \times DU)$$

be a pair of composable morphisms. Pasting the diagram associated to the twist category Tw directions and computing a homotopy limit yields the following diagram

$$\begin{tikzcd}
 & & & E \times_Y^h F \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & & \\
 & & E \arrow[dl, "f"'] \arrow[dr, "g"] & & F \arrow[dl, "h"'] \arrow[dr, "k"] & & \\
 & X \arrow[d, "p_1"'] & & Y \arrow[d, "q_1"] & & Z \arrow[d, "r_1"] & \\
 & \mathrm{Tw}_l^{op} \arrow[rr, "\mathrm{Tw}^{op}(\alpha)"'] & & \mathrm{Tw}_m^{op} \arrow[rr, "\mathrm{Tw}^{op}(\beta)"'] & & \mathrm{Tw}_n^{op} &
\end{tikzcd}$$

Now we have to check that the outer square given below commutes:

$$\begin{tikzcd}
 & & & E \times_Y^h F \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & & \\
 & & E \arrow[dl, "f"'] \arrow[dr, "g"] & & F \arrow[dl, "h"'] \arrow[dr, "k"] & & \\
 & X \arrow[d, "p_2"'] & & Y \arrow[d, "q_2"] & & Z \arrow[d, "r_2"] & \\
 & DS & & DT \arrow[ll, "Da"] & & DU \arrow[ll, "Db"] &
\end{tikzcd}$$

This follows by a direct diagram chase except for a non-trivial observation. Suppose $f_1, f_2 : E_1 \to E_2$ are a pair of functors and $k : E_2 \to DS$ is a functor mapping to a discrete category. Then $F_1 \simeq F_2$ implies that $kF_1 = kF_2$. The homotopy pullback square at the top does not commute on the nose, and if we apply the observation, the proof is straightforward.

Thus we can write that the composite is

$$[(\beta, b, h, k, F)] \circ [(\alpha, a, f, g, E)] = [(\beta\alpha, ab, f\pi_1, k\pi_2, E \times_Y^h F)]$$

The well-definedness of the composition follows from the universal property of homotopy pullback.

The associativity of the composition follows from the commutativity of homotopy pullbacks of categories up to canonical isomorphisms.

Clearly, we have a projection map $\pi : \mathcal{S} \to \triangle \times \Gamma$ that projects onto the first two coordinates.

Proposition 3.2.11. *The projection $\pi : \mathcal{S} \to \triangle \times \Gamma$ is a fibration of categories.*

Proof. First we show that the functor π is a fibration. Given an object $(n, T, q : Y \to \mathrm{Tw}_n^{op} \times DT)$, and a morphism (α, a) where $\alpha : [m] \to [n]$ is an order preserving map and $S \xleftarrow{a} T$ is a map of finite pointed sets. Let $p : \alpha^*Y \to \mathrm{Tw}_m^{op}$ be the pullback of q_1 along $\mathrm{Tw}^{op}(\alpha)$. Then composing the canonical map $g : \alpha^*Y \to Y$ with Daq_2 gives a map $\alpha^*Y \to DS$.

Now, we show that p is pullback preserving. Since p is obtained as a pullback of fibrations, note that the fibre of p at x is the fibre of q at $[\mathrm{Tw}(\alpha)](x)$ and morphisms are also pulled back from the image of $\mathrm{Tw}(\alpha)$. Thus the square of groupoids induced by a square σ in Tw_m is the square that is induced by the image of σ. Since the functor q is pullback preserving and all squares in Tw_n are pullback squares, the functor p is pullback preserving.

Next we will prove that $[(\alpha, a, id, g, \alpha^*Y)]$ is a cartesian lift. Suppose we are given the following diagrams.

$$\begin{tikzcd}
Z \arrow[d, "r_1"'] & F \arrow[l, "l"'] \arrow[d, "m"] \\
\mathrm{Tw}_l^{op} \arrow[dr, "\tilde{\beta}"'] & Y \arrow[d, "q_1"'] \\
 & \mathrm{Tw}_n^{op}
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[d, "r_2"'] & F \arrow[l, "l"'] \arrow[d, "m"] \\
DU & Y \arrow[d, "q_2"] \\
 & DT \arrow[ul, "b"]
\end{tikzcd}$$

where $\tilde{\beta} = \mathrm{Tw}^{op}(\beta)$. We will follow this tilde notation for the rest of the proof. Suppose we are also given maps $\gamma : \mathrm{Tw}_l^{op} \to \mathrm{Tw}_m^{op}$ and a map of finite pointed sets $c : S \to U$ such that $\tilde{\alpha}\tilde{\gamma} = \tilde{\beta}$ and $ca = b$. In order to show that the map $[(\alpha, a, id, g, \alpha^*Y)]$ is a cartesian morphism, we have to show there is a unique equivalence class $[(\gamma, c, x, y, X)]$

(shown by a dashed arrow below) that makes the diagrams below commute.

$$\begin{tikzcd}
 & & Z \arrow[ddd, dotted, "r_1"] & & \\
X \arrow[urr, dashed, "x"] \arrow[d, dashed, "y"'] & & \alpha^*Y \arrow[dll, "id"] \arrow[drr, "g"'] & & F \arrow[ull, "l"'] \arrow[d, "m"] \\
\alpha^*Y \arrow[dd, "p_1"'] & & & & Y \arrow[dd, "q_1"] \\
 & & \mathrm{Tw}_l^{op} \arrow[dll, "\tilde{\gamma}"] \arrow[drr, "\tilde{\beta}"] & & \\
\mathrm{Tw}_m^{op} \arrow[rrrr, "\tilde{\alpha}"'] & & & & \mathrm{Tw}_n^{op}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & Z \arrow[ddd, dotted, "r_2"] & & \\
X \arrow[urr, dashed, "x"] \arrow[d, dashed, "y"'] & & \alpha^*Y \arrow[dll, "id"] \arrow[drr, "g"'] & & F \arrow[ull, "l"'] \arrow[d, "m"] \\
\alpha^*Y \arrow[dd, "p_2"'] & & & & Y \arrow[dd, "q_2"] \\
 & & DU & & \\
DS \arrow[urr, "c"] & & & & DT \arrow[ull, "b"'] \arrow[llll, "a"]
\end{tikzcd}$$

First we focus on the diagram on the left. First we note that by composition in the category, we require

$$X \times_{\alpha^*Y} \alpha^*Y \simeq F.$$

This forces $X \equiv F$ over Z, Y and fixes the equivalence class. Thus if the dashed morphism in $calS$ exists, then it is unique. Choosing $X = F$, $x = l$ and $y = (\tilde{\gamma}r_1, m)$ (the map into the pullback is written via the components of the structure maps and this means $gy = m$) makes the left diagram commute. We have to simply check the compatibility with the right diagram to prove existence, so we focus on it. So we check whether the left pentagon commutes:

$$\begin{aligned}
cp_2y &= caq_2gy \\
&= caq_2m \\
&= bq_2m = r_2l
\end{aligned}$$

Thus the functor p is a cartesian lift and this establishes $\pi : \mathcal{S} \to \triangle \times \Gamma$ is a fibration. □

We have shown that the functor $\pi : \mathcal{S} \to \triangle \times \Gamma$ is a fibration. It has not yet been verified that the fibrations satisfy Segal conditions.

3.2.4 Double categories of spans and bispans

In this subsection, we apply the horizontal categorification functor (see Construction 2.1.58) to obtain a strict double category from the fibrations in Subsections 3.2.1, 3.2.2 and 3.2.3.

Construction 3.2.12. Consider the Segal fibration of spans, $p : Sp\mathcal{C} \to \triangle$ (from Construction 3.2.1). Pick a cleavage s (see Definition 2.1.16) which is a collection of cartesian lifts $s_{(\alpha,F)} : (m, \alpha^* F) \to (n, F)$. Let the components of $s_{(\alpha,F)}$ be $(\alpha, \eta_{(\alpha,F)})$ where $\eta_{(\alpha,F)} : \alpha^* F \to F \circ Tw(\alpha)$ is an isomorphism by Lemma 3.2.2. From Construction 2.1.23, we obtain a Segal category $P_s : \triangle^{op} \to \mathbf{Cat}$. The pseudofunctor P_s is described as follows:

1. On objects $P_s([n]) = Sp_n$ is the category of functors $\underline{Fun}(Tw_n, \mathcal{C})$.
2. On morphisms $\alpha : [m] \to [n]$,

$$P_s(\alpha) = (\_) \circ Tw(\alpha).$$

3. Let $\alpha : [m] \to [n], \beta : [n] \to [p]$, then the pseudofunctorial coherence $\phi_{\beta,\alpha} F : \alpha^*(\beta^* F) \to (\beta \circ \alpha)^* F$ is the unique isomorphism that satisfies

$$\eta_{(\beta\circ\alpha,F)} \phi_{\beta,\alpha} F = \eta_{(\beta,\alpha^* F)} \eta_{(\alpha,F)}.$$

   Such an isomorphism exists because $(\beta \circ \alpha, \eta_{(\beta\circ\alpha,F)})$ is a cartesian morphism.

Suppose we choose the cleavage $\alpha^* F = F \circ Tw(\alpha), \eta_{(\alpha,F)} = id$. Construction 2.1.58 produces a strict double category $c_h X$ out of a Segal category X. We apply this procedure to obtain a double category of spans $\mathbb{S}pan(\mathcal{C}) := c_h(P_s)$:

1. The category of objects of $\mathbb{S}pan(\mathcal{C})$ is the category

$$P_s(0) = \underline{Fun}(Tw_0, \mathcal{C}) = \mathcal{C}.$$

2. From Construction 2.1.38, the categorification $c(Obj \circ P_s)$ is a category whose objects are objects of $\mathcal{C}$ and morphisms are spans in $\mathcal{C}$. Now consider an object $F : Tw_2 \to \mathcal{C}$ of $P_s([2])$ (the image of F is shown in Figure 3.2). The boundaries are indicated in Figure 3.3, and the definition of categorification declares the red span in the figure as the composite of the black spans. From Construction 2.1.58, we have to quotient the double category freely generated by the morphisms $P_s([2])$ (as 2-morphisms of the double category) quotiented by a congruence specified in *loc.cit* in the given order:
   (a) It can be checked that the vertical composition of natural transformations is the vertical composition in the double category.

$$\begin{tikzcd}
 & & F(02) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & F(01) \arrow[dl] \arrow[dr] & & F(12) \arrow[dl] \arrow[dr] & \\
F(00) & & F(11) & & F(22)
\end{tikzcd}$$

FIGURE 3.2: An object of $P_s([2])$ from Construction 3.2.12.

(b) Given two loose 1-morphisms $F, G : Tw_1 \to \mathcal{C}$, a 2-morphism is a natural transformation $\eta : F \to G$ is shown (in green) in Figure 3.4. Note that the dashed arrow is uniquely determined by the rest of the data using the universal properties of pullback squares.

(c) The identity 2-morphism of a loose 1-morphism is the identity natural transformation.

(d) The identity 2-morphism of a tight 1-morphism is the identity morphism in $\mathcal{C}$.

Thus, we reproduce a strict version of the double category of spans constructed in Grandis-Pare [26, §3.2] where we have identified a composition of loose 1-morphisms up to isomorphisms of spans. The Grandis-Pare pseudo double category of spans is discussed in Example 2.1.48.

By applying the horizontal bicategory functor (see Definition 2.1.47), we get a strict bicategory of spans $\mathcal{H}(\mathbb{S}pan(\mathcal{C}))$. This is a strict version of Benabou's bicategory of bispans discussed in [6, §2.6].

The Segal fibration of spans of groupoids is discussed next. But we keep it brief since the construction is almost similar to Construction 3.2.12. The only change is that pullbacks are replaced by homotopy pullbacks.

Construction 3.2.13. Consider the restriction of the Segal fibration $\pi : \mathcal{S} \to \triangle \times \Gamma$, from Construction 3.2.5, over $\triangle \times D1$. Pick a cleavage s (see Definition 2.1.16) which is a collection of cartesian lifts $s_{(\alpha,p)} : (m, \alpha^* X) \to (n, p : X \to Tw_n^{op})$ (the coordinate of Γ is suppressed).

By Grothendieck construction (see 2.1.23), the fibration $p : X \to Tw_n$ corresponds to a pseudofunctor $P : Tw_n \to$ **Gpds** after choosing a cleavage and the special condition (see Definition 3.2.4) ensures that P preserves pullback. A choice in the construction is

$$\begin{tikzcd}
& F(12) \arrow[dl] \arrow[dr] & \\
F(11) & & F(22)
\end{tikzcd}$$

$\partial_0^* F$

$$\begin{tikzcd}
& F(01) \arrow[dl] \arrow[dr] & \\
F(00) & & F(11)
\end{tikzcd}$$

$\partial_2^* F$

$$\begin{tikzcd}
& F(02) \arrow[dl] \arrow[dr] & \\
F(00) & & F(22)
\end{tikzcd}$$

$\partial_1^* F$

Figure 3.3: The boundaries of F from Figure 3.2

$$\begin{tikzcd}
& & F(02) \arrow[r, dashed] \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & G(02) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\
& F(01) \arrow[r] \arrow[dl] \arrow[dr] & G(01) \arrow[dl] \arrow[dr] & F(12) \arrow[r] \arrow[dl] \arrow[dr] & G(12) \arrow[dl] \arrow[dr] & \\
F(00) \arrow[r] & G(00) & F(11) \arrow[r] & G(11) & F(22) \arrow[r] & G(22)
\end{tikzcd}$$

Figure 3.4: A morphism of $P_s([2])$ between black colored F and blue colored G is shown in green.

$$\begin{tikzcd}
A_{00} & A_{01} \arrow[l] \arrow[r] & A_{11} \\
E_{00} \arrow[u] \arrow[d] & E_{01} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & E_{11} \arrow[u] \arrow[d] \\
A'_{00} & A'_{01} \arrow[l] \arrow[r] & A'_{11}
\end{tikzcd}$$

FIGURE 3.5: A 2-cell in the double category of bispans from Construction 3.2.14.

the cleavage from the first paragraph of the proof of Proposition 3.2.6. The cartesian lifts of this cleavage is α^*X is the fibre product $X \times_{Tw_m^{op}} Tw_n^{op}$.

By repeating the rest of the procedure from Construction 3.2.12, we obtain a strict double category of spans of groupoids. The composition of 1-morphisms is given by homotopy pullbacks, and the composites are identified up to the equivalence of spans of groupoids.

The fibration of bispans of groupoids, if it satisfies Segal conditions, should correspond to a strict double category.

Construction 3.2.14. Consider the restriction of the Segal fibration $\pi : \mathcal{S} \to \triangle \times \Gamma$, from Construction 3.2.10, over $\triangle \times D1$. Pick a cleavage s (see Definition 2.1.16) which is a collection of cartesian lifts $s_{(\alpha,p)} : (m, \alpha^*X) \to (n, p : X \to Tw_n^{op})$ (the coordinate of Γ is suppressed). The component of the lift is a span: $(f_{\alpha,p}, E_{\alpha,p}, g_{\alpha,p})$.

By Grothendieck construction (see 2.1.23), the fibration $p : X \to Tw_n$ corresponds to a pseudofunctor $P : Tw_n \to$ **Gpds** after choosing a cleavage and the special condition (see Definition 3.2.4) ensures that P preserves pullback. A choice in the construction is the cleavage from the first paragraph of the proof of Proposition 3.2.11. The cartesian lifts of this cleavage is α^*X is the span: $(id, X \times_{Tw_m^{op}} Tw_n^{op}, c)$ where $c : X \times_{Tw_m^{op}} Tw_n^{op} \to X$ is the canonical map out of the fibre product. While applying the horizontal categorification function, the 2-morphisms are freely generated from morphisms of $P_s([1])$ modulo an equivalence relation. Figure 3.5 shows the image of the pseudofunctor of a morphism in $P_s([1])$.

All the morphisms in the above construction seem to indicate that if π from the above construction is a Segal fibration, then we can obtain a strict double category of bispans of groupoids.

Inspired by the Construction 3.2.14, we briefly describe the definition of the bicategory of bispans. This bicategory has been described in great detail by Rebro [53, Theorem 8].

Construction 3.2.15. Let $\mathcal{C}$ be a category with finite limits. Define a bicategory of bispans $\mathbf{Bispan}(\mathcal{C})$ as follows:

1. The objects of $\mathbf{Bispan}(\mathcal{C})$ are objects of $\mathcal{C}$ which we will denote by $A, B, C, \ldots$.
2. The 1-morphisms of $\mathbf{Bispan}(\mathcal{C})$ are spans in $\mathcal{C}$ like

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}$$

3. The 2-morphisms of $\mathbf{Bispan}(\mathcal{C})$ are bispans in $\mathcal{C}$ as shown below (with commuting squares) modulo an equivalence relation.

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
A & P \arrow[u] \arrow[d] & B \\
 & Y \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Two spans $X \leftarrow P \rightarrow Y$ and $X \leftarrow P' \rightarrow Y$ are equivalent if there exists an isomorphism $P \rightarrow P'$ such that the following triangles commute.

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, "\cong"'] & X \\
Y & P' \arrow[l] \arrow[u]
\end{tikzcd}$$

4. The 1-morphisms are composed by choosing a pullback. The composition of 1-morphisms is not associative on the nose.

5. The vertical morphism is the usual composition of spans in a category. The composition of horizontal morphisms is shown below.

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & & & & X \times_B Y \arrow[dl] \arrow[dr] & \\
A & E \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B & F \arrow[l] \arrow[r] \arrow[u] \arrow[d] & C & = & A & E \times_B F \arrow[u] \arrow[d] & C \\
& X' \arrow[ul] \arrow[ur] & & Y' \arrow[ul] \arrow[ur] & & & & X' \times_B Y' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

3.3 Miscellaneous properties of bispans

In this subsection, we record a useful lemma that will be used in the next chapter. Then we discuss the symmetries of the bicategory of bispans.

A very useful lemma which is recorded in [30] is the following:

Lemma 3.3.1. *A span of the form* $a \xleftarrow{f} b \xrightarrow{g} c$ *is invertible iff* f *and* g *are invertible.*

Now we discuss the flipping symmetry. The equivalence $\mathrm{Flip}_n : [n]^{op} \to [n]$ given by

$$\mathrm{Flip}_n(k) = n - k,$$

induces an involution of the Tw functor. Thus we have a $(\mathbb{Z}/2\mathbb{Z})^2$ action on the bicategory of bispans. The generating flips will be called horizontal flip (denoted by HFlip) and vertical flip (denoted by VFlip). Essentially HFlip flips 1-morphisms only, VFlip flips 2-morphisms only and the composition flips both. The figures below indicate the action of the Flip functors.

$$\begin{tikzcd}
& x \arrow[dl, red] \arrow[dr, orange] & & & & & x \arrow[dl, orange] \arrow[dr, red] & \\
a & X \arrow[u] \arrow[d, cyan] & b & {} \arrow[r, mapsto, "\text{HFlip}"] & {} & b & X \arrow[u] \arrow[d, cyan] & a \\
& y \arrow[ul, blue] \arrow[ur, magenta] & & & & & y \arrow[ul, magenta] \arrow[ur, blue] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & x \arrow[dl] \arrow[dr] & & & & & y \arrow[dl] \arrow[dr] & \\
a & X \arrow[u] \arrow[d] & b & {} \arrow[r, mapsto, "\text{VFlip}"] & {} & a & X \arrow[u] \arrow[d] & b \\
 & y \arrow[ul] \arrow[ur] & & & & & x \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Note that VFlip and HFlip preserve identity morphisms.

Chapter 4

Double Beck Chevalley condition

Let $\mathcal{C}$ be a category with finite limits. In Chapter 1, the category Span($\mathcal{C}$) is defined in Construction 1.1.9 and it is characterized by a universal property involving Beck-Chevalley functors in Theorem 1.1.14. In this chapter, the bicategory of bispans in $\mathcal{C}$ (see Construction 3.2.15) is analogously characterized via a universal property. The characterization of bicategory of bispans is the main result of this thesis.

First, let's take a look at the universal property of a bicategory of *spans*. Historically, Hermida's paper contains the first written account of the universal property of the bicategory of spans[1] [31]. The universal property is in terms of the conditions written down in the '60s by Beck and Chevalley, which went unpublished. In [13], Dawson, Pare and Pronk discuss a universal property for a pseudo double category of spans.

We start by noting that a morphism $f : A \to B$ in $\mathcal{C}$ can be interpreted as a span in two different ways:

$$\begin{tikzcd} & A \arrow[dl, "id"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "id"] & \\ B & & A \end{tikzcd}$$

$$u_*(f) \qquad\qquad\qquad u^*(f)$$

These two ways give a pair of functors $u^*, u_* : \mathcal{C} \to \mathbf{Span}(\mathcal{C})$. It can be checked that $u_*(f)$ is left adjoint to $u^*(f)$ in $\mathbf{Span}(\mathcal{C})$. In fact, the pair of pseudofunctors satisfy additional conditions listed in the following definition.

---

[1] This bicategory is the horizontal bicategory of the pseudo double category of spans discussed in Example 2.1.48.

Definition 4.0.1. Let $\mathcal{C}$ be a category with finite limits and $\mathcal{D}$ be a bicategory. A pair of pseudofunctors

$$(\_)^*, (\_)_* : \mathcal{C} \to \mathcal{D}$$

is called a *Beck-Chevalley pair* if

1. the former pseudofunctor is contravariant, the 2-functors agree on objects,
2. for any morphism $f : x \to y$, the morphisms f_* is left adjoint to f^* and the tuple $(f_*, f^*, \eta_f, \epsilon_f)$ is the adjunction data with unit η_f and counit ϵ_f.
3. for any pullback square in $\mathcal{C}$:

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "m"] \\
\bullet \arrow[r, "n"'] & \bullet
\end{tikzcd}$$

   the "Beck-Chevalley morphism"

$$b_*a^* \xrightarrow{\eta_n b_* a^*} n^*n_*b_*a^* \simeq n^*m_*a_*a^* \xrightarrow{n^*m_*\epsilon_a} n^*m_*$$

   is an isomorphism.

Hermida showed that the natural pair of maps $\mathcal{C} \xrightarrow{u^*,u_*} \mathbf{Span}(\mathcal{C})$ is universal among all Beck-Chevalley pairs out of $\mathcal{C}$. The precise statement can be found in Theorem 4.1.3.

In Chapter 3, we constructed the Segal fibration over $\triangle \times \Gamma$ of spans of groupoids and constructed a fibration that is a candidate for Segal fibration over $\triangle \times \Gamma$ of bispans of groupoids. Following Hermida's characterization of spans via a universal property, it is natural to ask for an analogous universal property of the Segal fibration of bispans. The primary motivation for such a characterization in this thesis is "quantizing an extended field theory" (cf. Definition 1.1.12 and "quantizing a field theory" in Subsection 1.1.2 from the Chapter 1).

The first step is to characterize the bicategory of bispans, *i.e. ignoring the symmetric monoidal structure.* Recall that conjecturally symmetric monoidal bicategories are the same as pinched Segal fibrations over $\triangle \times \Gamma$ (See Conjecture 2.4.6). Since a symmetric monoidal bicategory has a high number of coherences and coherence laws, we worked with Segal fibrations in the last chapter. In contrast, bicategories have significantly fewer coherences, and we can calculate using string diagrams (see Subsection 4.2.1), so we can work with bicategories without using the Segal fibration picture.

A bicategory in which all 2-morphisms are invertible is called a $(2,1)$-*category*. An example of a $(2,1)$ category is the bicategory of groupoids, functors and natural transformations. Although a *category* $\mathcal{C}$ is considered throughout this chapter, it can be remarked that the arguments throughout the chapter can be adapted for a $(2,1)$-category $\mathcal{C}$ with homotopy pullbacks (see Construction 2.1.26 for an example). The proof of the main theorem is a construction and verification of a 2-functor using string diagrams which works for $(2,1)$-category. For the proofs involving the bicategory of bispans, it may be assumed that the coherence 2-cells are suppressed.

Thus, in this chapter, we consider the bicategory of **Bispan**$(\mathcal{C})$ described in Construction 3.2.15. Let $\mathcal{C}$ be a category with finite limits. Recall that **Bispan**$(\mathcal{C})$ is a bicategory with the following morphisms:

- the objects of **Bispan**$(\mathcal{C})$ are the objects of $\mathcal{C}$.
- a 1-morphism $(f,X,g) : A \to B$ is a span $f : X \to A, g : X \to B$. The composition of 1-morphisms is by pullbacks.
- the set of 2-morphisms $(s,P,t) : (f,X,g) \to (f',X',g')$ is the data of the form

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A & P \arrow[u, "s"] \arrow[d, "t"'] & B \\
 & X' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

  modulo an equivalence relation: $(s,P,t) : (f,X,g) \to (f',X',g')$ is related to $(s',P',t') : (f,X,g) \to (f',X',g')$ if there is an isomorphism $k : P \to P'$ such that $s'k = s$ and $t'k = t$.

Note that there are a pair of functors $\mathcal{I}_*, \mathcal{I}^* : \mathrm{Span}(\mathcal{C}) \to \mathbf{Bispan}(\mathcal{C})$ which agree on objects and 1-morphisms. On 2-morphisms

$$\mathcal{I}_*(f : X \to Y) = [(id, X, f)],\ \mathcal{I}^*(f : X \to Y) = [(f, X, id)].$$

We obtain a pair of pseudofunctors $i_*, i^* : \mathcal{C} \to \mathbf{Bispan}(\mathcal{C})$ defined as $i_* = \mathcal{I}_* u_*$ and $i^* = \mathcal{I}^* u^*$. This pair of pseudofunctors (i_*, i^*) satisfy a more general version of Beck-Chevalley conditions called Double Beck-Chevalley conditions (see Definition 4.1.15). It is shown in Theorem 4.3.10 that the pair (i_*, i^*) is universal among all Double Beck-Chevalley pseudofunctors out of $\mathcal{C}$.

Section 4.1 discusses the pair of embeddings of the category $\mathcal{C}$ into the bicategory of spans and the embedding of the bicategory of spans into the bicategory of bispans. The conditions needed to be imposed on the pullback and pushforward to obtain a universal property of bispans are called "Double Beck-Chevalley" conditions, and the conditions are recorded in Definition 4.1.15. Theorem 4.1.18 checks the Double Beck-Chevalley conditions for the pair of embeddings of $\mathcal{C}$ into bispans of $\mathcal{C}$. The universal property is stated in Theorem 4.3.10. The proof of the theorem uses string diagrams. In Section 4.2, we draw the adjunctions, Double Beck-Chevalley conditions in string diagrams. In Section 4.3, we prove the universal property of bispans.

4.1 Hierarchy of Spans

We will discuss this characterization in a systematic manner paying attention to the following hierarchy of spans:

$$\mathcal{C} \hookrightarrow \mathbf{Span}(\mathcal{C}) \hookrightarrow \mathbf{Bispan}(\mathcal{C})$$

A typical 2-morphism in each case is shown below:

$$\begin{tikzcd}
a \arrow[r] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
& x \arrow[dl] \arrow[dr] \arrow[dd] & \\
a & & b \\
& y \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& x \arrow[dl] \arrow[dr] & \\
a & X \arrow[l] \arrow[r] \arrow[u] \arrow[d] & b \\
& y \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

A detailed description of the bicategories $\mathbf{Span}(\mathcal{C})$ and $\mathbf{Bispan}(\mathcal{C})$ for $\mathcal{C} = \mathrm{Gpds}^f$ can be found in a Segal fibrational manner in Section 3.2.2 and Section 3.2.3. An explicit description of the bicategory of bispan is discussed in Construction 3.2.15. The bispan bicategory in general is discussed in Construction 3.2.15.

We would like to know sufficient conditions that must be imposed on the functors F_*, F^* shown below so that the dotted extensions exist:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "()_*{,}()^*"] \arrow[rr, hook, bend left, "i_*{,}i^*"] \arrow[rrd, "F_*{,}F^*"'] & \mathbf{Span}(\mathcal{C}) \arrow[r, hook, "\mathcal{I}_*{,}\mathcal{I}^*"] \arrow[rd, dotted] & \mathrm{BiSp}(\mathcal{C}) \arrow[d, dotted] \\
 & & \mathcal{D}
\end{tikzcd}$$

Remark 4.1.1. The bicategory $\mathbf{Span}(\mathcal{C})$ has a fully faithful map $\mathcal{I}_*$ to $\mathbf{Bispan}(\mathcal{C})$ where $\mathcal{I}_*$ is identity on objects and 1-morphisms. On 2-morphisms, we have

$$\begin{tikzcd}
 & x \arrow[dl] \arrow[dr] \arrow[dd, "l"] & & & & & x \arrow[dl] \arrow[dr] \arrow[d, equal] & \\
a & & b & {} \arrow[r, mapsto] & {} & a & x \arrow[d, "l"] & b \\
 & y \arrow[ul] \arrow[ur] & & & & & y \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Note that we also have $\mathcal{I}^*$ that is contravariant on the morphism categories which sends

$$\begin{tikzcd}
 & x \arrow[dl] \arrow[dr] \arrow[dd, "l"] & & & & & y \arrow[dl] \arrow[dr] & \\
a & & b & {} \arrow[r, mapsto] & {} & a & x \arrow[u, "l"'] & b \\
 & y \arrow[ul] \arrow[ur] & & & & & x \arrow[ul] \arrow[ur] \arrow[u, equal] &
\end{tikzcd}$$

For the lift $\mathcal{F} : \mathbf{Span}(\mathcal{C}) \to \mathcal{D}$ to exist, it suffices that the pair F_*, F^* satisfies the Beck-Chevalley conditions.

Before we move ahead, we need the notion of a Beck-Chevalley morphism.

Definition 4.1.2. Given a pullback square σ shown below,

$$\begin{tikzcd}
w \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner", very near end] & y \arrow[d, "h"] \\
x \arrow[r, "g"'] & b
\end{tikzcd}$$

the *Beck-Chevalley morphism* is the induced morphism

$$BC[\sigma] : q_* p^* \to h^* g_*$$

defined by

$$\begin{tikzcd}
F_*(q)F^*(p) \arrow[r, dotted, "BC[\sigma]"] \arrow[d, "\eta_h F_*(q)F^*(p)"'] & F^*(h)F_*(g) \\
F^*(h)F_*(h)F_*(q)F^*(p) \arrow[r, "F^*(h)\theta F_*(p)"] & F^*(h)F_*(g)F_*(p)F^*(p) \arrow[u, "F^*(h)F_*(g)\epsilon_p"']
\end{tikzcd}$$

The following theorem is due to Beck and Chevalley and can be found in [31, A.2].

Theorem 4.1.3. *Let $F_*, F^* : \mathcal{C} \to \mathcal{D}$ be a covariant and a contravariant pseudofunctor that agrees on objects and identity morphisms. Suppose the following "Beck-Chevalley" conditions hold:*

1. *For every morphism $f : a \to b$ in $\mathcal{C}$, we have a functorial adjunction*

$$F_*(f) \dashv F^*(f).$$

2. *Given a pullback square σ in $\mathcal{C}$, the induced Beck-Chevalley morphism $BC[\sigma]$ is an isomorphism.*

Then there exists a unique 2-functor $\mathcal{F}_* : \mathbf{Span}(\mathcal{C}) \to \mathcal{D}$ such that $\mathcal{F}_*(f)_* = F_*(f)$ and $\mathcal{F}_*(f)^* = F^*(f)$ and on 2-morphisms

$$\mathcal{F}_*(l :_f x_g \to_h y_k) = F_*(k)\epsilon_l F^*(h) \circ \theta_{1*}\theta_2^*$$

where we have used a counit of the adjunction $F_*(l) \dashv F^*(l)$ and the functorial coherences $F_*(g) \simeq_{\theta_{1*}} F_*(k)F_*(l)$ and $F^*(f) \simeq_{\theta_2^*} F^*(l)F^*(h)$.

Remark 4.1.4. The functors F_*, F^* are generally termed as pushforward functor and pullback functors respectively.

With Theorem 4.1.3, we have a characterisation of functors $\mathcal{F}_* : \mathbf{Span}(\mathcal{C}) \to \mathcal{D}$. Next, we ask for conditions that allow us to lift $\mathcal{F}_*$ to a dotted arrow as shown:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[rrd, "F_*,F^*"'] & \mathbf{Span}(\mathcal{C}) \arrow[r, hook, "\mathcal{I}_*,\mathcal{I}^*"] \arrow[rd, "\mathcal{F}_*"] & \mathrm{Bisp}(\mathcal{C}) \arrow[d, dotted] \\
 & & \mathcal{D}
\end{tikzcd}$$

However, if $F : \mathbf{Bispan}(\mathcal{C}) \to \mathcal{D}$ is a 2-functor, then $F\mathcal{I}_*$ and $F\mathcal{I}^*$ are both 2-functors out of the span bicategory. These functors on further restriction to $\mathcal{C}$ satisfy a pair of Beck-Chevalley conditions and the adjunction condition is strengthened as we will see now.

Construction 4.1.5. The category $\mathcal{C}$ can be embedded into $\mathbf{Bispan}(\mathcal{C})$ in two ways, denoted by i_*, i^*:

1. Let $f : c \to c'$ be a morphism of $\mathcal{C}$, then define i_* as identity on objects and define

   $$i_*(f) := c \xleftarrow{id} c \xrightarrow{f} c'.$$

   By our choice of pullbacks, we note that $i_*(gf) = i_*(g)i_*(f)$ and thus it is a strict functor.

2. Let $f : c \to c'$ be a morphism of $\mathcal{C}$, then define i^* as identity on objects and define

   $$i^*(f) := c' \xleftarrow{f} c \xrightarrow{id} c.$$

   By our choice of pullbacks, we note that $i^*(gf) = i^*(f)i^*(g)$ and thus it is a strict contravariant functor.

These functors interact in a special way with each other which we now establish in a series of propositions. Let us denote the span by a bimodule type notation:

$$\begin{tikzcd} & & & x \arrow[dl, "f"'] \arrow[dr, "g"] & \\ {}_f x_g & = & a & & b \end{tikzcd}$$

.

Remark 4.1.6. Given a span $x_* =_f x_g$, consider the transpose span $x^* =_g x_f$ The compositions $x_* x^*$ and $x^* x_*$ have natural maps with appropriate identities as shown below:

$$\begin{tikzcd}
id_a & & & a \arrow[dl, "id_a"'] \arrow[dr, "id_a"] & & & id_a \arrow[dd, Rightarrow, "\eta"] \\
 & = & a & x \arrow[u, "f"] \arrow[d, "\triangle"'] & a & = & \\
x^*x_* \arrow[uu, Rightarrow, "\epsilon^!"] & & & x \times_b x \arrow[ul, "f\pi_1"] \arrow[ur, "f\pi_2"'] & & & x^*x_*
\end{tikzcd}$$

$$\begin{tikzcd}
id_b & & & b \arrow[dl, "id_b"'] \arrow[dr, "id_b"] & & & id_b \arrow[dd, Rightarrow, "\eta^!"'] \\
 & = & b & x \arrow[u, "g"] \arrow[d, "\triangle"'] & b & = & \\
x_*x^* \arrow[uu, Rightarrow, "\epsilon"'] & & & x \times_a x \arrow[ul, "g\pi_1"] \arrow[ur, "g\pi_2"'] & & & x_*x^*
\end{tikzcd}$$

Note that $\epsilon^! = \mathrm{VFlip}(\eta)$ and $\eta^! = \mathrm{VFlip}(\epsilon)$.

It turns out that the morphism x_* and its transpose interact in a special way: x^* is both the left and right adjoint of x_*. This motivates the following definition.

Definition 4.1.7. In a bicategory $\mathcal{B}$, an *ambijunction* (or an *ambidextrous adjunction*) is a tuple

$$(f : a \to b, g : b \to a, \eta : id_a \to gf, \epsilon : fg \to id_b, \eta^! : id_b \to fg, \epsilon^! : gf \to id_a)$$

such that (f, g, η, ϵ) is the data of the adjunction $f \dashv g$ and $(g, f, \eta^!, \epsilon^!)$ is the data of the adjunction $g \dashv f$.

Further if $\epsilon^!\eta = id$ and $\epsilon\eta^! = id$ then we say that the ambijunction is *special.* If only the second condition is satisfied, then we say it is *partly special.*

Proposition 4.1.8. *The tuple* $(x_*, x^*, \eta, \epsilon, \eta^!, \epsilon^!)$ *forms an ambidextrous adjunction. Further if* x_* *is invertible, then the ambijunction is special.*

Proof. We have to verify the triangle identities.

1. First we can check

$$\begin{tikzcd}
x_* \arrow[r, "x_*\eta"] \arrow[dr, "id"'] & x_*x^*x_* \arrow[d, "\epsilon x_*"] \\
 & x_*
\end{tikzcd}$$

So we have to show the composite of the following morphism is identity:

$$\begin{tikzcd}
x_*id \arrow[dd, "x_*\eta"] & & a \times_a x \\
& & x \times_a x \arrow[u, "(f{,}id)"'] \arrow[d, "(\triangle{,}id)"] \\
x_*x^*x_* \arrow[dd, "\epsilon x_*"] & = & x \times_a x \times_a x \\
& & x \times_b x \arrow[u, "(id{,}\triangle)"'] \arrow[d, "(id{,}g)"] \\
idx_* & & x \times_b b
\end{tikzcd}$$

The above morphism can be simplified:

$$\begin{tikzcd}
& x \times_a x \arrow[dl, "\pi_2"'] \arrow[dr, "(\triangle{,}id)"] & & x \times_b x \arrow[dl, "(id{,}\triangle)"'] \arrow[dr, "\pi_1"] & \\
x & & x \times_b x \times_a x & & x
\end{tikzcd}$$

Since the diagram

$$\begin{tikzcd}
x \arrow[r, dotted, "\triangle"] \arrow[d, dotted, "\triangle"'] & x \times_a x \arrow[d, "(\triangle{,}id)"] \\
x \times_b x \arrow[r, "(id{,}\triangle)"'] & x \times_b x \times_a x
\end{tikzcd}$$

is a pullback square, we can compose the morphism above to obtain the composite

$$\begin{tikzcd}
& & x \arrow[dl, "\triangle"'] \arrow[dr, "\triangle"] & & & & & x \arrow[ddl, "id"'] \arrow[ddr, "id"] & \\
& x \times_a x \arrow[dl, "\pi_2"'] & & x \times_b x \arrow[dr, "\pi_1"] & & = & & & \\
x & & & & x & & x & & x
\end{tikzcd}$$

where we have used the fact that $\triangle = (id, id)$.

2. The proof of the other zigzag identity for the adjunction $x_* \dashv x^*$ is obtained by applying HFlip to case 1.

3. Applying VFlip $=: V$ to case 1, we obtain

$$\begin{tikzcd}
Vx_* & Vx_*Vx^*Vx_* \arrow[l, "V(Vx_*\eta)"'] \\
& x_* \arrow[u, "V(\epsilon Vx_*)"'] \arrow[ul, "V(id)"]
\end{tikzcd}$$

Now we use the fact that V is identity on 1-morphism but flips the order of composition of 2-morphisms. Further, we use $\epsilon^! = \text{VFlip}(\eta)$ and $\eta^! = \text{VFlip}(\epsilon)$ and the fact that V is an involution to obtain,

$$\begin{tikzcd}
x_* \arrow[r, "\eta^! x_*"] \arrow[dr, "id"'] & x_*x^*x_* \arrow[d, "x_*\epsilon^!"] \\
& x_*
\end{tikzcd}$$

4. The proof of the other zigzag identity can be obtained by applying HFlip to the above calculation.

If $x_* =_f x_g$ is an equivalence, then by Lemma 3.3.1, the morphisms f, g are isomorphisms. Now the 2 composable 1 morphism $(\epsilon^!, \eta)$ is given by

$$\begin{tikzcd}
& a \arrow[ddl] \arrow[ddr] & \\
& x \arrow[u] \arrow[d] & \\
a & x \times_b x \arrow[l] \arrow[r] & a \\
& x \arrow[u] \arrow[d] & \\
& a \arrow[uul] \arrow[uur] &
\end{tikzcd}$$

Using the fact that $x \times_{x \times_b x} x \simeq x$, we the 2-morphism c is

$$\begin{tikzcd}
& a \arrow[dl, "id"'] \arrow[dr, "id"] & \\
a & x \arrow[u, "f"] \arrow[d, "f"'] & a \\
& a \arrow[ul, "id"] \arrow[ur, "id"'] &
\end{tikzcd}$$

Since f is invertible and the diagram

$$\begin{tikzcd}
& x \arrow[dl, "f"'] \arrow[dr, "f"] \arrow[dd, "f"] & \\
a & & a \\
& a \arrow[ul, "id"] \arrow[ur, "id"'] &
\end{tikzcd}$$

commutes, the 2-morphism ${}_fx_f$ is equal to id_{id_a}. Thus we have shown that $\epsilon^! \eta = id_{id_a}$. The other case $\epsilon\eta^! = id_{id_b}$ is very similar where b replaces a and g replaces f in the calculation. ☐

Corollary 4.1.9. *Let $f : c \to c'$ be a morphism in $\mathcal{C}$. The morphisms $i_*(f), i^*(f)$ form a part of an functorial ambijunction data and the ambijunction is special when f is an isomorphism.*

Proof. The statement of ambijunction and the speciality when f is an isomorphism is a direct corollary of Proposition 4.1.8. The functoriality of the ambijunction has to be proven. This follows from the fact that

$$i_*(gf) = i_*(g)i_*(f), \; i^*(gf) = i^*(f)i^*(g)$$

along with the relation of flip functors with $i_* = \mathrm{HFlip} \circ i^*$ and the fact that composition of adjunction is an adjunction. ☐

Remark 4.1.10. Since

$$\begin{tikzcd}
& x \arrow[dl, "f"'] \arrow[dr, "g"] & \\
a & & b
\end{tikzcd}
=
\begin{tikzcd}
& & x \arrow[dl, equal, "id"] \arrow[dr, equal, "id"'] & & \\
& x \arrow[dl, "f"'] \arrow[dr, equal, "id"'] & & x \arrow[dl, equal, "id"] \arrow[dr, "g"] & \\
a & & x & & b
\end{tikzcd}$$

we see that an arbitrary 1-morphism ${}_fx_g$ can be written as $i_*(g)i^*(f)$. Thus the image of the pair of functors i_*, i^* generate all 1-morphisms of **Bispan**$(\mathcal{C})$ via composition.

We will now show that the Beck-Chevalley condition stated in Theorem 4.1.3 is obeyed by the inclusions i_*, i^*.

Proposition 4.1.11. *The functors i_*, i^* satisfy Beck-Chevalley conditions for 1-morphisms.*

Proof. We have already checked the adjunction condition in Proposition 4.1.8.

Now we have to prove the “base change” property: Given a pullback square σ

$$\begin{tikzcd}
x \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner", very near end] & a \arrow[d, "h"] \\
b \arrow[r, "g"'] & c
\end{tikzcd}$$

we will demonstrate that the Beck-Chevalley morphism $BC[\sigma]$ is an isomorphism.

The Beck-Chevalley morphism is

$$BC[\sigma] := i^*(h)i_*(g)\epsilon_p \circ i^*(h)\theta i_*(p) \circ \eta_h i_*(q)i^*(p).$$

Using the adjunction data from Remark 4.1.6, we compute $BC[\sigma]$ as shown in Figure 4.1.

$$\begin{tikzcd}
id_a q_* p^* \arrow[dd] & x \times_x x \times_a a & x \arrow[d, equal] \\
 & x \times_x x \times_a x \arrow[u, "{(id,id,q)}"'] \arrow[d, "{(id,id,\triangle)}"] & x \arrow[d, "{(id,q)}"] \\
g^* g_* q_* p^* \arrow[dd] & x \times_x x \times_a a \times_c a & x \times_c a \arrow[d, equal] \\
 & x \times_x x \times_a a \times_c a \arrow[u, "{(id,id,id,id)}"'] \arrow[d, "{(id,id,p\pi_2,id)}"] & x \times_c a \arrow[d, equal] \\
g^* f_* p_* p^* \arrow[dd] & x \times_x x \times_b b \times_c a & x \times_c a \arrow[d, equal] \\
 & x \times_b b \times_c a \arrow[u, "{(\triangle,id,id)}"'] \arrow[d, "{(p,id,id)}"] & x \times_c a \arrow[d, "{(p,id)}"] \\
g^* f_* id_b & b \times_b b \times_c a & b \times_c a
\end{tikzcd}$$

FIGURE 4.1: The columns are the Beck-Chevalley composable morphism, the computation in the category of bispan and a simplification.

Composing the spans in the final column we get:

$$\begin{tikzcd}
x \arrow[r, "{(id,q)_*}", "\shortmid" marking] & x \times_c a \arrow[r, equal, "\shortmid" marking] & x \times_c a \arrow[r, "{(p,id)}", "\shortmid" marking] & b \times_c a
\end{tikzcd}$$

Thus the Beck-Chevalley morphism is equal to

$$x \xrightarrow{(p,q)_*} b \times_c a$$

which is an isomorphism since σ is a pullback square. $\square$

For 2-morphisms, we have a similar story.

Remark 4.1.12. A 2-morphism in the bispan bicategory can be decomposed as

$$\begin{tikzcd} & x \arrow[dl, "f"'] \arrow[dr, "g"] & \\ a & X \arrow[u, "l"] \arrow[d, "m"'] & b \\ & y \arrow[ul, "h"] \arrow[ur, "k"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} & x \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\ & X \arrow[u, "l"] \arrow[d, equal] & \\ a & X \arrow[l, "{fl,hm}"] \arrow[r, "{gl,km}"] \arrow[d, equal] & b \\ & X \arrow[d, "m"'] & \\ & y \arrow[uul, "h"] \arrow[uur, "k"'] & \end{tikzcd}$$

So we see that an arbitrary 2-morphism ${}_lX_m$ can be written as $\mathcal{I}_*(m)\mathcal{I}^*(l)$. Thus the image of the pair of functors $\mathcal{I}_*, \mathcal{I}^*$ generate all 2-morphisms of **Bispan**$(\mathcal{C})$ via composition.

From Theorem 4.1.3, we know that

$$\mathcal{F}_*(l :_f x_g \to_h y_k) = (F_*(k)\epsilon_l F^*(h)) \circ (\theta_{1*}\theta_2^*)$$

Since $\mathcal{I}^* = \mathcal{I}_* \circ \mathrm{VFlip}$, given $\mathcal{F}_*, \mathcal{F}^*$ out of **Span**$(\mathcal{C})$ to $\mathcal{D}$ such that

$$F\mathcal{I}_* = \mathcal{F}_*, \;\; F\mathcal{I}^* = \mathcal{F}^*,$$

we can explicitly write

$$\begin{aligned} F({}_lX_m) &= F(\mathcal{I}_*(m)\mathcal{I}^*(l)) \\ &\simeq F(\mathcal{I}_*(m))F(\mathcal{I}^*(l)) \\ &= \mathcal{F}_*(m)F_*(\mathrm{Vflip}(l)) \\ &= \Big((F_*(k)\epsilon_m F^*(h)) \circ (\theta'_{1*}\theta_2'^{*})\Big) \circ \Big((\theta''_{1*}\theta_2''^{*}) \circ (F_*(g)\eta^!_l F^*(f))\Big) \\ &= \Big((F_*(k)\epsilon_m F^*(h)) \circ (\theta_{1*}\theta_2^*) \circ (F_*(g)\eta^!_l F^*(f))\Big) \end{aligned}$$

where we have used the properties of the vertical flip functor. We have also used the fact that coherences θ_1, θ_2 that witnesses $fl = hm$ and $gl = km$ are the compositions of the dashed coherences. So the 2-functors $F\mathcal{I}_*, F\mathcal{I}^*$ determine the action of F on all morphisms and objects.

Now the natural question is what constraints do the functors $\mathcal{F}_*, \mathcal{F}^*$ satisfy?

Proposition 4.1.13. *The pair of functors $\mathcal{I}_*, \mathcal{I}^* : \mathbf{Span}(\mathcal{C}) \to \boldsymbol{Bispan}(\mathcal{C})$ satisfies the following "Vertical Beck-Chevalley" condition:*
Given a pullback square in $\mathcal{C}$:

$$\begin{tikzcd}
x \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near end] & a \arrow[d, "g"] \\
b \arrow[r, "f"'] & c
\end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd}
g_*g^* \arrow[r, "\epsilon_g"] \arrow[dd, "g_*\eta^{!}_q g^*"'] & id_c \arrow[dr, "\eta^{!}_f"] & \\
 & & f_*f^* \\
g_*q_*q^*g^* \arrow[r, "\theta_*\theta^*"'] & f_*p_*p^*f^* \arrow[ur, "f_*\epsilon_p f^*"'] &
\end{tikzcd}$$

Proof. The bottom path can be computed using the adjunction data

$$\begin{tikzcd}
g_*id_cg^* \arrow[dd, "g_*\eta^{!}_q g^*"] & a \times_a a \times_a a & a \\
 & a \times_a x \times_a a \arrow[u, "{(id,q,id)}"] \arrow[d, "{(id,\triangle,id)}"'] & x \arrow[u, "q"] \arrow[d, equal] \\
g_*q_*q^*g^* \arrow[dd, "\theta_*\theta^*"] & a \times_a x \times_x x \times_a a \arrow[d, equal] & x \arrow[d, equal] \\
 & a \times_a x \times_x x \times_a a \arrow[d, "{(p\pi_2,\pi_2,\pi_3,p\pi_3)}"'] & x \arrow[d, equal] \\
f_*p_*p^*f^* \arrow[dd, "f_*\epsilon_p f^*"] & b \times_b x \times_x x \times_b b & x \arrow[d, equal] \\
 & b \times_b x \times_b b \arrow[u, "{(id,\triangle,id)}"] \arrow[d, "{(id,p,id)}"'] & x \arrow[d, "p"'] \\
f_*f^* & b \times_b b \times_b b & b
\end{tikzcd}$$

and the composable arrow composes to the following morphism:

$$\begin{tikzcd}
 & x \arrow[dl, "q"] \arrow[dr, "p"] & \\
a & & b
\end{tikzcd}$$

Similarly we compute the top path:

$$\begin{tikzcd}
g_*g^* \arrow[ddd, "\epsilon_g"] & a \times_a a & a \arrow[d, equal] \\
 & a \arrow[u, "\triangle"] \arrow[d, "q"'] & a \\
 & c & c \arrow[u, "q"'] \arrow[d, "p"] \\
id_{c*} \arrow[ddd, "\eta^!_f"] & b \arrow[u, "p"] \arrow[d, "\triangle"'] & b \arrow[d, equal] \\
 & b \times_b b & b \\
 & & \\
f_*f^* & &
\end{tikzcd}$$

and the composable arrow composes to the following morphism:

$$\begin{tikzcd}
 & x \arrow[dl, "q"] \arrow[dr, "p"] & \\
a & & b
\end{tikzcd}$$

Since both morphisms are equal, the vertical Beck-Chevalley diagram commutes. ☐

There is an involved constraint that arises out horizontal functoriality of 2-morphisms.

Proposition 4.1.14. *The pseudofunctors i_*, i^* satisfy the "Horizontal Beck-Chevalley" condition: Given a diagram in $\mathcal{C}$ and a limit v over it:*

$$\begin{tikzcd}
u \arrow[r, "\pi_2^x"] \arrow[d, "\pi_1^x"] \arrow[dr, phantom, "\sigma_1"] \arrow[dr, phantom, "\lrcorner", very near end] & x_2 \arrow[d, "g_1"] & c \arrow[l, "n"'] \arrow[d, "p"] \\
x_1 \arrow[r, "f_1"] & b & y_2 \arrow[l, "g_2"] \\
a \arrow[u, "l"] \arrow[r, "m"'] & y_1 \arrow[u, "f_2"] & w \arrow[l, "\pi_1^y"] \arrow[u, "\pi_2^y"'] \arrow[ul, phantom, "\sigma_2"] \arrow[ul, phantom, "\ulcorner", very near end]
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, "\pi_2^x"] \arrow[d, "\pi_1^x"'] & x_2 & c \arrow[l, "n"'] \arrow[d, "p"'] \\
x_1 & v \arrow[ul, "S"] \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"'] \arrow[dr, "T"] & y_2 \\
a \arrow[u, "l"] \arrow[r, "m"'] & y_1 & w \arrow[l, "\pi_1^y"] \arrow[u, "\pi_2^y"']
\end{tikzcd}$$

where the squares σ_1, σ_2 are pullback squares and $BC[\sigma_1], BC[\sigma_2]$ denote the associated Beck-Chevalley isomorphisms. Then the following diagram commutes:

$$\begin{tikzcd}
n^*\pi^x_{2*}\pi^{x*}_1 l_* \arrow[r, "BC[\sigma_1]"] \arrow[d, "n^*\pi^x_{2*}\eta^!_S\pi^{x*}_1 l_*"'] & n^*g_1^*f_{1*}l_* \arrow[r, "\theta^*\theta_*"] & p^*g_2^*f_{2*}m_* & p^*\pi^y_{2*}\pi^{y*}_1 m_* \arrow[l, "BC[\sigma_2]"'] \\
n^*\pi^x_{2*}S_*S^*\pi^{x*}_1 l_* \arrow[d, "n^*\theta_*\theta_*l_*"'] & & & p^*\pi^y_{2*}T_*T^*\pi^{y*}_1 m_* \arrow[u, "p^*\pi^y_{2*}\epsilon_T\pi^{y*}_1 m_*"'] \\
n^*n_*\pi_{2*}\pi_1^*l^*l_* \arrow[r, "\epsilon^!_n\pi_{2*}\pi_1^*\epsilon^!_l"'] & \pi_{2*}\pi_1^* \arrow[rr, "\eta_p\pi_{2*}\pi_1^*\eta_m"'] & & p^*p_*\pi_{2*}\pi_1^*m^*m_* \arrow[u, "p^*\theta_*\theta^*m_*"']
\end{tikzcd}$$

Proof. We will compute the morphisms along top path and bottom path. We will show they are equal. In order to simplify the long calculation, we first note that 2-morphism spans with both legs isomorphisms are equivalent to identity morphism. So the functorial coherences θ_*/θ^* and Beck Chevalley isomorphisms are essentially identities and thus they wont be computed.

In the computation of bottom composable arrow (in the commutative diagram in question), we have directly shown the simplified expressions in Figure 4.2. Since Beck-Chevalley isomorphisms are equal to identity, we have to show that the above composable arrow in Figure 4.2 is identity when composed. Using

$$u \simeq x_1 \times_a x_2,\ w \simeq y_1 \times_a y_2,\ v = b \times_a c,$$

we can write

$$\begin{aligned} b \times_{x_1} u \times_{x_2} c &\simeq b \times_{x_1} (x_1 \times_a x_2) \times_{x_2} c \\ &\simeq b \times_a c \\ &\simeq v \end{aligned}$$

and

$$b \times_{x_1} v \times_{x_2} c \simeq b \times_{x_1} b \times_a c \times_{x_2} c.$$

Now we compute the composable morphism as

$$\begin{tikzcd}
& b \times_{x_1} b \times_a c \times_{x_2} c \arrow[dl, "(p_1,p_4)"'] \arrow[dr, equal] & & b \times_a c \arrow[dl, "(\triangle,\triangle)"'] \arrow[dr, "(\triangle,\triangle)"] & & b \times_{y_1} b \times_a c \times_{y_2} c \arrow[dl, equal] \arrow[dr, "(p_1,p_4)"] & \\
b \times_a c & & b \times_{x_1} b \times_a c \times_{x_2} c & & b \times_{y_1} b \times_a c \times_{y_2} c & & b \times_a c
\end{tikzcd}$$

$$\begin{tikzcd}
n^*\pi^x_{2*}id_u\pi_1^{x*}l_* \arrow[d, "n^*\pi^x_{2*}\eta^!_S\pi_1^{x*}l_*"'] \\
n^*\pi^x_{2*}S_*S^*\pi_1^{x*}l_* \arrow[d, equal] \\
n^*n_*\pi_{2*}\pi_1^*l^*l_* \arrow[d, "\epsilon^!_n\pi_{2*}\pi_1^*\epsilon^!_l"'] \\
\pi_{2*}\pi_1^* \arrow[d, "\eta_p\pi_{2*}\pi_1^*\eta_m"'] \\
p^*p_*\pi_{2*}\pi_1^*m^*m_* \arrow[d, equal] \\
p^*h_{2*}T_*T^*\pi_2^{y*}m_* \arrow[d, "p^*h_{2*}\epsilon_T\pi_2^{y*}m_*"'] \\
p^*\pi^y_{2*}id_w\pi_1^{y*}m_*
\end{tikzcd}
\qquad
\begin{tikzcd}
b \times_{x_1} u \times_{x_2} c \\
b \times_{x_1} v \times_{x_2} c \arrow[u, "(id{,}S{,}id)"'] \arrow[d, equal] \\
b \times_{x_1} v \times_{x_2} c \arrow[d, equal] \\
b \times_{x_1} v \times_{x_2} c \\
v \arrow[u, "(\pi_1{,}id{,}\pi_2)"'] \arrow[d, equal] \\
v \arrow[d, equal] \\
v \arrow[d, "(\pi_1{,}id{,}\pi_2)"] \\
b \times_{y_1} v \times_{y_2} c \arrow[d, equal] \\
b \times_{y_1} v \times_{y_2} c \arrow[d, equal] \\
b \times_{y_1} v \times_{y_2} c \arrow[d, "(id{,}T{,}id)"] \\
b \times_{y_1} w \times_{y_2} c
\end{tikzcd}$$

FIGURE 4.2: Computation of the long path of HBC commutative diagram

Simplifying this, we obtain the identity morphism of $b \times_a c$ which is what we set out to show. □

The properties listed in Propositions/Theorems 4.1.8, 4.1.3, 4.1.13 and 4.1.14 can be collected into a single definition.

Definition 4.1.15. Given a category $\mathcal{C}$ and a bicategory $\mathcal{B}$, a tuple $(F_*, F^*, \eta, \epsilon, \eta^!, \epsilon^!)$ where

- $F_* : \mathcal{C} \to \mathcal{B}$, $F^* : \mathcal{C}^{op} \to \mathcal{B}$ are pseudofunctors that agree on objects.

- for every morphism $f : a \to b$, the tuple $(F_*(f), F^*(f), \eta_f, \epsilon_f)$ is the adjunction data for an adjunction $F_*(f) \dashv F^*(f)$ and $(F^*(f), F_*(f), \eta^!_f, \epsilon^!_f)$ is the adjunction data for an adjunction $F^*(f) \dashv F_*(f)$.

is called a *double Beck-Chevalley functor data* if it satisfies the following axioms:

1. (Ambijunction) The tuple

$$(F_*(f), F^*(f), \eta_f, \epsilon_f, \eta^!_f, \epsilon^!_f)$$

   is an ambijunction and the ambijunction data is functorial in f.

   Further, if f is an isomorphism, then the ambijunction is partly special. For every object $a \in \mathcal{C}$, the ambijunction for id_a is

$$(F_*(id_a), F^*(id_a), \eta_a, \epsilon_a, \eta^!_a, \epsilon^!_a)$$

   is given by

$$\eta_a = \theta^*_a \theta_{*a} = (\epsilon^!_a)^{-1},\ \eta^!_a = \theta_{*a}\theta^*_a = (\epsilon_a)^{-1},$$

   where θ_*, θ^* are the identity coherences of F_*, F^*.

2. (BC) Given a pullback square in $\mathcal{C}$:

$$\begin{tikzcd} x \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near end] & a \arrow[d, "g"] \\ b \arrow[r, "f"'] & c \end{tikzcd}$$

   Let σ_* denote the invertible 2-cell that witnesses $g_*q_* \simeq_{\sigma_*} f_*p_*$, then the Beck-Chevalley morphism

$$BC[\sigma_*] : q_*p^* \to g^*f_*$$

   is an isomorphism.

3. (VBC) Given a pullback square in $\mathcal{C}$:

$$\begin{tikzcd} x \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near end] & a \arrow[d, "g"] \\ b \arrow[r, "f"'] & c \end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd}
g_*g^* \arrow[r, "\epsilon_g"] \arrow[d, "g_*\eta^!_q g^*"'] & id_c \arrow[dr, "\eta^!_f"] & \\
 & & f_*f^* \\
g_*q_*q^*g^* \arrow[r, "\sigma_*\sigma^*"'] & f_*p_*p^*f^* \arrow[ur, "f_*\epsilon_p f^*"'] &
\end{tikzcd}$$

FIGURE 4.3: Vertical 2-functoriality constraint

4. (HBC) Given a diagram in $\mathcal{C}$ and the limit v over it:

$$\begin{tikzcd}
u \arrow[r, "\pi_2^x"] \arrow[d, "\pi_1^x"] \arrow[dr, phantom, "\sigma_1"] \arrow[dr, phantom, "\lrcorner", very near end] & x_2 \arrow[d, "g_1"] & Y \arrow[l, "n"'] \arrow[d, "p"] \\
x_1 \arrow[r, "f_1"] & b & y_2 \arrow[l, "g_2"] \\
X \arrow[u, "l"] \arrow[r, "m"'] & y_1 \arrow[u, "f_2"'] & w \arrow[l, "\pi_1^y"] \arrow[u, "\pi_2^y"'] \arrow[ul, phantom, "\sigma_2"] \arrow[ul, phantom, "\ulcorner", very near end]
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, "\pi_2^x"] \arrow[d, "\pi_1^x"'] & x_2 & Y \arrow[l, "n"'] \arrow[d, "p"'] \\
x_1 & v \arrow[ul, "S"] \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"'] \arrow[dr, "T"] & y_2 \\
X \arrow[u, "l"] \arrow[r, "m"'] & y_1 & w \arrow[l, "\pi_1^y"] \arrow[u, "\pi_2^y"']
\end{tikzcd}$$

where the squares σ_1, σ_2 are pullback squares, let $BC[\sigma_1], BC[\sigma_2]$ denote the associated Beck-Chevalley isomorphisms. Then the following diagram commutes:

$$\begin{tikzcd}
n^*\pi^x_{2*}\pi_1^{x*}l_* \arrow[r, "BC{[\sigma_1]}"] \arrow[d, "n^*\pi^x_{2*}\eta^!_S\pi_1^{x*}l_*"'] & n^*g_1^*f_{1*}l_* \arrow[r, "\theta^*\theta_*"] & p^*g_2^*f_{2*}m_* & p^*\pi^y_{2*}\pi_1^{y*}m_* \arrow[l, "BC{[\sigma_2]}"'] \\
n^*\pi^x_{2*}S_*S^*\pi_1^{x*}l_* \arrow[d, "n^*\theta_*\theta_*l_*"'] & & & p^*\pi^y_{2*}T_*T^*\pi_1^{y*}m_* \arrow[u, "p^*\pi^y_{2*}\epsilon_T\pi_1^{y*}m_*"] \\
n^*n_*\pi_{2*}\pi_1^*l^*l_* \arrow[rr, "\epsilon^!_n\pi_{2*}\pi_1^*\epsilon^!_l"'] & & \pi_{2*}\pi_1^* \arrow[r, "\eta_p\pi_{2*}\pi_1^*\eta_m"'] & p^*p_*\pi_{2*}\pi_1^*m^*m_* \arrow[u, "p^*\theta_*\theta^*m_*"]
\end{tikzcd}$$

FIGURE 4.4: Horizontal 2-functoriality constraint

Definition 4.1.16. A pair of pseudofunctors $F_*, F^* : \mathcal{C} \to \mathcal{D}$ are called *a pair of Double Beck-Chevalley pseudofunctors* if these functors are part of a double Beck-Chevalley functor data $(F_*, F^*, \eta, \epsilon, \eta^!, \epsilon^!)$.

Remark 4.1.17. The first condition is mostly motivated by the properties of the bispan category mentioned in the corollary of Proposition 4.1.8. The speciality of the ambijunction is needed for the welldefinedness of the functor and the identity ambijunction condition is a technical assumption that seems harmless for the examples we have in mind.

The second and the third condition is a consequence of Theorem 4.1.3 and is need to prove horizontal and vertical functoriality of 1-morphisms and 2-morphisms.

Using Theorem 4.1.3, the first three conditions imply the existence of a pair of functors $\mathcal{F}_*, \mathcal{F}^* : \mathbf{Span}(\mathcal{C}) \to \mathcal{D}$.

The condition in Figure 4.4 will be used to prove the horizontal functoriality of 2-morphisms. The functoriality conditions can be simply paraphrased as a coherence condition that states that various equivalent ways of composing the "push-pull" functors are equivalent.

Theorem 4.1.18. *The pseudofunctors $i_*, i^* : \mathcal{C} \to$ **Bispan**$(\mathcal{C})$ (from Construction 4.1.5) are a pair of Double Beck-Chevalley 2-functors.*

Proof. We have to construct a tuple $(i_*, i^*, \eta, \epsilon, \eta^!, \epsilon^!)$ of double Beck-Chevalley functor data.

Constructions:

We have already defined the adjunction data in Remark 4.1.6.

$$\begin{tikzcd}
id_a & & & a \arrow[dl, "id_a"'] \arrow[dr, "id_a"] & & & id_a \arrow[dd, Rightarrow, "\eta"] \\
 & = & a & x \arrow[u, "f"] \arrow[d, "\triangle"'] & a & = & \\
x^*x_* \arrow[uu, Rightarrow, "\epsilon^!"] & & & x \times_b x \arrow[ul, "f\pi_1"] \arrow[ur, "f\pi_2"'] & & & x^*x_*
\end{tikzcd}$$

$$\begin{tikzcd}
id_b & & & b \arrow[dl, "id_b"'] \arrow[dr, "id_b"] & & & id_b \arrow[dd, Rightarrow, "\eta^!"'] \\
 & = & b & x \arrow[u, "g"] \arrow[d, "\triangle"'] & b & = & \\
x_*x^* \arrow[uu, Rightarrow, "\epsilon"'] & & & x \times_a x \arrow[ul, "g\pi_1"] \arrow[ur, "g\pi_2"'] & & & x_*x^*
\end{tikzcd}$$

Checking the various axioms of Double Beck-Chevalley condition:

1. (Ambijunction) For any $f : a \to b$ in $\mathcal{C}$, it has been verified in the corollary to Proposition 4.1.8 that the tuple $(i_*(f), i^*(f), \eta_f, \epsilon_f, \eta^!_f, \epsilon^!_f)$ is an ambijunction that is functorial in f. Further the ambijunction was proved to be partly special when f is an isomorphism. Since i_*, i^* both send identity morphisms to identity 1-morphisms, the identity coherences are trivial. The adjunction data for $id_a \dashv id_a$ is also trivial and therefore axiom for identity ambijunction holds.
2. (BC) The Beck-Chevalley condition has been checked in the Proposition 4.1.11.
3. (VBC)The Vertical Beck-Chevalley condition has been checked in Proposition 4.1.13.
4. (HBC) The Horizontal Beck-Chevalley condition has been checked in Proposition 4.1.14.

□

Remark 4.1.19. Summarizing the situation: If we have a 2-functor $F : \textbf{Bispan}(\mathcal{C}) \to \mathcal{D}$, we can get a pair of pseudo-functors $\mathcal{F}_*, \mathcal{F}^*$ by composing F with $\mathcal{I}_*, \mathcal{I}^*$ respectively. From Remark 4.1.12, we see that

$$F({}_fX_g) = F(\mathcal{I}_*(g)\mathcal{I}^*(f)) \simeq \mathcal{F}_*(g)\mathcal{F}^*(f).$$

From Theorem 4.1.3, we know that further restriction of the pair $\mathcal{F}_*, \mathcal{F}^*$ should satisfy a pair of Beck-Chevalley conditions.

We can get a pair of pseudo-functors $F_*, F^* : \mathcal{C} \to \mathcal{D}$ by composing F with the pair of double Beck-Chevalley pseudofunctors i_*, i^* respectively. From Remark 4.1.10, we see that

$$F({}_fx_g) = F(i_*(g)i^*(f)) \simeq F_*(g)F^*(f).$$

Such a composition is called “pull-push” functor.

Further, i_*, i^* agree on objects and for any object a in the category $\mathcal{C}$, since we have $i_*(id_a) = i^*(id_a)$, $i_*(id_{id_a}) = i^*(id_{id_a})$, the following must be true:

$$F_*(a) = F^*(a),\ \ F_*(id_a) = F^*(id_a),\ \ F_*(id_{id_a}) = F^*(id_{id_a}).$$

Now we ask the following natural question: *What are the conditions that should be imposed on the pseudofunctors*

$$F_*, F^* : \mathcal{C} \to \mathcal{D},$$

so that it induces a 2-functor $F : \textbf{\textit{Bispan}}(\mathcal{C}) \to \mathcal{D}$ *such that* $Fi_* = F_*$, $Fi^* = F^*$*?* The answer: F_*, F^* should be a pair of double Beck-Chevalley pseudofunctors. This is the main theorem of this chapter (Theorem 4.3.10). The proof of this theorem is in Section 4.3. In the Section 4.2, we describe the basic moves allowed in proofs using string diagrams and we also present the Double Beck-Chevalley condition in a diagrammatic form.

4.2 Basic moves

The proof of Theorem 4.3.10 will use calculations by manipulating string diagrams. A move is an equation of diagrams. In Subsection 4.2.1, a quick introduction to string diagrams is given. Especially the zig-zag laws followed by adjunctions (Figure 4.5) and coherence laws followed by pseudofunctors are depicted as moves (Figure 4.6). These are standard moves in the theory.

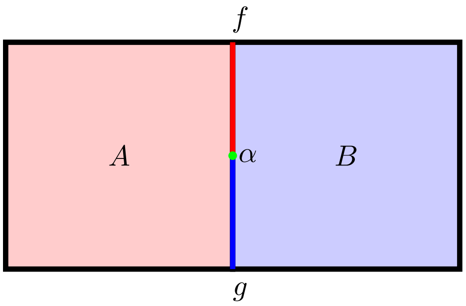
In Subsection 4.2.2, we present the string diagrams for the Double Beck-Chevalley data (see Definition 4.1.15). The Beck-Chevalley isomorphism in the definition is depicted in Figure 4.10, speciality (of the ambijunction) move is depicted in Figure 4.11. The vertical Beck-Chevalley move is shown in 4.12 and the horizontal Beck-Chevalley move is shown in 4.13.

4.2.1 String diagram notations

In this subsection, we setup the notations for string diagrams.

Given a bicategory $\mathcal{D}$, the string diagram notations for the concepts in a bicategory are shown below:

A 2-morphism $\alpha : f \to g$ between 1-morphisms $f, g : A \to B$ is shown below:



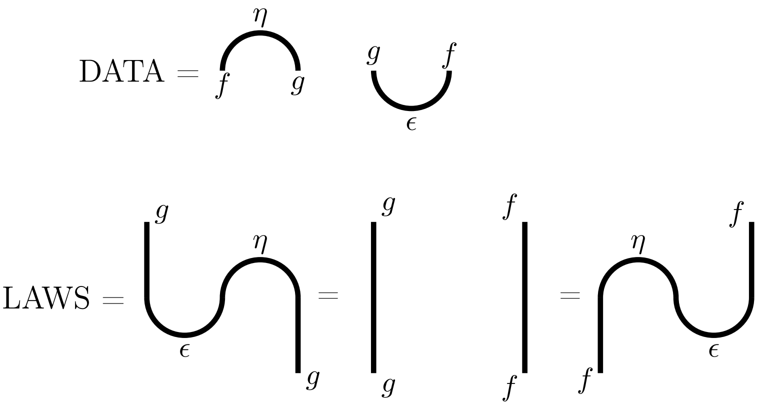


FIGURE 4.5: Adjunction moves

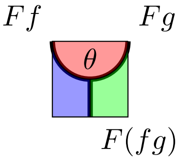


FIGURE 4.6: Coherence isomorphism of a 2-functor

The associator and unitor coherences are suppressed in string diagrams. For instance, identity 2-morphism is denoted by no dot and identity morphism is denoted by an empty region.

The order of composable horizontal morphisms is from left to right in the string diagram and the composition of vertical 2-morphisms is from top to down. For example, to depict

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

we will draw f to the left of g.

The adjunction data (f, g, η, ϵ) and laws are denoted in Figure 4.5.

Given a 2-functor between strict bicategories $F : \mathcal{C} \to \mathcal{D}$, the composition coherence is shown in Figure 4.6.

The invertiblity of the functorial coherences are captured by the string diagram equations in Figure 4.7.

The compositional and unitor coherence laws are depected in Figure 4.8.

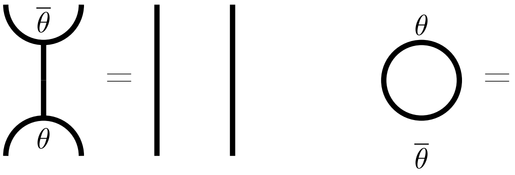


Figure 4.7: Two conditions denoting invertibility of functorial coherences

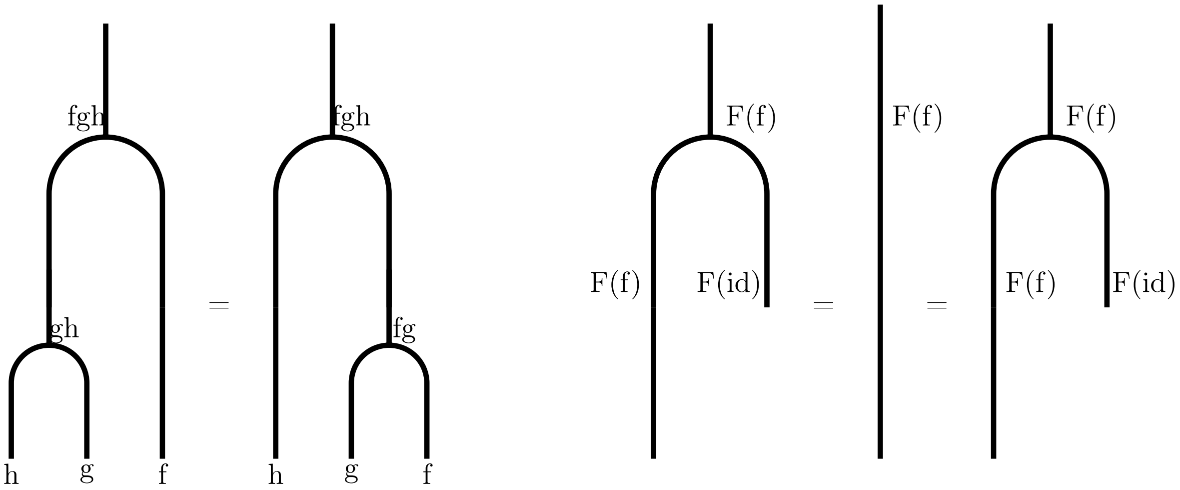


Figure 4.8: Functorial coherence laws

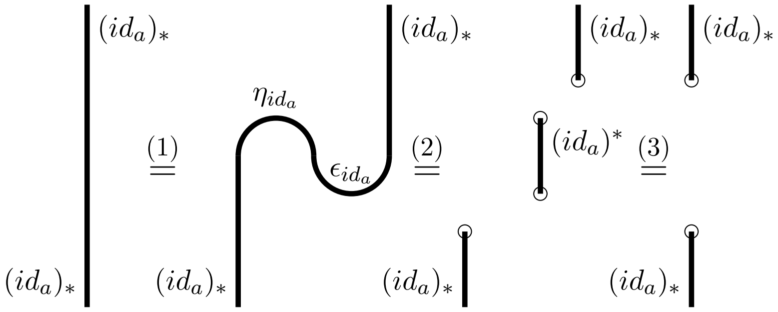


FIGURE 4.9: Breaking pushforward and pullback of identity morphism

4.2.2 Ambijunction and Beck-Chevalley moves

The string diagrams for the data of double Beck-Chevalley conditions is discussed in this subsection.

The ambijunction data is essentially the data of a pair of adjunctions. The identity ambijunction data specified in Definition 4.1.15 allows us to “break" $F^*(id)$ and $F_*(id)$. For the case of $F_*(id_a)$, this is shown in Figure 4.9. The steps use the zig-zag property of adjunction, definition of identity ambijunction and invertibility of identity coherence.

We will now specify the Beck-Chevalley morphism. Given a pullback square in $\mathcal{C}$:

$$\begin{tikzcd}
x \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near end] & a \arrow[d, "g"] \\
b \arrow[r, "f"'] & c
\end{tikzcd}$$

Let σ_* denote the invertible 2-cell that witnesses $g_*q_* \simeq_{\sigma_*} f_*p_*$ and $t := gq = fp$, then the string diagram for the Beck-Chevalley morphism

$$BC[\sigma_*] : q_*p^* \to g^*f_*$$

is shown in Figure 4.10.

If $f : a \to b$ is invertible, then the special ambijunction condition entails $\epsilon_f \eta^!_f$ is identity and the string diagram for this is shown in Figure 4.11.

The vertical Beck-Chevalley condition (stated in Definition 4.1.15) in terms of string diagrams is described in Figure 4.12.

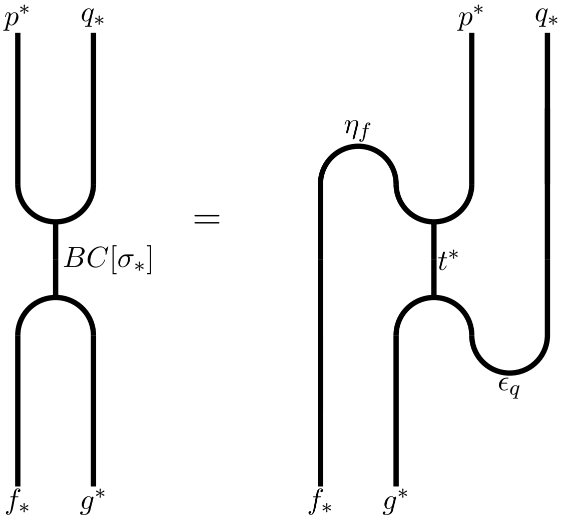


FIGURE 4.10: Beck-Chevalley morphism $BC[\sigma_*]$

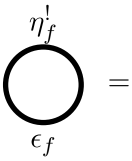


FIGURE 4.11: The partly special ambijunction condition gives a move where the circle on left hand side can be deleted.

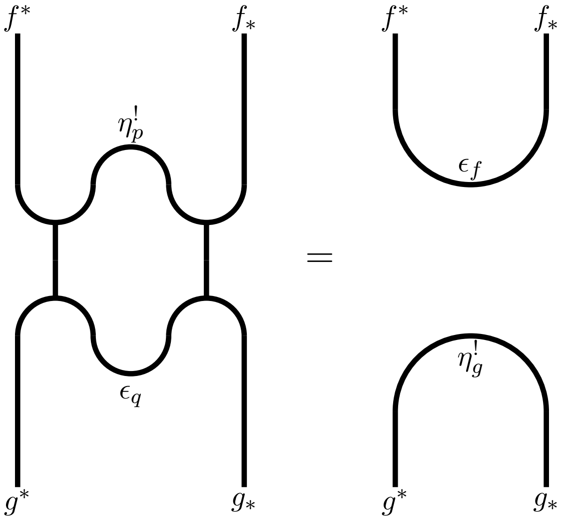


FIGURE 4.12: Vertical BC condition

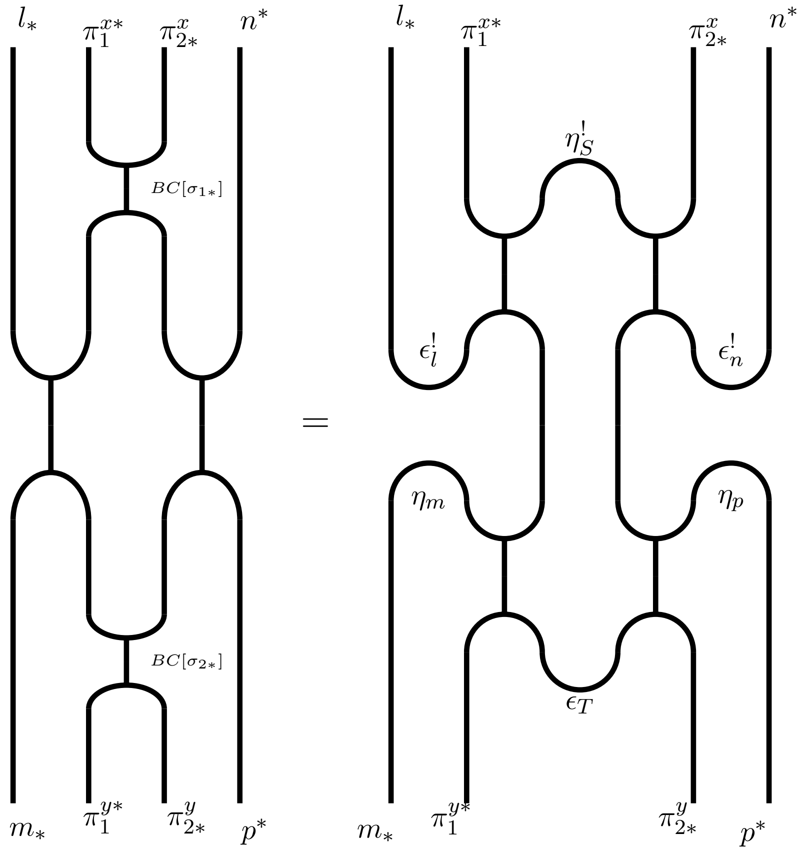


FIGURE 4.13: Horizontal BC condition

The horizontal Beck-Chevalley condition (stated in Definition 4.1.15) in terms of string diagrams is described in Figure 4.13.

4.3 Proof of the main theorem

In this section, we characterise the universal property of bispans in Theorem 4.3.10. Given a double Beck-Chevalley functor data $F_*, F^* : \mathcal{C} \to \mathcal{D}$, Construction 4.3.1 specifies the data of the 2-functor $F : \mathbf{Bispan}(\mathcal{C}) \to \mathcal{D}$. First, F is a well defined functor on the morphism category of $\mathbf{Bispan}(\mathcal{C})$ (Lemma 4.3.5).

Construction 4.3.1. Given a 2-functor that satisfies the Double Beck-Chevalley conditions given in Definition 4.1.15, we construct a 2-functor

$$F : \mathbf{Bispan}(\mathcal{C}) \to \mathcal{D}.$$

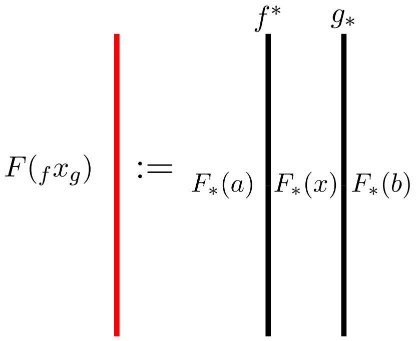


FIGURE 4.14: String diagram for $g_* f^*$

Define F on objects as

$$F(c) = F_*(c) = F^*(c).$$

Given a 1-morphism ${}_f x_g : a \to b$,we define

$$F({}_f x_g) := F_*(g) F^*(f)$$

which we will abbreviate as $g_* f^*$ as usual.

Given a 2-morphism ${}_l X_m :_f x_g \to_h y_k$, the action of F on 2-morphisms is given by

$$F({}_l X_m) = (k_* \epsilon_m h^*) \circ (\theta_{*gl} \theta^*_{fl}) \circ (g_* \eta^!_l f^*).$$

The 1-morphism and 2-morphism above are shown in string diagram notation in Figure 4.14 and Figure 4.15.

Given a 2-composable arrow in **Bispan**($\mathcal{C}$)

$$\begin{tikzcd}
 & & x_1 \times_b x_2 \arrow[dl, "\pi_1^x"'] \arrow[dr, "\pi_2^x"] & & \\
 & x_1 \arrow[dl, "\alpha"'] \arrow[dr, "f"] & & x_2 \arrow[dl, "g"'] \arrow[dr, "\beta"] & \\
a & & b & & c
\end{tikzcd}$$

and its composition

$$\begin{tikzcd}
 & x_1 \times_b x_2 \arrow[dl, "\alpha\pi_1^x"'] \arrow[dr, "\beta\pi_2^x"] & \\
a & & c
\end{tikzcd}$$

the composition coherence of the 2-functor F is depicted in Figure 4.8.

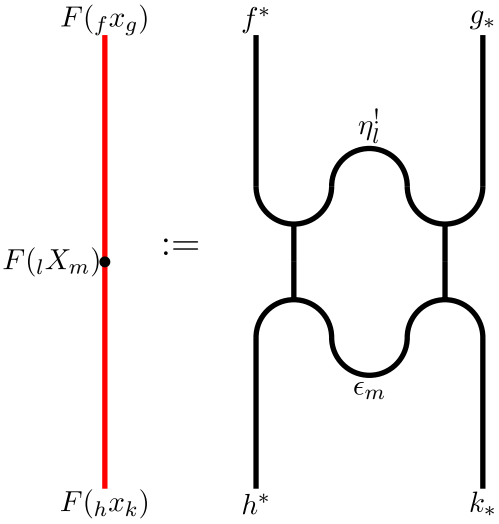


FIGURE 4.15: String diagram for 2-morphism assigned to F

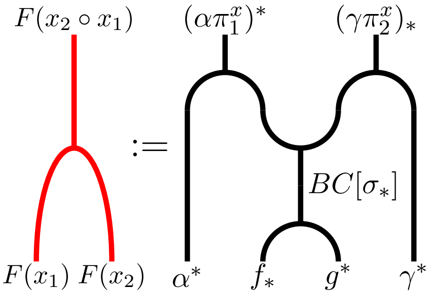


FIGURE 4.16: Composition coherence of F

Let $a, b \in \mathcal{C}$, now we prove that

$$F : \mathbf{Bispan}(\mathcal{C})(a, b) \to \mathcal{D}(Fa, Fb)$$

is a functor in a series of lemmas. Lemma 4.3.2 shows that F is well defined on 2-morphisms, Lemma 4.3.3 shows that F preserves identity of 1-morphisms and Lemma 4.3.4 shows that F preserves composition of 2-morphisms. Thus we conclude F is a functor on the morphism category (in Lemma 4.3.5).

The 2-morphisms in the bicategory of bispans is defined up to isomorphisms of diagrams. If F, in Construction 4.3.1, is a 2-functor, then F should be well defined. We prove this in the next lemma.

Lemma 4.3.2. *The assignment*

$$F : \boldsymbol{Bispan}(\mathcal{C})(a, b) \to \mathcal{D}(Fa, Fb)$$

given in Construction 4.3.1 is well defined on 2*-morphisms.*

Proof. Suppose ${}_lX_m$, ${}_{l'}X'_{m'}$ are related, then there exists an isomorphism $\alpha : X \to X'$ in $\mathcal{C}$ such that

$$l = l'\alpha, m = m'\alpha.$$

In order to show that $F(X) = F(X')$, we use the string diagram description of $F(X), F(X')$ given in Construction 4.3.1. We start with $F(X)$ on the left hand side in Fgure 4.17 and the last diagram can be simplified to by $F(X')$.

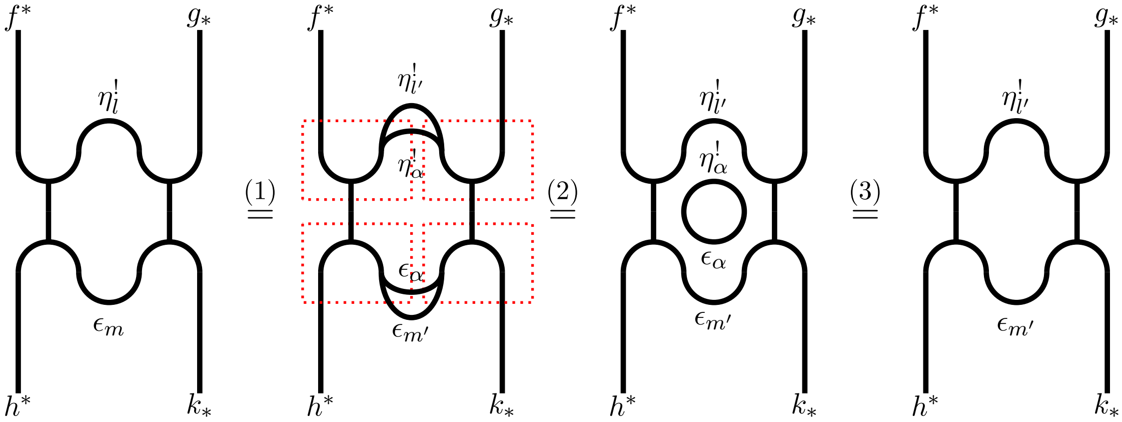


FIGURE 4.17: Well definedness of the 2-functor

The reasons for the equality (numbered in Figure 4.17) is given below:

1. Follows from the definition of X', specifically from

$$l = l'\alpha, m = m'\alpha.$$

2. Follows from 2-functor's composition coherence condition presented in string diagram form in Figure 4.8. This condition is applied 4 times as indicated by the dotted red boxes in the figure.

3. Follows from the fact that α is an isomorphism and hence the partly special condition of the ambijunction data for α as shown in Figure 4.11.

This proves that the assignment is well defined. ☐

Lemma 4.3.3. *The assignment*

$$F : \boldsymbol{Bispan}(\mathcal{C})(a, b) \to \mathcal{D}(Fa, Fb)$$

given in Construction 4.3.1 preserves identity of every object in $\boldsymbol{Bispan}(\mathcal{C})(a, b)$*.*

Proof. Let ${}_f x_g$ be an object $\mathbf{Bispan}(\mathcal{C})(a, b)$, then the identity morphism is ${}_{\mathrm{id}}x_{\mathrm{id}}$. Since identity is an isomorphism, by the ambijunction axiom in Definition 4.1.15, the ambijunction data is special. The proof that F preserves identity 2-cell is shown in Figure 4.18.

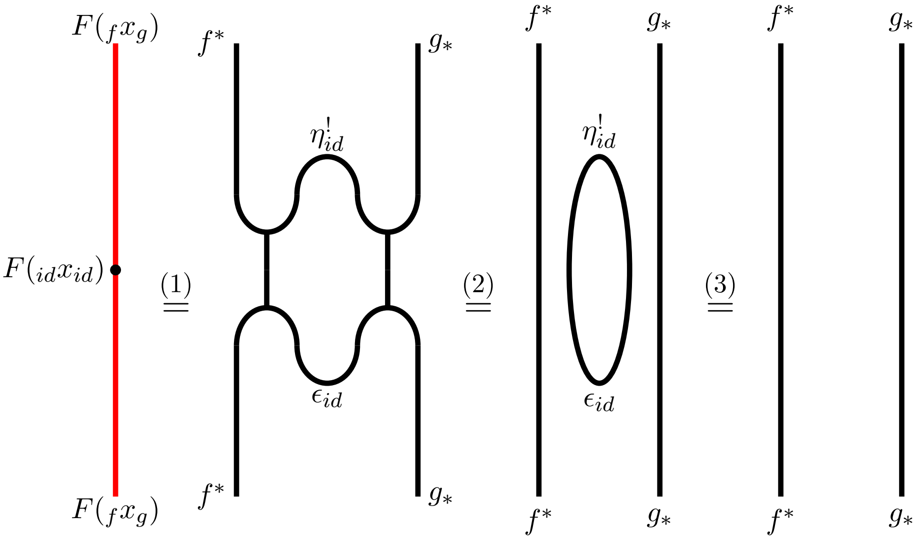


FIGURE 4.18: Proof of F preserves identity 2-morphisms.

The following list explains the reasons for every step of the equality in Figure 4.18.

1. From Construction 4.3.1, the 2-cell assignment $F({}_{\mathrm{id}}x_{\mathrm{id}})$ is given by the first equality.
2. This equality follows from the invertibility of the coherence $id^* f^* \simeq_\theta f^*$ (See Figure 4.7.
3. The third equality from the speciality of the ambijunction specified in Figure 4.11.

□

Lemma 4.3.4. *The assignment*

$$F : \boldsymbol{Bispan}(\mathcal{C})(a,b) \to \mathcal{D}(Fa, Fb)$$

given in Construction 4.3.1 preserves composition in the category $\boldsymbol{Bispan}(\mathcal{C})(a,b)$.

Proof. Let ${}_f x_g, {}_h y_k, {}_l z_m$ be three objects in **Bispan**$(\mathcal{C})(a,b)$, and let

$${}_p X_q : x \to y,\ {}_r Y_s : y \to z$$

be a pair of composable arrows. The composition in the bicategory of bispans is given by

$$\begin{tikzcd}
x & & x \\
X \arrow[u, "p"] \arrow[d, "q"'] & & \\
y & = & X \times_y Y \arrow[uu, "p\pi_1"'] \arrow[dd, "s\pi_2"] \\
Y \arrow[u, "r"] \arrow[d, "s"'] & & \\
z & & z
\end{tikzcd}$$

We need to show that

$$F({}_{p\pi_1} X \times_y Y_{s\pi_2}) = F({}_r Y_s) \circ F({}_p X_q).$$

We start with the left hand side of the above equality

In Figure 4.19, the first equality uses the definition of F. The second equality uses the functoriality of the adjunction data.

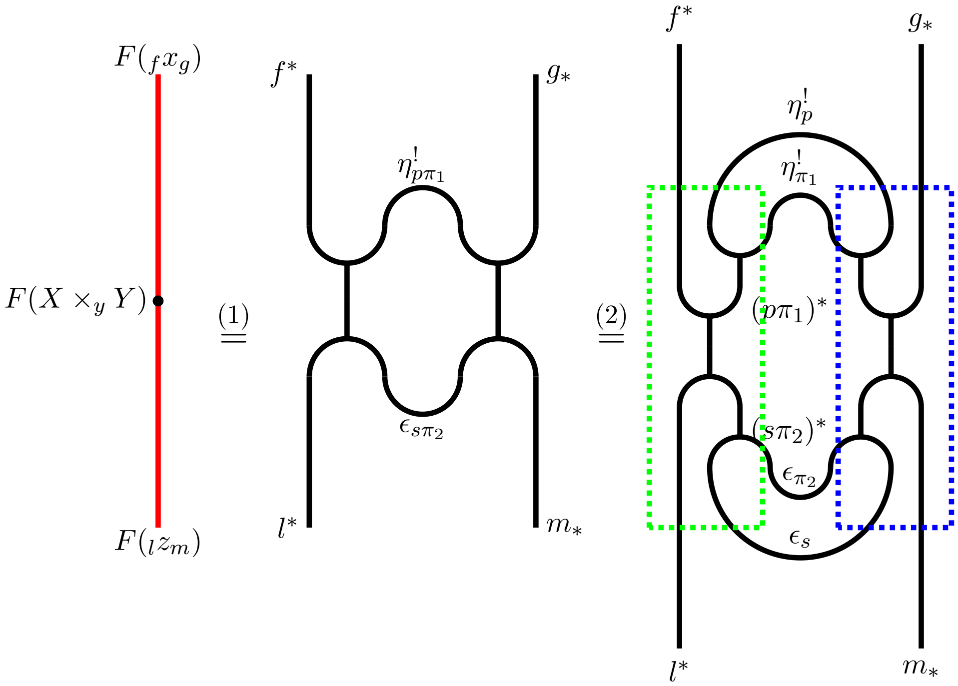


FIGURE 4.19: Vertical composition of 2-cells

Now we focus on the green dotted rectangle in the figure. Using the 2-functoriality of F^*, we can write the composition of 2-morphisms in the rectangle as:

$$\begin{aligned}
(p\pi_1)^* f^* &\overset{\theta_1}{\simeq} \pi_1^*(fp)^* \\
&= \pi_1^*(hq)^* \\
&\overset{\overline{\theta}_2}{\simeq} \pi_1^* q^* h^* \\
&\overset{\theta_3}{\simeq} \pi_2^* r^* h^* \\
&\overset{\theta_4}{\simeq} \pi_2^*(hr)^* \\
&= (\pi_2)^*(ls)^* \\
&\overset{\overline{\theta}_4}{\simeq} (s\pi_2)^* l^*
\end{aligned}$$

We can use the 2-functoriality of F_* to replace the diagram in the dotted red box in Figure 4.19.

So the string diagram in the last step can be written as another string diagram shown in Figure 4.20. Note that we have omitted the composition coherences in the diagram and the equality follows from the vertical Beck-Chevalley axiom shown in Figure 4.12.

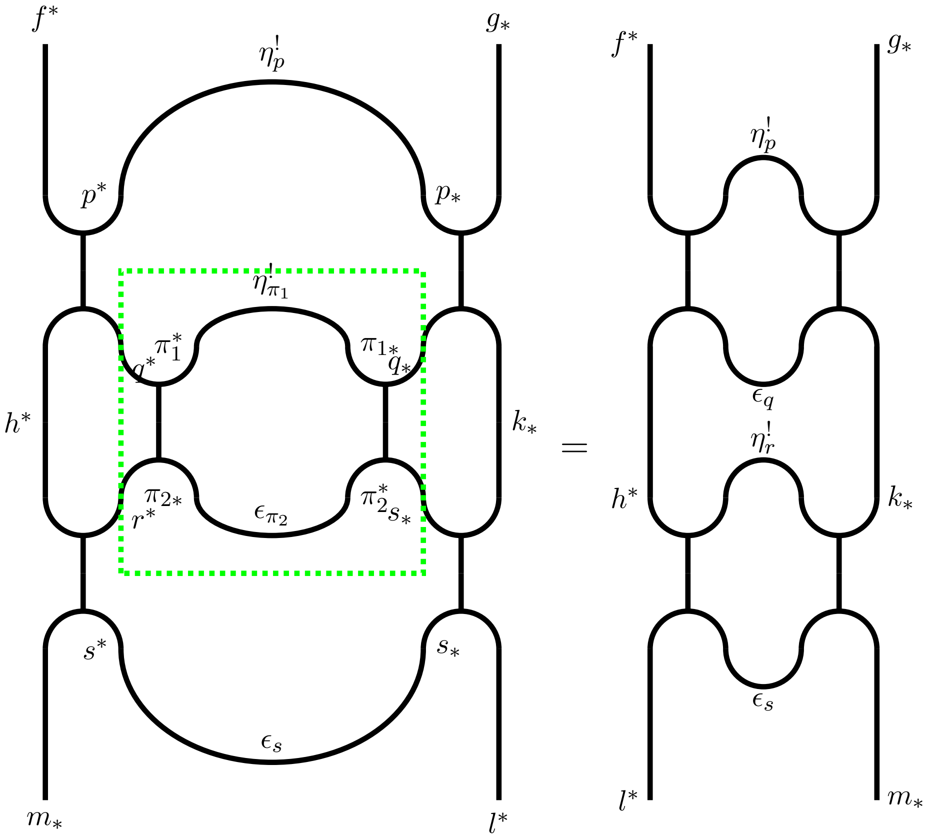


FIGURE 4.20: Final step of the proof of Proposition 4.3.4

□

Lemma 4.3.5. *The assignment*

$$F : \boldsymbol{Bispan}(\mathcal{C})(a, b) \to \mathcal{D}(Fa, Fb)$$

given in Construction 4.3.1 is a functor.

Proof. The proof follows from Propositions 4.3.2, 4.3.3, 4.3.4.

□

Next we check the naturality of the composition coherence proposed in Figure 4.16.

We need two observations that will aid the proof of Lemma 4.3.7. They are recorded as propositions below.

Proposition 4.3.6. *Let a, b, c, d, e be 1-morphisms in a category $\mathcal{C}$, and suppose $cb = ed$, then for a 2-functor $F : \mathcal{C} \to \mathcal{D}$, the move shown in Figure 4.21 is allowed.*

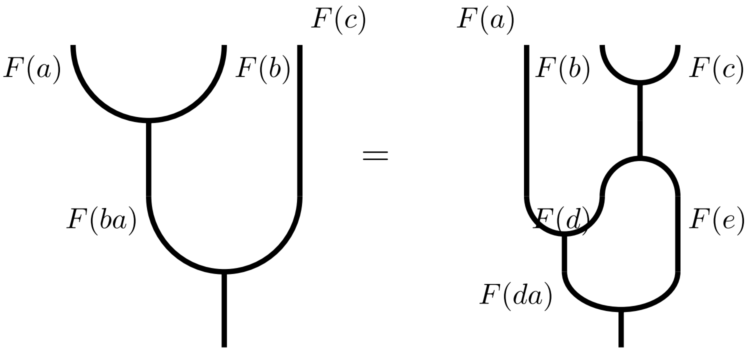


FIGURE 4.21: Proposition 4.3.6

Proof. The string diagrammatic steps of the proof are shown in Figure 4.22.

The steps of the proof are explained below:

1. This follows from 2-functorial composition axiom given in Figure 4.16 and the hypothesis $bc = de$. We use the invertibility of the composition coherence shown in Figure 4.7.
2. This follows from the composition axiom applied to the dotted red rectangle.

□

Lemma 4.3.7. *The composition coherence given in Construction 4.3.1 is a natural transformation.*

Proof. Given a pair of composable 1-morphisms and a pair of horizontally composable 2-morphisms between them as shown below:

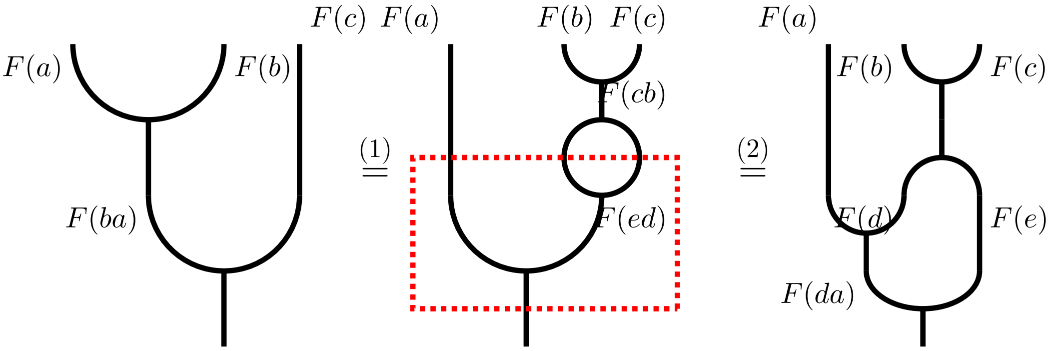


FIGURE 4.22: Proof of Proposition 4.3.6

$$\begin{tikzcd}
 & x_1 \arrow[dl, "\alpha"'] \arrow[dr, "f_1"] & & x_2 \arrow[dl, "g_1"'] \arrow[dr, "\gamma"] & \\
a & X \arrow[u, "l"] \arrow[d, "m"'] & b & Y \arrow[u, "n"'] \arrow[d, "p"] & c \\
 & y_1 \arrow[ul, "\beta"] \arrow[ur, "f_2"'] & & y_2 \arrow[ul, "g_2"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

compositions involve choices of pullbacks. The composition is given by

$$\begin{tikzcd}
 & x_1 \times_b x_2 \arrow[dl, "\alpha\pi_1^x"'] \arrow[dr, "\gamma\pi_2^x"] & \\
a & X \times_b Y \arrow[u, "S"'] \arrow[d, "T"] & c \\
 & y_1 \times_b y_2 \arrow[ul, "\beta\pi_1^y"] \arrow[ur, "\delta\pi_2^y"'] &
\end{tikzcd}$$

Note that we have the following equations: $S = (l\pi_1, n\pi_2)$, $T = (m\pi_1, n\pi_2)$ where π_1, π_2 are the projections of $X \times_b Y$ to X, Y respectively.

We have to check the following naturality square:

$$\begin{tikzcd}
F(x_2 \circ x_1) \arrow[r, "\theta_x"] \arrow[d, "F(Y\times_b X)"'] & F(x_2) \circ F(x_1) \arrow[d, "F(Y)\circ F(X)"] \\
F(y_2 \circ y_1) \arrow[r, "\theta_y"'] & F(y_2) \circ F(y_1)
\end{tikzcd}$$

where θ represent the compositional coherences.

The naturality coherence involves checking the equality of the string diagram shown in Figure 4.23.

We substitute the definitions of F in the left hand side of the equality shown in Figure 4.23 to obtain the string diagram in Figure 4.24.

The adjunction zig-zag is used in the red dotted box and their companion counits below. In the blue dotted box, the invertible Beck-Chevalley isomorphism and inverse can be substituted to obtain the Figure 4.25

The Horizontal Beck-Chevalley condition described in Figure 4.13, is replaced in the dotted red box and the adjunctions are straightened to obtain the string diagram shown in Figure 4.26.

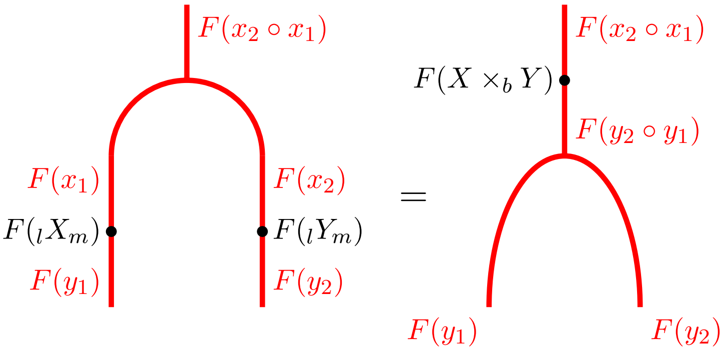


FIGURE 4.23: Naturality of the composition coherence

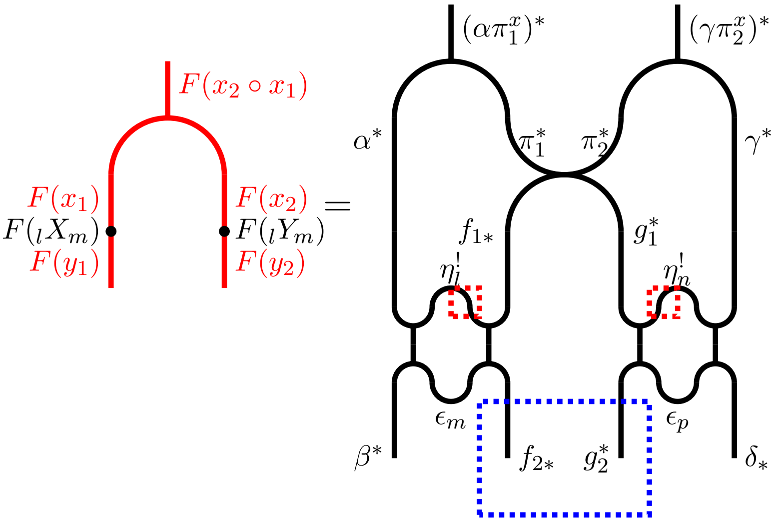


FIGURE 4.24: Naturality check: Step 1.

Now using Proposition 4.3.6 for the dotted red box and three other symmetric strings along with invertiblity of composition coherences (see Figure 4.7), we get the string diagram in Figure 4.27.

This step finishes the proof of naturality. □

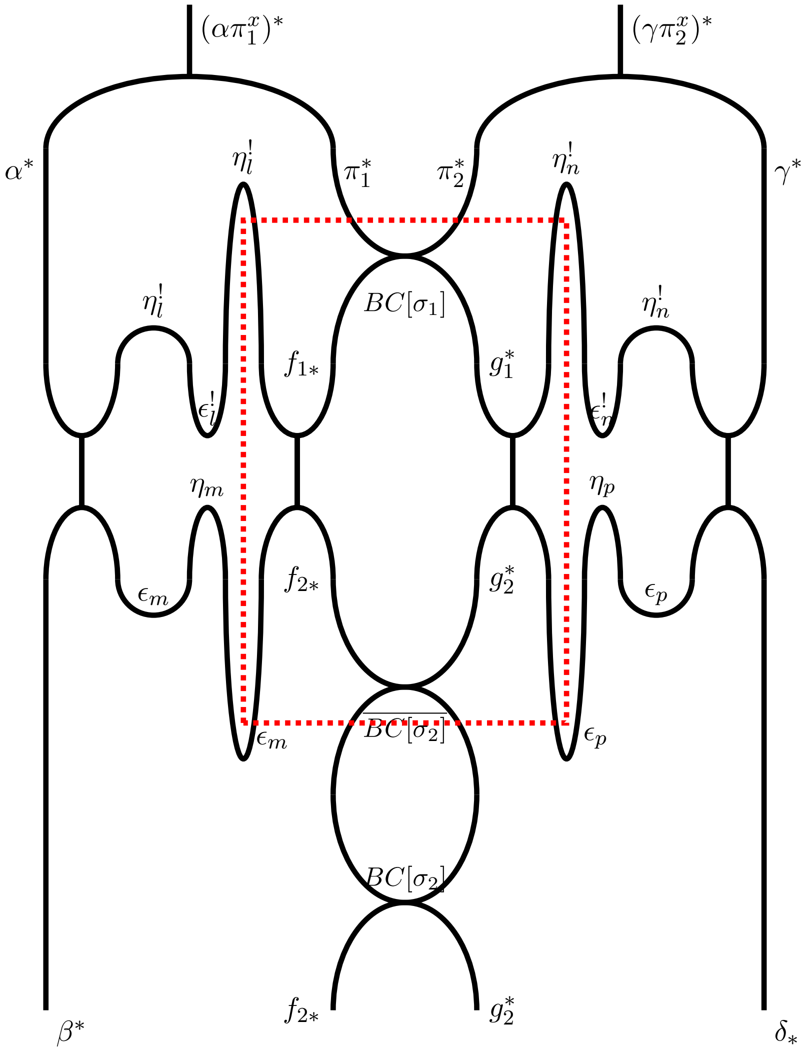


FIGURE 4.25: Naturality check: Step 2.

Finally we have to check the composition and identity coherence axioms. Given a 3-composable arrow

$$\begin{tikzcd}
 & & & u \arrow[dl, "l_1"'] \arrow[dr, "r_1"] \arrow[d, phantom, "\vee" near start] & & & \\
 & & v \arrow[dl, "l_2"'] \arrow[dr, "r_2"] \arrow[d, phantom, "\vee" near start] & & w \arrow[dl, "l_3"'] \arrow[dr, "r_3"] \arrow[d, phantom, "\vee" near start] & & \\
 & x_1 \arrow[dl, "f"'] \arrow[dr, "g"] & & x_2 \arrow[dl, "h"'] \arrow[dr, "k"] & & x_3 \arrow[dl, "m"'] \arrow[dr, "n"] & \\
a & & b & & c & & d
\end{tikzcd}$$

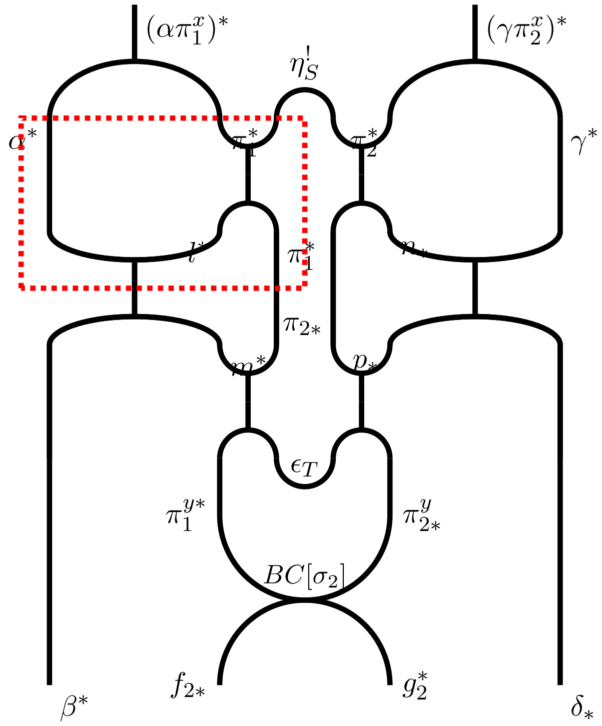


FIGURE 4.26: Naturality check: Step 3.

the composition axiom is checked in Lemma 4.3.8. In terms of string diagrams, we have to check the left diagram in Figure 4.28.

Lemma 4.3.8. *The proposed assigment F given in Construction 4.3.1 satisfies the composition axiom.*

Proof. We will assume the three composable arrow is the one given above. Using the functorial coherence specified in Figure 4.16, we can write

In Figure 4.29, the dotted rectangles can be replaced using the composition axiom to obtain the string diagram in Figure 4.30.

Since $l_3r_1 = r_2l_1$, we can replace the $(l_3r_1)^*$ line segment by $(r_2l_1)^*$. And then using composition axiom a couple o times, we arrive at the string diagram in Figure 4.31.

Now we zig-zag r_2^* and isotope the string diagram to obtain the Figure 4.32.

In Figure 4.32, the 1-morphisms in the dotted red box can be composed. After using composition axiom for $(kr_2l_1)^*$ and $(kl_3r_1)^*$, then the invertibility of the composition coherence of $(kl_3)^*$, composition axiom $m^*r_3^*r_1^*$ and functoriality of the adjunction data for r_3r_1 , we get the required final string diagram. □

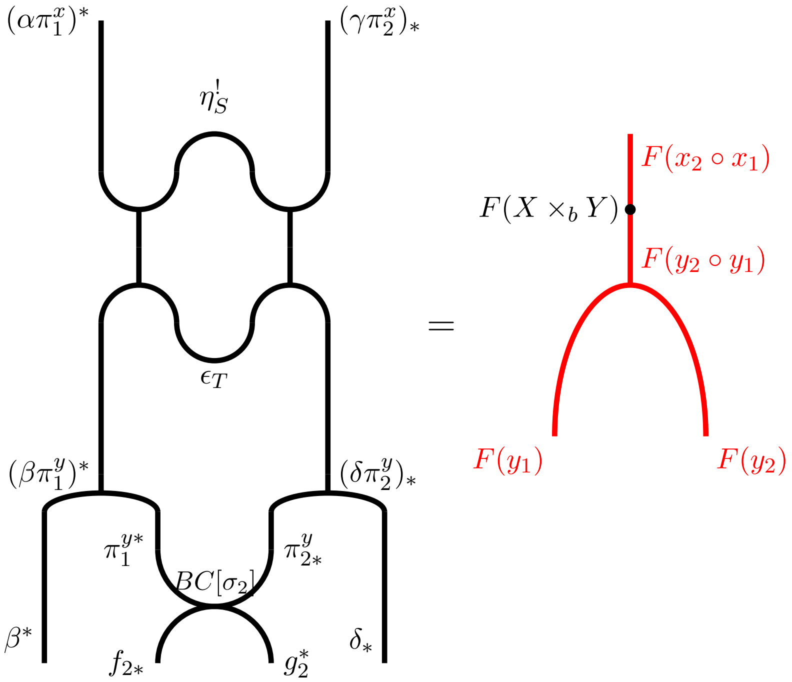


FIGURE 4.27: Naturality check: Step 4.

Next we check that the identity axiom (the equality on the right in Figure 4.28).

Lemma 4.3.9. *The proposed assigment F given in Construction 4.3.1 satisfies the identity axiom.*

Proof. Given an 1-morphism ${}_f x_g$ composed with identity on the left as shown below,

$$\begin{tikzcd}
 & & x \arrow[dl, "f"'] \arrow[dr, "id_a"] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & a \arrow[dl, "id_a"'] \arrow[dr, "id_a"] & {} & x \arrow[dl, "f"] \arrow[dr, "g"] & \\
a & & a & & b
\end{tikzcd}$$

the string diagram for the identity coherence axiom is shown in Figure 4.33,

The ambidexterity axiom stated in Definition 4.1.15 specifies the ambidexterity data associated with an identity morphism. Using this data, we simplify the computation in Figure 4.34.

The final equality in Figure 4.34 uses the identity coherence axioms for $F_*(g)$ in the dotted blue box and $F^*(f)$ in the dotted green box. The string in the dotted red box

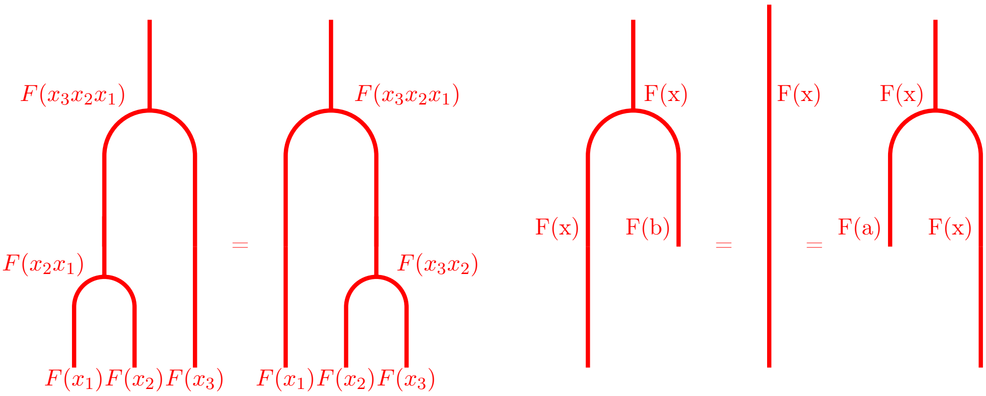


FIGURE 4.28: Functorial coherence axioms checked in Propositions 4.3.9 and 4.3.8

vanishes because of the invertibility of the identity coherence. One final application of the identity coherence axiom to $F^*(f)$ finishes this step.

For the other identity coherence axiom, the composition of 1-morphism is shown below:

$$\begin{tikzcd}
 & & x \arrow[dl, "id_x"'] \arrow[dr, "g"] & & \\
 & x \arrow[dl, "f"'] \arrow[dr, "g"] & & b \arrow[dl, "id_b"] \arrow[dr, "id_b"] & \\
a & & b & & b
\end{tikzcd}$$

The string diagram proof of the identity coherence axiom is proved in Figure 4.35.

In the first step, we use the breaking of identity morphism property (proved in Figure 4.9) on $(id_x)^*$ in the blue dotted box. Then we use the identity coherence axiom on $F^*(g)$ for the dotted green box and on $F_*(g)$ for the dotted red box. We also apply identity coherence axiom for F^*f on the left most connected string. In the last equality we use the zig-zag identity to straighten g_*. □

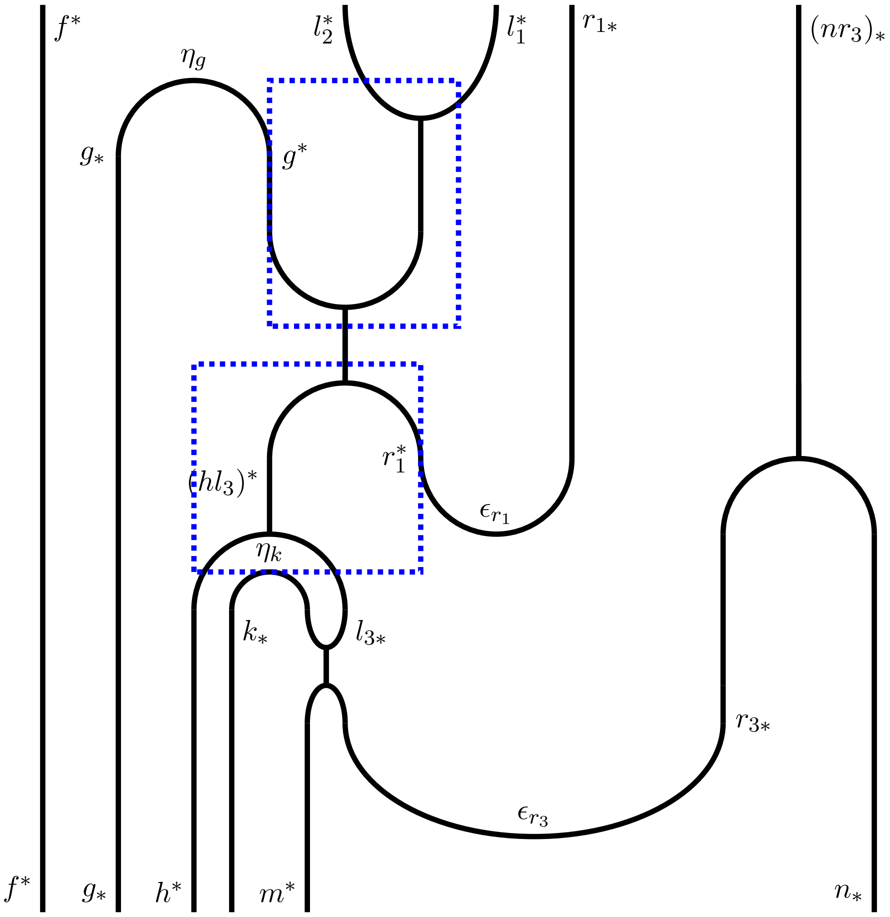


FIGURE 4.29: Step 1 of the composition coherence 2-morphism on RHS of the first equality in Figure 4.28.

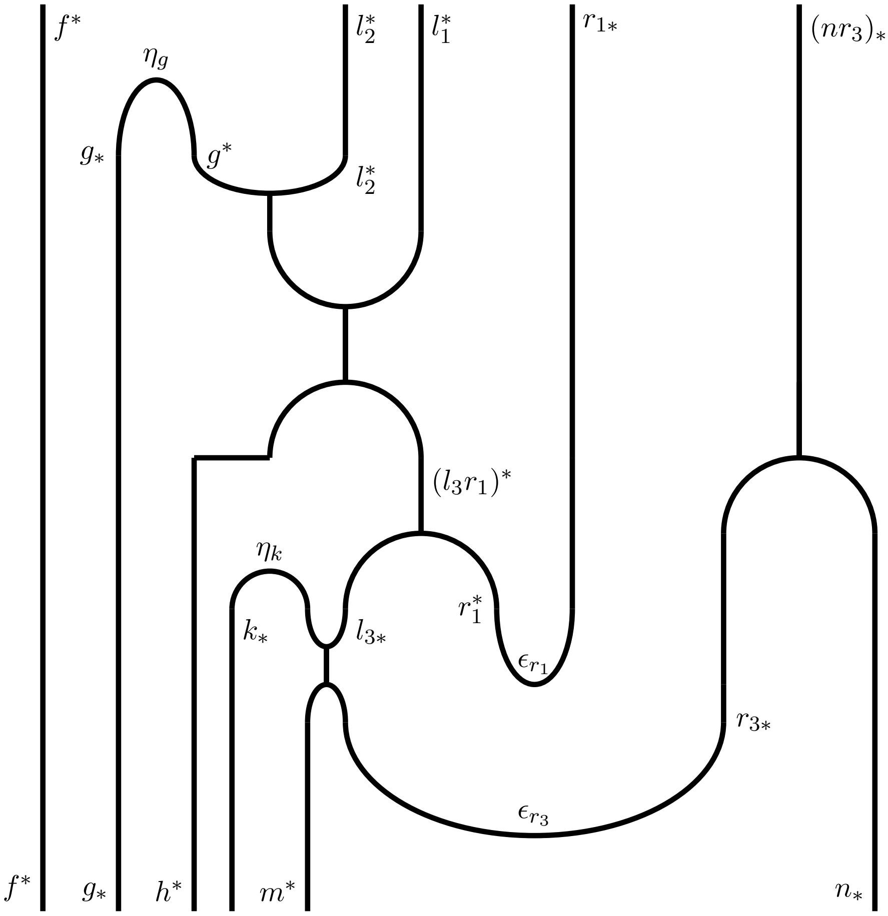


FIGURE 4.30: Step 2 of the composition coherence 2-morphism on RHS of the first equality in Figure 4.28.

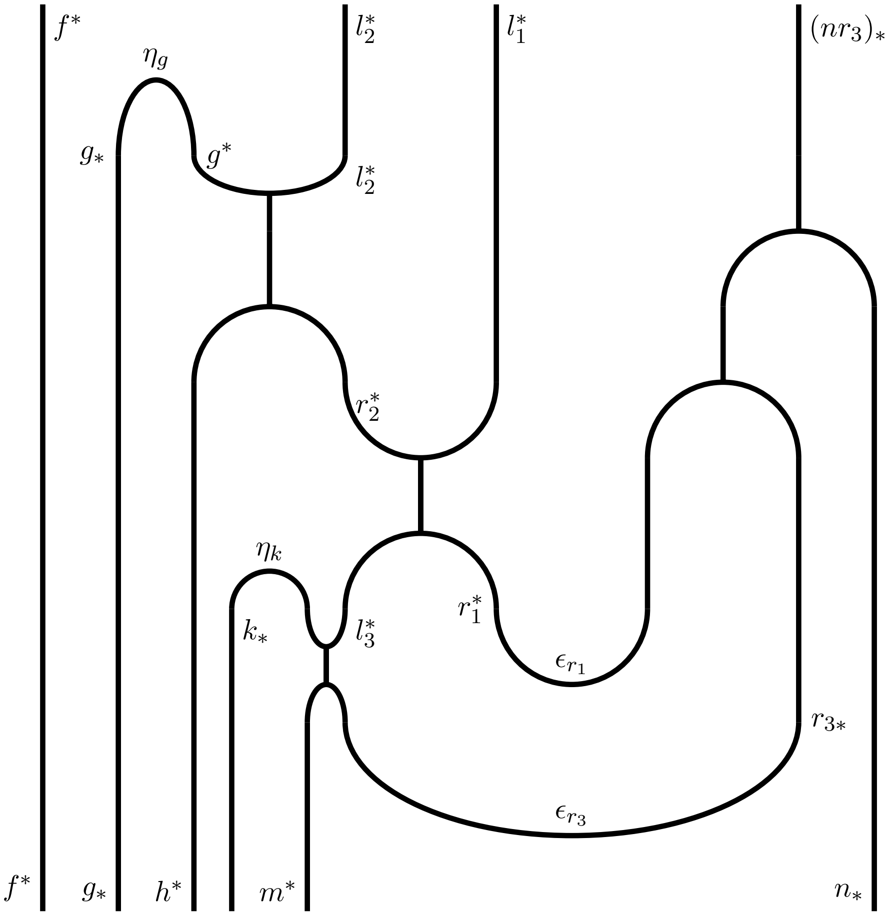


FIGURE 4.31: Step 3 of the composition coherence 2-morphism on RHS of the first equality in Figure 4.28.

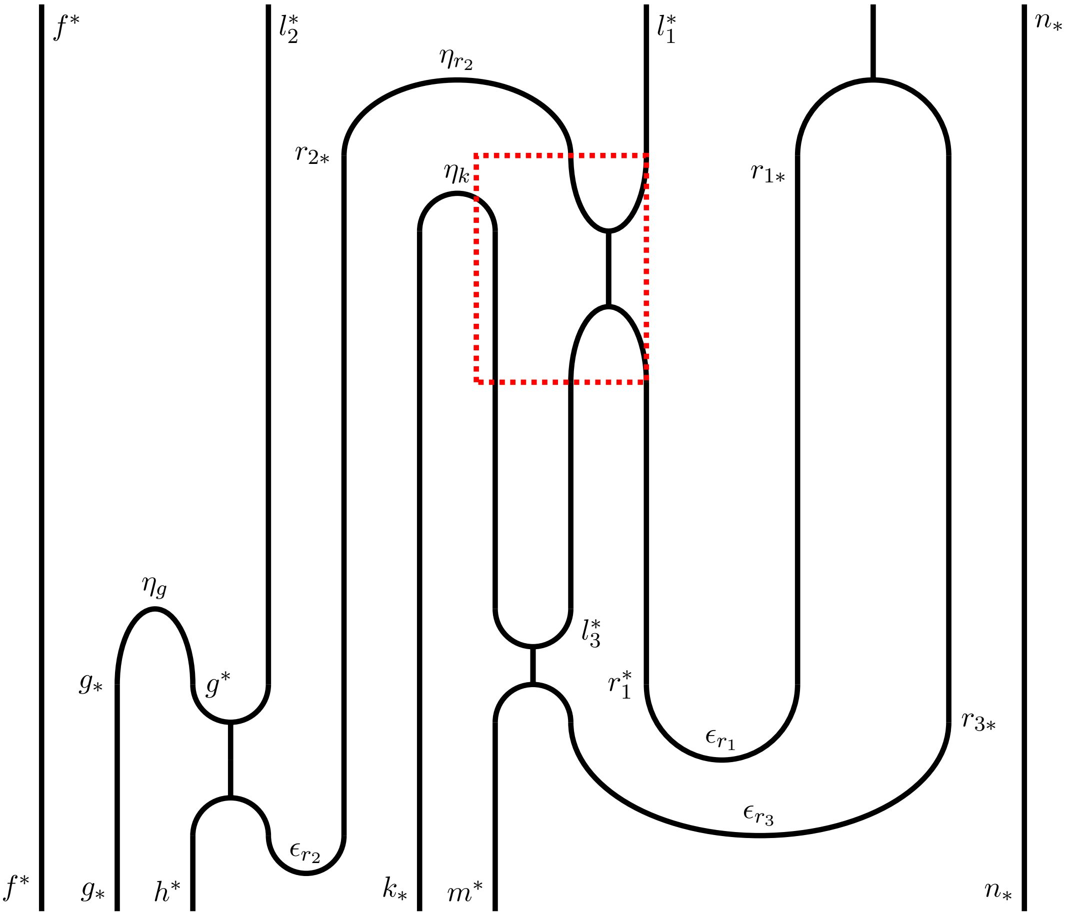


FIGURE 4.32: Step 4 of the composition coherence 2-morphism on RHS of the first equality in Figure 4.28.

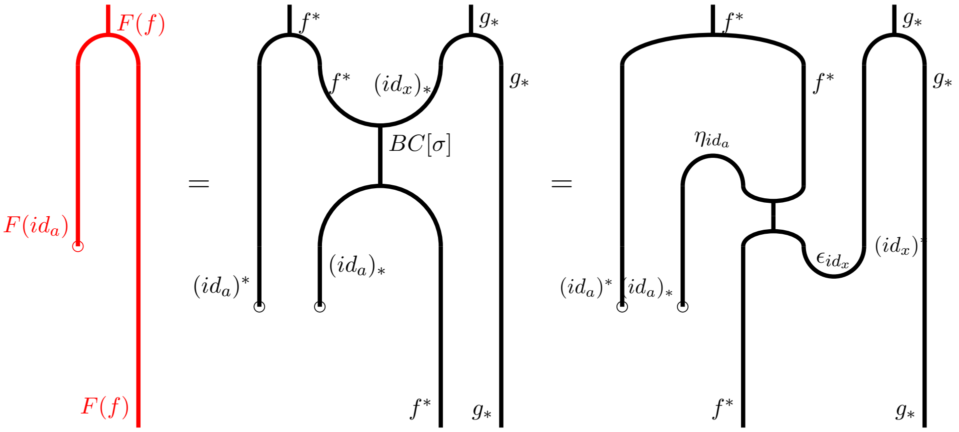


FIGURE 4.33: Step 1 of identity axiom check

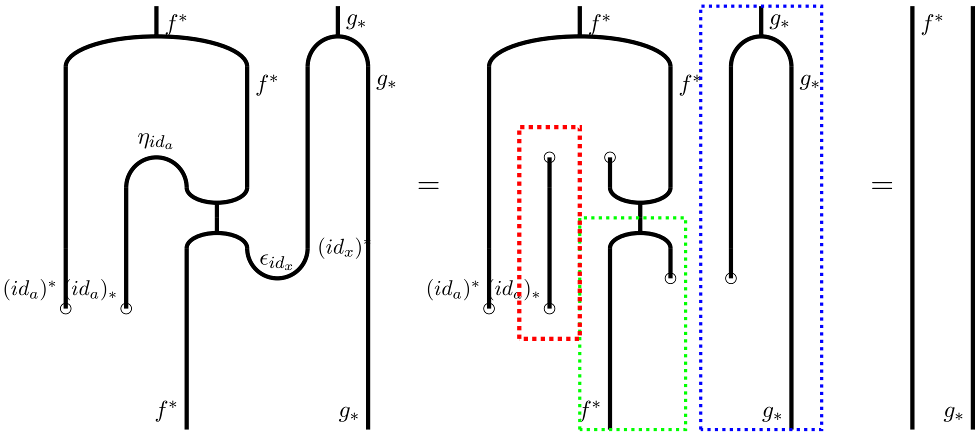


FIGURE 4.34: Step 2 of the identity axiom check

Theorem 4.3.10 (Main theorem)**.** *Let $\mathcal{C}$ be a category with pullbacks and $\mathcal{D}$ be a bicategory. Given any pair of Double Beck-Chevalley 2-functors (see Definition 4.1.16) $F_*, F^* : \mathcal{C} \to \mathcal{D}$ with Double Beck-Chevalley data (see Definition 4.1.15)*

$$(F_*, F^*, \eta, \epsilon, \eta^!, \epsilon^!),$$

there exists a 2-functor

$$F : \textbf{\textit{Bispan}}(\mathcal{C}) \to \mathcal{D}$$

such that $F\mathcal{I}_* = \mathcal{F}_*$, $F\mathcal{I}^* = \mathcal{F}^*$.

Proof. The tentative functor is constructed in Construction 4.3.1. Note that by construction

$$F\mathcal{I}_* = \mathcal{F}_*, F\mathcal{I}^* = \mathcal{F}^*$$

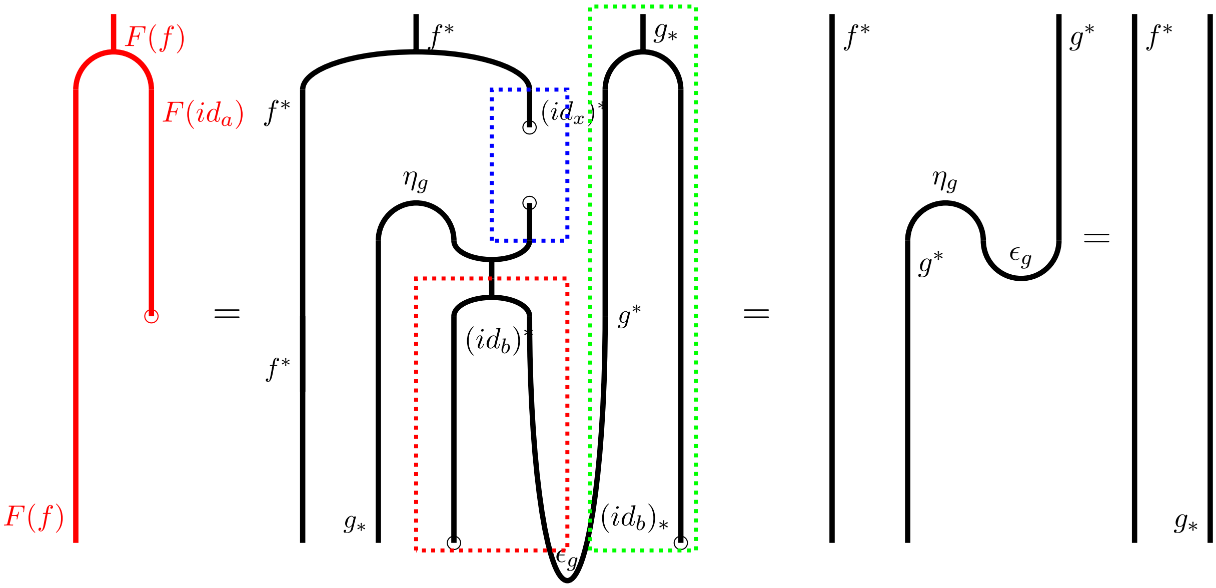


FIGURE 4.35: Identity Coherence Axiom.

holds. Given objects a, b in the Bispan category, Lemma 4.3.5 shows that F is a functor on the category of morphisms. This checks that F preserves composition of vertical 2-morphisms and identity morphism of all 1-morphisms. The naturality of the compositional coherence of F is checked in Lemma 4.3.7. This checks that F preserves composition of horizontal 2-morphisms upto coherence isomorphisms. Finally Lemmas 4.3.8, 4.3.9 check the composition and identity axioms for coherences of F. □

Remark 4.3.11. Note that the uniqueness of the 2-functor follows from the uniqueness of the 2-functor in Hermida's Theorem 4.1.3. Indeed, the Double Beck-Chevalley conditions contain the conditions of Hermida's theorem. Thus we have unique pair of functors $\mathcal{F}_*, \mathcal{F}^*$ out of the bicategory of Spans in $\mathcal{C}$ to $\mathcal{D}$ satisfying the conditions in the above Theorem. If $F\mathcal{I}_* \simeq \mathcal{F}_*$, $F\mathcal{I}^* \simeq \mathcal{F}^*$, then the 2-functor is uniquely determined since on the level of 2-morphisms, every span can be written as a composition of push and pull of a 2-morphism in Bispan. This is similar to the discussion in Construction 4.1.5.

Chapter 5

Applications to gauge theories

In the last chapter, we characterized bispans in a category $\mathcal{C}$ (with pullbacks) using double Beck-Chevalley functors (see Theorem 4.3.10). The double Beck-Chevalley functor data (see Definition 4.1.15) was inspired by Morton's work [49].

For the choice of $\mathcal{C}$ as the category whose objects are skeletal groupoids with morphisms as functors, we obtain an example of double Beck-Chevalley functor data that arises in Morton's work (discussed in Subsection 5.1.5). Recall that the definition of double Beck-Chevalley functor data has 4 axioms: Ambidexterity, Beck-Chevalley, Vertical Beck-Chevalley, Horizontal Beck-Chevalley (see Definition 4.1.15). One of the requirements in the ambidexterity axiom is that the ambijunction associated to an isomorphism is special. This condition is verified in the proof of Theorem 5.1.35 and the rest of the axiom verification is directly from Morton. Thus using the universal property of the bicategory of bispans, Theorem 5.1.35 gives a 2-functor $\tilde{Q}$ which matches Morton's 2-functor (described in Construction 5.1.36). Morton's example is an example of an extended prequantization 2-functor (see Definition 1.1.29).

Whitehead introduced the notion of a *crossed module* (Definition 5.1.14) as an algebraic way to model 2-types. In [47] Moerdijk and Svensson construct a 2-groupoid WX, called *a Whitehead 2-groupoid*, from a given CW complex X (see Construction 5.1.9). They show that the functor W has a right adjoint N (a nerve functor), and together, they form a Quillen equivalence (see Theorem 5.1.11). Thus, the homotopy theory of 2-types is shown to be equivalent to the homotopy theory of 2-groupoids, establishing the Grothendieck's homotopy hypothesis for the 2 dimensional case (see Subsection 1.1.4). This result induces an equivalence between pointed 2-groupoids and pointed connected 2-types. Further, the homotopy theory of pointed 2-groupoids is proven to be equivalent to the homotopy theory of crossed modules [68]. Thus, we have three homotopy theories pairwise equivalent to each other, furnishing three descriptions of 2-groups as shown in Figure 5.1.

$$\begin{tikzcd}
 & 2-Gpds_* \arrow[ddl, shift left, "|N(\_)|"] \arrow[ddr, shift left, "C"'] & \\
 & 2Groups & \\
2types_* \arrow[uur, shift left, "W"] \arrow[rr, shift left, "\mathcal{W}"] & & CrsMod \arrow[uul, shift left, "*//(\_)"'] \arrow[ll, shift left, "|(\_)|"]
\end{tikzcd}$$

FIGURE 5.1: Three equivalent descriptions of 2-groups.

The notion of a derived internal hom object is well known (see Definition 5.1.12). Given a pair of CW complexes A, B, one can construct a derived internal hom object, a mapping space, $Maps(A, B)$. Given a pair of 2-groupoids $\mathcal{A}$, $\mathcal{B}$, Noohi defines a derived mapping 2-groupoid and shows that the nerve of this 2-groupoid is homotopy equivalent to $Maps(N\mathcal{A}, N\mathcal{B})$ (see Theorem 5.1.13). He also constructs a derived mapping 2-groupoid between a pair of crossed modules (Definition 5.1.26). We compute $\tilde{Q}(Maps(S^n, X))$ for $n = 1, 2$ in Subsection 5.2.

Organization

In Section 5.1, we discuss the homotopy theory of 2-groupoids, crossed modules and 2-types (in Subsections 5.1.2, 5.1.3). In particular, we discuss a crossed module description of the mapping space $Maps$. We also recall Morton's 2-functor $\tilde{Q}$ (in Construction 5.1.36). In Section 5.2 we compute the value of

$$\tilde{Q}(\pi_{\leq 1} Maps(S^n, X))$$

for $n = 1, 2$ using the crossed module description.

5.1 Preliminaries

In this section, there are no new results except Theorem 5.1.35. In Subsection 5.1.1, we quickly recall the basics of model categories. In the next three subsections, we explore three equivalent descriptions of 2-groups as shown in Figure 5.1 and we follow [51]. Moerdijk and Svensson's pair of functors $W : sSets \leftrightarrows 2 - Gpd : N$, shown in Figure 5.1, is constructed in Subsection 5.1.2. Theorem 5.1.11 claims that the pair N, W is a Quillen equivalence when considering 2-types instead of all simplicial sets. Noohi's derived mapping 2-groupoid (Definition 5.1.12) and the result that the nerve of this mapping 2-groupoid has the right homotopy type, can also be found in this subsection (Theorem 5.1.13). In Subsection 5.1.3, we introduce the model category of crossed modules. The pair of functors $*//(\_) : CrsMod \leftrightarrows 2 - Gpd_* : C$ shown in Figure 5.1 is constructed in this subsection and Theorem 5.1.19 claims that these functors induce an equivalence of homotopy theories. Noohi's derived mapping 2-groupoid of crossed modules (Definition 5.1.26) and the Whitehead crossed module (Definition 5.1.20) is discussed in Subsection 5.1.4.

In Subsection 5.1.5, we discuss Kapranov-Voevodsky 2-vector spaces and show that Morton's construction [50] gives us an example of a double Beck-Chevalley functor data in Theorem 5.1.35. This is the only new result in this section.

5.1.1 Model categories

We quickly recall the basic definitions in model category theory from [24, Chapter II].

Definition 5.1.1. A model category $\mathcal{C}$ is a category equipped with three classes of morphisms called fibrations, cofibrations and weak equivalences, respectively. These morphism classes and the category $\mathcal{C}$ are supposed to satisfy the following axioms.

1. (CM 1) The category $\mathcal{C}$ has all limits and colimits.
2. (CM 2) Let f, g, h be 1-morphisms such that $f = g \circ h$. If any two of these is a weak equivalence, so is the third.
3. (CM 3) If a morphism f is a retract of a morphism g, and g is weak equivalence, fibration or cofibration, then so is f.
4. (CM 4) Suppose that we are given a commutative solid arrow where i is a cofibration and p is a fibration:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

   Then the dotted arrow exists making the triangles commute if i or p is a weak equivalence.

5. (CM 5) Any morphism $f : A \to B$ can be factored as $f = p \circ i$ where p is a fibration, i is a cofibration and further the factorization can be chosen so that any one of the pair p, i is a weak equivalence.

If there is a weak equivalence $f : A \to B$, then we say A and B are weakly equivalent.

We will not compare distinct model structures on the same category in this chapter so we suppress the three classes of morphisms and say $\mathcal{C}$ is a model category.

By (CM1) in Definition 5.1.1, a model category $\mathcal{C}$ has an initial object ϕ and a terminal object $*$.

Definition 5.1.2. An object X of $\mathcal{C}$ is *fibrant* if the canonical map $X \to *$ is a fibration. Similarly, an object X of $\mathcal{C}$ is *cofibrant* if the canonical map $\phi \to X$ is a cofibration.

Construction 5.1.3. Let X be an object of a model category $\mathcal{C}$, then by (CM5) the canonical map $X \to *$ can be factored as $X \xrightarrow{i} RX \xrightarrow{p} *$ with i chosen as a weak equivalence. Thus, we have a fibrant object RX that is weakly equivalent to X. Such an object RX is called a *fibrant replacement* for X. Similarly, given any object X in a model category $\mathcal{C}$, there is a *cofibrant replacement* QX for X.

It turns out that we can define a notion of *homotopy* between morphisms in a model category (see [24, Chapter II.1]) and this leads to a definition of a homotopy category of a model category.

Construction 5.1.4. Let X, Y be two objects in a model category $\mathcal{C}$, define

$$Hom_{\mathbf{Ho}\mathcal{C}}(X,Y) = Hom_{\mathcal{C}}(QX, RY)/\sim$$

where $\sim$ is the homotopy relation in the model category $\mathcal{C}$. We get the *homotopy category* $\mathbf{Ho}(\mathcal{C})$ whose objects are objects of $\mathcal{C}$ and the set of morphisms is given by $Hom_{\mathbf{Ho}\mathcal{C}}$. The composition is induced from the composition of morphisms in $\mathcal{C}$.

Clearly, we have a functor $j : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ that is the identity on objects and sends a morphism to its equivalence class. It can be shown that j maps weak equivalences to isomorphisms (see Whitehead's theorem at [24, §II.1.10]). The category $\mathbf{Ho}(\mathcal{C})$ is initial among all functors out of $\mathcal{C}$ that map weak equivalences to isomorphisms [24, §II.1.11].

Given a functor $F : \mathcal{C} \to \mathcal{A}$ out of a model category that sends weak equivalences between cofibrant objects to isomorphisms, one can construct a left Kan extension LF (dotted arrow) (see [24, Lemma II.7.3])

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "j"'] & \mathcal{A} \\
\mathbf{Ho}(\mathcal{C}) \arrow[ur, dashed, "LF"'] &
\end{tikzcd}$$

Similarly, we have a right Kan extension RF if F maps weak equivalences between fibrant objects to isomorphisms.

Theorem 5.1.5. *[24, Theorem 7.7] Let $\mathcal{C}, \mathcal{D}$ be model categories. If an adjunction $F \dashv G$ with left adjoint $F : \mathcal{C} \to \mathcal{D}$ is such that F preserves weak equivalences between cofibrant objects and G preserves weak equivalences between fibrant objects, then $LF : \mathbf{Ho}(\mathcal{C}) \to \mathbf{Ho}(\mathcal{D})$ and $RG : \mathbf{Ho}(\mathcal{D}) \to \mathbf{Ho}(\mathcal{C})$ exists with $LF \dashv RG$.*

Further, suppose that for any cofibrant $X \in \mathcal{C}$ and any fibrant $A \in \mathcal{D}$, $X \to GA$ is a weak equivalence iff its adjoint $FX \to A$ is a weak equivalence. Then the LF and RG are adjoint equivalences and thus $\mathbf{Ho}(\mathcal{C}) \simeq \mathbf{Ho}(\mathcal{D})$.

A morphism that is both a fibration and a weak equivalence is called a *trivial fibration*. A morphism that is both a cofibration and a weak equivalence is called a *trivial cofibration*.

Definition 5.1.6. Let $\mathcal{C}$ and $\mathcal{D}$ denote two model categories. An adjunction $F \dashv G$ with $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ is called a *Quillen adjunction* if L preserves cofibrations and trivial cofibrations (or equivalently R preserves fibrations and trivial fibrations).

It turns out that given a Quillen adjunction $F \dashv G$, we have an adjunction $LF \dashv RG$ on the homotopy categories [24, Lemma II.7.9].

An example is the Quillen equivalence of simplicial sets and compactly generated weak Hausdorff spaces. Let $Sing : Top \to sSets$ denote the *singular complex* functor. It's left adjoint *geometric realization* is denoted by $|(\_)|$. We won't describe the model structures here and assume the reader knows it. A detailed discussion can be found in [24, Chapter I].

In Subsections 5.1.2, 5.1.3, we describe the examples of model categories relevant to this chapter.

5.1.2 Homotopy theory of 2-groupoids and 2-types

We know that a groupoid is a category in which all the morphisms are invertible. Thus, a groupoid has objects and invertible 1-morphisms. A 2-groupoid is a bicategory that has invertible 1-morphisms and invertible 2-morphisms (see Definition 2.1.3).

Definition 5.1.7. Let $Gpds$ denote the category of groupoids and functors between them. $2 - Gpds$ will denote the category of strict 2-groupoids and strict 2-functors. $2 - Gpds_*$ will denote the full subcategory of $2 - Gpds$ with objects as one-object 2-groupoids.

Throughout this chapter, we will need the notion of a *homotopy groupoid.*

Definition 5.1.8. Let $\mathcal{X}$ be a 2-groupoid, the *homotopy groupoid* $Ho(\mathcal{X})$ has the same objects as $\mathcal{X}$. Given two objects x, y, the 1-morphisms of $Ho(\mathcal{X})$ are 1-morphisms of $\mathcal{X}$ modulo the following relation: $f \sim g$ if there is a 2-morphism $\eta : f \to g$. The composition is induced from the composition of 1-morphisms in $\mathcal{X}$.

Alternatively, given a groupoid $\mathcal{C}$, let $i\mathcal{C}$ denote $\mathcal{C}$ with identity 2-morphisms. The left adjoint to the functor $i : Gpds \to 2 - Gpds$ is Ho.

Moerdijk and Svensson have constructed a model structure on the category $2 - Gpds$ of strict 2-groupoids and 2-functors with weak equivalences as 2-functors that are essential surjective and fully faithful (see Definition 2.1.6). Since Noohi has proved that the homotopy theories of 2-groupoids and strict 2-groupoids are equivalent (see [51, Proposition 9.8]), we may not always specify which kind of 2-groupoids are under discussion. In [47, §2], Moerdijk and Svensson consider a nerve functor $N(\_) : 2 - Gpds \to sSets$ described in [47, §2]. They also construct a strict 2-groupoid out of a simplicial set which we describe now.

Construction 5.1.9. [47, Example 1.2, (2.5)] Given a simplicial set X, let X_i denote the ith skeleton of X. Define $\mathcal{W}(X)$ as a 2-groupoid with objects as elements of X_0 and 1-morphisms as homotopy classes of paths in X_1 between points of X_0. Given a pair of 1-morphisms $f, g : x \to y$, the set of 2-morphisms is given by the set of maps $\alpha : I \times I \to X$ modulo an equivalence relation. A map α is a 2-cell which is bounded by vertical paths that are constant paths at x, y and horizontal paths are given by 1-morphisms f, g, as shown below.

$$\begin{tikzcd}
x \arrow[r, dash, "f"] \arrow[d, dash, "c_x" description] \arrow[dr, phantom, "\alpha"] & y \arrow[d, dash, "c_y" description] \\
x \arrow[r, dash, "g"] & y
\end{tikzcd}$$

We say $\alpha \sim \alpha'$ if there is a homotopy $H : I \times I \times I \to X$ between them that is constant on X_0, X_1. Composition is obtained by concatenation of cells along common boundaries which is strictly associative and unital.

Construction 5.1.10. Given a space X, the groupoid obtained by identifying 2-isomorphic 1-morphisms in WX (from Construction 5.1.9) is called the *fundamental path groupoid* of X and is denoted by $\pi_{\leq 1}(X)$. The objects of $\pi_{\leq 1}(X)$ are elements of the 0-skeleton of a CW complex X, and morphisms are homotopy classes of paths in X_1 between points of X_0. The composition is the concatenation of paths.

We quickly note that, for a CW complex X, the homotopy groupoid of WX (Definition 5.1.8) is the fundamental path groupoid of X.

They prove the following theorem. The following summary is from [51, Theorem 5.6].

Theorem 5.1.11. *[47, §2] The pair of functors* $W : sSets \overset{2}{\leftrightarrows} -Gpd : N$ *is a Quillen adjunction with* $W \dashv N$ *where* $\mathcal{W}$ *is from Construction 5.1.9. It satisfies the following properties:*

1. *The adjoints preserve weak equivalences.*
2. *For every* 2*-groupoid* $\mathcal{X}$*, the counit of the adjunction* $WN\mathcal{X} \to \mathcal{X}$ *is a weak equivalence.*
3. *For every simplicial set* X *such that* $\pi_n X = 0$ *for* $n \geq 3$*, the unit of the adjunction* $X \to NWX$ *is a weak equivalence.*

So, in particular, the functor $RN : \mathbf{Ho}(2 - Gpd) \to \mathbf{Ho}(sSets)$ *induces an equivalence of categories onto the full subcategory of* 2*-types.*

Loosely speaking, 2-groupoids are a categorical description of 2-types.

There is a well-known monoidal closed model structure in the category of $sSets$. See [24, Chapter II.2] for details regarding monoidal closed model structure. For our purposes, it suffices to note that there is a monoidal structure along with an internal hom object and these structures are compatible with the model structure. From [51, Definition 5.9]

Definition 5.1.12. Let $\mathcal{C}$ be a monoidal closed model category with an internal hom object denoted by $\underline{\mathcal{H}om}$. Let X, Y be objects of $\mathcal{C}$, then the *derived hom-object* $\underline{\mathcal{RH}om}(X, Y)$ is defined as $\underline{\mathcal{H}om}(QX, RY)$ where QX is the cofibrant replacement of X and RY is the fibrant replacement of Y (see Construction 5.1.3).

In Noohi's paper[51], it is noted that the nerve functor N is not full. Further, the $sSet$ model category has an internal hom object that gives a derived mapping space. In simple words, there is a space of maps between spaces. Noohi shows that we can endow the model category of 2-groupoids with a monoidal structure so that there is an internal hom, and then he proves that the nerve functor preserves the derived mapping space.

In more details: In case of 2-groupoids, for a specific monoidal structure[1], and the associated internal hom of 2-groupoid: $\underline{\mathcal{H}om}(\mathcal{X}, \mathcal{Y})$ is a 2-groupoid with strict 2-functors as objects, pseudo natural transformations as 1-morphisms and modifications as 2-morphisms. See [51, §4] for a discussion on two different monoidal model structures. The above monoidal structure is chosen since it behaves well with the nerve functor, as shown by this theorem:

Theorem 5.1.13. *[51, Proposition 7.9] Let $\underline{Hom}$ denote the derived internal hom object of sSets and $\underline{\mathcal{RH}om}$ denote the derived internal hom object of $2-Gpds$. If $\mathcal{X}, \mathcal{Y}$ are 2-groupoids, then there is a natural homotopy equivalence between simplicial sets*

$$N\underline{\mathcal{RH}om}(\mathcal{X}, \mathcal{Y}) \simeq \underline{Hom}(N\mathcal{X}, N\mathcal{Y})$$

Note that Theorem 5.1.13 and Theorem 5.1.11 shows that if X and Y are 2-types, then geometric realization $|\underline{\mathcal{RH}om}(\mathcal{W}Y, \mathcal{W}X)|$ is homotopy equivalent to derived mapping space $Maps(X, Y)$ where WX is the Whitehead 2-groupoid from Definition 5.1.9.

5.1.3 Crossed Modules

In this subsection, we describe the homotopy theory of crossed modules, define the derived mapping 2-groupoid between two crossed modules.

Definition 5.1.14. [68, §2] A *crossed module* $\mathcal{G}$ is a tuple (G_2, G_1, d, ρ) where G_2, G_1 are groups, $d : G_2 \to G_1$ is a group homomorphism, $\rho : G_1^{op} \to \mathrm{Aut}(G_2)$ is a group homomorphism (or equivalently a right action of G_1 on G_2 via group automorphisms) such that the following conditions are satisfied:

1. (equivariance) $d(h^g) = d(h)^g$ for all $h \in G_2$ and $g \in G_1$.
2. (Peiffer) $h_2^{d(h_1)} = h_2^{h_1}$ for all $h_1, h_2 \in G_2$.

where x^y represents the adjoint (i.e. conjugation) action whenever x, y belong to the same group and when $x \in G_2, y \in G_1$, then $x^y := \rho(y)(x)$. We will denote G_1 by $\mathrm{Dom}(\mathcal{G})$ and G_2 by $\mathrm{Tar}(\mathcal{G})$.

A morphism of crossed modules $f : \mathcal{G} \to \mathcal{H}$ is a pair of group homomorphisms $f_i : G_i \to H_i$ for $i = 1, 2$ (called *components*) such that $df_1 = f_2d$ and

$$f_2(\rho(g_1)(g_2)) = \rho(f_1(g_1))(f_2(g_2)).$$

---

[1] the Gray tensor product for the cognoscenti.

The category whose objects are crossed modules and morphisms are morphisms of crossed modules is denoted by *CrsMod*.

Given a crossed module $d : G_2 \to G_1$, define the kernel of d as $\pi_0(\mathcal{G})$ and the cokernel of d as $\pi_1(\mathcal{G})$.

Definition 5.1.15. Let $\phi : \mathcal{G} \to \mathcal{H}$ be a morphism of crossed modules with components (ϕ_1, ϕ_2).

1. The morphism ϕ is a *weak equivalence* if it induces isomorphisms on π_1 and π_2.
2. A morphism ϕ is a *fibration* if the components ϕ_1, ϕ_2 are both surjective.

There is a model structure on the category *CrsMod* with fibrations and weak equivalences specified in Definition 5.1.15 (see [51, §6]).

Proposition 5.1.16. *[51, Proposition 6.2, 6.3] Two important facts about the model category CrsMod:*

1. *Let $\phi : \mathcal{G} \to \mathcal{H}$ be a morphism of crossed modules with components (ϕ_1, ϕ_2). The morphism ϕ is a trivial fibration iff the commutative square*

$$\begin{tikzcd} G_2 \arrow[r, "\phi_2"] \arrow[d, "d"'] & H_2 \arrow[d, "d"] \\ G_1 \arrow[r, "\phi_1"'] & H_1 \end{tikzcd}$$

   is a pullback square. Note that this means every crossed module is fibrant.

2. *A crossed module $d : G_2 \to G_1$ is cofibrant iff G_1 is a free group.*
3. *Given a crossed module $d : G_2 \to G_1$ and a surjective homomorphism $F_1 \to G_1$ with F_1 a free group, the crossed module $G_2 \times_{G_1} F_1 \to F_1$ is a cofibrant replacement for the crossed module $d : G_2 \to G_1$.*

Now we construct a one object strict 2-groupoid from a crossed module (see [51, §3.3]).

Construction 5.1.17. Given a crossed module $\mathcal{G}$ in CrsMod specified by $d : G_2 \to G_1$ with action denoted by ρ, denote $*//\mathcal{G}$ as the strict 2-groupoid with one object $*$ and the category of morphisms is specified as follows:

1. Objects$(*//\mathcal{G}(*, *)) = G_1$.

$$\begin{tikzcd} \bullet \arrow[r, bend left, "id"{name=U1}] \arrow[r, bend right, "x"'{name=D1}] \arrow[Rightarrow, from=U1, to=D1, "\alpha"] & \bullet \arrow[r, bend left, "id"{name=U2}] \arrow[r, bend right, "y"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\beta"] & \bullet \arrow[r, phantom, "="] & \bullet \arrow[r, bend left, "id"{name=U3}] \arrow[r, bend right, "yx"'{name=D3}] \arrow[Rightarrow, from=U3, to=D3, "\beta\alpha"] & \bullet \end{tikzcd}$$

FIGURE 5.2: Composition for G_2 in Construction 5.1.18

2. Given two elements $f, g \in G_1$, a morphism of $*//\mathcal{G}(*, *)$ is a pair $\rho = (a, f)$ with $a \in G_2$ such that $da = f^{-1}g$.

3. The vertical composition of 2-morphisms $(a, f) : f \to g, (b, g) : g \to h$ is given by (ba, f). The horizontal composition of 1-morphisms is the group multiplication in G_1. The horizontal composition of 2-morphisms $(a, f) : f \to g$ and $(b, h) : h \to k$ is given
$$(b, h) \circ (a, f) := (b\rho(h)(a), hf).$$

Next, we construct a crossed module from a pointed 2-groupoid.

Construction 5.1.18. Let $\mathcal{X}$ denote a pointed strict 2-groupoid.

1. Define CX_1 as the set of 1-morphisms of $\mathcal{X}$ with group multiplication as composition. Composition is associative since $\mathcal{X}$ is a strict 2-groupoid.

2. Define CX_2 as all the 2-morphisms with source as the identity 1-morphism. For the group multiplication, we use horizontal composition as shown in Figure 5.2.

3. Define $d_C : CX_2 \to CX_1$ sends a 2-morphism to its target. Figure 5.2 shows that d is a group homomorphism.

4. Let x be an element of CX_1 and α be an element of CX_2, define the element α^x to be the composite of the 2-cell depicted in Figure 5.3. Note α^{-1} represents the horizontal inverse in the figure.

From Figure 5.3, the equivariance axiom is trivially checked. The Pieffer identity follows from the middle four exchange.

Recall the weak equivalences of a crossed module from Definition 5.1.15 and weak equivalences of pointed 2-groupoids are 2-functors that are essential surjective and fully faithful (see Definition 2.1.6).

Theorem 5.1.19. *[51, §3] The pair of functors*

$$*//(\_) : CrsMod \leftrightarrows 2 - Gpd_* : C$$

$$\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left, "x", ""{name=A1, below}] \arrow[r, bend right, "x"', ""{name=B1, above}] \arrow[from=A1, to=B1, Rightarrow, "\alpha"] & \bullet \arrow[r, bend left, "id", ""{name=A2, below}] \arrow[r, bend right, "y"', ""{name=B2, above}] \arrow[from=A2, to=B2, Rightarrow, "\beta"] & \bullet \arrow[r, bend left, "x^{-1}", ""{name=A3, below}] \arrow[r, bend right, "x^{-1}"', ""{name=B3, above}] \arrow[from=A3, to=B3, Rightarrow, "\alpha^{-1}"] & \bullet \end{tikzcd}$$

FIGURE 5.3: Action of G_2 on G_1 in Construction 5.1.18

from Constructions 5.1.17 and 5.1.18 preserve weak equivalences and induce an equivalence of homotopy theories.

Although we defined a 2-group as a connected 2-type, from Theorem 5.1.11 and Theorem 5.1.19 shows that that there are 3 descriptions for homotopy theory of 2-groups which are equivalent to each other.

5.1.4 Mapping space of crossed modules

In this subsection, we aim to define Noohi's derived mapping 2-groupoid between crossed modules and describe the homotopy equivalence with mapping spaces.

First, let us describe Whitehead's construction of a crossed module from a CW pair (X, A) [68, §2]. Given a pointed CW complex $(X, *)$ and subcomplex A, the boundary map between relative homotopy groups

$$\partial : \pi_2(X, A, *) \to \pi_1(A, *)$$

is a crossed module. See [52, §2.1.2].

Definition 5.1.20. Given a CW pair (X, A), the boundary map

$$\partial : \pi_2(X, A, *) \to \pi_1(A, *)$$

can be upgraded to a crossed module called the *fundamental crossed module of the pair* (X, A).

Given a pointed CW complex $(X, *)$ with first skeleton X_1, the crossed module associated with the pair (X, X_1) is called the *Whitehead crossed module* $\mathcal{W}(X, X_1)$.

Note that $\mathcal{W}(X, X_1)$ is a cofibrant crossed module. It can be checked that the Whitehead crossed module $\mathcal{W}X$ of a space X is *isomorphic* to the composite $\mathcal{W}(X) := C \circ W(X)$ where W is given in Construction 5.1.9 and C is given in Construction 5.1.18.

$$\begin{tikzcd}
{} \arrow[r] & \cancelto{0}{\pi_2(X_1,*)} \arrow[r] & \pi_2(X,*) \arrow[d] \\
\pi_1(X,*) \arrow[d] & {\color{red}\pi_1(X_1,*)} \arrow[l] & {\color{red}\pi_2(X,X_1,*)} \arrow[l, red, "\partial"'] \\
\cancelto{0}{\pi_1(X,X_1,*)} \arrow[r] & {}
\end{tikzcd}$$

FIGURE 5.4: LES of pairs. The red arrow is part of the data of a Whitehead crossed module.

Proposition 5.1.21. *Let* $(X,*)$ *be a pointed connected* 2*-type with 1-skeleton* X_1 *and let*

$$\mathcal{W}(X,X_1) = (\partial : \pi_2(X,X_1,*) \to \pi_1(X_1,*))$$

denote the Whitehead crossed module (from Definition 5.1.20). Then the kernel of ∂ *is the homotopy group* $\pi_2(X,*)$ *and the cokernel of* ∂ *is the homotopy (fundamental) group* $\pi_1(X,*)$.

Proof. Consider the long exact sequence of relative homotopy groups applied to $X_1 \hookrightarrow X$ shown in Figure 5.4. Since X_1 is a 1-type, $\pi_2(X_1,*) = 0$ and since the quotient X/X_1 is a wedge of spheres, $\pi_1(X,X_1,*) = 0$. Thus the kernel of ∂ is isomorphic to $\pi_2(X,*)$ and cokernel of ∂ is isomorphic to $\pi_1(X,*)$. □

The computations of Whitehead crossed modules of a circle and oriented surfaces are shown in the next two examples.

Example 5.1.22. If we choose the standard CW structure of the circle S^1 with a 0-cell and a 1-cell, then $\mathcal{W}S^1$ is the discrete crossed module $d : 0 \to \mathbb{Z}$.

Example 5.1.23. If we choose the standard CW structure of an orientable surface Σ_g of genus g, with one 0-cell, $2g$ 1-cells and one 2-cell, then $\mathcal{W}\Sigma_g$ is

$$d : \mathbb{Z} \to F_{2g}$$

which provides a generator-relations presentation of $\pi_1(\Sigma_g)$. Note that F_{2g} is the free group on $2g$ generators.

Following [51, §6], we define the derived mapping 2-groupoid between crossed modules. Morphisms of crossed modules have already been defined in Definition 5.1.14. Now, we define weak transformations and modifications.

Definition 5.1.24. [51, Section 6.1] Let $p, q : \mathcal{H} \to \mathcal{G}$ be a pair of morphisms between crossed modules. A *weak natural transformation* $t : p \to q$ is given by a pair (a, θ) where $a \in G_1$ and $\theta : H_1 \to G_2$ is a crossed homomorphism (i.e. $\theta(hh') = \theta(h)^{p_1(h')}\theta(h')$) satisfying the rules:

1. $p_1(h)^a d_G(\theta(h)) = q_1(h)$ for all $h \in H_1$.
2. $p_2(x)^a \theta(d_H(x)) = q_2(x)$ for all $x \in H_2$.

A *strict natural transformation* is a weak natural transformation with $a = 1$.

Note that if θ is trivial, then a weak natural transformation is simply conjugation by a, i.e. $p = a^{-1}qa$, which we will write as $p = q^a$. Next, we define modifications between weak transformations.

Definition 5.1.25. [51, Definition 8.6] Let $p, q : \mathcal{H} \to \mathcal{G}$ be a pair of morphisms between crossed modules and let the pairs $(a, \theta), (b, \sigma)$ denote a pair of weak natural transformations from p to q. A *modification* from (a, θ) and (b, σ) is an element $\mu \in G_2$ that has the following properties:

1. $ad\mu = b$.
2. $\mu\sigma(x) = \theta(x)\mu^{q_1(x)}$ for all $x \in H_1$.

The morphisms, weak natural transformations and modifications of crossed modules can be assembled into a 2-groupoid $\underline{\mathcal{H}om}_{CrsMod}$ [51, Definition 8.5,8.6].

Recall that the Whitehead crossed module $\mathcal{W}X$ of a CW complex X is cofibrant, and every crossed module is fibrant. So we have the following definition by using the equivalence of crossed modules and pointed 2-groupoids and the definition of derived mapping 2-groupoid from Definition 5.1.12.

Definition 5.1.26. Let $\mathcal{H}, \mathcal{G}$ be crossed modules, then the derived mapping 2-groupoid $\underline{Hom}_{Noohi}(\mathcal{H}, \mathcal{G})$ is defined as $\underline{\mathcal{H}om}_{CrsMod}(Q\mathcal{H}, \mathcal{G})$ where $Q\mathcal{H}$ is a cofibrant replacement of $\mathcal{H}$.

The following theorem shows that Definition 5.1.26 has the correct homotopy type.

Theorem 5.1.27. *[51, §8] For any connected pointed 2-types Z, let $\mathcal{W}Z$ denote the associated Whitehead crossed module. Then we have a homotopy equivalence of topological spaces*

$$|\underline{Hom}_{Noohi}(\mathcal{W}M, \mathcal{W}X)| \simeq Maps(M, X).$$

5.1.5 An example of a DBC functor data

In this section, we quickly describe the bicategory $KV2Vect$ of Kapranov-Voevodosky 2-vector spaces following Morton [50]. Inspired by [50, §5], we construct an example of a double Beck-Chevalley functor data out of the category of groupoids and with target $KV2Vect$ (Construction 5.1.34). In Theorem 5.1.35, we check that double Beck-Chevalley functor data that we have constructed satisfy the axioms: Ambidexterity, Beck-Chevalley, Vertical Beck-Chevalley and Horizontal Beck-Chevalley and thus by Theorem 4.1.15 we obtain a canonical 2-functor $\tilde{Q}$. In Construction 5.1.36, we explicitly specify the 2-functor that arises from the Theorem 4.1.15. It agrees with Morton's 2-functor (called Λ in *loc.cit*) in [50, §7.1]. In Corollary 5.1.37, the value of $\tilde{Q}$ is computed on groupoids at various levels of morphisms.

Definition 5.1.28. A Kapranov-Voevodsky (KV) two-vector space is a semisimple linear additive category with finitely many simple objects. A *two-linear map* is a linear functor.

The symbol KV2Vect denotes the 2-category of Kapranov-Voevodsky two-vector spaces, two-linear maps and natural transformations.

Example 5.1.29. Let G be a finite group. The category $\mathrm{Rep}(G)$ of linear representations of a group is a KV two-vector space. The semisimplicity follows from Maschke's theorem, and the number of simple objects is finite since it is equal to the number of conjugacy classes in a finite group.

Example 5.1.30. Let $\mathcal{G}$ be a finite groupoid. It can be shown that every finite groupoid G is equivalent to a coproduct of one object groupoids (aka groups):

$$\mathcal{G} \simeq \coprod_{x \in \pi_0(G)} *//\mathrm{Aut}(x)$$

where $\pi_0(C)$ is the set of isomorphism classes of objects of a category C and $\mathrm{Aut}(x)$ is the group of automorphisms of x in C. Since G is a finite groupoid, $\pi_0(G)$ is finite, and the groups $\mathrm{Aut}(x)$ are finite. Thus

$$\mathrm{Fun}(\mathcal{G}, Vect) \simeq \prod_{x \in \pi_0(G)} \mathrm{Rep}(\mathrm{Aut}(x)).$$

Thus, the category

$$\mathrm{Rep}(\mathcal{G}) = \mathrm{Fun}(\mathcal{G}, Vect)$$

of linear representations of the finite groupoid is a KV two-vector space. Note that the finite is crucial; otherwise, the category will not be generated by finitely many simple objects.

A *finite 2-groupoid* is a bicategory with a finite number of invertible 1-morphisms and 2-morphisms and a finite number of objects. An *essentially finite 2-groupoid* is a 2-groupoid biequivalent to a finite 2-groupoid.

Remark 5.1.31. If $\mathcal{G}$ is a finite 2-groupoid (see Definition 2.1.3), then the category of representations of $\mathcal{G}$,

$$\mathrm{Fun}(\mathcal{G}, Vect)$$

factors through the homotopy groupoid of $\mathcal{G}$ (Definition 5.1.8), denoted by $\overline{\mathcal{G}}$. Thus

$$\mathrm{Fun}(\mathcal{G}, Vect) \simeq \mathrm{Fun}(\overline{\mathcal{G}}, Vect).$$

Example 5.1.32. Let G be a finite group. Given a $\#G$-tuple of vector spaces V_g indexed by elements g of G, define

$$V = \underset{g \in G}{\oplus} V_g$$

Then V is a G-graded vector space. The G-graded vector spaces form a linear category $\mathrm{Vect}[G]$, and the direct sum can be defined levelwise. Scalar multiplication with a vector space W is defined by

$$(W \otimes V)_g = W \otimes V_g.$$

It has $\#G$ simple objects; thus, $\mathrm{Vect}[G]$ is a KV two-vector space. A monoidal structure can be defined by

$$(V \otimes W)_g = \bigoplus_{g_1 g_2 = g} V_{g_1} \otimes W_{g_2}.$$

This monoidal category is often called a *group* 2-*algebra*(since it categorifies a group algebra).

Example 5.1.33. Let Vect denote the category of vector spaces. Then, for any natural number n, the category Vect^n of n-tuples of vector spaces is a two-vector space. It follows from the fact that it has n simple objects given by a copy of $\mathbb{C}$ in the ith coordinate for $1 \leq i \leq n$. Given a collection of vector spaces F_{ij} for $1 \leq i \leq m, 1 \leq j \leq n$. A matrix functor

$$F : \mathrm{Vect}^n \to \mathrm{Vect}^m$$

can be defined as

$$F(V)_i = \overset{n}{\underset{j=1}{\oplus}} F_{ij} \otimes V_j.$$

The functor F is linear and thus, it is a two-linear map. Given linear transformations $t_{ij} : F_{ij} \to G_{ij}$, a matrix natural transformation $t : F \to G$ can be induced by summing

$$F_{ij} \otimes V_j \xrightarrow{t_{ij}\otimes id} G_{ij} \otimes V_j$$

The collection of Vect^n for different natural numbers n, matrix functors and matrix natural transformations form a 2-category which we will denote by $\mathrm{KV2Vect}_c$ (following [36]). The monoidal structure on $\mathrm{KV2Vect}_c$ is analogous to the tensor product of $\mathbb{C}^m$.

Example 5.1.33 is canonical since it is well known that the 2-category KV2Vect is equivalent $\mathrm{KV2Vect}_c$ (see [50, §3.2]).

Given a matrix 2-linear map F, define $F^\dagger$ as

$$F^\dagger_{ij} = F^\vee_{ji}$$

where $V^\vee$ is the dual of the vector space V. The two-linear map $F^\dagger$ is an ambidextrous adjoint to F ([50, Theorem 2, page 672]). This fact follows from the Hom-tensor adjunction in the category of vector spaces.

The key ingredient in the computations below, Construction 5.1.34, is the isomorphism between the left Kan extension $f_*F(y)$ and the right Kan extension $f^!F(y)$ of $F : Y \to \mathrm{Vect}$. Since every groupoid is equivalent to a coproduct of one-object groupoids, the computations in Construction 5.1.34 will follow from considering the one-object groupoid case which we discuss now.

Let $Y = *//G$, $X = *//H$ and thus a functor $f : Y \to X$ is given by a group homomorphism $G \to H$ that induces an algebra homomorphism $f : \mathbb{C}[G] \to \mathbb{C}[H]$. Given a functor $F : X \to Vect$, we have an $\mathbb{C}[H]$-module $F(*) = F$. The pullback functor f^*F is given by the abelian group F again but with a $\mathbb{C}[G]$ action via f. This functor is, therefore, the restriction of representation F. The left and the right adjoint to restriction are given by

$$f_*F = \mathbb{C}[H] \otimes_{\mathbb{C}[G]} F,$$

$$f^!F = \mathrm{Hom}_{\mathbb{C}[G]-\mathrm{Mod}}(\mathbb{C}[H], F).$$

The Nakayama isomorphism $N : f^!F \to f_*F$ is given by the formula:

$$\phi \to \frac{1}{\#G} \sum_{h\in H} h^{-1} \otimes \phi(h)$$

Construction 5.1.34. We will denote the functor category $\mathrm{Fun}(X, \mathrm{Vect})$ by Vect^X in order to stress the analogy with the vector space of functions $\mathbb{C}^X$. Let $f : Y \to X$ be a map of finite groupoids. Then the pullback functor $f^* : \mathrm{Vect}^X \to \mathrm{Vect}^Y$ given by

$$f^*F = F \circ f.$$

is a two-linear map with an ambidextrous adjoint given by the "pushforward" f_*. In section 4.2 of [49], Morton notes that for $F \in \mathrm{Vect}^Y$, the pushforward f_*F can also be written as left Kan extension $Lan_f F$. More accurately, using the pointwise Kan extension formula and decomposition of a groupoid into a coproduct of groups, we get

$$f_*F(x) = \bigoplus_{[y]|f(y)\simeq x} \mathbb{C}[Aut(x)] \otimes_{\mathbb{C}[Aut(y)]} F(y)$$

where $[y]|f(y) \simeq x$ denotes the set of all isomorphism classes of objects y in Y such that the condition $f(y) \simeq x$ holds.

Now we describe the identity coherence $\theta_* : (id_X)_*F \to F$ for a groupoid X, for $x \in X$,

$$\theta_{*x} : \mathbb{C}[Aut(x)] \otimes_{\mathbb{C}[Aut(x)]} F(x) \to F(x)$$

is given by the canonical isomorphism

$$g \otimes v \mapsto gv.$$

The other coherence θ^* for the pullback is just identity.

Following Morton, we will assume all the groupoids are skeletal. Thus, if x is isomorphic to y in the groupoid, then $x = y$. This assumption will be used in the formulae below. We will now specify the ambidextrous adjunction data $(f^*, f_*, \eta, \epsilon, \eta^!, \epsilon^!)$:

1. The unit

   $$\eta_F : F \to f^*f_*F(y) = \bigoplus_{[y']|f(y)\simeq f(y')} \mathbb{C}[Aut(f(y))] \otimes_{\mathbb{C}[Aut(y')]} F(y')$$

   is given by

   $$(\eta_F(y))(v) = \bigoplus_{[y']|f(y)\simeq f(y')} 1 \otimes v.$$

2. The counit

   $$\epsilon_G(x) : \bigoplus_{[y]|f(y)\simeq x} \mathbb{C}[Aut(x)] \otimes_{\mathbb{C}[Aut(y)]} G(x) \to G(x)$$

is given by

$$\bigoplus_{[y]|f(y)\simeq x} g_y \otimes v \mapsto \sum_{[y]|f(y)\simeq x} g_y v.$$

3. The unit

$$\eta^!_G(x) : G(x) \to f_* f^* G(x) = \bigoplus_{[y]|f(y)\simeq x} \mathbb{C}[Aut(x)] \otimes_{\mathbb{C}[Aut(y)]} G(x)$$

is given by

$$v \mapsto \bigoplus_{[y]|f(y)\simeq x} \frac{1}{\#Aut(y)} \sum_{g\in Aut(x)} g^{-1} \otimes g(v)$$

4. The counit

$$\epsilon^!_F(y) : \bigoplus_{[y']|f(y)\simeq f(y')} \mathbb{C}[Aut(f(y))] \otimes_{\mathbb{C}[Aut(y')]} F(y') \to F(y)$$

is given by

$$\bigoplus_{[y']|f(y)\simeq f(y')} g_{y'} \otimes v_{y'} \mapsto \frac{\#Aut(y)}{\#Aut(f(y))} g_y(v_y).$$

The proof of the following theorem is assembled from Morton's work [50]. The only thing missing is the speciality of the ambijunction for the equivalence of skeletal groupoids. Note that this was crucial to prove that the 2-functor underlying quantization is well-defined.

Theorem 5.1.35. *The data $(f^*, f_*, \eta, \epsilon, \eta^!, \epsilon^!)$ described in Construction 5.1.34 is a double Beck-Chevalley functor data.*

Proof. First we prove axiom 1 from Definition 4.1.15 for the data $(f^*, f_*, \eta, \epsilon, \eta^!, \epsilon^!)$ constructed in Construction 5.1.34. As mentioned in Construction 5.1.34, Morton has already shown that this is the data of the ambidextrous adjunction. The speciality of the ambijunction and the compatibility with identity coherences remains to be seen. First, we prove the speciality of the ambijunction. Let $f : Y \to X$ be an equivalence of skeletal groupoids. We quickly note that f induces a bijection on isomorphism classes of objects (by essential subjectivity, skeletalness and using the inverse functor). It also induces an isomorphism on the automorphism group of an object in Y (by fully faithfulness). We will use these observations below. Now we check $\epsilon\eta^! = id$. Using the details from

Construction 5.1.34, and the observations above for equivalence, we see that

$$\begin{aligned}\epsilon_G(x)\eta^!_G(x)(v) &= \epsilon_G(x)\left(\frac{1}{\#Aut(x)}\sum_{g\in Aut(x)} g^{-1}\otimes gv\right)\\ &= \frac{1}{\#Aut(x)}\sum_{g\in Aut(x)} g^{-1}(gv)\\ &= v.\end{aligned}$$

Next we check $\eta^!\epsilon = id$ using the details from Construction 5.1.34, and the observations above for an equivalence,

$$\begin{aligned}\epsilon^!_F(y)\eta_F(y)(v) &= \epsilon^!_F(y)(1\otimes v)\\ &= \frac{\#Aut(y)}{\#Aut(f(y))}v\\ &= v.\end{aligned}$$

Now we consider $(id^*, id_*, \eta, \epsilon, \eta^!, \epsilon^!)$ and the identity coherence θ_* defined in Construction 5.1.34. Clearly, $\epsilon^! = \epsilon = \theta_*$ and $\eta^! = \eta = \theta_*^{-1}$. Recall that θ^* is identity; thus, we have checked axiom 1. Morton has checked the remaining axioms in [50], and we indicate them here. Axiom 2 of Definition 4.1.15 discusses the Beck-Chevalley isomorphism condition. This check is quite involved and can be found in the proof of Theorem 3 (page 685) in [50]. Axiom 3 is checked in the proof of Lemma 4 (page 692), and Axiom 5 is checked in the proof of Lemma 5 (page 694) in *loc.cit.* □

Since Theorem 5.1.35 checks that the hypotheses of Theorem 4.3.10 hold, we obtain a 2-functor

$$\tilde{Q} : \mathbf{Bispan}(\mathrm{Gpds}^f) \to \mathrm{KV2Vect}$$

The description of this 2-functor is given below in Construction 5.1.36. We recover Morton's 2-functor, which is no surprise since our work is motivated by his construction.

Construction 5.1.36. From the double Beck-Chevalley functor data from Construction 5.1.34 and the 2-functor construction from Construction 4.3.1 we obtain a 2-functor $\tilde{Q}$ with the following properties:

1. On objects: Let X be a finite groupoid, then

$$\tilde{Q}(X) = \mathrm{Vect}^X.$$

2. On 1-morphisms: Let ${}_fX_g : A \to B$ denote a span of groupoids. Then
$$\tilde{Q}({}_fX_g) = g_*f^*.$$

3. On 2-morphisms: Let ${}_lW_m :_f X_g \rightarrow_h Y_k$ with X, Y as morphisms from A to B. Then
$$\tilde{Q}({}_lW_m) := (k_*\epsilon_m h^*) \circ (\theta_{*gl}\theta^*_{fl}) \circ (g_*\eta^!_l f^*).$$
The 2-morphism $\tilde{Q}({}_lW_m)F$ is from g_*f^*F to k_*h^*F for a functor $F \in \mathrm{Vect}^A$, i.e.
$$\tilde{Q}({}_lW_m)F : \bigoplus_{[x]|g(x)\simeq b} \mathbb{C}[Aut(b)] \underset{\mathbb{C}[Aut(x)]}{\otimes} F(f(x)) \to \bigoplus_{[y]|k(y)\simeq b} \mathbb{C}[Aut(b)] \underset{\mathbb{C}[Aut(y)]}{\otimes} F(h(y))$$
Given a fixed isomorphism class $[x] \in X$ such that $g(x) = b$ and a fixed isomorphism class $[y] \in Y$ such that $k(y) = b$, the matrix entry $[\tilde{Q}({}_lW_m)F]_{y,x}$ is
$$[\tilde{Q}({}_lW_m)F]_{y,x}(\alpha \otimes v) = \bigoplus_{[w]|mw=y} \frac{1}{\#Aut(w)} \sum_{\beta \in Aut(x)} \alpha g(\beta^{-1}) \otimes \beta \cdot v$$

Therefore we recover Morton's 2-functor Λ from [50, §7.1].

A specific case of Construction 5.1.36 is the value of $\tilde{Q}$ on a groupoid at various morphism levels.

Corollary 5.1.37. *From Construction 5.1.36, it follows that:*

1. *On object X, $\tilde{Q}(X) = Vect^X$.*

2. *On morphism $* \xleftarrow{!} X \xrightarrow{!} *$,*
$$\tilde{Q}({}_!X_!) = \mathbb{C}^{\pi_0(X)}.$$

3. *On 2-morphism*
$$\begin{tikzcd}
 & \bullet \arrow[dl] \arrow[dr] & \\
\bullet & X \arrow[u] \arrow[d] & \bullet \\
 & \bullet \arrow[ul] \arrow[ur] &
\end{tikzcd}$$
$$\tilde{Q}({}_\bullet X_\bullet) = \#X,$$
where $\#X$ is the groupoid cardinality of X.

Remark 5.1.38. Note that the groupoid cardinality arose from the factor in the Nakayama isomorphism. This factor was essential to verifying the speciality of ambidextrous adjunction in axiom 1 (see proof of Theorem 5.1.35).

5.2 Computations

Given a 2-type X, we compute the fundamental path groupoids of the mapping space $Maps(S^n, X)$ for $n = 1, 2$ (where S^n is the n-dimensional sphere) in Subsection 5.2.1. We compute the value of $\tilde{Q}$ on the fundamental path groupoid of the mapping spaces in Subsection 5.2.2.

5.2.1 Groupoid presentations of certain mapping spaces

In this subsection, we first define a notion of conjugacy classes (Definition 5.2.3) and class functions of a crossed module (Definition 5.2.4).Given a crossed module $\mathcal{G}$, there is an associated 2-type $X(\mathcal{G}) := |N(*//\mathcal{G})|$ (follows from results in Section 5.1).

We describe the fundamental path groupoid of a free loop space of $X(\mathcal{G})$ in terms of $\mathcal{G}$ in Proposition 5.2.6. It can be seen that the isomorphism classes of objects of this groupoid is the set of conjugacy classes of the crossed module $\mathcal{G}$.

Given a space X, the fundamental group $\pi_1(X, *)$ acts on higher homotopy groups $\pi_n(X, *)$ for $n > 1$. In particular, this induces an action groupoid $\pi_2(X)//\pi_1(X)$. In Proposition 5.2.7, we prove that the fundamental path groupoid of $Maps(S^2, X)$ is equivalent to $\pi_2(X)//\pi_1(X)$.

Definition 5.2.1. Given a crossed module $\mathcal{G}$, for $a, b \in G_1$, we say a is conjugate to b if there exists $\alpha \in G_2$ and $c \in G_1$ such that

$$d\alpha = a^{-1}c^{-1}bc = a^{-1}b^c.$$

Proposition 5.2.2. *The conjugate relation is an equivalence relation on G_1.*

Proof. The reflexivity follows from choosing α and c as identity.

Now, we prove symmetry. If $a \sim_{\{\alpha,c\}} b$,then

$$d\alpha = a^{-1}c^{-1}bc.$$

Inverting the equation and rearranging we get

$$d((\alpha^{-1})^c) = c(d\alpha^{-1})c^{-1} = b^{-1}cbc^{-1},$$

and thus $b \sim a$.

For transitivity, if $d\alpha = a^{-1}b^x$ and $d\beta = b^{-1}c^y$ then note that

$$d(\alpha\beta^x) = d(\alpha)d(\beta^x) = a^{-1}b^x(b^{-1})^x c^{yx} = a^{-1}c^{yx}.$$

□

Definition 5.2.3. Given a crossed module $\mathcal{G}$, the set of conjugacy classes of $\mathcal{G}$ is defined as the set of equivalence classes of the conjugate relation:

$$C_{\mathcal{G}} := G_1/\sim .$$

Given a morphism $\phi : \mathcal{H} \to \mathcal{G}$, the induced morphism between conjugacy classes is given by $C_\phi := \phi_1$.

Definition 5.2.4. Given a crossed module $\mathcal{G}$, the vector space of complex-valued functions on the set $C_{\mathcal{G}}$ is called the space of class functions and is denoted by $\mathrm{Cl}(\mathcal{G})$. It can be viewed as functions on $\mathrm{Dom}(\mathcal{G})$, which are constant on conjugacy classes.

Construction 5.2.5. Let $\mathcal{G} = (G_2, G_1, d)$ be a crossed module . We construct a groupoid of conjugacy classes of $\mathcal{G}$ denoted by $\mathcal{C}_{\mathcal{G}}$.

1. The objects of $\mathcal{C}_{\mathcal{G}}$ are objects of G_1.
2. Given a pair of objects x, y, the morphisms are defined as

$$\mathcal{C}_{\mathcal{G}}(x,y) = \frac{\{(a,\theta)|x^a d\theta = y\}}{\sim}$$

where $(a,\theta) \sim (a',\theta')$ if there exists $\mu \in G_2$ such that $ad\mu = a'$ and $\mu\theta' = \theta\mu^y$.

3. The composition is given by the formula:

$$[(b,\sigma)] \circ [(a,\theta)] := [(ab, \theta^b\sigma)]$$

Note that the isomorphism class of objects of $\mathcal{C}_{\mathcal{G}}$ is the set of conjugacy classes of $\mathcal{G}$ (see Definition 5.2.3).

Proposition 5.2.6. *Let $\mathcal{G}$ be a crossed module, and let $X = |\mathcal{G}|$ be its geometric realization. If $LX := Maps(S^1, X)$ denotes the free loop space of X, then*

$$\pi_{\leq 1}(LX) \simeq \mathcal{C}_{\mathcal{G}}$$

where $\mathcal{C}_{\mathcal{G}}$ is conjugacy groupoid of $\mathcal{G}$ constructed in Construction 5.2.5. The connected components of the free loop space are in bijection with the set of conjugacy classes $C_{\mathcal{G}}$ of $\mathcal{G}$.

Proof. From Example 5.1.22, we know that $\mathcal{W}S^1$ is the cofibrant crossed module $\partial : 0 \to \mathbb{Z}$. Now we compute $\mathcal{C}_{\mathcal{G}} := \underline{Hom}_{Noohi}(\mathcal{W}S^1, G)$ ((see Definition 5.1.26)).

1. Objects of the 2-groupoid are morphisms of crossed modules:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "\partial"'] & G_2 \arrow[d, "d"] \\ \mathbb{Z} \arrow[r, "g"'] & G_2 \end{tikzcd}$$

   We will identify the homomorphism g with $g(1)$ (i.e. $g(1) = g$). So, the objects of the 2-groupoid can be identified with elements of G_1.

2. Weak natural transformations are the morphisms (see Definition 5.1.24) between g and g', which is given by a pair (a, θ) satisfying a pair of axioms. First, we note that the crossed homomorphism $\theta : \mathbb{Z} \to G_2$ is determined by $\theta(1)$ which we will denote by θ again. The second axiom gives $\theta(0) = 1_{G_2}$. The first axiom gives the following relation: for $a \in G_1$,

$$g^a d\theta = g'.$$

   We quickly note a morphism $(a, \theta) : g \to g'$ iff g' is conjugate to g in the sense of Definition 5.2.1.

3. The 2-morphisms are given by modifications (see Definition 5.1.25) $\mu : (a, \theta) \to (b, \sigma)$ given by $\mu \in G_2$ such that

$$ad\mu = b, \mu\sigma = \theta\mu^{g'}.$$

The composition of 1-morphism and the horizontal composition of 2-morphism is the wreath product multiplication of G_2 and G_1. The vertical composition of 2-morphisms is simply the product of the group elements.

The equivalence classes of objects of $\pi_{\leq 1}(LX)$ is

$$\pi_0(LX) = \pi_0(\mathcal{C}_{\mathcal{G}}) = C_{\mathcal{G}},$$

is the set of conjugacy classes of G as given in Definition 5.2.3. □

Recall the definition of an action groupoid (Definition 1.1.31). An interesting remark is the following: If we think of the right action of H on A is given by a functor $\underline{A} : (*//G)^{op} \to Sets$ with $\underline{A}(*) = A$, then the category of elements[2] of $\underline{A}$ is the groupoid $A//H$.

Proposition 5.2.7. *Let $\mathcal{G}$ be a crossed module and $X = |\mathcal{G}|$ be its geometric realization. Then*

$$\pi_{\leq 1}(Maps(S^2, X)) \simeq \pi_2(X,*)//\pi_1(X,*)$$

where $\pi_2(X,*)//\pi_1(X,*)$ represents the action groupoid (see Definition 1.1.31) for the action of $pi_1(X,*)$ on $\pi_2(X,*)$

Proof. From Example 5.1.23, we know that $\mathcal{W}S^2$ is a cofibrant crossed module $\partial : \mathbb{Z} \to 0$. Now we compute Noohi's derived mapping 2-groupoid, $\underline{Hom}_{Noohi}(\mathcal{W}(S^2), \mathcal{G})$ (see Definition 5.1.26).

1. Objects of the 2-groupoid are morphisms of crossed modules:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "g"] \arrow[d, "\partial"'] & G_2 \arrow[d, "d"] \\
0 \arrow[r] & G_1
\end{tikzcd}$$

   We will identify the homomorphism g with $g(1)$ (i.e. $g(1) = g$). Since $dg = 0$, we see that $g \in G_2$ should belong to kernel of d. So the objects of the 2-groupoid can be identified with elements of $\ker(d)$.

2. The morphisms are given by weak natural transformations (see Definition 5.1.24) between g and g' which is given by a pair (a, θ) satisfying a pair of axioms. First, we note that the crossed homomorphism $\theta : 0 \to G_2$ forces $\theta(0) = 1$. The first axiom is redundant and the second axiom gives the following relation: for $a \in G_1$,

$$g^a = g'.$$

   We quickly note that a morphism $a : g \to g'$ iff g' is in orbit of g.

---

[2] from Definition 1.1.31.

3. The 2-morphisms are given by modifications (see Definition 5.1.25) $\mu : a \to b$ given by $\mu \in G_2$ such that
$$ad\mu = b.$$

From item (3) in the above list, a pair of 1-morphisms a, b are isomorphic iff they map to the same element in the cokernel of d. Thus the homotopy groupoid (Definition 5.1.8) is the action groupoid of the action of cokernel of d on the kernel of d.

From Proposition 5.1.21, we see that cokernel of d is $\pi_1(X, *)$ and kernel of d is $\pi_2(X, *)$. Thus, we are done. □

5.2.2 Values of the tentative DW theory

Given CW complex X and $n = 1, 2$, the groupoid $\pi_{\leq 1}(Maps(S^n, X))$ is computed in the previous section. In Corollary 5.1.37, Morton's 2-functor $\tilde{Q}$ is computed on groupoids at various morphism levels (in the bicategory of bispans of groupoids). Composing these constructions, we can compute $\tilde{Q}(\pi_{\leq 1}(Maps(M, X))$ at various levels.

In the following proposition, the only non-trivial computation is the value of $\tilde{Q}(\pi_{\leq 1} LX)$ computed at the level of 1-morphisms.

Proposition 5.2.8. *Let X be the geometric realization of the crossed module $G = (G_2, G_1, d)$.*

1. *Let pt denote the unique connected 0-dimensional manifold, then on the level of objects*
$$\tilde{Q}(\pi_{\leq 1}(Maps(pt, X))) \simeq Rep(\pi_1(X)).$$

2. *Let S^1 denote the 1-dimensional manifold that is the unit circle. Then on the level of 1-morphisms*
$$\tilde{Q}(\pi_{\leq 1}(Maps(S^1, X))) \simeq Cl(G).$$

Proof. We will use the formulae in Corollary 5.1.37 to calculate $\tilde{Q}$ of a groupoid at various morphism levels.

1.
$$\tilde{Q}(\pi_{\leq 1}(Maps(pt, X))) \simeq \tilde{Q}(\pi_{\leq 1}(X)) \simeq \mathrm{Vect}^{*//\pi_1(X,*)} = \mathrm{Rep}(\pi_1(X, *)).$$

   Since X is a pointed connected CW complex, the fundamental path groupoid $\pi_{\leq 1}(X)$ is equivalent to the one-object groupoid $*//\pi_1(X, *)$.

2.
$$\tilde{Q}(\pi_{\leq 1}(Maps(S^1, X))) \simeq \tilde{Q}(\mathcal{C}_{\mathcal{G}}) = \mathbb{C}^{C_{\mathcal{G}}} = \mathrm{Cl}(G).$$

The first equality follows from Proposition 5.2.6. The second equality follows from the computation of $\tilde{Q}$ on one-morphism in Corollary 5.1.37. The third equality from Definition 5.2.4.

□

Remark 5.2.9. Assuming the extended DW TQFT for 2-groups exists, results of Proposition 5.2.8 can be considered as the values of an extended 2-dimensional TQFT on a point and a circle. The value on a point is consistent with Martins and Porter's result [17, Theorem 276]. Indeed, the value of the circle in *loc.cit* is the vector space of class functions on $\pi_1(X, *)$.

In the following proposition, the results of $\tilde{Q}(\pi_{\leq 1} Maps(S^n, X))$ are computed at the level of objects for $n = 1$ and the level of 1-morphisms for $n = 2$.

Proposition 5.2.10. *Let X be the geometric realization of the crossed module $G = (G_2, G_1, d)$. Let $\mathbb{C}[\mathcal{C}_G]$ denote the groupoid algebra associated with the conjugacy groupoid from Construction 5.2.5.*

1. *Let S^1 denote the unit circle, then on the level of objects*
$$\tilde{Q}(\pi_{\leq 1}(Maps(S^1, X))) = Rep(\mathbb{C}[\mathcal{C}_G]).$$

2. *Let S^2 denote the two-dimensional unit sphere, then on the level of 1-morphisms*
$$\tilde{Q}(\pi_{\leq 1}(Maps(S^2, X))) = \mathbb{C}^{\pi_2(X,*)//\pi_1(X,*)},$$
where $\pi_2(X, *)//\pi_1(X, *)$ represents the action groupoid of π_1 action on π_2 (see Definition 1.1.31).

Proof. 1.
$$\tilde{Q}(\pi_{\leq 1}(Maps(S^1, X))) \simeq \tilde{Q}(\mathcal{C}_{\mathcal{G}}) \simeq \mathrm{Rep}(\mathbb{C}[\mathcal{C}_{\mathcal{G}}]).$$

The equalities above follow from Proposition 5.2.1 and Corollary 5.1.37 respectively.

2.
$$\tilde{Q}(\pi_{\leq 1}(Maps(S^2, X))) \simeq \tilde{Q}(\pi_2(X, *)//\pi_1(X, *)) = \mathbb{C}^{\pi_2(X,*)//\pi_1(X,*)}.$$

which follows from Proposition 5.2.7 and Corollary 5.1.37 respectively.

□

Remark 5.2.11. Assuming the extended DW TQFT for 2-groups exists, results of Proposition 5.2.10 can be considered as the values of an extended three-dimensional TQFT on a circle and a sphere. The values match Martins-Porter's results (see [17, Theorem 283]). Definition 281 in *loc.cit* matches the 2-groupoid described in Construction 5.2.5.

Remark 5.2.12. It is expected that the value of the tentative extended 3-dimensional DW theory on a circle is a modular tensor category (mentioned as far back as in [37]). If a quantization functor is constructed, we can compute the quasitriangular Hopf algebraic structure on the algebra $\mathbb{C}[\mathcal{C}_{\mathcal{G}}]$. From this computation, we hope to find the Drinfeld double of a 2-group.

Bibliography

[1] Michael Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186, 1988.

[2] John C. Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *J. Math. Phys.*, 36(11):6073–6105, 1995.

[3] Igor Baković. Fibrations of bicategories. Available at the webpage `https://www2.irb.hr/korisnici/ibakovic/groth2fib.pdf`.

[4] Paul Balmer and Ivo Dell'Ambrogio. *Mackey 2-functors and Mackey 2-motives*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2020.

[5] C. Barwick. On the Q construction for exact quasicategories. *arXiv e-prints*, page arXiv:1301.4725, January 2013.

[6] Jean Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, Berlin, 1967.

[7] John Bourke. The category of categories with pullbacks is cartesian closed. *arXiv e-prints*, page arXiv:0904.2486, April 2009.

[8] Morton Brown. Locally flat imbeddings of topological manifolds. *Ann. of Math. (2)*, 75:331–341, 1962.

[9] Ronald Brown and Philip J. Higgins. The classifying space of a crossed complex. *Math. Proc. Cambridge Philos. Soc.*, 110(1):95–120, 1991.

[10] Damien Calaque and Claudia Scheimbauer. A note on the (∞, n)-category of cobordisms. *Algebr. Geom. Topol.*, 19(2):533–655, 2019.

[11] Aurelio Carboni and Peter Johnstone. Connected limits, familial representability and Artin glueing. volume 5, pages 441–459. 1995. Fifth Biennial Meeting on Category Theory and Computer Science (Amsterdam, 1993).

[12] Gian Luca Cattani and Glynn Winskel. Profunctors, open maps and bisimulation. *Math. Structures Comput. Sci.*, 15(3):553–614, 2005.

[13] Robert Dawson, Robert Paré, and Dorette Pronk. The span construction. *Theory Appl. Categ.*, 24:No. 13, 302–377, 2010.

[14] Robbert Dijkgraaf and Edward Witten. Topological gauge theories and group cohomology. *Comm. Math. Phys.*, 129(2):393–429, 1990.

[15] Charles Ehresmann. Catégories doubles et catégories structurées. *C. R. Acad. Sci. Paris*, 256:1198–1201, 1963.

[16] Josep Elgueta. Representation theory of 2-groups on Kapranov and Voevodsky's 2-vector spaces. *Adv. Math.*, 213(1):53–92, 2007.

[17] João Faria Martins and Timothy Porter. A categorification of Quinn's finite total homotopy TQFT with application to TQFTs and once-extended TQFTs derived from strict omega-groupoids. *arXiv e-prints*, page arXiv:2301.02491, January 2023.

[18] Thomas M. Fiore, Simona Paoli, and Dorette Pronk. Model structures on the category of small double categories. *Algebr. Geom. Topol.*, 8(4):1855–1959, 2008.

[19] Daniel S. Freed. Higher algebraic structures and quantization. *Comm. Math. Phys.*, 159(2):343–398, 1994.

[20] Daniel S. Freed, Michael J. Hopkins, Jacob Lurie, and Constantin Teleman. Topological quantum field theories from compact Lie groups. In *A celebration of the mathematical legacy of Raoul Bott*, volume 50 of *CRM Proc. Lecture Notes*, pages 367–403. Amer. Math. Soc., Providence, RI, 2010.

[21] Daniel S. Freed and Frank Quinn. Chern-Simons theory with finite gauge group. *Comm. Math. Phys.*, 156(3):435–472, 1993.

[22] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[23] Nora Ganter and Mikhail Kapranov. Representation and character theory in 2-categories. *Adv. Math.*, 217(5):2268–2300, 2008.

[24] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[25] R. Gordon, A. J. Power, and Ross Street. Coherence for tricategories. *Mem. Amer. Math. Soc.*, 117(558):vi+81, 1995.

[26] Marco Grandis and Robert Pare. Limits in double categories. *Cahiers Topologie Géom. Différentielle Catég.*, 40(3):162–220, 1999.

[27] Alexander Grothendieck. *Fondements de la géométrie algébrique.* Secrétariat mathématique, 1962.

[28] Alexander Grothendieck. *Revêtements étales et groupe fondamental. Fasc. II: Exposés 6, 8 à 11*, volume 1960/61 of *Séminaire de Géométrie Algébrique.* Institut des Hautes Études Scientifiques, Paris, 1963.

[29] Alexandre Grothendieck. *Pursuing stacks (à la poursuite des champs). Vol. I*, volume 20 of *Documents Mathématiques (Paris) [Mathematical Documents (Paris)].* Société Mathématique de France, Paris, 2022. Edited by Georges Maltsiniotis.

[30] Rune Haugseng. Iterated spans and classical topological field theories. *Math. Z.*, 289(3-4):1427–1488, 2018.

[31] Claudio Hermida. Representable multicategories. *Adv. Math.*, 151(2):164–225, 2000.

[32] Alexander E. Hoffnung. Spans in 2-Categories: A monoidal tricategory. *arXiv e-prints*, page arXiv:1112.0560, December 2011.

[33] Niles Johnson and Donald Yau. *2-dimensional categories.* Oxford University Press, Oxford, 2021.

[34] André Joyal and Ross Street. Pullbacks equivalent to pseudopullbacks. *Cahiers Topologie Géom. Différentielle Catég.*, 34(2):153–156, 1993.

[35] D. B. Kaledin. Adjunction in 2-categories. *Uspekhi Mat. Nauk*, 75(5(455)):101–152, 2020.

[36] M. M. Kapranov and V. A. Voevodsky. 2-categories and Zamolodchikov tetrahedra equations. In *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 177–259. Amer. Math. Soc., Providence, RI, 1994.

[37] M. Kontsevich. Rational conformal field theory and invariants of 3-dimensional manifolds., December 1988. A copy of paper is on the webpage `https://www.ihes.fr/~maxim/TEXTS/3TQFT_and_RCFT.pdf`.

[38] Stephen Lack. Icons. *Appl. Categ. Structures*, 18(3):289–307, 2010.

[39] Stephen Lack and Simona Paoli. 2-nerves for bicategories. *K-Theory*, 38(2):153–175, 2008.

[40] Tom Leinster. Basic Bicategories. *arXiv Mathematics e-prints*, page math/9810017, October 1998.

[41] F. E. J. Linton. Autonomous categories and duality of functors. *J. Algebra*, 2:315–349, 1965.

[42] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.

[43] Saunders MacLane. *Categories for the working mathematician.* Graduate Texts in Mathematics, Vol. 5. Springer-Verlag, New York-Berlin, 1971.

[44] Saunders MacLane and J. H. C. Whitehead. On the 3-type of a complex. *Proc. Nat. Acad. Sci. U.S.A.*, 36:41–48, 1950.

[45] Michael A. Mandell. An inverse K-theory functor. *Doc. Math.*, 15:765–791, 2010.

[46] A. D. Mednyh. Determination of the number of nonequivalent coverings over a compact Riemann surface. *Dokl. Akad. Nauk SSSR*, 239(2):269–271, 1978.

[47] Ieke Moerdijk and Jan-Alve Svensson. Algebraic classification of equivariant homotopy 2-types. I. *J. Pure Appl. Algebra*, 89(1-2):187–216, 1993.

[48] Jeffrey C. Morton. Double bicategories and double cospans. *J. Homotopy Relat. Struct.*, 4(1):389–428, 2009.

[49] Jeffrey C. Morton. Cohomological twisting of 2-linearization and extended TQFT. *J. Homotopy Relat. Struct.*, 10(2):127–187, 2015.

[50] Jeffrey Colin Morton. Two-vector spaces and groupoids. *Appl. Categ. Structures*, 19(4):659–707, 2011.

[51] Behrang Noohi. Notes on 2-groupoids, 2-groups and crossed modules. *Homology Homotopy Appl.*, 9(1):75–106, 2007.

[52] Timothy Porter. The crossed menagerie (14 chapter version), September 2019. Available on the webpage `https://ncatlab.org/timporter/files/menagerie14.pdf`.

[53] Franciscus Rebro. Building the bicategory $\mathrm{Span}_2(\mathcal{C})$. *arXiv e-prints*, page arXiv:1501.00792, January 2015.

[54] Christopher John Schommer-Pries. *The classification of two-dimensional extended topological field theories.* ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–University of California, Berkeley.

[55] Graeme Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[56] Graeme Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577. Cambridge Univ. Press, Cambridge, 2004.

[57] Amit Sharma. Symmetric monoidal categories and Γ-categories. *Theory Appl. Categ.*, 35:Paper No. 14, 417–512, 2020. [Paper number previously given as 13].

[58] Michael Shulman. Framed bicategories and monoidal fibrations. *Theory Appl. Categ.*, 20:No. 18, 650–738, 2008.

[59] Michael A. Shulman. Constructing symmetric monoidal bicategories. *arXiv e-prints*, page arXiv:1004.0993, April 2010.

[60] Carlos Simpson. *Homotopy theory of higher categories*, volume 19 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2012.

[61] Noah Snyder. Mednykh's formula via lattice topological quantum field theories. In *Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones' 60th birthday*, volume 46 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 389–398. Austral. Nat. Univ., Canberra, 2017.

[62] Germán Stefanich. Higher sheaf theory I: Correspondences. *arXiv e-prints*, page arXiv:2011.03027, November 2020.

[63] Ross Street. Categorical structures. In *Handbook of algebra, Vol. 1*, volume 1 of *Handb. Algebr.*, pages 529–577. Elsevier/North-Holland, Amsterdam, 1996.

[64] Zouhair Tamsamani. Sur des notions de n-catégorie et n-groupoïde non strictes via des ensembles multi-simpliciaux. *K-Theory*, 16(1):51–99, 1999.

[65] Bertrand Toen. Dualité de tannaka supérieure i: Structures monoidales, June 2000. Available on the webpage `https://perso.math.univ-toulouse.fr/btoen/files/2015/02/tan.pdf`.

[66] Angelo Vistoli. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005.

[67] Linde Wester Hansen and Michael Shulman. Constructing symmetric monoidal bicategories functorially. *arXiv e-prints*, page arXiv:1910.09240, October 2019.

[68] J. H. C. Whitehead. Combinatorial homotopy. II. *Bull. Amer. Math. Soc.*, 55:453–496, 1949.

[69] Simon Willerton. The twisted Drinfeld double of a finite group via gerbes and finite groupoids. *Algebr. Geom. Topol.*, 8(3):1419–1457, 2008.

[70] Edward Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.

[71] Qiaochu Yuan. Surfaces and the representation theory of finite groups. Available on the webpage `https://math.berkeley.edu/~qchu/TQFT.pdf`.