Principle of Holography of Information and Asymptotic Symmetries

A thesis
submitted to the

Tata Institute of Fundamental Research
Mumbai, India - 400005

for the degree of

Doctor of Philosophy
in Physics

by

Chandramouli Chowdhury

(Advisor: Suvrat Raju)

International Centre for Theoretical Sciences
Tata Institute of Fundamental Research
Bengaluru, India - 560089

September, 2023

(Final Version after Viva Submitted in June, 2024)

DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgment of collaborative research and discussions.

The work was done under the guidance of Professor Suvrat Raju at the Tata Institute of Fundamental Research.

Chandramouli Chowdhury

Date: 10 June 2024

In my capacity as the formal supervisor of record of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

(Suvrat Raju)

Date: 12 June 2024

List of publications

List of Publications during PhD towards thesis (2018 - present)

1. Recovering information in an asymptotically flat spacetime in quantum gravity *with* Olga Papadoulaki (Perimeter Institute). **Arxiv**: 2203.07449. **Journal**: Classical and Quantum Gravity - IOP Science.

2. The Asymptotic Structure of Gravity in Higher Even Dimensions *with* Ruchira Mishra (U. Chicago) and Siddharth Prabhu (TIFR, Mumbai). **Arxiv**: 2201.07813. **Journal**: Under review.

3. Holography from the Wheeler-DeWitt equation *with* Victor Godet (ICTS, Bangalore), Olga Papadoulaki (ICTP, Trieste) and Suvrat Raju (ICTS, Bangalore). **Arxiv**: 2107.14802. **Journal**: JHEP.

4. A physical protocol for observers near the boundary to obtain bulk information in quantum gravity *with* Olga Papadoulaki (ICTP, Trieste) and Suvrat Raju (ICTS, Bangalore). **Arxiv**: 2008.01740. **Journal**: SciPost Phys.

List of Publications during PhD apart from thesis (2018 - present)

1. Revisiting Logarithmic Correction to Five Dimensional BPS Black Hole Entropy *with* A. H. Anupam, Ashoke Sen (ICTS, Bangalore). **Arxiv**: 2308.00038.

2. Logarithmic Correction to BPS Black Hole Entropy from Supersymmetric Index at Finite Temperature *with* A. H. Anupam, P. V. Athira, Ashoke Sen (ICTS, Bangalore). **Arxiv**: 2306.07322.

3. Analytic Results for Loop-Level Momentum Space Witten Diagrams *with* Kajal Singh (HRI, India). **Arxiv**: 2305.18529.

4. Generalized BMS algebra in higher even dimensions *with* Anupam A. H. (ICTS, Bangalore), Arpan Kundu (IMSc, Chennai). **Arxiv**: 2209.06839. **Journal**: Phys. Rev. D.

5. On loop celestial amplitudes for gauge theory and gravity *with* Soner Albayrak (Yale U.) and Savan Kharel (Williams Coll. and Chicago U.). **Arxiv**: 2007.09338. **Journal**: Phys. Rev. D.

6. Study of momentum space scalar amplitudes in AdS spacetime *with* Soner Albayrak (Yale U.) and Savan Kharel (Williams Coll.). **Arxiv**: 2001.06777. **Journal**: Phys. Rev. D.

7. New relation for Witten diagrams *with* Soner Albayrak (Yale U. and Caltech) and Savan Kharel (Yale U. and Williams Coll.). **Arxiv**: 1904.10043. **Journal**: JHEP.

List of Publications related to UG projects (2014 - 2018)

1. Conformal vacuum and the fluctuation-dissipation theorem in a de Sitter universe and black hole spacetimes *with* Ashmita Das (IIT Guwahati), Surojit Dalui (IIT Guwahati) and Bibhas Ranjan Majhi (IIT Guwahati). **Arxiv**: 1902.03735. **Journal**: Phys.Rev.D.

2. Spectral Representation of Thermal OTO Correlators *with* Soumyadeep Chaudhuri (ICTS, Bangalore) and R. Loganayagam (ICTS, Bangalore). **Arxiv**: 1810.03118. **Journal**: JHEP.

3. Unruh-DeWitt detector in the presence of multiple scalar fields: A toy model *with* Ashmita Das (IIT Guwahati) and Bibhas Ranjan Majhi (IIT Guwahati). **Arxiv**: 1711.05103. **Journal**: Eur. Phys. J. Plus.

4. Fluctuation-dissipation relation in accelerated frames *with* Ananya Adhikari (IIT Guwahati), Krishnakanta Bhattacharya (IIT Guwahati) and Bibhas Ranjan Majhi (IIT Guwahati). **Arxiv**: 1707.01333. **Journal**: Phys. Rev. D.

Acknowledgements

I would take this opportunity to thank several people who have either directly or indirectly been associated with this thesis.

First and foremost, I would like to thank my advisor and collborator Suvrat Raju for his constant support, patience and encouragement from day 1. His ability to see through the chaos and arrive at a conclusion via physical insight has always amazed me and conversations with him about various subjects in physics have helped me understand something with a new perspective. I greatly acknowledge the impact he has had on me as a physicist and a person and also on several sections of this thesis. Apart from being a great physicist and a wonderful person, he is a very competitive squash player and I will miss our matches at the court.

I have also been very lucky to meet several people at the right time, with one such person being Alok Laddha. Everything that I know about the subject of Asymptotic Symmetries is owed to him and despite having never written a paper together I have learned a lot by conversing with him at various times. He has been a great friend over the years and has helped me get out of the canonical grad school depression at various stages.

Right from the very start I had been interested in the subject of scattering amplitudes and I would like to thank Savan Kharel for being extremely patient with all the dumb questions I had about the subject and also on collaborating on a number of papers. Through this, I have also interacted with Soner Albayrak whose abilities with mathematica have really impressed me and I enjoyed our collaborations very much.

Towards the end of my journey at ICTS, I wanted to start exploring some directions away from holography and I was very lucky to meet and start working with Ashoke Sen, on the subject of logarithmic corrections to the black hole entropy, about a year before I left. I really admire his way of thinking about problems and his exceptional computational abilities have made me realize the importance of working through every step of a calculation and building physical insights through calculating.

I have spent a significant time in the last couple of years of my stay working with Anupam A. H. on several papers and I really enjoyed our collaborations and arguments regarding minus signs and factors of 2! I acknowledge him for his patience and deeply respect his ideologies about several things in life and politics.

I have also had the privilege of working with several other collaborators and in particular I would like to thank Olga, Ruchira, Siddharth, Arpan, Kajal, and Victor for playing a major role in all papers I have written during the last 5 years.

Given that ICTS is a small institute you end up interacting with nearly every one and in particular I would like to thank the members of the string theory group from whom I have learned a lot and also shared a good friendship over the last 5 years, namely: Priyadarshi, Tuneer, Joydeep, Akhil, Omkar, Shivam, Pushkal, Soumyadeep, Sudip, Ashik H., Godwin, Kaustabh, Nava, Pronobesh, Loga and many others! I would also like to thank my office mates for several gaming sessions (outside the office of course!) and hilarious conversations to keep the spirits up: Aditya Sharma, Rajarshi and Saurav. I would also like thank my other friends on campus, Saikat, Jeetu, Shrashti, Saumav, Bikram, Shridhar, Nirnoy, Divya, Junaid, Aditya, Devang, for making my stay at ICTS very memorable and pleasant. I would also like to thank Rajarshi, Sugan, Bhanu, Andrew, Anup, Manpreet for playing squash. I would also like to thank Abhijit Gadde and Prayush Kumar for being on my thesis monitoring committee and for their useful feedback from time to time.

I would like to thank several staff members at ICTS for making my stay productive and all the other administrative hassles easy. In particular Balaji, Raju, Naga and Bharath for the wonderful canteen service, Nidhi, Pavana, Shalini and Veena for their assistance regarding any academic matter, Jeeva, Madhulika, Ramya for all the admin and grant-related issues and especially Suresh, Mahindra and the sports, maintenance and security staff at the institute and the hostels. Without them, it would be very difficult to do physics smoothly on a day-to-day basis.

I would also like to thank my friends outside ICTS and academia Chandan, Alok, Manaswi, Mohit for helping me connect with the outside world.

Finally, I would like to thank my parents for their support and acknowledge all the sacrifices they have made in order for me to be where I am now.

Abstract

In these works, we explore the unusual nature in which gravity localizes information. We draw inspiration from older work on this subject and work in a regime where effective field theory is valid. We show how it is possible for observers to decode information by staying near the boundary of a Cauchy slice in both AdS and flat spacetime. This is known as the principle of holography of information. We construct explicit physical protocols by which observers at the boundary of AdS and flat spacetime can interact with a state, measure its energy, and then reconstruct the state using this data. We emphasize the necessity of gravity in this entire procedure and argue why the construction of such protocols is forbidden in other quantum field theories. We also demonstrate how solutions of the gravitational constraint equations satisfy the principle of holography of information. Although there are similar constraints present in gauge theories, we show how they fail to give rise to a similar structure.

In another part of the thesis, we studied the symmetries of non-linear general relativity for flat spacetime in even dimensions near null infinity. This analysis is necessary for understanding the phase space of general relativity near null infinity and also allows one to understand how things like the memory effect etc. generalize to the non-linear theory. Although most of our formulas are explicitly valid in six dimensions, they are easily extendable to all even dimensions.

Contents

Chapter 1

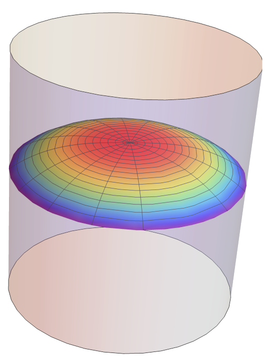
Introduction

Understanding the properties of quantum gravity has been an outstanding problem in modern physics and is still not understood in its full glory. While there have been several approaches in attacking the problem, most notably string theory, there are still some corners that are ill-understood. With the remarkable discovery of the gauge/gravity duality [1], also known as the AdS/CFT correspondence or the holographic duality, our understanding of the interplay between gravitational and gauge theories has been revolutionized. The AdS/CFT correspondence states that gravitational theories in higher-dimensional Anti-de Sitter (AdS) spacetime are dual to certain strongly coupled conformal field theories (CFTs) living on the boundary of AdS. This is a concrete example demonstrating the behavior of gauge theories at strong coupling. An implication of this duality is known as the *holographic principle*. This posits that the information available in the bulk of spacetime can be encoded in its boundary. Via the AdS/CFT correspondence, we then have the statement that the information stored in the bulk of AdS is also available in the CFT living at its boundary. The duality, however, is more powerful and it also maps the operators of quantum gravity in the bulk with the operators of the boundary CFT. There is a dictionary relating the operators on both sides and using that one can compute and compare correlation functions on both sides. This has several implications, such as allowing one to study the properties of quantum gravity by using the tools of quantum field theory. This has enabled one to understand several properties of quantum gravity that were unknown before. These include the notion of entanglement entropy, black hole physics, etc.

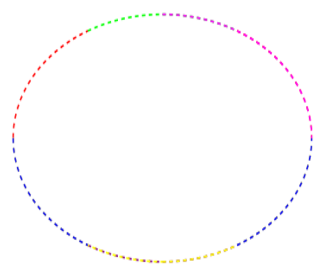
In this thesis, we explore the role of gravity in the holographic principle.

In all known examples of the duality the theory of gravity on the quantum side typically arises from a string theory and therefore it is natural to ask if the holographic principle relies on string theory or if is a more general property of quantum gravity. In particular, we find that one aspect of the holographic principle, known as the *principle of holography of information* only relies on the presence of gravity in the bulk and does not directly require any stringy properties [2]. This principle states that for any theory of quantum gravity in AdS or flat spacetime, all information that is available in the bulk of a Cauchy slice is also available at its boundary.

For theories of quantum gravity in AdS, this is a fairly well-understood statement as it has been one of the central implications of the AdS/CFT correspondence [1, 3]. A manifestation of this duality is the principle of holography of information. The principle of holography of information is a weaker statement by itself as compared to the AdS/CFT correspondence but also relies on weaker assumptions and not on any specific UV completion of gravity. However, it is still an implication of the fact that a theory of quantum gravity in AdS is holographic [4]. This principle can be summarized in a diagrammatic form as done in figure 1.1.



(a) Information on a Cauchy slice in the bulk



(b) Holographic storage of the information at the boundary of the Cauchy slice.

Figure 1.1: The cartoon figure shows how all information present in the bulk Cauchy slice (left) is also present at its boundary (right). On the left figure, we have a disk whose profile takes the form of a rainbow and on the right, we show how that rainbow gets smeared on the boundary of the Cauchy slice.

This version of holography also exists in flat spacetime where it was argued by Laddha, et. al in [5] that all information about massless excitations in flat spacetime is available near the past of future null infinity $\mathcal{I}^+_-$ or the future of past null infinity $\mathcal{I}^-_+$. This means that in a theory of gravity, all information that is available in the bulk of a Cauchy slice is also available at the boundary of the slice (see figure 1.2).

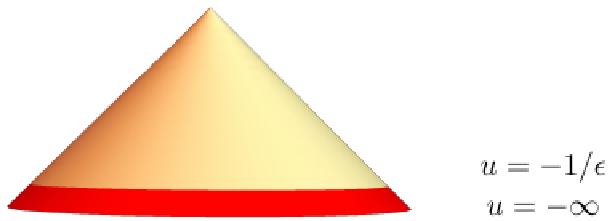


Figure 1.2: Holographic nature of $\mathcal{I}^+$.

This also has an interesting consequence for the black hole information paradox in flat spacetime. The most common version of the information paradox states that there is no way to recover the exact information present inside a black hole[1]. This is because we know via Hawking [7] that a black hole in flat spacetime will radiate energy in the form of Hawking radiation, which is thermal in nature. Therefore all information about what fell into the black hole disappears once the black hole has completely radiated which would be in conflict with expectations about unitary evolution in quantum mechanics. There have been several approaches that try to diagnose this paradox in clever ways and we refer the reader to [2] for a discussion on them. The paradox appears because of the incorrect assumption that the Hilbert space in quantum gravity factorizes in the same way it does in an ordinary QFT. However, as first argued in [5], by carefully analyzing states in a theory of quantum gravity it is found that all information about a system is already present at a cut near null infinity. Therefore, there is no notion of information going into a black hole as it was already present at the boundary. Hence the principle of holography of information provides a new resolution to the black hole information paradox.

It is a natural question to ask whether such a principle also exists in theories without gravity. As discussed in the literature [8], the presence of *split*

[1]For recent discussions related to this via the island approach in alternative theories of gravity, we refer the reader to [6].

states makes it impossible for other theories to be holographic. Physically, the main obstruction to this principle is the presence of local gauge invariant operators in theories without dynamical gravity. In particular, for gauge theories, this is related to the presence of negative charges, which are absent in gravity (since the only charge in gravity, energy, is positive in nature).

In both scalar and gauge theories in the absence of dynamical gravity, it is possible to construct operators in the bulk of spacetime that commute with *every* operator on the boundary. This implies that it is impossible for a boundary observer at a time slice to distinguish between two states in the bulk that differ by the action of a local unitary operator. Such obstructions are not possible in a theory with dynamical gravity as it is not possible to completely localize an operator. This is related to the uncertainty principle, which states that in order to localize a state in position space, one needs to delocalize it in momentum space. Since energy (which is the charge in gravity) is a component of the momentum, it is therefore impossible to completely localize a state that has finite energy. Therefore, the absence of an uncertainty principle for other charges (for example, the electrical charge in QED) allows one to create localized excitations in a theory without dynamical gravity.

In the works included in this thesis (see chapters 2, 3 and 4), we avoid any reference to a particular choice of a theory of quantum gravity and build our ideas in effective field theory. Thus, the regime we explore only allows for perturbative excitations on a given state. We then show that there are certain properties satisfied by quantum gravity at low energies that allow one to decode a state by making measurements at the boundary of a time slice. These properties are listed below,

1. Since any finite energy excitation in the bulk must leave an imprint at the boundary due to the uncertainty principle, this implies that there are no local gauge invariant operators in gravity.

2. The Hamiltonian of the theory is a boundary term and can be expressed in terms of the metric fluctuations at the boundary.

3. In effective field theory we can explicitly compute the Hamiltonian and show that it is a positive operator as it is quadratic in the degrees of freedom [9].

In [9, 10, 11] we have mostly studied states which lie in the vacuum sector of the theory, i.e., they are built by acting with low-energy operators on a vacuum state. For states in this sector, it can be shown that by performing certain measurements at the boundary of a Cauchy slice, observers can obtain complete information about the state in the bulk [10]. The observers measure the energy of the state by measuring the gravitational flux at the boundary, so the presence of gravity is crucial in this analysis. We can demonstrate this process with some simple equations. To aid our discussion we refer to the figure below to show the location of the observers

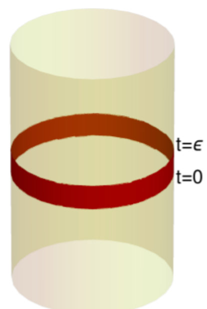


Figure 1.3: Location of observers in global AdS.

Physical Protocols for Decoding Information

As local descriptions of the bulk are not gauge invariant, it is often convenient to describe the state in terms of observables located at the boundary. For doing this we use the fact that any local operator in the bulk can be expressed in terms of boundary operator using the HKLL prescription. This is done by smearing the local operator in the bulk with a kernel having support in the time interval $[0, \pi]$ (which is the light crossing time in global AdS in units where the AdS length is set to 1). This allows one to define a Hilbert space in terms of gauge invariant boundary operators [4, 5]. Thus, any generic state on the Cauchy slice can be expressed by the action of an operator on the vacuum state smeared with a function at the boundary, $f(t, \Omega)$. Therefore, one can express a generic state in terms of some convenient basis operators smeared with such functions. A simple example of this is the construction of a state by the action of a single trace operator $O(t, \Omega)$ on the vacuum state,

$$|\psi\rangle = \int_0^\pi dt \int d\Omega\, O(t, \Omega) f(t, \Omega) |0\rangle \,. \tag{1.1}$$

This is just a particular choice of labeling the state and many other choices are possible. Another convenient choice for labeling the state is to use energy eigenstates as done in [10]. The role of the observers located in the thin time band $[0, \epsilon]$ (described by the red region in figure 1.3) is to measure the function $f(t, \Omega)$ by performing physically motivated measurements[2]. By physically motivated measurement we refer to actions performed by the observers that do not violate unitarity and causality. More concretely the observers are given two kinds of powers. One, to interact with the state and act on it with unitaries (comprising of operators smeared in the thin band $[0, \epsilon]$) and the other, measuring the energy of the system[3]. In [10] we have shown how given some prior about the state, which in general has a more complicated structure than (1.1), these two powers are sufficient for the observers to completely decode the state.

There exists a similar protocol for making measurements in flat spacetime near null infinity. This procedure utilizes the description of the Hilbert space in terms of asymptotic observables near null infinity as described in [5] and the protocol in AdS described above [11]. Once again the observers have to be given two kinds of powers to interact with the state and following very similar prescriptions they can completely decode the information content of the state. The crucial difference between the flat space example and the AdS example is that one should be careful about the basis in which the state is parametrized. In the AdS case, it is natural to choose the basis state as those of energy eigenstates, as they provide a discrete basis for parametrizing fluctuations. However, it is not possible to extend that directly to flat spacetime as energy is not discrete. To circumvent this issue, we introduced a discrete basis, which is similar to the usual Fock space basis, at null infinity and then described how one can use physical protocols to decode the state completely. This version of the protocol only works for massless particles in flat spacetime and it is not yet clear how to extend this to massive particles [11].

[2]By relaxing this requirement, it is also possible to show that the function $f(t, \Omega)$ can be measured by evaluating correlation functions of the form $\langle\psi|HO(t_0, \Omega)|\psi\rangle$, with $t_0 \in [0, \epsilon]$.

[3]Note that this measurement has to be performed in a quantum mechanical manner. We emphasize that this is not a measurement that is possible in the current universe and would require the presence of meta-observers who span over the boundary at a given time.

Holography from Wheeler-DeWitt equation

In another project, we have demonstrated how the principle of holography is applicable for any state that solves the Wheeler-DeWitt equation perturbatively in G_N [9]. The Wheeler-DeWitt equation is a key equation in quantum gravity that describes the quantum state of the universe as a whole, rather than just the behavior of individual particles. Solving the Wheeler-DeWitt equation is often viewed as a complementary approach to quantum gravity, in which the gravitational excitations are not thought of as strings but as solutions to a particular wave equation. This equation imposes the gravitational constraints (often known as the *Gauss law*) as a quantum equation,

$$\mathcal{H}\Psi = 0 \tag{1.2}$$

where Ψ is the wave functional of the system containing information about the gravitational and matter degrees of freedom and $\mathcal{H}$ is the Hamiltonian constraint (viewed as an operator). In our work, we have solved this equation (along with the momentum constraint $\mathcal{H}_i\Psi = 0$) perturbatively in G_N. The principle of holography of information in this context is the statement that if two solutions of these equations are the same near the boundary of AdS, then they must be the same everywhere. By two wave functionals being the same near the boundary, we simply mean that all correlation functions consisting of operators near the boundary (including the Energy operator) should be equal. In our work we show how these solutions satisfy the principle of holography of information in AdS, thereby resting the principle (for any theory of quantum gravity) on a firmer footing.

Holography of Information and Asymptotic Symmetries

Having established the principle of information holography in flat space in four dimensions [5], a natural question was on how this could be extended to higher dimensions. This led us to study the phase space of general relativity in flat spacetime in higher (even) dimensions in greater detail. This falls under the study of *asymptotic symmetries*. These are symmetries typically corresponding to two kinds of transformations, namely, *supertranslations* and *superrotations*. We first review these and then state how our results are connected.

Starting with the seminal work of Bondi-Metzner-Sachs (BMS) and van der Burg in the 1960's, we have learnt a great deal about the asymptotic structure of gravity in four dimensions [12, 13, 14, 15]. While originally studied because of its inherent importance in understanding the nature of gravity, the study of asymptotic symmetries is crucial in formulating the principle of holography of information in asymptotically flat spacetimes. In recent years, the work by Strominger and collaborators [16, 17, 18] has been of central importance for reviving the interest in the study of asymptotic symmetries, and also for uncovering a remarkable connection between asymptotic symmetries, soft theorems and the memory effect.

A new kind of symmetry that was obtained by analyzing the asymptotic structure of gravity is known as *Supertranslations*. These are a type of symmetry transformation that acts at null infinity in flat spacetime. They were first introduced Bondi, Metzner and Sachs in the 1960s as the *BMS group*. The BMS group contains the usual boosts, rotation, and translations along with an infinite-dimensional subgroup called supertanslation subgroup [13]. This describes the asymptotic symmetry group of flat spacetime in the far field region of an isolated gravitating system.

Supertranslations are angle-dependent translations at null infinity and can be seen as a generalization of ordinary translations. They can be thought of as functions on the *celestial sphere*, which parametrizes null infinity[4]. They can also be viewed as an extension of ordinary translations to the asymptotic region of spacetime. They are transformations that shift the retarded time at null infinity by a finite amount, and they are generated by a set of vector fields that satisfy certain algebraic relations, known as supertranslation algebra.

They also have an important phenomenological relevance which was uncovered via the work of Strominger, et. al [18]. They are connected with the gravitational soft theorem, which relates the behavior of scattering amplitudes in the limit of soft particle emissions to the asymptotic symmetries of spacetime, including supertranslations. Hence they play an important role in the study of gravitational radiation, as they can affect the scattering amplitudes of massless particles in the far-field region. Specifically, they are related to soft radiation, which corresponds to particles that are emitted or absorbed at null infinity with zero energy or momentum. It was found that the S-matrix is invariant under supertranslations and that this symmetry leads to a set of Ward identities that constrain the form of the S-matrix [18].

Along with supertranslations, there is another kind of extension allowed

[4] In the usual notation of the flat spacetime, the Celestial sphere represents the $\mathbb{S}^2$ sphere parameterized by (θ, ϕ) .

for the BMS group at null infinity. While supertranslations can be viewed as a generalization of the ordinary translations at null infinity, it is also possible to generalize Lorentz transformations. These are known as *superrotations*. Superrotations are a type of symmetry transformation that is similar to supertranslations but involve rotations of the celestial sphere at null infinity in addition to supertranslations [19]. Later, it was also shown that they were connected to gravitational scattering amplitudes in four-dimensional spacetime, where they were found to be related to soft graviton emission [20, 21].

More specifically, superrotations are transformations that involve parametrizing the null generators of null infinity, while preserving their intersection angles. In other words, they involve rotating the directions in which null rays are pointing at null infinity while keeping the overall structure of the null cones fixed. Superrotations are characterized by an infinite number of functions that parameterize the rotation angles at each point on the celestial sphere. They allow for diffeomorphisms of the celestial sphere and can be viewed as a generalization of the Lorentz transformation at null infinity. Superrotations have been shown to be an infinite-dimensional symmetry group of the S-matrix in gravitational scattering amplitudes, just like supertranslations. They are related to the subleading soft graviton emission. The study of superrotations has led to new insights into the structure of scattering amplitudes, as well as connections between gravitational scattering and other areas of physics, such as conformal field theory and holography.

Analyzing the structure of the Hilbert space in the presence of supertranslations was played an important role in demonstrating the principle of holography of information for massless particles in 4-dimensional flat space [5]. Motivated by this, we wanted to understand how this result generalizes to higher dimensions and this required a thorough analysis of the phase space of non-linear general relativity in higher dimensions.

The first step towards this was taken in [22] where the authors analyzed the symmetries of null infinity in higher even dimensions for linearized general relativity. In our work [23, 24], we have studied symmetries of the full non-linear theory in (even) dimensions greater than four near null infinity and constructed the charges corresponding to supertranslations and superrotations. One of the central results of our work is that we identify the correct variables to expand the gravitational fluctuations in the presence of supertranslations and superrotations and evaluate the corresponding conserved charges.

In [23], we discuss supertranslations in (even) higher dimensions, specifically focusing on the six-dimensional case. We first specify the free data of the theory, which are given by the first two subleading coefficients in the large-r expansion of g_{ab} (the metric of the Celestial sphere). The graviton is defined by a combination of this free data (which are the subleading and sub-subleading components in r of g_{ab}, denoted by C_{ab} and D_{ab}), which is denoted by $\tilde{D}_{ab}$. This redefined field $\tilde{D}_{ab}$ contains the radiative data,

$$\tilde{D}_{ab} = D_{ab} - \frac{1}{4} C^c_a C_{bc} - \frac{1}{16} q_{ab} C_{cd} C^{cd} \ . \tag{1.3}$$

This redefinition does not affect the News tensor ($\partial_u D_{ab}$), but is necessary for the graviton to have the correct asymptotic fall-offs and further, for the theory to admit a finite symplectic form in the full non-linear theory. This redefinition is also required even if one is studying the linearized theory about a non-trivial Minkowski background ($C_{ab} \neq 0$). In quantum theory, this implies that the graviton gets redefined depending on the vacuum one is working with.

We have also computed the supertranslation charge using the covariant phase space formalism [23]. For this, we first evaluate the symplectic form of the theory and find that the symplectic form uniquely splits into two parts: one which is finite and characterizes the radiation of the system, and the other, carrying information about the vacuum structure of the theory, which is divergent (when integrated along $\mathcal{I}^+$). As the name suggests, the entire radiation content of the theory is contained in the finite & radiative part. The split is unambiguous and the ambiguity is fixed by noting that only the radiative part leads to the correct Noether charge (as implied from the soft theorems) and also from a spacetime variational analysis. In fact, upon trying to construct the charge from the divergent piece, we notice that it cannot be expressed as a total variation[5]. Therefore, the radiative symplectic form at null infinity is uniquely defined and it also helps us understand the canonically conjugate variables in the theory. As we point out, the symplectic form in the non-linear theory is a simple generalization of known results [26], and therefore the Noether charge can be obtained by following similar steps. We also compute the Bondi mass and evaluate the supertranslation charge using that. We find an exact matching between the two expressions – the one via the radiative symplectic form and the one via the Bondi mass. The Noether charge in the full non-linear theory is a simple generalization of the result in

[5]Recent work in [25] has discussed this issue in greater detail.

the linearized theory about $C_{ab} = 0$. Upon combining with the results in [11] it is easy to see that we end up getting the expected Ward identity and therefore, the same structure for the Weinberg Soft theorem. As a toy model, we have also studied QED in higher dimensions and demonstrated how the gravitational case can be thought of as a generalization of those results. Given the similarity between the structure of the phase space constructed out of the correct variables, and that of the phase space in 4-dimensions, the entire analysis of [5] can be trivially generalized to that of higher even dimensions which naturally establishes the principle of holography of information in all even dimensions.

In more recent work[6], we explored the connections between the subleading soft theorems and asymptotic symmetries, enabling us to construct the superrotation charge in higher dimensions [24]. This subject is also interesting from the point of view of flat space holography, as it serves as a guiding principle on the correct phase space variables one should be using in higher dimensions. Like in the case of supertranslations, we find that for defining superrotations in higher dimensions, we need a redefinition of the radiative data, similar to the equation given in equation (1.3). In terms of the redefined variables it is possible to construct the superrotation charge and also relate this to the subleading soft theorem.

The thesis is organized as follows: In chapters 2 and 3, we demonstrate the physical protocols for detecting states in AdS and flat space-time respectively. In chapter 4 we show how the principle of holography of information follows from the solutions of the Wheeler-DeWitt equation. In chapter 5 we shift gears and analyze asymptotic symmetries in higher even dimensions. We add some appendices that aid the analysis in the main chapters.

[6]The paper [24] is present in the thesis of my collaborator (Arpan Kundu) who is a graduate student at IMSC, Chennai, at the time of writing this thesis and is not to be taken into account for my thesis. Its details are only stated in the introduction in brevity for completion.

Chapter 2

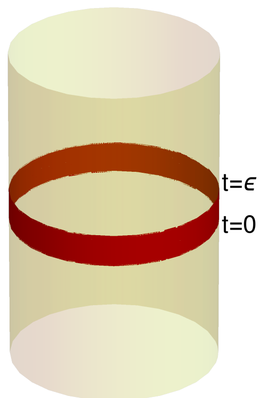
Recovering Information in AdS

2.1 Introduction

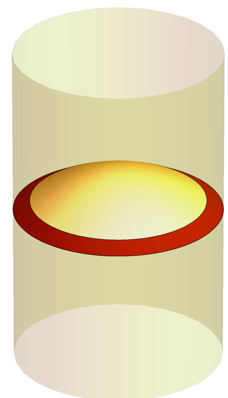
A recent paper [5], which extended previous ideas [4, 27, 28, 29], argued that a careful consideration of low-energy quantum gravity already suggests that information about the interior of a spacetime can be obtained from measurements near its boundary. This result holds both in asymptotically AdS spacetimes and in asymptotically flat spacetimes. In this chapter, we present a concrete physical protocol that allows observers near the boundary of global AdS to extract information about the bulk, when the system is in a low-energy state.

Consider a set of observers who live near the boundary of global AdS. The observers are spread out all over the sphere near the boundary. However, they are equipped with detectors that function only in the time-band $[0, \epsilon]$, and so they cannot make any measurements beyond this time-interval. The geometry of our setup is shown in Figure 2.1a. Restricting the observers to a time-band near the boundary provides a precise version of a physically-similar picture, shown in Figure 2.1b, where one allows the observers to explore the spacetime on a single time-slice but not enter the bulk region for $r < \cot(\frac{\epsilon}{2})$. We explain the setup in more detail in section 2.2.

The observers live in a pure state of the theory that is well-described by an excitation of quantum fields above the global AdS vacuum. We would like to understand how much information such observers can glean about this excited bulk state. We will follow the textbook framework used in discussions of quantum information. We assume that the observers have access to a



(a)



(b)

Figure 2.1: On the left, we have the precise setup of this chapter. The observers can make manipulations and measurements in a small time band (shaded in red) near the boundary of AdS. This is physically similar to the picture on the right, where the observers can explore an annular region near the boundary of a Cauchy slice that runs through the bulk.

number of identically prepared systems. The observers can manipulate each system by acting with local unitary operators, and make measurements of Hermitian operators in their region of spacetime. They can also classically communicate the results of their measurements to one-another.

In a local quantum field theory, it is clear that the observers cannot determine much about the bulk state since a large part of the spacetime is just inaccessible to them. It is commonly believed that leading gravitational effects should not modify this conclusion significantly, i.e., a common belief is that in the presence of weak gravity, unless the observers do something that is extraordinarily complicated, they should still not be able to learn much about the bulk state.

But, in this chapter, we would like to present a surprising conclusion. When leading gravitational interactions are switched on, even if the observers are allowed to act only with simple unitaries, and make only simple measurements, it is possible to develop a protocol that allows the observers to completely determine the state of bulk quantum fields.

For the impatient reader, we immediately explain the main technical idea. First, consider the simplest example. Say that the observers are given the task of verifying whether the state in the bulk — which we denote by $|g\rangle$ and normalize using $\langle g|g\rangle = 1$ — is the global AdS vacuum, which we denote by $|0\rangle$, or some other state. In a local quantum field theory, this task is impossible: in a local QFT, the states $|0\rangle$ and $U_{\text{bulk}}|0\rangle$, where U_{bulk} is a unitary operator localized near the middle of AdS on some time slice, are completely indistinguishable if one is restricted to observations near the boundary on the same time slice. But, in the presence of gravity, the observers can use gravitational effects to measure the Hamiltonian H, which is just given by integrating a particular component of the metric on the boundary sphere. In global AdS, the possible answers for H are quantized. We assume that the observers can measure H with sufficient accuracy to distinguish the first excited state in global AdS from the vacuum. Since H is an operator in the quantum theory, by the standard rules of quantum mechanics, the observers can get different possible results for their measurement. For this task, all they need to do is to determine the relative frequency with which their measurement yields 0. By the Born rule, this is given by $\langle g|P_0|g\rangle$ where $P_0 = |0\rangle\langle 0|$ is the projector onto the vacuum. If this is 1, they know the global state is the vacuum, and otherwise it is not.

The skeptic might feel that this is a special case since the vacuum is uniquely identified by a conserved charge. So we now give the observers a harder task. Let X be some simple Hermitian operator accessible to the observers, and consider the state $|X\rangle = X|0\rangle$. The observers are now given the task of determining if the global state $|g\rangle$ coincides with $|X\rangle$. Once again, in a local quantum field theory, this task is impossible. The observers cannot distinguish between $|X\rangle$ and $U_{\text{bulk}}|X\rangle$. But this task can be accomplished using gravitational effects.

For simplicity, let us assume that the observers determine, using the procedure above, that $\langle 0|g\rangle = 0$. Then they can complete their task using a two step process.

1. First, they act on the state with a one-parameter family of unitary operators, $U = e^{iJX}$, for various values of J, near $J = 0$.

2. After this manipulation, they measure the energy, and determine the relative-frequency with which they obtain 0, to second order in J.

A very simple calculation yields the expected answer. Expanding the unitary operator to second order in J,

$$\langle g|e^{-iJX}P_0e^{iJX}|g\rangle = \langle g|(1-iJX-\frac{1}{2}J^2X^2)P_0(1+iJX-\frac{1}{2}J^2X^2)|g\rangle + \mathrm{O}\big(J^3\big)\,. \tag{2.1}$$

Using the fact that $P_0 = |0\rangle\langle 0|$ and that $\langle 0|g\rangle = 0$ and also that $X|0\rangle = |X\rangle$ we see that

$$\langle g|e^{-iJX}P_0e^{iJX}|g\rangle = J^2|\langle g|X\rangle|^2 + \mathrm{O}\big(J^3\big)\,. \tag{2.2}$$

So the protocol used by the observers directly yields the magnitude of the overlap of $|g\rangle$ with $|X\rangle$. If this is $\langle X|X\rangle^{\frac{1}{2}}$, the observers know the two states are the same, and they differ otherwise.

The protocol that we outline in section 2.3 is just a souped-up version of the two examples that we provided above. Due to the entanglement present in the vacuum, it turns out that states of the form $X|0\rangle$, where X is a low-energy Hermitian operator, near the boundary generate a basis for the entire low-energy Hilbert space. By repeating the procedure above for various choices of the operator X, we show that the observers can unambiguously identify the bulk state. The technical subtlety is simply that the Hilbert space is a complex vector space, whereas Hermitian operators form a real vector space; so, we need to do work to extract some phases from within the state. We also address other special cases, including the case where $\langle g|0\rangle \neq 0$ in section 2.3.

We emphasize that even in a local quantum field theory, operators of the form $X|0\rangle$ form a basis. But this property of the vacuum (called cyclicity) is of absolutely no use to near-boundary observers in their task of gleaning bulk information without the operator P_0, which they can access only in a theory of gravity. In fact, in section 2.4, we explain that without gravity, the near-boundary observers obtain exactly zero information about the state near the center of AdS. Even in a non-gravitational gauge theory, the so-called "split property" tells us that it is possible to arrange matters so that every observation made by the near-boundary observers yields the same result that it would have yielded in the vacuum state, even if the state near the center of the AdS is completely different. Likewise, if one retains gravity but takes the classical limit, the observers are again unable to discern details of the bulk state. (See section 3.3 of [5] and also the discussion in [30].)

In this chapter, we do not explore the implications of our analysis for black holes. But it appears to us that the effect above is closely tied to a key conclusion that has emerged from studies of black hole evaporation: the idea that degrees of freedom in the interior of the black-hole have a dual description in the exterior. In the context of discussions of the black hole information paradox a few years ago [31, 32], this point was emphasized in [33] and in parallel work [34, 35]. Subsequently, a similar point was also made in the ER=EPR proposal [36]. When one considers black holes in asymptotically flat space, an extrapolation of these arguments suggests [5] that information about the microstate can always be extracted by measurements outside the black hole. On the other hand, a different setup has also been studied in a number of recent papers [37, 38, 39, 40, 41] which consider black holes in AdS that evaporate into baths without dynamical gravity. Even here, precisely in line with previous expectations, the important physical point is that at late times the black-hole interior is described by degrees of freedom in the bath. Neglecting this identification leads to paradoxes, not only about black holes but even about empty AdS [42]. The analysis in this chapter and in [4] is relevant to this story because it provides a clear physical origin of these effects that identify degrees of freedom in one region with those in another region, in a Lorentzian setting, and without invoking any indirect arguments.

The results in this chapter also imply that when standard quantum information measures are applied to the geometry shown in Figure 2.1b, then the answers obtained after including the effects of dynamical gravity are very different from the answers without dynamical gravity. In particular, since information about the entire Cauchy slice is already present in the dark-red "annular region" of Figure 2.1b, the von Neumann entropy of that region is zero when the global state is pure. Similarly if one considers two different states, then the relative entropy of the states with respect to the "algebra" of the annular region is the same as the relative entropy with respect to the "algebra" on the full Cauchy slice.

In [43], it was suggested that usual notions of quantum information-localization could be recovered by considering only "simple operators". This was also the motivation behind the introduction of the "little Hilbert space" [44] or "code-subspace" [45]. But the surprising aspect of the analysis here is that it is performed entirely within the code subspace. Note that these comments are not in contradiction with the results of [46, 47] since the annular region of Figure 2.1b is not the entanglement wedge of any region on the boundary. We discuss this issue, and also some possible caveats in section 2.5.

The analysis presented here can be thought of as a perturbative check of the idea that holography [1, 3, 48] is implicit in gravity, even from a low-energy perspective. We focus on AdS to avoid some of infrared intricacies of flat space. But we will return to a consideration of flat space in the next chapter.

An outline of this chapter is as follows. We frame the problem precisely in section 2.2. Section 2.3 contains the central part of our analysis, and we explain how bulk information can be extracted from manipulations and measurements near the boundary. We explain in section 2.4 why this protocol fails in theories without gravity, and conclude with a discussion of some implications and subtleties in section 2.5. We are frequently asked whether our protocol would work in the presence of global symmetries, and so we include a special example showing how to deal with global symmetries in section 2.3.

2.2 Setup

In this section, we clearly outline our physical setting, the task that the near-boundary observers are given, and the precise powers that they have.

We will consider a spacetime that, asymptotically, tends to global AdS_{d+1}.

$$ds^2 \underset{r\to\infty}{\longrightarrow} -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-1}^2. \tag{2.3}$$

In our setup, gravity is weak, but it is dynamical. The AdS radius sets a natural energy scale and, in these units, the scale at which gravitational effects become strong is denoted by $N = G^{\frac{-1}{(d-1)}}$, where G is the low-energy Newton's constant. The physical assumption in our analysis is that the low-energy predictions obtained by straightforwardly quantizing gravity are not affected by UV-effects.

There may be additional dynamical fields in the theory, including string-theoretic excitations. We will assume for simplicity, as is standard in AdS/CFT discussions, that there is no hierarchy of interactions and all tree-level interactions are controlled by the parameter $\frac{1}{N}$, but this assumption can be relaxed as explained in Appendix A. The detailed field-content of the theory will not be important and, apart from the graviton, we will use massive scalars below to discuss the effect of dynamical fields. If ϕ is such a field of mass m, then we will consider normalizable excitations of this field with a boundary value

$$\phi(t, r, \Omega) \underset{r\to\infty}{\longrightarrow} \frac{1}{r^{\Delta}} O(t, \Omega), \tag{2.4}$$

where $\Delta = \frac{d}{2} + \sqrt{(\frac{d}{2})^2 + m^2}$. The operators $O(t, \Omega)$ restricted to the time-interval $t \in [0, \epsilon]$ are the natural observables in this setup.

The low-energy Hilbert space can be obtained by quantizing the dynamical fields. A standard analysis tells us that this Hilbert space contains a unique vacuum, $|0\rangle$, that is separated from the lowest excited state by a gap proportional to the AdS scale.

The global state that the observers are meant to probe is denoted by $|g\rangle$ throughout this chapter. This is a pure state. It is also a low-energy state in the following sense. We introduce a UV-scale Λ. This scale defines what we mean by "low energy" whenever we use the term below. Λ is a user-defined scale with the property that it is parametrically smaller than the Planck scale: $\Lambda \ll N$. Then we demand that

$$1 - \| P_{E<\Lambda} |g\rangle \|^2 \ll 1, \tag{2.5}$$

where $P_{E<\Lambda}$ is the projector onto states with energy lower than Λ. Note that the condition above allows the the state to have some small components of high-energy. Such high-energy tails are invariably generated by the action of local unitaries. But these tails will not be of interest to us.

2.2.1 Task of the observers

The observers have a simple task. They need to determine the global state $|g\rangle$ up to a given accuracy. More precisely, the observers are challenged with identifying a state $|g_{\text{est}}\rangle$ such that

$$1 - |\langle g_{\text{est}} | g\rangle|^2 \leq \delta, \tag{2.6}$$

where δ is a parameter with the property that $\delta \ll 1$ but $\delta \gg \frac{1}{N}$. This means that we require the observers to determine the state to a high-level of accuracy, but not such a high-level that the accuracy competes with the ratio of the cosmological scale to the Planck scale.

Of course, if the observers are allowed to directly probe the bulk, this is a straightforward task. But, as we describe below, the observers are restricted to a near-boundary region and it is only by using uniquely gravitational effects that they will be able to perform this task.

2.2.2 Abilities and limitations of the observers

The observers are given access to a number of identically prepared systems, all in the state $|g\rangle$. They are allowed to conduct multiple experiments and collate the results of these experiments in order to identify the close-enough state $|g_{\text{est}}\rangle$. One way to envision the setup is to think of a larger spacetime, in which local patches are well approximated by global AdS. These local patches are then prepared in identical states. The observers make measurements in each local patch, and then travel across the larger spacetime to collate the measurements from different patches.

We emphasize that the need for identical copies is not special to our protocol but is a very basic requirement in any quantum-information analysis. Since measurements are probabilistic, a single copy of the system cannot be used to determine its state. In particular, even bulk correlation functions, which are expectation values of products of operators, can only be measured by averaging the results of measurements in identical systems. Therefore even if the observers were to explore the bulk to obtain information, and not use our protocol at all, they would still require multiple identical copies in order to be able to fix the bulk state.

The interesting restrictions arise in the sort of manipulations and measurements that the observers are allowed to make, and we describe these in turn below.

Allowed manipulations

In quantum mechanics, it is standard to allow observers to manipulate the system by acting with unitary operators. Note that it is not permissible to "act" on a state with arbitrary Hermitian operators. But Hermitian operators can be added to the Hamiltonian of the theory, and this results in a unitary transformation of the state.

We want to remain within the realm of the low-energy theory. We will do this by allowing the observers to modify the state through only low-energy simple unitaries. The allowed unitaries depend on a parameter J and we demand that

$$U = 1 + iJX + \mathrm{O}\big(J^2\big)\,, \tag{2.7}$$

where X is a low-energy, Hermitian operator from the time band $[0,\epsilon]$ near the boundary of AdS. The operator X must also be a simple operator which means that when it is expressed as a polynomial in the elementary observables

$O(t,\Omega)$ of (2.4) it does not involve any terms of degree higher than Λ. We will only consider this unitary in the vicinity of $J = 0$. In (2.7) the reason that we have not written the $\mathrm{O}(J^2)$ term explicitly is that, as we show below, it drops out of the analysis.

The response of the system under the action of the unitary above can be written as a <u>modification of the state</u>, $|g\rangle \to |g_{\text{mod}}\rangle$ where

$$|g_{\text{mod}}\rangle = |g\rangle + iJX|g\rangle + \mathrm{O}\big(J^2\big)\,. \tag{2.8}$$

We pause to mention two points.

1. There are several physical ways of generating the unitary action (2.7), and our protocol is insensitive to the method used. In the introduction, to keep the notation simple, we simply used the unitary e^{iJX}. But, physically it may be more natural for the observers to turn on a source near the boundary that deforms the Hamiltonian in the time-interval $[0,\epsilon]$ by a term, $-Jx(t)$, where $\int_0^\epsilon x(t)dt = X$. The precise effect of this deformation is an action by the unitary $\mathcal{T}\left\{e^{i\int Jx(t)dt}\right\}$ where $\mathcal{T}$ is the time-ordering symbol. But to first order this unitary also coincides with (2.7), which is all that we require.

2. We will consider many different manipulations of the state below. But, in order to avoid introducing a plethora of symbols, the relevant unitary is always denoted by U and the the modified state is always represented by $|g_{\text{mod}}\rangle$. So, the notation $|g_{\text{mod}}\rangle$ may refer to different states below, but in each case it will appear immediately after we explain the manipulation that produces it.

Allowed measurements

The observers are allowed to perform measurements of low-energy operators near the boundary that are localized in the time-band $[0,\epsilon]$. We are particularly interested in a measurement of the <u>energy</u> through the metric. In a gravitational theory in AdS, when Fefferman Graham gauge is chosen near the boundary, the energy of the state is given by the subleading falloff of the metric [49].

$$H = \frac{d}{16\pi G}\lim_{r\to\infty} r^{d-2}\int h_{00}(r,t,\Omega)d^{d-1}\Omega, \tag{2.9}$$

where $h_{\mu\nu}$ is the deviation of the metric from the global AdS metric displayed in (2.3). This is just a manifestation of the Gauss law. (A gauge-invariant expression for the energy can be found in [50].) We would like to make a few comments.

1. Since the energy is a delocalized observable, it can be measured in two ways. First, we may consider a team of observers spread out at very large r and all points of the sphere. Each of these observers measures the local value of the metric, and the observers then add their results to obtain the expression for (2.9). Alternately, a single observer may use multiple identically prepared systems, make measurements at different points on the sphere in different systems, and then add up her results.

2. The energy is a quantum mechanical observable. This means that, except in energy eigenstates, its measurement does not yield a definite value. However, in global AdS, the possible values obtained upon measuring the energy are quantized since energy eigenstates are separated by a gap that is proportional to the inverse AdS length.[1] Both the quantum fluctuations, and the quantized nature of the energy will be useful for us.

3. When the observers measure the energy, there is some probability that they might obtain the answer 0. By the standard rules of quantum mechanics, in the state $|g_{\text{mod}}\rangle$, this probability is given by

$$\langle g_{\text{mod}}|P_0|g_{\text{mod}}\rangle, \tag{2.10}$$

where $P_0 = |0\rangle\langle 0|$. We will be interested in this probability to obtain 0, not only in the original state, but also after we have manipulated the state.

4. We do not require arbitrary accuracy in the measurements above. For instance, as in any quantum mechanical system, to truly measure (2.10) to arbitrary accuracy would require an infinite number of systems. Here we will be satisfied if the observers can make measurements so that

$$|\langle g_{\text{mod}}|P_0|g_{\text{mod}}\rangle_{\text{measured}} - \langle g_{\text{mod}}|P_0|g_{\text{mod}}\rangle_{\text{true}}| \ll \delta, \tag{2.11}$$

where δ is the accuracy scale set in the task.

[1]This can be seen by straightforwardly quantizing fields about global AdS and examining the low-energy spectrum. From the perspective of AdS/CFT, we note that the spectrum of energy levels in global AdS is dual to the spectrum of operator-dimensions in the boundary theory, which is expected to be discrete.

2.3 Protocol to extract information

We now describe how the observers near the boundary can complete the task described in section 2.2, using the manipulations and measurements described there. First we describe the main idea, and then describe a more detailed algorithm that covers some subtleties and exceptional cases.

2.3.1 The main idea

In describing the main idea, we will make certain simplifying assumptions. However, we emphasize that our procedure is completely general, and in the next subsection we account for all possible special cases.

For simplicity, consider a state where $\langle 0|g\rangle = 0$, i.e. the state that the observers are given has no overlap with the vacuum itself. If the state is not of this form to start with, we explain below how the observers can perform a simple preliminary manipulation to ensure this. Now consider the combined effect of acting with a unitary (as displayed in (2.7)) followed by the measurement of the energy and a determination of the relative frequency with which this energy-measurement yields 0 as displayed in (2.10).

This relative frequency is easy to compute in perturbation theory. Using equation (2.8), we see that the relative frequency with which the measurement of the energy yields 0 is then given by

$$\langle g_{\rm mod}|P_0|g_{\rm mod}\rangle = J^2|\langle 0|X|g\rangle|^2 + {\rm O}\big(J^3\big) = J^2|\langle X|g\rangle|^2 + {\rm O}\big(J^3\big)\,. \tag{2.12}$$

Note that the second order term above arises from combining two of the $\mathrm{O}(J)$ terms displayed in Eqn. (2.8). Also, as advertised above, we note that we were justified in neglecting the $\mathrm{O}(J^2)$ term in Eqn. (2.8); that term does not enter into the leading result above, since $\langle 0|g\rangle = 0$. We draw the reader's attention to the notation that we have used

$$|X\rangle = X|0\rangle, \tag{2.13}$$

since we will use the same notation multiple times below.

As described in Appendix A.1, if X varies over the set of low-energy Hermitian operators that are localized near the boundary, then the set of states (2.13) already yields a complete basis for the low-energy Hilbert space. This may sound like an unfamiliar statement, but this is simply related to the state-operator map that is familiar from the study of quantum field theories.

We emphasize that this is not just a formal property of the theory. In Appendix A.1 we show how to construct the operators dual to a particular state explicitly.[2]

The upshot is that the simple physical process described above is already sufficient to tell us the absolute value of the overlap of $|g\rangle$ with a set of basis vectors in the theory.

Now this process by itself is not sufficient to tell us the phase of this overlap. But we can determine the phase as follows. Say that the observers have two reference Hermitian operators, X_r and X_i. These operators have the property that the matrix elements of these operators between the state $|g\rangle$ and the vacuum are both non-zero and purely real and purely imaginary respectively.

$$\langle g|X_r\rangle = \mathrm{Re}\left(\langle g|X_r\rangle\right) \neq 0; \qquad \langle g|X_i\rangle = i\mathrm{Im}\left(\langle g|X_i\rangle\right) \neq 0. \tag{2.14}$$

Here, as above, we have used the notation $|X_r\rangle = X_r|0\rangle$ and likewise for $|X_i\rangle$. We explain below how the observers can generate such reference operators without much difficulty.

Then the phase of $\langle g|X\rangle$ can be fixed easily for all other operators. The observers first act with the unitary

$$U = 1 + iJ(X_r + X) + \mathrm{O}\big(J^2\big)\,, \tag{2.15}$$

and then by measuring the energy, as above, they determine the expectation value of P_0 in the modified state to second order in J.

This measurement allows the observers to determine $|\langle g|X_r\rangle + \langle g|X\rangle|^2$ through direct measurement. But note

$$\mathrm{Re}\left(\langle g|X\rangle\right) = \frac{|\langle g|X_r\rangle + \langle g|X\rangle|^2 - \langle g|X_r\rangle^2 - |\langle g|X\rangle|^2}{2\langle g|X_r\rangle}. \tag{2.16}$$

[2] We have found that this point often causes confusion, and so we would like to repeat that the set of states in (2.13) would form a basis for the Hilbert space even in a QFT without gravity. But this property by itself would not enable the observers near the boundary to obtain any information about the behaviour of the state $|g\rangle$ in the bulk in a local QFT. In gravity we are assisted by the fact that the process of measuring the energy, and determining the frequency with which the measurement yields 0, corresponds, mathematically, to the insertion of a one-dimensional projector P_0 in (2.12). In non-gravitational theories, including gauge-theories, as we explain in section 2.4, the measurement of any operator near the boundary always corresponds to an infinite-dimensional projector, and so the near-boundary observers can obtain no information about the bulk.

Since all terms on the right hand side are known, the observers can use this to determine $\mathrm{Re}(\langle g|X\rangle)$. This still leaves a sign ambiguity in $\mathrm{Im}(\langle g|X\rangle)$. This can be fixed by acting with the unitary

$$U = 1 + iJ(X_i + X) + \mathrm{O}(J^2), \tag{2.17}$$

and measuring P_0 in the modified state. The observers use this to determine $|\langle g|X_i\rangle + \langle g|X\rangle|^2$ and then note that

$$\mathrm{Im}(\langle g|X\rangle) = \frac{|\langle g|X_i\rangle + \langle g|X\rangle|^2 - |\langle g|X_i\rangle|^2 - |\langle g|X\rangle|^2}{2\mathrm{Im}(\langle g|X_i\rangle)}. \tag{2.18}$$

Note that the left hand sides of equations (2.16) and (2.18) are subject to a strong consistency check: upon squaring and adding, they must yield $|\langle g|X\rangle|^2$, which is known independently.

By using this procedure, the observers can determine the overlap of $|g\rangle$ with any state of the form (2.13). Since such states form a basis, this completely determines $|g\rangle$.

2.3.2 Details of the protocol

We now fill in some of the details that we omitted above, address various possible exceptions, and explain how to generate the reference operators used above. We start by explaining how the observers can perform an initial manipulation on the state to remove its overlap with the vacuum.

Preliminary step: Determination of $|\langle 0|g\rangle|^2$ and initial manipulation

First, the observers make a number of measurements of the energy on their identically prepared systems, without performing any manipulation at all. The probability that they obtain the answer 0 is given by

$$\langle g|P_0|g\rangle = |\langle 0|g\rangle|^2. \tag{2.19}$$

By performing a sufficient number of measurements so that the relative frequency tends to the probability, they are able to determine $|\langle 0|g\rangle|^2$ to any desired accuracy. If $|\langle 0|g\rangle|^2 = 0$, the observers proceed to the next step.

Otherwise, the observers only need to perform a simple manipulation of the state. They act with a simple local unitary that takes

$$|g\rangle \to \mathfrak{U}^z|g\rangle, \tag{2.20}$$

so that $\langle 0|\mathfrak{U}^z|g\rangle = 0$. The construction of this unitary is described in greater detail in Appendix A.2. But the main idea is very simple: it is easy to make two states in a large Hilbert space orthogonal by altering the state of a single degree of freedom. Here, let $O(t, \Omega)$ be the boundary value of a bulk propagating field as displayed in (2.4). By smearing this field with two suitably chosen functions, f_1, f_2 which have limited support both in time — so that they vanish outside $t \in [0, \epsilon]$ — and in space — so that they vanish outside a small region on the sphere — we can find operators that satisfy the Heisenberg algebra and thereby isolate a simple-harmonic degree of freedom. The local unitary described in Appendix A.2 acts only on this simple harmonic degree of freedom. This is already sufficient to make the state $\mathfrak{U}^z|g\rangle$ orthogonal to the vacuum. The unitary may inject some energy into the state, but the state remains a low-energy state.

Note that if the observers can reconstruct the state $\mathfrak{U}^z|g\rangle$, since they know the unitary, $\mathfrak{U}^z$, they can back-calculate the state $|g\rangle$. In the discussion below, we will assume that this unitary operation has been performed. Rather than introducing separate notation for the case where $\langle 0|g\rangle = 0$ from the start, and for the case where the observes are required to act with an additional unitary, we will simply assume that $\langle 0|g\rangle = 0$ in all equations below.

Determination of reference operators

We now show how the observers can find two operators that satisfy the relations (2.14).

The operator X_r is particularly easy to find. By trial and error and by using the protocol described above, the observers need to find only one Hermitian operator with the property that $|\langle g|X_r\rangle|^2 \neq 0$. Since the overall phase of $|g\rangle$ is physically meaningless, the observers can immediately choose the convention that

$$\langle g|X_r\rangle = |\langle g|X_r\rangle|, \tag{2.21}$$

which satisfies the first part of (2.14).

The observers can now find the operator X_i as follows. Consider the set of all states at a given energy. We denote these states by $|E, \{\ell\}\rangle$ where we have separated the energy eigenvalue, E, from other possible quantum-numbers of the state that we have clubbed into $\{\ell\}$. Then, for each state at this energy, it is always possible to find two Hermitian operators, $X_{E,\{\ell\}}$ and $Y_{E,\{\ell\}}$ near the boundary, which have the property that

$$\left(X_{E,\{\ell\}} + iY_{E,\{\ell\}}\right)|0\rangle = |E,\{\ell\}\rangle. \tag{2.22}$$

Note that the operator on the left-hand side is not Hermitian due to the factor of i that multiplies $Y_{E,\{\ell\}}$.

We pause to mention an important physical point. The observers do not need to find the exact operators $X_{E,\{\ell\}}$ and $Y_{E,\{\ell\}}$ that satisfy equation (2.22). It is acceptable for them to attain a level of accuracy that is controlled by the error δ that appears as part of the task-specification in equation (2.6). In particular, the explicit construction of such operators in Appendix A.1 does not keep track of $\mathrm{O}\left(\frac{1}{N}\right)$ corrections, which is not a problem because $\delta \gg \frac{1}{N}$.

Now, by manipulating the state with various unitaries, as indicated in the table below, the observers can obtain a number of physical quantities.

Unitary Manipulation	**Quantity Inferred**
$1 + iJX_{E,\{\ell\}} + \mathrm{O}(J^2)$	$\left\|\langle g\|X_{E,\{\ell\}}\rangle\right\|^2$
$1 + iJY_{E,\{\ell\}} + \mathrm{O}(J^2)$	$\left\|\langle g\|Y_{E,\{\ell\}}\rangle\right\|^2$
$1 + iJ\left(X_{E,\{\ell\}} + X_r\right) + \mathrm{O}(J^2)$	$\mathrm{Re}\left(\langle g\|X_{E,\{\ell\}}\rangle\right)$ and $\left\|\mathrm{Im}\left(\langle g\|X_{E,\{\ell\}}\rangle\right)\right\|$
$1 + iJ\left(Y_{E,\{\ell\}} + X_r\right) + \mathrm{O}(J^2)$	$\mathrm{Re}\left(\langle g\|Y_{E,\{\ell\}}\rangle\right)$ and $\left\|\mathrm{Im}\left(\langle g\|Y_{E,\{\ell\}}\rangle\right)\right\|$
$1 + iJ\left(X_{E,\{\ell\}} + Y_{E,\{\ell\}}\right) + \mathrm{O}(J^2)$	$\mathrm{Im}\left(\langle g\|X_{E,\{\ell\}}\rangle\right)\mathrm{Im}\left(\langle g\|Y_{E,\{\ell\}}\rangle\right)$

(2.23)

In some cases, the quantity on the right column in the table above may be related to the direct observable through some simple algebra. For instance, in the last line above, we have

$$\begin{aligned}|\langle g|X_{E,\{\ell\}}\rangle + \langle g|Y_{E,\{\ell\}}\rangle|^2 = |\langle g|X_{E,\{\ell\}}\rangle|^2 + |\langle g|Y_{E,\{\ell\}}\rangle|^2 + 2\mathrm{Re}(\langle g|X_{E,\{\ell\}}\rangle)\mathrm{Re}(\langle g|Y_{E,\{\ell\}}\rangle) \\ + 2\mathrm{Im}(\langle g|X_{E,\{\ell\}}\rangle)\mathrm{Im}(\langle g|Y_{E,\{\ell\}}\rangle).\end{aligned} \tag{2.24}$$

The observers can determine the product of the imaginary parts since they know all other terms in the equation above: they obtain the left hand side of the equation through direct measurement and know the other terms on the right from previous measurements.

The table above leaves the observers with an ambiguity in the sign of the imaginary part of the overlap of $|g\rangle$ with the various basis vectors. This is because, in each case they know the real part and only the magnitude of the overlap. But since the last line in the table tells them about the product of the imaginary parts, this is a correlated ambiguity. Once they infer the sign of the imaginary part of a single overlap, they can immediately infer the signs of all the other imaginary parts.

This single sign can be fixed as follows. Through a measurement of the energy, the observers can also determine

$$\langle g|P_E|g\rangle = \sum_{\{\ell\}} |\langle g|E, \{\ell\}\rangle|^2. \tag{2.25}$$

where the sum runs over all states at that energy. The physical process for determining this is simply to measure the energy in the state $|g\rangle$ and determine the relative frequency with which the result E is obtained. This directly yields the left hand side. But we have

$$\sum_{\{\ell\}} |\langle g|E, \{\ell\}\rangle|^2 = \sum_{\{\ell\}} |\langle g|X_{E,\{\ell\}}\rangle|^2 + |\langle g|Y_{E,\{\ell\}}\rangle|^2 + 2C_E, \tag{2.26}$$

where we have defined

$$C_E = \sum_{\ell} \left(\mathrm{Re}\langle g|Y_{E,\{\ell\}}\rangle \mathrm{Im}\langle g|X_{E,\{\ell\}}\rangle - \mathrm{Re}\langle g|X_{E,\{\ell\}}\rangle \mathrm{Im}\langle g|Y_{E,\{\ell\}}\rangle\right). \tag{2.27}$$

As we pointed out above, the signs of the all the imaginary parts are correlated. Therefore simply from the measurements in (2.23), we already know $|C_E|$. From the measurement of $\langle g|P_E|g\rangle$ we can fix the sign of C_E and this immediately fixes the sign of all the imaginary parts.

The reference operator X_i can then be generated using any operator with a non-zero imaginary part in its matrix element between $|g\rangle$ and the vacuum. If $X_{E,\{\ell\}}$ is such an operator,

$$X_i = X_{E,\{\ell\}} - \frac{\mathrm{Re}\left(\langle g|X_{E,\{\ell\}}\rangle\right)}{\langle g|X_r\rangle} X_r. \tag{2.28}$$

Complete protocol

Once the observers have determined a set of reference operators, they can then proceed to systematically determine the overlap of $|g\rangle$ with any set of basis states as explained in section 2.3.1.

For instance, they may choose to use the basis of energy eigenstates. For each energy eigenstate, they find the operator dual to it so that it can be written in the form (2.22). They then perform the deformations of the Hamiltonian given in the Table in (2.23). In addition, they require a manipulation by the following unitary

$$U = 1 + iJ(X_i + X_{E,\{\ell\}}) + \mathrm{O}\big(J^2\big)\,. \tag{2.29}$$

A measurement of the energy following this unitary then immediately allows the observers to read off $\mathrm{Im}\left(\langle g|X_{E,\ell}\rangle\right)$ as explained near equation (2.18).

Together with the other physical quantities obtained by the deformations displayed in Equation (2.23), this allows the observers to completely determine $\langle g|E,\{\ell\}\rangle$ — both in magnitude and in phase. Proceeding in this manner for each separate energy eigenstate, below the UV-cutoff Λ, the observers completely determine the state $|g\rangle$ to the desired accuracy. A flowchart describing the entire process, including the verification described in the next section is shown in Figure 2.2.

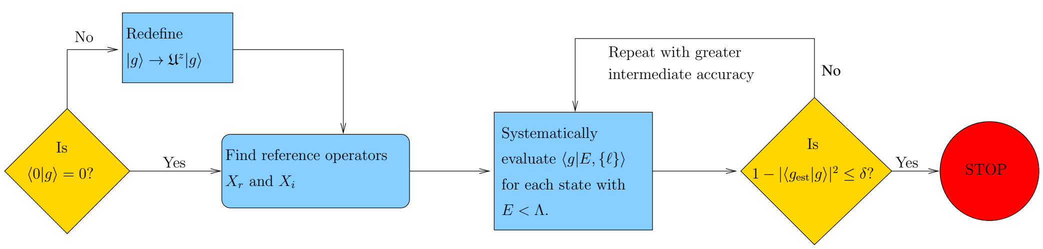


Figure 2.2: A flowchart describing the key steps in our protocol.

We would like to emphasize two obvious points.

1. There are various degeneracies in the energy-spectrum, but the observers can independently determine the overlap with each separate energy eigenstate. Each eigenstate is associated with a unique set of Hermitian operators $X_{E,\{\ell\}}$ and $Y_{E,\{\ell\}}$.

2. The observers can successfully perform their procedure, even if two energy eigenstates are related by a global symmetry. Since the states are distinct, the pair of Hermitian operators associated with the two energy eigenstates by equation (2.22) are also distinct. We give an explicit example in section 2.3.3.

Verification

In principle, it is sufficient for the observers to determine the overlap of the state with all energy eigenstates. However, since this involves a number of operations, errors may accumulate in this process. So we now explain how the observers can also verify that they have successfully completed the task, in only a few steps.

At the end of the process above, the observers have the following estimate of the state

$$|g_{\rm est}\rangle = (X_{\rm est} + iY_{\rm est})\,|0\rangle, \tag{2.30}$$

where

$$\begin{aligned} X_{\rm est} &= \sum_{E,\{\ell\}} {\rm Re}\left(\langle E,\{\ell\}|g\rangle\right) X_{E,\{\ell\}} - {\rm Im}\left(\langle E,\{\ell\}|g\rangle\right) Y_{E,\{\ell\}} \\ Y_{\rm est} &= \sum_{E,\{\ell\}} {\rm Im}\left(\langle E,\{\ell\}|g\rangle\right) X_{E,\{\ell\}} + {\rm Re}\left(\langle E,\{\ell\}|g\rangle\right) Y_{E,\{\ell\}} \end{aligned} \tag{2.31}$$

Now the observers act with a two-parameter deformation of the Hamiltonian

$$U = 1 + i\left(J_1 X_{\rm est} + J_2 Y_{\rm est}\right) + {\rm O}\big(J_1^2\big) + {\rm O}\big(J_2^2\big)\,, \tag{2.32}$$

followed by an energy measurement and a determination of the frequency with which this yields 0. As explained above, by determining the ${\rm O}(J_1^2)\,, {\rm O}(J_2^2)\,, {\rm O}(J_1 J_2)$ terms in this observation, the observers obtain the values of

$$\alpha = |\langle g|X_{\rm est}\rangle|^2; \quad \beta = |\langle g|Y_{\rm est}\rangle|^2; \quad \gamma = {\rm Re}\left(\langle g|X_{\rm est}\rangle\, {\rm Re}\left(\langle g|Y_{\rm est}\rangle + {\rm Im}\left(\langle g|X_{\rm est}\rangle\, {\rm Im}\left(\langle g|Y_{\rm est}\rangle\,.\right.\right.\right.\right. \tag{2.33}$$

Now with α, β, γ defined as above

$$|\langle g|X_{\rm est}\rangle + i\langle g|Y_{\rm est}\rangle|^2 = \alpha + \beta \pm 2\sqrt{\alpha\beta - \gamma^2}. \tag{2.34}$$

The observers can easily fix the sign of the square-root. The first check is that with one of the two possible signs, the observers should obtain exactly 1 for the right hand side. This is already an extremely strong check on the estimates of the observers. But the observers can additionally verify that this is the correct sign by using the reference operators above to independently determine the real and imaginary parts of the overlap between $|g\rangle$ and $|X_{\rm est}\rangle$ and $|Y_{\rm est}\rangle$.

2.3.3 An example with global symmetries

In the discussion above, we have not assumed the absence of global symmetries. It may be, for other reasons [51, 52], that global symmetries do not exist in a theory of quantum gravity. But this issue does not affect our protocol. The main physical point is that our protocol involves not only measurements of the energy but also of correlators of the Hamiltonian with other dynamical fields. These correlators can break the degeneracy between states related by global symmetries.

To demonstrate this, we now give an example of how the observers can identify the state, in a situation where the bulk theory does have global symmetries. In addition to the field of mass m dual to a boundary operator as displayed in equation (2.4), say that we have another field $\widetilde{\phi}$ of exactly the same mass, dual to a boundary operator $\widetilde{O}$.

Then the low-energy theory contains two states of the same energy: $\frac{d}{2} + \sqrt{(\frac{d}{2})^2 + m^2}$. Denoting the normalized states by $|\Delta\rangle$ and $|\widetilde{\Delta}\rangle$, we put the system in a state

$$|g\rangle = a|\Delta\rangle + b|\widetilde{\Delta}\rangle, \tag{2.35}$$

with $|a|^2 + |b|^2 = 1$. The task of the observers is to determine the complex number $\frac{b}{a}$. (The overall phase of the state is meaningless.)

Both energy eigenstates can be written as

$$|\Delta\rangle = (X_\Delta + iY_\Delta)\,|0\rangle, \qquad |\widetilde{\Delta}\rangle = \left(\widetilde{X}_\Delta + i\widetilde{Y}_\Delta\right)|0\rangle. \tag{2.36}$$

It is possible to find explicit real functions $f_R(t,\Omega)$ and $f_I(t,\Omega)$, as explained in Appendix A.1, that are supported on $t \in [0,\epsilon]$ and satisfy

$$\begin{aligned} X_\Delta &= \int O(t,\Omega) f_R(t,\Omega) dt d^{d-1}\Omega; \qquad & Y_\Delta &= \int O(t,\Omega) f_I(t,\Omega) dt d^{d-1}\Omega \\ \widetilde{X}_\Delta &= \int \widetilde{O}(t,\Omega) f_R(t,\Omega) dt d^{d-1}\Omega; \qquad & \widetilde{Y}_\Delta &= \int \widetilde{O}(t,\Omega) f_I(t,\Omega) dt d^{d-1}\Omega. \end{aligned} \tag{2.37}$$

Due to the global symmetry, the same functions f_R and f_I appear for both states in the right hand side of the equation above, but notice that $|\Delta\rangle$ is produced by applying O to the vacuum, whereas $|\widetilde{\Delta}\rangle$ is produced by applying $\widetilde{O}$ to the vacuum.

As a result of the global symmetry, the states generated by the operators above satisfy

$$\langle X_\Delta | Y_\Delta \rangle = \langle \widetilde{X}_\Delta | \widetilde{Y}_\Delta \rangle; \qquad \langle X_\Delta | X_\Delta \rangle = \langle \widetilde{X}_\Delta | \widetilde{X}_\Delta \rangle; \qquad \langle Y_\Delta | Y_\Delta \rangle = \langle \widetilde{Y}_\Delta | \widetilde{Y}_\Delta \rangle;$$
$$\langle X_\Delta | \widetilde{X}_\Delta \rangle = \langle X_\Delta | \widetilde{Y}_\Delta \rangle = \langle Y_\Delta | \widetilde{X}_\Delta \rangle = \langle Y_\Delta | \widetilde{Y}_\Delta \rangle = 0. \tag{2.38}$$

Using the protocol above, the observers can first find the ratio of the magnitudes of the coefficients:

$$\frac{|b|}{|a|} = \frac{|\langle \widetilde{X}_\Delta | g \rangle|}{|\langle X_\Delta | g \rangle|}. \tag{2.39}$$

Second, they choose the convention for the overall phase of $|g\rangle$ so that $\langle X_\Delta | g \rangle = |\langle X_\Delta | g \rangle|$. This is equivalent to fixing

$$a = \frac{|\langle X_\Delta | g \rangle|}{\langle X_\Delta | \Delta \rangle}. \tag{2.40}$$

The phase of b can also be found using the operator X_Δ, from (2.37), in the role of the reference operator X_r. (The reference operator X_i will not be required in this case.) Using the method above, the observers can determine the values of

$$\text{Re}(\langle \widetilde{X}_\Delta | g \rangle) = \text{Re}(b \langle \widetilde{X}_\Delta | \widetilde{\Delta} \rangle); \quad \text{Re}(\langle \widetilde{Y}_\Delta | g \rangle) = \text{Re}(b \langle \widetilde{Y}_\Delta | \widetilde{\Delta} \rangle). \tag{2.41}$$

Since we already know the magnitude of b, the two equations above unambiguously tell us the phase of b, and therefore the complex number $\frac{b}{a}$ as required.

2.4 Local quantum field theories

In section 2.3 we explained how, in a theory of gravity, the near-boundary observers could identify the bulk state. In this section, we explain that in a local quantum field theory in AdS, not only are the observers unable to identify the bulk state, their ignorance about the state near the middle of AdS is complete. This is a consequence of the so-called "split property" of local quantum field theories that we review below. We start with a discussion of QFTs without any gauge fields and then include gauge fields.

2.4.1 Local QFTs without gauge fields

In a local QFT, our treatment can be a little more rigorous because we no longer need to make any distinction between simple and complicated operators. Let $\phi_i(t,r,\Omega)$ be the set of local quantum fields with the boundary conditions that

$$\phi_i(t,r,\Omega) \underset{r\to\infty}{\longrightarrow} \frac{1}{r^{\Delta_i}} O_i(t,\Omega). \tag{2.42}$$

Then we define two algebras. The first is

$$\mathcal{A}_{[0,\epsilon]} = \text{span of}\{O_{i_1}(t_1,\Omega_1)\dots O_{i_n}(t_n,\Omega_n)\}, \qquad t_i \in [0,\epsilon], \tag{2.43}$$

with no constraint on the coordinates Ω_i and no limit on n. In a local QFT, this algebra is precisely the same as the algebra of bulk fields on the time-slice $t = \frac{\epsilon}{2}$ with the radial coordinate in the range $r \in [\cot\left(\frac{\epsilon}{2}\right), \infty)$. We can also define an algebra of commuting operators

$$\mathcal{A}_{\text{bulk}} = \text{span of}\{\phi_{i_1}(t=\frac{\epsilon}{2}, r_1, \Omega_1)\dots\phi_{i_n}(t=\frac{\epsilon}{2}, r_n, \Omega_n)\}, \qquad r_i < \cot(\frac{\epsilon}{2}) - \chi. \tag{2.44}$$

Here χ is a small parameter that separates the causal wedge of the time band $[0,\epsilon]$ from the support of the operators that are elements of $\mathcal{A}_{\text{bulk}}$. The support of the two algebras is shown in Figure 2.3.

Note that the support of the operators in $\mathcal{A}_{\text{bulk}}$ is in a region that is space-like to the time-band $[0,\epsilon]$ on the boundary. Therefore, by microcausality, we have

$$[A_1, A_2] = 0, \qquad \forall\ A_1 \in \mathcal{A}_{[0,\epsilon]},\ A_2 \in \mathcal{A}_{\text{bulk}}. \tag{2.45}$$

We give the near-boundary observers the same task as in section 2.2. Their powers are that they are allowed to act with an arbitrary unitary from $\mathcal{A}_{[0,\epsilon]}$ and make arbitrary projective measurements from the same algebra.

First, we note an obvious point. Let $P_1, U_1 \in \mathcal{A}_{[0,\epsilon]}$ be, respectively, an arbitrary projector and arbitrary unitary from the algebra of the time band. Let $U_{\text{bulk}} \in \mathcal{A}_{\text{bulk}}$ be an arbitrary unitary from the commuting algebra. Then we have

$$\langle g|U_1^\dagger P_1 U_1|g\rangle = \langle g|U_{\text{bulk}}^\dagger U_1^\dagger P_1 U_1 U_{\text{bulk}}|g\rangle. \tag{2.46}$$

This is an exact relation by microcausality. So the observers cannot distinguish the state $|g\rangle$ from $U_{\text{bulk}}|g\rangle$ by any combination of possible manipulations and measurements that they are allowed to make.

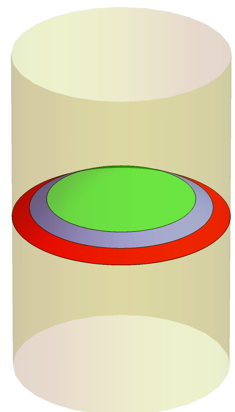


Figure 2.3: In a local QFT, the algebra $\mathcal{A}_{\text{bulk}}$ supported in the inner region (green) exactly commutes with the algebra $\mathcal{A}_{[0,\epsilon]}$ supported in the outermost region (red). The "split property" of local QFTs tells us that when the regions are separated by a small "collar" (blue region), the wavefunctions of the two regions can be prepared absolutely independently. The discussion of section 2.3 tells us that in a theory with dynamical gravity, split states do not exist for the configuration above.

The reader might wonder if the observers can get at least "some" information about the bulk state. But even this turns out to be impossible by virtue of the so-called split property [53]. The split property can be phrased as follows. Let $|\Psi_1\rangle$ and $|\Psi_2\rangle$ be two arbitrary states in the Hilbert space. Then the split property tells us that it is possible to find a state $|g\rangle$, so that

$$\langle g|A_1 A_2|g\rangle = \langle \Psi_1|A_1|\Psi_1\rangle\langle \Psi_2|A_2|\Psi_2\rangle, \qquad \forall\ A_1 \in \mathcal{A}_{[0,\epsilon]};\ A_2 \in \mathcal{A}_{\text{bulk}}. \quad (2.47)$$

This tells us that it is possible to find a global state $|g\rangle$ with the following properties.

1. For all observations made near the boundary, $|g\rangle$ looks exactly like $|\Psi_1\rangle$.

2. For all observations made in the bulk, $|g\rangle$ looks exactly like $|\Psi_2\rangle$.

3. The results of observations made jointly near the boundary and in the bulk are uncorrelated.

Split states can be constructed explicitly in local QFTS as described in [54].

We briefly contrast these results with gravity. The analysis of section 2.3 implies that, for the geometrical configuration of Figure 2.1b — where a region is completely surrounded by its complement — split states do not exist in gravity, at least within the low-energy Hilbert space. This is closely tied to the fact that there are no local gauge-invariant operators in gravity. So, unlike local QFTs, it is simply not possible to find a local unitary U_{bulk} that commutes with operators near the boundary and changes the state in the interior of the spacetime.[3]

2.4.2 Non-gravitational gauge theories

We now turn to non-gravitational gauge theories. Very superficially, it might appear that in such theories the near-boundary observers could obtain some information about the bulk by taking advantage of the conserved charges that are defined by boundary integrals. However, this turns out not to be the case. In particular, the procedure of section 2.3 cannot be repeated at all because there is no analogue of the "projector on the vacuum" which projects onto a unique state. In contrast, the "projector onto states of zero charge" in a non-gravitational theory projects onto an infinite-dimensional subspace of states.

This is physically related to the fact that non-gravitational gauge theories have exactly local gauge-invariant operators. For instance, consider a small Wilson loop that is localized at time $t = \frac{\epsilon}{2}$ and is confined in the radial coordinate to $r < \cot(\frac{\epsilon}{2}) - \chi$ as we discussed above. This Wilson loop furnishes an example of a unitary operator U_{bulk} that commutes with all physical manipulations and measurements that can be made in the near-boundary region.

There is no unique way to identify the algebra associated with a region in gauge theories. This point was discussed extensively in [55, 56, 57, 58]. Physically if one regulates the theory on a lattice, then the "link" variables cut through the boundary of any region. So one has to decide whether to count them as part of the region or not. The different possible choices lead to different centers for the algebra of the region.

[3] We believe that even in gravity, it should be possible to find split states for geometrical configurations where the region and its complement both include a part of the asymptotic boundary. This is consistent with the fact that, in such configurations, it is possible to find commuting algebras by dressing operators from the region and its complement to different parts of the asymptotic boundary. But this topic is beyond the scope of this chapter.

However, since we have a "collar region" that separates the algebra inside from the algebra outside, the different choices of center should not affect the validity of equation (2.47), even in gauge theories. This means that once again, in the absence of gravity, the observers have no information about the state near the center of AdS.

Note that the observers outside can determine the charge in the union of the collar region and the interior region. This is the form in which the split property for gauge theories is stated in [59]. But since the collar can hold an arbitrary amount of charge, if one focuses attention only on the interior region, then the near-boundary observers have no information just as in a local QFT without gauge fields.

Special case: information with priors

We now discuss a special case where the observers are given strong priors about the possible global state. In this special case, the observers can use long-range gauge fields to obtain information about the interior. We caution the reader that both the task given to the observers and the prior information available to them in this problem is quite different from the discussion in the rest of the chapter. So we urge the reader not to confuse the discussion in this subsection with the general discussion in the rest of the chapter.

We again consider two bulk fields, ϕ and $\widetilde{\phi}$ with the same mass m and boundary values O and $\widetilde{O}$ as in section 2.3.3. The difference is that we switch off dynamical gravity but we gauge the global $SO(2)$ symmetry. We also fix a gauge so that it is meaningful to speak of the value of the fields at a bulk point $\phi(t, r, \Omega)$. When the symmetry is gauged, the charge, Q, is an element of the algebra $A_{[0,\epsilon]}$ with commutation relations

$$[Q, \phi(t, r, \Omega)] = i\widetilde{\phi}(t, r, \Omega); \quad [Q, \widetilde{\phi}(t, r, \Omega)] = -i\phi(t, r, \Omega). \tag{2.48}$$

We now consider the situation where the observers are told ahead of time that the global state is of the form

$$|g\rangle = \exp\left[i\lambda \int \Big(f(r, \Omega)\phi(t = \frac{\epsilon}{2}, r, \Omega) + \widetilde{f}(r, \Omega)\widetilde{\phi}(t = \frac{\epsilon}{2}, r, \Omega)\Big)\, dr d^{d-1}\Omega\right] |0\rangle, \tag{2.49}$$

where $f(r, \Omega)$ and $\widetilde{f}(r, \Omega)$ have support for $r < \cot\frac{\epsilon}{2} - \chi$. Moreover, we allow the observers to explore this state for different values of λ near $\lambda = 0$. Note that, by fiat, we have disallowed the action of additional bulk unitaries on the state. The observers are given the task of determining the real functions $f(r,\Omega)$ and $\widetilde{f}(r,\Omega)$.

The observers now act with a unitary

$$U = 1 + iJ \int \left[O(t,\Omega)h(t,\Omega) + \widetilde{O}(t,\Omega)\widetilde{h}(t,\Omega) \right] dt d^{d-1}\Omega + \mathrm{O}\big(J^2\big) . \quad (2.50)$$

In the modified state, the observers measure the global charge and compute the expectation value of Q^2 to first order in J and to first order in λ.

A simple calculation yields

$$\langle g_{\rm mod}|Q^2|g_{\rm mod}\rangle = J\lambda \left(\langle f,h\rangle + \langle \widetilde{f},\widetilde{h}\rangle \right) + \ldots \quad (2.51)$$

where $\ldots$ denotes higher order terms and the inner-product $\langle,\rangle$ is defined through

$$\begin{aligned}\langle f,h\rangle &\equiv \int \langle 0|\{\phi(t=\frac{\epsilon}{2},r,\Omega),O(t',\Omega')\}|0\rangle f(r,\Omega)h(t',\Omega')d^{d-1}\Omega d^{d-1}\Omega' dr dt' \\ &= \mathcal{N}_{\Delta}\int f(r,\Omega)h(t',\Omega')\left[\frac{1}{\sqrt{1+r^2}\cos(t'-\frac{\epsilon}{2})-r\Omega\cdot\Omega'}\right]^{\Delta} d^{d-1}\Omega d^{d-1}\Omega' dr dt'. \end{aligned} \quad (2.52)$$

In the second line above we have substituted the explicit form of the two-point function, and $\mathcal{N}_{\Delta}$ is an unimportant numerical factor.

The reader will notice that something interesting has happened. On the right hand side above, we have a convolution, using the bulk to boundary propagator, of the function f with h and separately of the function $\widetilde{f}$ with $\widetilde{h}$. Since the observers can choose any pair of functions $h,\widetilde{h}$ with support in the time-band $t'\in[0,\epsilon]$, this is sufficient to completely reconstruct f.

We will show this by proving that that there is no non-zero function f so that $\langle f,h\rangle = 0$, for all functions h with support in $t\in[0,\epsilon]$. This inner-product is just the real part of the overlap of the bra $\int\langle 0|\phi(t=\frac{\epsilon}{2},r,\Omega)f(r)dr d\Omega$ with the ket $\int O(t',\Omega')h(t',\Omega')dt'd\Omega'|0\rangle$. In fact the overlap has no imaginary part because the operators are spacelike separated.[4] But, by the arguments of Appendix A.1, it is impossible to choose f to make this overlap vanish for all possible choices of h.

[4]While bulk operators like ϕ may fail to commute with boundary operators like O even at spacelike separation due to the Wilson lines that stretch from ϕ to the boundary, this effect appears only at the next order in perturbation theory in the two-point function.

It is instructive to understand what is happening in this example. The operators ϕ and $\widetilde{\phi}$ are not local operators since, in a gauge-invariant formalism, they would have Wilson lines stretching out to the boundary. The procedure above took advantage of these Wilson-line "tails" to extract the functions f and $\widetilde{f}$. The procedure was successful because the observers were given the prior that the state was of the form (2.49). Without the prior, the observers could have made no progress because, as explained above, in a non-gravitational gauge theory, it is possible to hide information from the near-boundary observers by acting with local gauge invariant operators in the bulk.

However, in the gravitational setting, since there are no local gauge-invariant operators, one cannot change the state in the interior without having some effect that propagates out to the boundary. This is the essential reason that in gravitational theories, the observers can determine the state in the bulk even without a prior of the form (2.49).

2.5 Conclusion and discussion

Main result

Our main result is that in a theory of quantum gravity in global AdS, observers near the boundary can extract information about the bulk through a physical process. This result is closely tied to the arguments in [4, 5]. These papers argued that in theories of gravity — with either asymptotically AdS or asymptotically flat boundary conditions — operators that probe the bulk have a dual representation as operators near the boundary. The innovation in this chapter is that we have provided a physical protocol by means of which bulk information can be extracted.

Our analysis assumed that observers near the boundary could perform the standard operations that are allowed in quantum mechanics — manipulations of the system through unitary operators and projective measurements. The unitary operators that the observers use in our protocol are in one to one correspondence with low-energy Hermitian operators in the near-boundary region. The key part of our protocol is that, in a theory of gravity, the observers can follow such a unitary with a measurement of the energy, and a determination of the relative frequency with which this measurement yields 0. By the logic sketched in the introduction, and then explained in greater detail in section 2.3, the observers can use this procedure to determine the state in the bulk.

Implications for quantum information measures

The von Neumann entropy, which is a commonly used quantum-information measure, precisely measures the uncertainty in the state that remains after an observer has extracted all possible information that can be obtained through manipulations by local unitaries and local measurements. But what we have shown here is that when the global state of the system is a low-energy state, the near-boundary observers can identify the state precisely with no uncertainty.

In a theory of gravity, since the spacetime can fluctuate, one must be careful about what one means by the von Neumann entropy of a region. One possibility is to define a region geometrically using a relational prescription and then consider quantum-information measures defined with respect to a set of simple operations confined to that region [43]. With this definition in mind, consider the region defined by the set of points $\{r, t = \frac{\epsilon}{2}, \Omega\}$ with $r > r_0$ and Ω arbitrary. This is the shaded annular region shown in Figure 2.1b with $r_0 = \cot(\frac{\epsilon}{2})$. This can be defined relationally as the causal wedge of the boundary time-band with $t \in [0, \epsilon]$. Then our analysis suggests that the von Neumann entropy of this region is 0. This is very different from the same quantity, as computed in a local quantum field theory, where one might expect an answer proportional to r_0^{d-1} after UV-regulation. This provides a striking example of how the von Neumann entropy is <u>not</u> a perturbative quantity. Turning on weak gravitational effects changes this quantity from a finite value to 0. It would be interesting to understand the relationship of this result with the results of [60].

A similar comment holds for the relative entropy, which quantifies how well two states can be distinguished using observations in a region. We can consider the relative entropy of the states on an annular region of the form above as a function of r_0. In a local quantum field theory, by the monotonicity of relative entropy, we would expect this answer to increase as r_0 decreases. The result of our analysis is that since the information content does not increase in a theory of gravity, the relative entropy is <u>constant</u> as a function of r_0.

It may be possible to find mathematical generalizations of the von Neumann entropy or relative entropy in the presence of gravity by generalizing the replica trick [61], or through some other method. But, if these generalizations are to have the usual physical interpretation in terms of the information available in a region then we believe that they should agree with the answers indicated above.

Also note that our result is not in contradiction with the RT [62, 63] or HRT formulas [64] or even their quantum corrected versions [46, 47]. In these setups, one always considers a subregion on the boundary dual to an entanglement wedge in the bulk. Our analysis is applicable to regions which include an entire time-slice of the boundary, and so it does not apply to cases where the region and its complement both include a part of the boundary.

Failure of the split property in gravity

A physical way to understand our result is that each bulk excitation leaves a distinctive imprint on the wavefunction near the boundary. This imprint can be read off by near-boundary observers using a set of physical manipulations and measurements. This is quite different from a local quantum field theory since it means that it is not possible to modify the wavefunction inside a bounded region without also modifying the wavefunction outside it.

But this immediately means that the "split property" fails in gravity, at least for the geometrical configuration where a region is surrounded by its complement. It is sometimes incorrectly stated [65] that gravitational effects allow one to read off the expectation value of the energy and other Poincare charges, but do not yield information beyond this. But as the analysis in this chapter shows, the constraints in gravity are far more powerful and allow one to obtain complete information about the interior.

We expect that the split property will be recovered if we consider a different kind of geometrical configuration, where the region and its complement both include some part of the asymptotic boundary.

No superluminal communication

A question that is often asked is as follows: what if an observer enters the bulk and sets off an "explosion" near the middle of AdS. Will the near-boundary observers not know about this on the same time-slice, and does this not lead to a protocol for superluminal communication? In fact, our protocol does not lead to a protocol for superluminal communication and this can be seen in two ways.

One obvious reason is that the near-boundary observers are spread out over the entire sphere at large r. In order to determine that an explosion has taken place, they need to travel around the sphere to combine their results and this itself takes at least as much time as the light-crossing time of AdS. So they cannot extract information about the explosion “any faster” than another set of observers who are allowed to enter the bulk and directly make bulk measurements.

But the second deeper reason is as follows. In a local quantum field theory, the precise way to check if an observer in the interior can send a signal faster than the speed of light to observers in the exterior is as follows. We consider two different wavefunctions of the theory so that at a given time, they differ inside some bounded region but are the same outside. We then allow the wavefunctions to evolve with time, and check if the region, within which they differ, grows faster than the speed of light. However, the failure of the split property alluded to above tells us that one cannot set up this experiment in gravity at all: if two wavefunctions differ inside, they will also differ outside. Said another way, the observer inside cannot transmit information superluminally to the near-boundary observers because the near-boundary region already contains a copy of all information from the interior!

We emphasize that it is still possible to ask questions about asymptotic causality. If we make a disturbance near the boundary at some point of time, it is still important to check that this disturbance does not travel to another part of the boundary through the bulk any faster than it can travel through the boundary. This kind of check was performed in [66] (see [67] for recent developments) and yields important constraints on bulk dynamics.

Implications for black holes

The analysis in this chapter has explicitly focused on low-energy states. In the case of black holes, one can ask two kinds of questions. First, consider the set of states in the vicinity of a given microstate. This is sometimes called the “little Hilbert space” [44] and it includes excitations in the black hole interior [68]. Then one can ask if the observers can extend the protocol described here to uniquely identify the excitation. In effective field theory, the spectrum of possible excitations about a black hole appears to be continuous because the gaps between energy levels in the vicinity of a black-hole microstate are too small to be seen perturbatively. Nevertheless, the analysis of [69, 70] suggests that our protocol can be extended to black holes, as we will explore in forthcoming work.

However, a second kind of question is as follows: can the observers determine the microstate of the black hole itself from observations in the exterior? It is clear that this question cannot be answered within the framework of this chapter where the observers are restricted to acting with simple unitaries. But this is not surprising. Any protocol to decode the microstate of the black hole — whether it involves making direct observations on the Hawking radiation or indirect observations through gravitational effects as described here — will necessarily require the observers to perform complex manipulations and very accurate measurements.

But, with the caveat above, it is interesting that — although such an analysis requires some careful extrapolations as discussed in [5] — it is, in principle, possible to extend this protocol all the way to black hole microstates.

Possible obstructions to the protocol

Our protocol leads to results that are in conflict with common intuitive notions of locality. So it is important to ask if a more careful consideration of the physics could reveal possible physical obstructions to the implementation of this protocol. We now sketch some possibilities in this direction.

The standard framework used in discussions of quantum information relies on a separation between the system and the observer. This is because both unitary manipulations and measurements require the observers to deform the Hamiltonian. We described in section 2.2 how our setup could be realized by embedding multiple copies of global AdS within a larger spacetime. But it may happen that this arrangement leads to subtleties. For instance, since the system is gravitational, the observers cannot decouple exactly from the system that they are observing.

We have also assumed that there is no obstruction to making measurements of the energy at the AdS scale. But it is possible that such measurements are difficult for some reason. Note that the specific issues discussed in [71] are not directly relevant for our protocol. The paper [71] was written in the context of flat space but even there, as described in Appendix B of [5], the Hamiltonian can be measured accurately by making smeared measurements over a region with large radial extent. This issue has not been studied in AdS, but even if such a smearing is required, it can be performed within the annular region of Figure 2.1b, which corresponds to an infinite range of the radial coordinate. But we cannot rule out the possibility that other effects in the "same universality class" are significant in our context.

We do not have any evidence that effects of the kind above are important. But it would be extremely interesting if they are, since this would teach us that, in gravitational systems and in cosmological settings, one must be cautious while using the standard rules for observables in quantum mechanics.

Chapter 3

Recovering Information in Flat Spacetimes

3.1 Differences with the AdS Protocol

As summarized in the previous chapter, in [10] we showed that a set of observers living on a thin time band in global Anti-de Sitter space (AdS) can determine a state in the bulk via a physical protocol in a theory of quantum gravity. It has been shown [5] that the knowledge of all possible correlation functions composed of operators near $\mathcal{I}^+_-$, in a particular state, allows one to reconstruct the state itself. However in this chapter, like in the previous one, we are restricting ourselves to measurements which can be performed by observers localized at $\mathcal{I}^+_-$ in a "physical experiment". For this purpose, we shall only consider a class of states $|\psi\rangle$ which are built by exciting the vacuum state $|0\rangle$ by low energy operators. There are more general classes of states as discussed in [72] but the examination of these and the construction of a protocol to detect them from $\mathcal{I}^+_-$ require more examination that is beyond the scope of the thesis.

We now discuss the difficulties in constructing such a protocol in flat spacetime as compared to AdS. Flat spacetime has an intricate IR structure due to the presence of an infinite dimensional symmetry [17, 73, 18] which leads to an infinitely degenerate vacuum state. Thus flat spacetime has multiple vacua in contrast to the AdS spacetime that has a unique vacuum.

Another major difficulty in flat spacetime is the lack of a discrete energy spectrum (as is the case for the AdS) and hence we have to modify the protocol described in the previous section. Since the energy is continuous in flat spacetime it is not useful to decompose a state $|\psi\rangle$ in terms of a basis spanned by energy eigenstates. However, we can still exploit the fact that there is a lower bound of the energy when measured from asymptotic infinity, which we renormalize to zero. Subsequently, we introduce a more convenient basis to decompose the states. This is called the *normal-ordered basis*, which is very similar to the usual Fock basis (see eq. (3.14)). Throughout this chapter we represent the matter fields with scalars but this restriction can be relaxed and we can consider fields of any spin (or even stringy excitations). By causality, any massless excitation can be expressed in terms of operators smeared over $\mathcal{I}^+$ and therefore the basis is constructed out of fields living at $\mathcal{I}^+$. A similar basis construction is also possible in AdS spacetime but in that case it was more convenient to work with an energy eigen basis [10].

In this chapter we shall restrict to measuring the energy of states built using *hard operators* (an operator with finite energy) on a particular vacuum state denoted by $|0\rangle$. Using the Born-rule, this reduces to computing expectation values of a product of unitary operators and the projector onto the vacuum state. We will show that this allows us to completely decode a state of the kind shown in eq.(3.14).

As discussed in the previous chapter, such a protocol does not violate causality. In flat spacetime, this is because, in order to measure the energy of a state, the observers are forced to be spread over the whole Celestial sphere since the energy is expressed as a surface integral of metric fluctuations over the entire Celestial sphere (see equation (3.15)). Hence each observer only measures a part of the metric fluctuation and in order to evaluate the energy, they have to meet and sum their results. This process clearly takes more than the light crossing time and therefore prevents any violation of causality.

We review the Hilbert space structure of flat spacetime in section 3.2, then explain how the protocol gets modified in flat space in section 3.3 and end with conclusions and discussions on various related ideas in section 3.4.

3.2 Hilbert Space of Flat Spacetime near Null Infinity

In this section, we review the asymptotic structure of Minkowski spacetime and how its information is encoded in data at null infinity [5, 14, 74]. Moreover, we will revisit how upon quantization these data give rise to the Hilbert space of the low energy effective theory.

The metric for an asymptotically flat space time near $\mathcal{I}^+$ can be written in the retarded Bondi coordinates [12, 13] as

$$ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}d\Omega^A d\Omega^B + rC_{AB}d\Omega^A d\Omega^B + \frac{2m_B}{r}du^2 + \gamma^{AC}D_C C_{AB}dud\Omega^B + \ldots, \tag{3.1}$$

where the *retarded time* $u = t - r$ with t being the time, r the radial direction and $\Omega = (\theta, \phi)$ the coordinates on the unit sphere S^2. The capital latin letters $A, \cdots$ take values on the unit sphere and γ_{AB} is the metric of the unit S^2. $C_{AB}(u, \Omega)$ is the shear field and it encodes the information about the radiative degrees of freedom. In this gauge $\gamma^{AB}C_{AB} = 0$, thus the shear is traceless. The radiative data are encoded at future null infinity $\mathcal{I}^+$ which has the topology $\mathbb{R} \times S^2$ and is parametrized by (u, Ω). The afforementioned unit S^2 is called the *Celestial sphere*. D_A is the covariant derivative with respect to the metric γ_{AB} and m_B is the *Bondi mass aspect*.

Throughout this chapter we shall work with matter fields which are massless scalar fields, however the results are easily generalizable to other massless fields. The large$-r$ fall-off of the scalar field is fixed by demanding the finiteness of its energy

$$\lim_{r\to\infty} \phi^{bulk}(u, r, \Omega) = \frac{1}{r}\phi(u, \Omega) + \mathcal{O}\Big(\frac{1}{r^2}\Big), \tag{3.2}$$

where $\phi(u, \Omega)$ encodes the classical radiative data of the matter field.

At null infinity the gravitational and matter data are not independent as they are related by the Hamiltonian constraint of General relativity. This can be explicitly seen from the uu-component of the Einstein equation which gives the evolution equation for the Bondi mass aspect

$$\partial_u m_B = \frac{1}{4}D^A D^B N_{AB} - \frac{1}{8}N_{AB}N^{AB} - 4\pi G T^{(0)}_{uu}, \tag{3.3}$$

where $N_{AB} = \partial_u C_{AB}$ is the Bondi News tensor and $T^{(0)}_{uu}$ is the leading order of the matter stress tensor, which for a scalar field is given by $\frac{1}{2}(\partial_u\phi)^2$. Thus m_B is a function of the radiative data and of an integration constant at $u = -\infty$ (the Bondi mass at $\mathcal{I}^+_+$ is zero since we do not consider massive particles here).

We move forward to introduce the phase space and the conserved charges. The radiative phase space is characterized in terms of the News and the Shear tensor. The Poisson brakets between the two was found to be [14, 74, 15]

$$\{N_{AB}(u,\Omega), C_{MN}(u',\Omega')\} = -\frac{16\pi}{\sqrt{\gamma}}\delta(u-u')\delta^2(\Omega-\Omega')\left[\gamma_{A(M}\gamma_{N)B} - \frac{1}{2}\gamma_{AB}\gamma_{MN}\right] . \tag{3.4}$$

This will be later promoted to a commutator upon quantizataion.

In the absence of massive particles, the conserved charges of the theory can be expressed in terms of the integration constant of m_B at $u \to -\infty$. These charges are known as the *supertranslation charge*

$$\mathcal{Q}_{l,m} = \frac{1}{4\pi G_N}\int \sqrt{\gamma}d^2\Omega\ Y_{l,m}(\Omega)m_B(u=-\infty,\Omega) \tag{3.5}$$

where $Y_{l,m}$ are the spherical harmonics on S^2. $\mathcal{Q}_{l=0,m=0}$ is the familiar ADM Hamiltonian. The supertranslation charges can be separated into a *soft* and a *hard* part by using eq.(3.3). This decomposition can also be thought of as a separation into terms involving linear and non-linear News

$$\mathcal{Q}^{soft}_{l,m} = -\frac{1}{16\pi G_N}\int_{-\infty}^{\infty} du\, d^2\Omega\, \sqrt{\gamma}Y_{lm}(\Omega)\left(D^A D^B N_{AB}\right) , \tag{3.6a}$$

$$\mathcal{Q}^{hard}_{l,m} = \frac{1}{16\pi G_N}\int_{-\infty}^{\infty} du\, d^2\Omega\, \sqrt{\gamma}Y_{lm}(\Omega)\left(\frac{1}{2}N_{AB}N^{AB} + 16\pi G_N T^{M(0)}_{uu}\right) . \tag{3.6b}$$

The action of the supertranslation generator on the radiative data at $\mathcal{I}^+$ is given by the following poisson brackets

$$\{C_{MN}(u,\Omega), \mathcal{Q}_{lm}\} = Y_{lm}(\Omega)\partial_u C_{MN}(u,\Omega) - 2\Big(D_M D_N Y_{lm}(\Omega) - \frac{1}{2}\gamma_{MN}D^2 Y_{lm}(\Omega)\Big) . \tag{3.7}$$

The radiative phase space of asymptotically flat spacetimes at future null infinity is given by free fields even at the non-linear level and the quantization of such a theory was developed by Ashetakar, et. al [14, 74, 15]. In the quantum theory one derives the following commutation relations (obtained by promoting the Poisson brackets above to commutators)

$$[N_{AB}(u,\Omega), N_{CD}(u',\Omega')] = 16\pi i G_N \frac{1}{\sqrt{\gamma}}\partial_u\delta(u-u')\delta^2(\Omega-\Omega')\left(\gamma_{A(C}\gamma_{D)B} - \frac{1}{2}\gamma_{AB}\gamma_{CD}\right) ,$$

$$[C_{AB}(u,\Omega), N_{CD}(u',\Omega')] = -8\pi i G_N \frac{1}{\sqrt{\gamma}} \mathrm{sign}(u-u')\delta^2(\Omega-\Omega') \left(\gamma_{A(C}\gamma_{D)B} - \frac{1}{2}\gamma_{AB}\gamma_{CD}\right) ,$$

where the tensors C_{AB}, N_{AB} have been promoted to operators.

3.2.1 Hilbert Space

The naive Hilbert space construction leads to states with divergent norms [74], but as we will present below (we refer the reader to section 2.3 of [5] for an extensive discussion) this can be resolved by defining the Hilbert space as a direct sum over Fock spaces. Each such Fock space is built on top of a specific vacuum defined by the soft part of the supertranslation charges[1].

The vacuum[2] is specified by the eigenvalue of the supertranslation charge with $l > 0$, i.e, the zero mode of the News

$$\mathcal{Q}_{lm} \left|\{s\}\right\rangle = s_{lm} \left|\{s\}\right\rangle \tag{3.8}$$

where $s_{lm} \in \mathbb{R}$ are also the eigenvalue of the soft part of the super-translation charge since the hard part annihilates the vacuum. Therefore in order to completely specify a vacuum state we need to specify the value of $\{s\} \equiv (s_{00}, s_{1-1}, s_{10}, \cdots)$. This means the vacuum is infinitely degenerate with a degeneracy of $\mathbb{R}^{\mathbb{Z}}$.

We normalize the soft vacua by using a Dirac-delta normalization[3]

$$\langle\{s\}|\{s'\}\rangle = \delta(\{s\} - \{s'\}) \equiv \prod_{lm} \delta(s_{lm} - s'_{lm}) \,. \tag{3.9}$$

By acting with the creation operators on each $|\{s\}\rangle$, we construct the Fock space $\mathcal{H}_{\{s\}}$. The total Hilbert space is given from the direct sum

$$\mathcal{H} = \bigoplus_{\{s\}} \mathcal{H}_{\{s\}} \ . \tag{3.10}$$

[1]We thank Alok Laddha for explaining many issues about the IR structure of flat spacetime.

[2]An equivalent construction of the vacuum state is by considering the eigenstates of the shear mode. See [75] for a detailed discussion.

[3]Another convenient choice for normalizing the vacuum is to use the Kronecker delta function.

To summarize, the Hilbert space of massless states is given by the direct sum of the Fock spaces built using excitations on all possible vacua by acting with operators at $\mathcal{I}^+$. In [5] it was shown that one can reconstruct the aforementioned Hilbert space by acting on all possible vacua with operators defined in a small cut near the past of future null infinity $\mathcal{I}^+_-$. These operators form an algebra which we symbolize as $\mathcal{A}_{-\infty,\epsilon}$ and comprise the set of all functions of operators $C_{AB}(u,\Omega)$, $\phi(u,\Omega)$, $m_B(u,\Omega)$ at $\mathcal{I}^+$ with $u \in (-\infty, -\frac{1}{\epsilon}]$. In this chapter we explain how – under certain assumptions – observers with access to operators at a small cut near $\mathcal{I}^+_-$ can reconstruct states by performing specific physical measurements.

3.2.2 Projector onto Vacuum state

Having defined the Hilbert space of the theory we now define the vacuum state $|0\rangle$ of interest. Since the Hilbert space is a direct sum of the superselection sectors (3.10), the vacuum $|0\rangle$ can be expressed as a superposition of the soft vacua $|\{s\}\rangle$

$$|0\rangle = \int_{-\infty}^{\infty} \Big(\prod_{l,m} ds_{l,m}\Big) q_{\{s\}} \,|\{s\}\rangle \equiv \int Ds \; q_{\{s\}} \,|\{s\}\rangle \tag{3.11}$$

where the smearing functions $q_{\{s\}}$ are chosen such that $|0\rangle$ is normalizable[4]. The vacuum is normalized as $\langle 0|0\rangle = 1$ and this constrains the smearing functions $q_{\{s\}}$

$$\int Ds |q_{\{s\}}|^2 = 1\,. \tag{3.12}$$

The explicit structure of $q_{\{s\}}$ is not required and we can chose any function which obeys the normalization above. By definition, the state $|0\rangle$ is annihilated by the annihilation operators in the Fock space and it is also renormalized such that it has zero energy. In this chapter we shall restrict to detecting states which are built by acting with hard operators on $|0\rangle$.

It will also be useful to define the projector onto states with zero energy[5]. Since the vacuum is the only state with zero energy in gravity, the projector

[4]Using a Kronecker-delta normalization in (3.9) would allow us to choose $|0\rangle$ equal to a particular value of $|\{s\}\rangle$ instead of smearing over all of them.

[5]It is more physical to consider a projector onto a thin band of energies near zero and it can be checked that such a projector (when appropriately normalized) tends to P_0 when the band size is close to zero.

onto states with zero energy is equivalent to the projector onto the vacuum state. The projector onto zero energy eigenstates can be expressed as [5]

$$P_0 = \int Ds \, |\{s\}\rangle \langle\{s\}| \,. \tag{3.13}$$

Since energy is measured using the ADM Hamiltonian, this projector is an element of the algebra of operators at $\mathcal{I}^+_-$.

3.3 Physical Protocol for detecting Massless particles in Flat spacetime

In this section we extend the main result presented in [10] for the case of quantum gravity coupled to massless fields in 3+1 dimensional flat spacetime[6]. As described in the previous section, the vacuum in flat spacetime is infinitely degenerate which leads to additional complications as compared to the AdS case. Therefore, to keep things simple we are going to study states which are built acting on vacuum $|0\rangle$ with hard operators. We shall discuss the implications of our protocol for more general states towards the end of the chapter.

The bulk state in general will be denoted by $|\psi\rangle$. All measurements are performed by observers who are located near $\mathcal{I}^+_-$. Like in the AdS case, the observers are given two kinds of abilities:

1. They can modify the state by acting on it with a unitary operator which has support on a small cut near $\mathcal{I}^+_-$.

2. They are allowed to measure the energy of the state or a state modified by the action of a unitary[7].

We will prove below that having these two abilities are enough for the observers to determine the state completely. This will help establish a physical protocol via which one can, in principle, design experiments which demonstrate the principle of holography of information.

Like in the AdS case, it is useful to work with a state that does not have an overlap with the vacuum $|0\rangle$. The same arguments of appendix A.2 can

[6]We will restrict to $3+1$ dimensional spacetime but it should be possible to generalize our results to any even dimensional spacetime.

[7]We assume that this measurement process does not induce a backreaction on the state.

be trivially extended to flat space as well. Henceforth, we shall assume that such a process has already been performed on a given state and $|\psi\rangle$ will denote states which do not have an overlap with the vacuum.

3.3.1 Basis used for construction

As emphasized before, one crucial difference between the construction in AdS [10] and flat spacetime is the absence of discrete energy eigenstates in the latter. This means that the energy eigenstates are a natural choice of basis for the expansion of the state in AdS but not in flat spacetime, as the energy is continuous. We therefore construct another basis called the *normal ordered basis* which allows us to reconstruct the state using a physical protocol[8]. It will be shown how this method allows us to follow similar steps for reconstruction as those in the AdS case. For simplicity, we first explain the protocol by working with a state which is built out of operators of a single flavour. We later extend this to states built with multiple flavours.

Any state $|\psi\rangle$ constructed out of a single flavoured field ϕ on top of the vacuum $|0\rangle$ can be expanded in the normal ordered basis as

$$|\psi\rangle = \int \sum_{n=1}^{\infty} \prod_{j=1}^{n} :\phi(u_j, \Omega_j): g_n(\vec{u}, \vec{\Omega}) d\vec{u} d\vec{\Omega} \, |0\rangle \tag{3.14}$$

where $g_n(\vec{u}, \vec{\Omega}) \equiv g_n(u_1, \Omega_1; u_2, \Omega_2; \cdots; u_n, \Omega_n)$ are certain smooth smearing functions. Here :: denotes *normal ordering* and it is defined by pushing all the creation operators in the expansion to the left and the annihilation operators to the right[9]. From the definition of the state $|\psi\rangle$ in (3.14) we see that $\langle 0|\psi\rangle = 0$. We show in the following subsections how this choice of basis allows us to compute the functions g_n in a *sieve procedure*, which means that g_n can be evaluated only after obtaining g_{n-1}.

The task of the observers is to determine the function $g_n(\vec{u}, \vec{\Omega})$ by performing certain kinds of measurements around $\mathcal{I}^+_-$. For example, using the

[8]The normal ordered basis can also be used in the AdS construction but we find that it is much more convenient to use the energy eigenstate basis in that case. It is important to note that this basis is formed out of continuous functions and there are certain subtle limitations in using this. These limitations are discussed in section 3.4.

[9]For example: $:\phi(u_1, \Omega_1) \cdots \phi(u_n, \Omega_n): |0\rangle \sim \int d\omega_1 \cdots d\omega_n e^{i\omega_1 u_1 + \cdots i\omega_n u_n} a^\dagger(\omega_1 \Omega_1) \cdots a^\dagger(\omega_n \Omega_n) |0\rangle$. The expansion of the field $\phi(u, \Omega)$ at $\mathcal{I}^+$ is derived in appendix A.4.

powers mentioned in section 3.3 . The observers are also allowed to manipulate the state by acting on it with a unitary operator located near a small cut at $\mathcal{I}^+_-$ and also measure the energy of the state. An expression for the energy in the Bondi gauge in $3+1$ dimensional flat spacetime is given as[10] (this is equal to $\mathcal{Q}_{l=0,m=0}$ as defined in (3.5))

$$E = \frac{1}{16\pi G_N} \int_{\mathcal{I}^+_-} \sqrt{\gamma} d^2\Omega \; m_B(u=-\infty, \Omega)\,. \tag{3.15}$$

Like in the AdS case, we shall assume that the Born rule is valid. Therefore, energy will be measured in a quantum sense with the energy of the vacuum state renormalized to 0. This means that the answer to "what is the frequency with which we obtain 0 upon measuring the energy of the state $|\psi\rangle$?" is given as[11]

$$\langle\psi|P_0|\psi\rangle \tag{3.16}$$

where P_0 is the projector onto the vacuum state $|0\rangle$ as defined in (3.13). For states of the form shown in eq.(3.14) we clearly have $\langle\psi|P_0|\psi\rangle = 0$. Henceforth, when we write that we measure the energy of the state, we always mean a measurement of the kind above. Notice that in order to measure the energy of the state, the observers need to be spread across the entire Celestial sphere as each of them only measures a part of the metric fluctuation m_B (which is suppressed by G_N, see eq.(3.3)). This ensures that there is no violation of causality (see section 3.4 for a discussion).

The observers are also allowed to modify the state by acting on it with some unitary U, and then measure the energy of the modified state $U|\psi\rangle$. The unitaries that we will be using are of the form

$$U_n = \exp\left[i \int_{-\infty}^{-1/\epsilon} d\vec{u}' d\vec{\Omega}' \prod_{j=1}^{n} f_n(\vec{u}', \vec{\Omega}') O(u'_j, \Omega'_j)\right]. \tag{3.17}$$

Here we denote the operators used by the observers with $O(u', \Omega')$ (although they are still the same field ϕ) with $f_n(u', \Omega')$'s being smearing functions

[10] A gauge invariant expression can be derived by following the procedure illustrated in [9].

[11] In all our measurements, we will only be concerned with the frequency with which the energy is zero. One can also consider projectors onto an energy band close to zero, but as discussed in section 3.4, such modifications do not alter the result. This means that as long as the energy of the state is within a range $[0, \delta]$, it will be assumed to be the vacuum state. For non-zero energies away from δ, such a measurement does not yield a useful result in flat spacetime as energy is continuous.

that localize these operators near $\mathcal{I}^+_-$, i.e, $u' \in (-\infty, -\frac{1}{\epsilon}]$. For these expectation values to be simple analytic functions it is useful to choose $O(u',\Omega') \sim u'\pi(u',\Omega')$, i.e, proportional to the conjugate momenta of the scalar fields. The factor of u' is added for the sake of maintaining dimensions but can always be absorbed by an appropriate choice of f.

We now show that a measurement of the form $\langle\psi|U_n^\dagger P_0 U_n|\psi\rangle$ will allow us to fix the functions g_n up to a phase factor. For this we first expand the unitaries up to the first order in f, i.e,

$$U_n = 1 + i\int d\vec{u}d\vec{\Omega}\prod_{j=1}^{n} f_n(\vec{u}',\Omega')O(u'_j,\Omega'_j) + \mathcal{O}(f^2)\,. \tag{3.18}$$

Henceforth, unless necessary, we shall suppress the $\mathcal{O}(f^2)$ terms in the expressions below.

Let us consider the measurement where we compute the energy of the state $U_1|\psi\rangle$ and compute the frequency with which we get zero. By the Born-rule, this is equivalent to computing $\langle\psi|U_1^\dagger P_0 U_1|\psi\rangle$. This correlator is equal to (we refer the reader to appendix A.3 for the details of this computation)

$$\langle\psi|U_1^\dagger P_0 U_1|\psi\rangle = \left|\int du du' d\vec{\Omega}d\vec{\Omega}'\; f_1(u',\Omega')g_1(u,\Omega)\,\langle 0|\phi(u,\Omega)O(u'_j,\Omega'_j)|0\rangle\right|^2. \tag{3.19}$$

The correlation function above allows us to determine the function $g_1(u,\Omega)$ up to a phase factor, which we denote by $e^{i\theta_1}$

$$\int du du' d\vec{\Omega}d\vec{\Omega}'\; f_1(u',\Omega')g_1(u,\Omega)\,\langle 0|\phi(u,\Omega)O(u'_j,\Omega'_j)|0\rangle = \sqrt{\langle\psi|U_1^\dagger P_0 U_1|\psi\rangle}e^{i\theta_1}\,.$$

In appendix A.5 we explain how the function g_1 can be reconstructed from such an integral equation. Since the overall phase of the state $|\psi\rangle$ is not a physically measurable quantity, we can choose it such that $e^{i\theta_1} = 1$. This integral equation completely fixes the function g_1 for us[12]. We refer the reader to appendix A.5 for further details.

In the following subsection 3.3.2 we explain how we obtain the functions g_n when $n \neq 1$ using appropriate correlation functions. Subsequently in subsections 3.3.3-3.3.5 we develop a physical protocol that a set of observers near $\mathcal{I}^+_-$ can use to recover the states.

[12] It is useful to contrast the operator U_1 with the operator X_r defined in the previous chapter.

3.3.2 Information recovery using correlation functions

We demonstrate a simple use of the normal ordered basis (3.14) by allowing ourselves to measure arbitrary expectation values. We show how one can easily obtain all g_n's having determined g_1, by measuring specific correlation functions. We note that this procedure has to be performed in a sieve-like manner, i.e, we can determine the value of g_n once we know the value of g_{n-1}. Let us consider the following correlation function $\langle\psi|U_1^\dagger P_0 U_n|\psi\rangle$ at $O(f^2)$

$$\langle\psi|U_1^\dagger P_0 U_n|\psi\rangle = \int d\vec{u}d\vec{u}'d\vec{\Omega}d\vec{\Omega}'dudu'd\Omega d\Omega' \sum_{j=1}^{n} f_1(u',\Omega')f_n(\vec{u}',\vec{\Omega}')g_1^*(u,\Omega)g_j(\vec{u},\vec{\Omega}) \times \langle 0|\phi(u,\Omega)O(u',\Omega')|0\rangle \langle 0|O_1\cdots O_n : \phi_1\cdots\phi_j : |0\rangle \tag{3.20}$$

where we use the shorthand notation $O_n \equiv O(u_1',\Omega_1';\cdots;u_n',\Omega_n')$, $\phi_n \equiv \phi(u_1,\Omega_1;\cdots;u_n,\Omega_n)$ and $g_n(\vec{u},\vec{\Omega}) \equiv g_n(u_1,\Omega_1;\cdots;u_n,\Omega_n)$. Therefore upon measuring $\langle\psi|U_1^\dagger P_0 U_n|\psi\rangle$ for all n, starting with $n=2$, we easily obtain the value for all $g_{n>1}(\vec{u},\vec{\Omega})$.

Such a measurement is not physically viable as the final answer is not real in general. However, it demonstrates a simple use of the normal ordered basis in order to extract information about the state in a sieve-like manner. It has been argued in [5] that the measurement of all possible correlation functions allows a complete reconstruction of the state and hence it was expected that such a procedure should exist.

3.3.3 Physical Protocol

In the following section we explain how we can obtain the functions g_n by performing measurements that are physically viable.

As explained in the subsection 3.3.1, we can obtain the function g_1 by measuring $\langle\psi|U_1^\dagger P_0 U_1|\psi\rangle$ and exploiting the freedom to choose the overall phase of the state $|\psi\rangle$. Since we can only fix the overall phase of the state once, this procedure will still leave a phase ambiguity $e^{i\theta_n}$ for all other $g_{n>1}$. To see this ambiguity explicitly, we first modify the state by acting on it with a unitary U_n, then measure its energy and see the frequency with which we get 0. Using the Born-rule this is equivalent to measuring $\langle\psi|U_n^\dagger P_0 U_n|\psi\rangle$.

Upon expanding this to $\mathcal{O}(f^2)$ we get,

$$\langle\psi|U_n^\dagger P_0 U_n|\psi\rangle = \left|\langle 0|\int d\vec{u}' d\vec{\Omega}' f_n(\vec{u}',\vec{\Omega}') O_1 \cdots O_n|\psi\rangle\right|^2. \quad (3.21)$$

By inverting this relation we obtain the correlation function which characterizes the phase ambiguity $e^{i\theta_n}$

$$\langle 0|\int d\vec{u}' d\vec{\Omega}' f_n(\vec{u}',\vec{\Omega}') O_1 \cdots O_n|\psi\rangle = \sqrt{\langle\psi|U_n^\dagger P_0 U_n|\psi\rangle} e^{i\theta_n}. \quad (3.22)$$

The advantage of expanding $|\psi\rangle$ in the normal ordered basis (3.14) becomes obvious in this step; using such an expansion ensures that the only g_i's contributing to the correlator on the LHS are for $i \leq n$. Hence we have to evaluate these correlators in a sieve-like procedure since only after one determines the value of g_n, one can determine g_{n+1}.

We will now demonstrate how the phase ambiguities $e^{i\theta_n}$ can be fixed by making a two-step measurement. This requires the action of two unitary operators on the state and then measuring its energy. This results in correlation functions of the form $\langle\psi|U_j^\dagger U_i^\dagger P_0 U_i U_j|\psi\rangle$. In the following subsections we explain how this fixes the value of θ_n completely, by first determining $\cos\theta_n$ and then the value of $\sin\theta_n$.

3.3.4 Determining $\cos\theta_n$

As shown above, the phase of g_1 is completely fixed by making a choice for the overall phase of the state $|\psi\rangle$. This will allow us to compute the value of $\cos\theta_n$ by performing a two-step measurement of the form $\langle\psi|U_n^\dagger U_1^\dagger P_0 U_1 U_n|\psi\rangle$ at $\mathcal{O}(f^2)$. A simple calculation shows

$$\cos\theta_n = \frac{\langle\psi|U_1^\dagger U_n^\dagger P_0 U_n U_1|\psi\rangle - \langle\psi|U_1^\dagger P_0 U_1|\psi\rangle - \langle\psi|U_n^\dagger P_0 U_n|\psi\rangle}{2\sqrt{\langle g|U_1^\dagger P_0 U_1|g\rangle}\sqrt{\langle g|U_n^\dagger P_0 U_n|g\rangle}}. \quad (3.23)$$

However, this does not completely fix θ_n since determination of $\cos\theta_n$ leaves us with an ambiguity for the sign of $\sin\theta_n$. In the following subsection we explain how we can fix the value of $\sin\theta_n$ for all n.

3.3.5 Determining $\sin\theta_n$

In order to fix $\sin\theta_n$ we just need one g_n whose phase $e^{i\theta_n}$ is not purely real. The fact that the phase of the function g_1 was chosen to be purely real allowed us to measure the value of $\cos\theta_n$. However in general we do not expect the phases of all g_n to be purely real. This can be checked by evaluating the value of $\cos\theta_n$ using eq.(3.23) and as long as for some $n = n_0$, $\cos\theta_{n_0} \neq \pm 1$, we can perform an analogues two-step measurement involving U_{n_0} to determine the sign of $\sin\theta_n$. Such an n_0 can be easily obtained by trial and error. Then, we can perform a measurement of the form $\langle\psi|U^\dagger_{n_0}U^\dagger_n P_0 U_n U_{n_0}|\psi\rangle$ at $\mathcal{O}(f^2)$

$$\langle\psi|U^\dagger_n U^\dagger_{n_0} P_0 U_{n_0} U_n|\psi\rangle = \langle\psi|U^\dagger_n P_0 U^\dagger_n|\psi\rangle + \langle\psi|U^\dagger_{n_0} P_0 U^\dagger_{n_0}|\psi\rangle \\ + 2\sqrt{\langle\psi|U^\dagger_n P_0 U_n|\psi\rangle}\sqrt{\langle\psi|U^\dagger_{n_0} P_0 U_{n_0}|\psi\rangle}\big(\cos\theta_n\cos\theta_{n_0} + \sin\theta_n\sin\theta_{n_0}\big)\,. \tag{3.24}$$

In this measurement, we end up with a correlated ambiguity in the phases of g_n and g_{n_0}, which implies that upon knowing the value of $\sin\theta_{n_0} \neq 0$ for any given n_0, we can easily determine the value of all other $\sin\theta_n$. We now explain how we can fix the value of $\sin\theta_{n_0}$.

Determining the sign of $\sin\theta_{n_0}$

As shown in the previous section, by performing certain simple measurements up to $\mathcal{O}(f^2)$, we decode a lot of information about the state $|\psi\rangle$. However, we are left with one final sign ambiguity concerning $\sin\theta_{n_0}$. Since this is only one sign ambiguity, we just need one measurement which can distinguish between $e^{i\theta_{n_0}}$ and $e^{-i\theta_{n_0}}$.

We shall work with the special case $n_0 = 2$ to demonstrate the procedure, but the method can be performed for a generic n_0 as well. Consider a measurement of the kind $\langle\psi|U^\dagger_1 P_0 U_1|\psi\rangle$, upon expanding it up to $\mathcal{O}(f^3)$ we get

$$\langle\psi|U^\dagger_1 P_0 U_1|\psi\rangle = \int du'_1 du'_2 d\Omega'_1 d\Omega'_2 f_1(u'_1,\Omega'_1) f_1(u'_2,\Omega'_2)\,\langle\psi|O(u'_1,\Omega'_1)P_0 O(u'_2,\Omega'_2)|\psi\rangle \\ + i\int d\vec{u}' d\vec{\Omega}' f_1(u'_1,\Omega'_1) f_1(u'_2,\Omega'_2) f_1(u'_3,\Omega'_3)\Big[\,\langle\psi|O_1 P_0 O_2 O_3|\psi\rangle - \langle\psi|O_2 O_3 P_0 O_1|\psi\rangle\,\Big] \tag{3.25}$$

where $d\vec{u}' d\vec{\Omega}' = du'_1 du'_2 du'_3 d\Omega'_1 d\Omega'_2 d\Omega'_3$ and $O_i = O(u'_i,\Omega'_i)$. As explained in the sections before, the terms at $\mathcal{O}(f^2)$ enable us to completely fix the function $g_1(u,\Omega)$. Let us now focus on the term at $\mathcal{O}(f^3)$ and expand $|\psi\rangle$ using (3.14),

$$
\begin{aligned}
&i\int d\vec{u}' d\vec{\Omega}' f_1(u_1',\Omega_1')f_1(u_2',\Omega_2')f_1(u_3',\Omega_3')\Big[\langle\psi|O_1P_0O_2O_3|\psi\rangle - \langle\psi|O_2O_3P_0O_1|\psi\rangle\Big] \\
&= i\int d\vec{u}' d\vec{\Omega}' du d\Omega d\vec{u} d\vec{\Omega}\; f_1(u_1',\Omega_1')f_1(u_2',\Omega_2')f_1(u_3',\Omega_3')g_1(u,\Omega) \\
&\quad\times\Big[g_2(\vec{u},\vec{\Omega})\,\langle 0|\phi(u,\Omega)O_1|0\rangle\langle 0|O_2O_3 :\phi_1\phi_2: |0\rangle - g_2^*(\vec{u},\vec{\Omega})\,\langle 0| :\phi_1\phi_2: O_2O_3|0\rangle\langle 0|O_1\phi(u,\Omega)|0\rangle\Big]
\end{aligned}
\tag{3.26}
$$

where we have used the notation $\phi_n = \phi(u_n,\Omega_n)$. Such a measurement does not allow us to reconstruct the full function g_2 since the only localizing functions appearing in this expression are dependent on $f_1(u',\Omega')$. However it is still sensitive to the sign of $\sin\,\theta_2$ since (see appendix A.4 for details on evaluating such correlators)

$$
\langle\psi|U_1^\dagger P_0 U_1|\psi\rangle \;\text{ at } \mathcal{O}(f^3)\; \sim (g_2 - g_2^*) \sim \sin\,\theta_2\,. \tag{3.27}
$$

Upon determining the sign of $\sin\,\theta_2$ we can determine all the other signs using (3.24). Such a construction can be easily extended for any $n_0 \neq 2$ as well. This allows us to determine the value of $\sin\,\theta_{n_0}$ and therefore fix the full wave function $|\psi\rangle$.

Notice that for this proof we are assuming that we have access to both even and odd g_n's, i.e, we need at least one of each g_{odd} and $g_{even} \neq 0$. This excludes the two special cases when the only non-zero g_n's are either made out of all even or all odd n. These cases can be dealt with in a similar manner as above but with a slight difference and are presented in appendix A.6.

3.3.6 Multiple flavors

We now explain how the results of the previous section can be extended to cases when we have multiple flavours[13]. To keep things simple, we illustrate the algorithm when we have two flavours only, but it can be trivially generalized for multiple flavours.

[13] Although we write all explicit formulas for scalar fields, they all trivially generalize to the fields with higher spins.

Consider the following state

$$|\psi\rangle = \int d\vec{u}d\vec{\Omega}\Big[-g_{00} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_{ij} :\phi_1\cdots\phi_i::\tilde{\phi}_1\cdots\tilde{\phi}_j:\Big]|0\rangle \tag{3.28}$$

where the term $-g_{00}$ ensures that $\langle 0|\psi\rangle = 0$ and ϕ, $\tilde{\phi}$ are two different flavours of scalar fields. We have omitted the dependence of g_{ij} and $\phi, \tilde{\phi}$ on (u, Ω) to avoid a clutter of notation. In a manner similar to the previous sections, we shall determine the values of all g_{ij} in a sieve-like procedure. The unitary operator used to modify the state is denoted as,

$$U_{ij} = \exp\left[i\int d\vec{u}'d\vec{\Omega}'\ f_{ij}(\vec{u}', \vec{\Omega}')O_1\cdots O_i\tilde{O}_1\cdots\tilde{O}_j\right] \tag{3.29}$$

where O and $\tilde{O}$, describe the same fields as ϕ and $\tilde{\phi}$, but they are localized near $\mathcal{I}^+_-$ (as ensured by f_{ij}).

We first explain how the functions g_{ij} are recovered using arbitrary correlation functions and then go on to demonstrate the same using a physical protocol. The steps performed are very similar to the ones in the case of a single flavour.

Arbitrary correlators

Here we extend the procedure presented in section 3.3.2 to determine the functions g_n for multiple flavours. We first have to measure the value of g_{10} by measuring $\langle\psi|U_{10}^{\dagger}P_0U_{10}|\psi\rangle$ at $\mathcal{O}(f^2)$. This allows us to measure g_{10} up to a phase $e^{i\theta_{10}}$. This phase θ_{10} can be set to zero by exploiting the fact that the overall phase of the state $|\psi\rangle$ is not a physically measurable quantity.

Upon determining g_{10} we can easily determine all the other g_{ij}'s in a sieve-like manner by evaluating the following correlation function

$$\begin{aligned}&\langle\psi|U_{10}^{\dagger}P_0U_{ij}|\psi\rangle\\ &= \int dV f_{10}f_{ij}g_{10}\sum_{m=0}^{i}\sum_{n=0}^{j} g_{mn}\langle 0|\phi_1O_1|0\rangle\langle 0|O_1\cdots O_i\tilde{O}_i\cdots\tilde{O}_j:\phi_1\cdots\phi_m::\tilde{\phi}_1\cdots\tilde{\phi}_n:|0\rangle\end{aligned} \tag{3.30}$$

with $dV = d\vec{u}d\vec{u}'d\Omega d\Omega'$ and we use the same shorthand notation for ϕ_i, etc. as defined earlier. Clearly this is not a physical protocol since the correlation functions in general are not real. However it demonstrates the use of the normal ordered basis to decode generic states. In the following section we explain how one can determine the value of the smearing functions by using a physical protocol.

Physical Protocol

We now explain how the analysis in the previous section can be extended to a physical protocol where the only kind of measurements allowed are the ones which give real answers. For a single flavour this is explained in section 3.3.3. Due to the conceptual similarity between the single and the multiple flavoured case it suffices to summarize the main steps without repeating the details. We remind the reader that this entire procedure is performed in a sieve-like manner, i.e, one should have measured the value of g_{mn} before determining $g_{i>m,j>n}$.

1. We first fix the value of g_{10} by exploiting the fact that the overall phase of the wave function is meaningless. This is exactly similar to fixing the value of g_1 in the single flavour case.

2. The general phase ambiguity in the function g_{ij} can be quantified by the measurement at $\mathcal{O}(f^2)$ of

$$\langle s| \int f_{ij} O_1 \cdots O_i \tilde{O}_1 \cdots \tilde{O}_j |\psi\rangle = \sqrt{\langle\psi|U_{ij}^\dagger P_0 U_{ij}|\psi\rangle} e^{i\theta_{ij}} \,. \tag{3.31}$$

   The only phase fixed till now is $\theta_{10} = 0$.

3. By measuring $\langle\psi|U_{ij}^\dagger P_0 U_{ij}|\psi\rangle$ we obtain the value of g_{ij} up to a phase factor $e^{i\theta_{ij}}$.

4. The value of cos θ_{ij} can be fixed by measuring $\langle\psi|U_{ij}^\dagger U_{10}^\dagger P_0 U_{10} U_{ij}|\psi\rangle$ at $\mathcal{O}(f^2)$. This uses the fact that we have fixed $\theta_{10} = 0$. Fixing the value of cos θ_{ij} does not completely fix θ_{ij} since it leaves us with a sign ambiguity for sin θ_{ij} $\forall\ i, j$.

5. In order to fix the value of sin θ_{ij} for all i, j, we first need to fix the value of sin θ_{ij} for some particular i, j. This can be done (for example) for $i = 2, j = 0$ by measuring $\langle\psi|U_{10}^\dagger P_0 U_{10}|\psi\rangle$ at $\mathcal{O}(f^3)$. Noting the analogy with the single flavoured case, the value of other sin θ_{ij} can be fixed by measuring $\langle\psi|U_{20}^\dagger U_{ij}^\dagger P_0 U_{ij} U_{20}|\psi\rangle$ $\forall\ i, j$.

The special cases where we have all even or all odd g's are similar to those of the single flavoured case and are shown in appendix A.6.

3.4 Discussions

Main Result

We presented a physical protocol that observers localized on a cut near $\mathcal{I}^+_-$, in a theory of quantum gravity coupled to massless matter in $3+1$ dimensional asymptotically flat spacetime, can use to detect bulk massless excitations on a given vacuum state $|0\rangle$. The protocol comprises the manipulation of the state with unitary operators and the measurement of its overlap with the vacuum. These measurements are performed in a manner which is quantum mechanical in nature and utilizes the Born-rule. The protocol presented in this chapter is an extension of a similar result for asymptotically AdS spacetimes [10] presented in chapter 2. The crucial difference between the two is of a technical nature. In the AdS case, we used the basis of energy eigenstates to expand any given state and this step was crucial for developing the protocol. Unlike AdS, the energy spectrum in flat spacetime is continuous, thus the energy eigen basis (which can be thought of as a band of energy) is not a convenient basis to expand the states. To overcome this difficulty, we introduced a normal ordered basis (3.14) to expand the states. Then the observers that are localized at a small cut near $\mathcal{I}^+_-$ can extract information in a sieve-like procedure.

Limitation of the Normal Ordered Basis

The state $|\psi\rangle$ when expanded in the normal ordered basis (3.14) is defined using smearing functions $g_n(\vec{u}, \vec{\Omega})$ which are continuous. As shown in appendix A.5, these functions are reconstructed from their moments. In principle, one needs infinitely many moments to reconstruct a function exactly, however, in any physical experiment we only have access to a finite number of moments. This introduces a natural cut-off in the accuracy with which the smearing functions can be reconstructed. For example, the protocol described in the main text fails when the smearing functions are highly oscillatory, i.e, non-smooth. This is because, it will not be possible for the observers to distinguish between two smearing functions which differ beyond a certain moment.

Effect of Noise

In all the measurements we assume that the energy of the vacuum state is a known quantity and is renormalized to zero. Therefore in most measurements that we perform, we end up with the projector onto the vacuum state P_0. However in general one can also consider projectors onto a small but finite energy P_δ and ask if the results change as we take $\delta \to 0$. It is simple to see that upon normalizing the projector correctly one does not end up getting different results from the ones obtained by using P_0. Thus, as long as the energy of a state is in between $[0, \delta]$, the state is assumed to be the vacuum state. Such projectors have also been considered in [76] in relation to the monogamy paradox in flat spacetime.

Construction vs Reconstruction

By using the Reeh-Schlieder theorem [5, 53, 54] it is possible to show that any state can be created by acting with hermitian operators, localized near $\mathcal{I}^+_-$, on the vacuum $|0\rangle$[14]. This is also possible in AdS [10, 4] where any state can be created by acting with operators, localized on a thin time band near the boundary, on the vacuum. Such a construction only requires the positivity of energy as measured from asymptotic infinity and is valid for non-gravitational QFTs as well.

However note that the process of detecting a state by being able to perform measurements only using operators localized at a small cut near the boundary is a non-trivial process as compared to creating the state. Reeh-Schlieder like constructions do not guarantee that the state can be reconstructed by performing measurements near the boundary of spacetime (both in flat spacetime and in AdS). One simple way to see this is by considering the case of a local QFT in flat spacetime. Even in this case, it is possible to construct any state of the QFT by acting with operators (near $\mathcal{I}^+_-$) on the vacuum. However, as argued in the main text, it is clearly not possible to reconstruct this state exactly since there exists local operators in the bulk which commute with observables at the boundary. Therefore, although it is typically possible to construct any state in flat spacetime by acting with operators near $\mathcal{I}^+_-$, reconstructing the state is only possible in a quantum theory of gravity.

[14] More generally this construction works by acting with an operator on a state which is cyclic and separating.

Detection of Memory Effect & States Built on Multiple Vacua

The protocol established in this chapter gives a procedure on how to determine hard excitations on a given vacuum state. However, it is possible to consider states which are built on multiple vacuum states, for example,

$$|\Psi\rangle \equiv \sum_{m=1}^{M}\sum_{n=1}^{\infty}\prod_{j=1}^{n}\int :\phi(u_j,\Omega_j):g_n^{(m)}(\vec{u},\vec{\Omega})d\vec{u}d\vec{\Omega}\,|0_{(m)}\rangle \ .$$

The main difficulty in decoding states of this kind is that the observers near $\mathcal{I}^+_-$ do not apriori know which vacuum states $|0_{(m)}\rangle$ are used to build $|\Psi\rangle$. If the observers are given priors about the particular $|0_{(m)}\rangle$'s that appear, they can use certain special unitaries constructed out of the superrotation charge [77] in order to bring the state $|\Psi\rangle$ to the form of (3.14). Without that information, it is not possible to detect the state using the protocol illustrated in the main text. In order to detect these states in the absence of priors, one would have to make use of the transition operators defined in [5]. However, it is not clear how one can measure such transition operators in a physical process.

There is a similar restriction for detecting the Memory effect [73, 78] as it concerns the transition from a given configuration of vacuum to another one. Such transitions typically occur in scattering processes from $\mathcal{I}^-$ to $\mathcal{I}^+$ but it is not a priori clear how one can study them by performing measurement only at $\mathcal{I}^+$ (or $\mathcal{I}^-$), we hope to address this in a future work.

Generalizations to Higher dimensions & Massive fields

Due to the similarities between supertranslations in four dimensions and other higher even dimensions [26, 79, 23] our results should easily generalize to higher even dimensions. However, generalizing this result to odd dimensions has to be explored in greater detail. Additionally, it would be interesting to extend such a protocol in the presence of massive fields and establish the principle of holography of information in that context.

Chapter 4

Holography From the WDW Equation

4.1 Introduction

It has been argued in a series of papers that theories of gravity localize quantum information very differently from local quantum field theories [5]. This argument can be encapsulated in a principle of holography of information: in a theory of quantum gravity, information that is available in the bulk of a Cauchy slice is also available near its boundary [2]. This principle can be made precise and proved in asymptotically AdS spacetimes and in four-dimensional asymptotically flat spacetimes. In [10, 11], a physical protocol was presented that exploited this effect to allow observers near the boundary of AdS and flat spacetime to extract information about low-energy states in the bulk without directly exploring the bulk. These protocols have been reviewed in the previous two chapters.

In the presence of a negative cosmological constant, these effects may be expected from the AdS/CFT conjecture [1, 48, 3]. But a study of how quantum gravity localizes information sheds light on the physical origin of holography for gravitational theories. It also indicates how holography should be extended beyond asymptotically AdS spacetimes to asymptotically flat spacetimes.

In this chapter, we present a direct perturbative analysis of the allowed wavefunctionals in a theory of gravity coupled to matter in an asymptotically AdS spacetime. We find that any two wavefunctionals that coincide at the boundary for an infinitesimal interval of time must also coincide in the bulk. This is a uniquely gravitational effect; wavefunctionals in a local quantum field theory do not have such a property[1].

In gravity, the metric is one of the dynamical degrees of freedom. In the Hamiltonian formalism, the degrees of freedom are divided into the metric on a spatial slice and its conjugate momentum, which is related to the extrinsic curvature of the slice. We consider theories that might have additional matter fields. The values of these fields on a spatial slice provide another set of canonical variables whose conjugate momenta are related to the time derivatives of these fields. A wavefunctional assigns a complex number to any specification of the metric and other fields on a spatial slice.

Not every wavefunctional is a valid state in a theory of gravity. A valid wavefunctional must take on the same value for configurations that can be related by a diffeomorphism that vanishes asymptotically. This leads to a set of constraints on the wavefunctional, of which the most important constraint is called the Wheeler-DeWitt (WDW) equation [80].

In this chapter, we present a direct perturbative analysis of the WDW equation. We build on an important old paper by Kuchar [81] who analyzed the solutions of the WDW equation about flat space in the free limit. We extend this analysis by expanding the constraints to leading nontrivial order in perturbation theory in the gravitational interaction in the presence of a negative cosmological constant. This analysis is already sufficient to reveal the remarkable property of these solutions alluded to above.

The structure of the constraints that we find can roughly be described as follows. The metric degrees of freedom can be divided into a longitudinal component, a transverse-traceless component and, what we call, a "T-component" that keeps track of the trace [82]. The transverse-traceless component can be freely specified, just like another dynamical field. Invariance of the wavefunctional under spatial diffeomorphisms fixes its dependence on the longitudinal component of the metric. The so-called Hamiltonian constraint, which imposes invariance of the state under diffeomorphisms that mix space and time, fixes the dependence of the wavefunctional on the T-component. We show that an important role is played by a specific integral of the Hamiltonian constraint on the entire Cauchy slice which relates the asymptotic T-component of the metric to the total energy of the transverse-traceless gravitons and matter-fields on the Cauchy slice.

[1]See appendix B.1 for a concrete example in QED

We prove that these constraints are sufficient to disallow any deformations of the wavefunctional which alters its form in the bulk without changing its boundary values. The reason can be understood as follows. A bulk deformation that changes the energy must necessarily also change the T-component of the metric near the boundary. So deformations that leave the asymptotic T-component unchanged can only "move" energy from one part of space to another and must have zero total energy. But the Heisenberg uncertainty principle tells us that an operator that implements such a deformation must be completely delocalized. Therefore, while such an operator may commute with the asymptotic metric, it must fail to commute with some other dynamical operator near the boundary. The final result is that correlators of the T-component of the metric and of other dynamical operators at the boundary of AdS for an infinitesimal amount of time completely fix the wavefunctional.

This result establishes, in the perturbative approximation, that one of the central aspects of holography follows from the constraints of gravity. The significance of this result can be illustrated by studying the contrast between gravitational and non-gravitational quantum field theories in AdS. Even in a non-gravitational theory, the specification of data on the entire timelike boundary of AdS is sufficient to reconstruct physics in the bulk. See Figure 4.1a. This is just a property of the causal structure and is not indicative of holography. What our result shows is that, in a gravitational theory, data on an infinitesimal time band on the boundary of AdS is already sufficient to reconstruct the state in the bulk. See Figure 4.1b.

We emphasize that in a non-gravitational theory, our final result could not possibly be true. The action of a unitary operator in a bulk at the same time would commute with all observations on this infinitesimal time band on the boundary by microcausality. Therefore, in a non-gravitational theory, it is impossible to distinguish a given state from the state obtained after the action of such a unitary.

Relationship to previous work. As mentioned above, it has already been argued previously [5, 2] that gravitational theories localize information very differently from ordinary quantum field theories. These previous arguments [5, 2], which built on [4, 83], relied on weak assumptions about the structure of the Hilbert space, and the nature of the gravitational Hamiltonian to arrive at nonperturbative results.

Although the analysis in this chapter is perturbative, it is more explicit.

(a) (b)

Figure 4.1: A common misunderstanding of "holography" is that it only tells us that data on the timelike boundary of AdS can be used to reconstruct physics at the bulk point P as shown in the left subfigure. But this statement follows from the causal structure of AdS and does not require holography. Gravitational theories are genuinely holographic. Here we show how, in gravity, data on an infinitesimal time band (right subfigure) can be used to reconstruct physics in the bulk.

We make no prior assumptions either about the Hilbert space or about the gravitational Hamiltonian. Instead, we explicitly construct the low-energy Hilbert space by studying solutions to the gravitational constraints and we explicitly show that such solutions must have correlations between a component of the asymptotic metric and the energy of the state. This analysis also reveals how the unusual localization of quantum information in quantum gravity is visible at the level of wavefunctionals.

The analysis in this chapter takes advantage of the infrared cutoff that is provided by global AdS boundary conditions. (See comment 6 in section 4.6.) For this reason the analysis presented here reproduces Result 5 of [5] — which pertains to asymptotically AdS spacetimes and was proved there using operator-theoretic techniques — but cannot immediately be used to make contact with Result 1 and Result 2 of [5], which apply to asymptotically-flat spacetime. We expect that it should be possible to generalize the proof of the holography of information presented here to address the infrared subtleties present in flat space.

The relationship between the bulk constraints and holography was also explored previously in [27, 28] and more recently in [30] (see also the prescient essay [84]) although the techniques used in this chapter are quite different. A radial version of the WDW equation was studied in the context of AdS/CFT [85], which was analyzed further in [86] and has proved to be useful in the context of the study of $T\overline{T}$ deformations [87, 88, 89, 90, 91, 92, 93, 94, 95] and bulk reconstruction [96]. Here our analysis is different since we are considering the conventional WDW equation that governs wavefunctionals on a Cauchy slice.

The Wheeler-DeWitt equation has been studied in the mini-superspace approximation — for which we found [97, 98, 99, 100, 101] useful — and also in the context of two-dimensional models [102, 103, 104, 105, 106] and in terms of the Ashtekar variables [107]. See [108, 109, 110] for a more detailed list of references. However, there has been relatively little work on a straightforward perturbative analysis of the equation in higher dimensions. As already mentioned, Kuchar [81] studied this problem at zeroth order in the gravitational constant, and here we will show that, even at leading order, the structure of the constraints is interesting and leads to surprising properties of the solutions.

The question of how the gravitational constraints affect the localization of quantum information was also studied, from another perspective, in [65, 111]. (See also [112, 59].) However, these papers reached the opposite conclusion from the one we will reach here: in [65, 111] it was claimed that it should be possible to perturbatively construct states that differ inside a bounded region but are asymptotically identical. It appears to us that this conclusion was reached because [65, 111] focused on the asymptotic gravitational field but failed to consider quantum correlators of the metric and the dynamical scalar field that was included in the analysis there. As we will see in section 4.6 this latter class of correlators, involving both the metric and dynamical fields, plays an important role and cannot be neglected.

As already stated, the results we derive here are valid for theories of gravity and do <u>not</u> have an analogue in non-gravitational gauge theories. To illustrate this difference, in Appendix B.1, we analyze the constraints in electromagnetism. We show that they are significantly weaker than the constraints in a theory of gravity. Consequently, QED and other nongravitational gauge theories localize information much like ordinary local quantum field theories and does not share the unusual constrained properties of gravitational wavefunctionals. Appendices B.2, B.3 and B.4 provide additional technical details.

4.2 Summary of this chapter

We now provide a concise summary of our results in this chapter. The equations in this section are all linked to corresponding equations in later sections, which provide a more detailed discussion of the physics.

When gravity is quantized using the canonical formalism, the physical states of the theory are given by wavefunctionals of the metric g_{ij} on a spatial slice, and the matter fields ϕ that obey the so-called Hamiltonian and momentum constraints,

$$\mathcal{H}(f)\Psi[g,\phi] = 0, \qquad \mathcal{H}_i(f)\Psi[g,\phi] = 0 \ . \tag{4.1}$$

These are the constraints displayed in equation (4.22) — where the conjugate momenta for the metric and the matter fields are denoted $\pi^{ij} = -i\frac{\delta}{\delta g_{ij}}$ and $\pi = -i\frac{\delta}{\delta \phi}$ — which have been smeared with a function f that vanishes at the boundary. The momentum constraint is linear in momenta while the Hamiltonian constraint is quadratic.

To study these constraints, we first expand the metric about a background AdS metric as $g_{ij} = \gamma_{ij} + \kappa h_{ij}$ where γ_{ij} is the AdS metric and $\kappa = \sqrt{8\pi G}$. We also introduce a corresponding momentum operator $\Pi^{ij} = -\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta h_{ij}}$ that is more appropriate for understanding perturbation theory. We further decompose this metric fluctuation as

$$h_{ij} = h_{ij}^{\mathrm{TT}} + h_{ij}^{\mathrm{L}} + h_{ij}^{\mathrm{T}}, \tag{4.2}$$

in terms of the transverse-traceless component, the longitudinal component, and what we term the "T-component". This decomposition was introduced about flat space in [82], and we generalize it to AdS. The precise definition of the three components is given in equation (4.40). Similarly, the conjugate momentum can be decomposed as $\Pi^{ij} = \Pi^{ij}_{\mathrm{T}} + \Pi^{ij}_{\mathrm{TT}} + \Pi^{ij}_{\mathrm{L}}$ and we show below equation (4.50) that each component is the canonical momentum associated with the corresponding metric component.

We then expand the constraints in perturbation theory. It is convenient to set the AdS scale, $\ell = 1$ as done in the rest of the thesis, and treat κ as a small dimensionless parameter that allows us to organize the perturbative expansion. The validity of perturbation theory then requires that any numbers that emerge from the action of derivative operators on the wavefunctional should not scale with $\frac{1}{\kappa}$ and we ensure this below.

At leading order in κ, the momentum constraint implies that the wavefunctional $\Psi[h,\phi]$ is independent of h^{L}. This is simply the statement that the wavefunctional should be invariant under linearized spatial diffeomorphisms. At next order, it gives

$$\left(-2\nabla_j \Pi^{ij}_{\mathrm{L}} + \kappa\, Q^i\right) \Psi[h,\phi] = 0 \ , \tag{4.3}$$

where Q^i is quadratic in the canonical variables and is given in (4.61). We have $\Pi^{ij}_{\mathrm{L}} = -\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta h^{\mathrm{L}}_{ij}}$ so the second order momentum constraint determines the dependence of Ψ in h^{L}_{ij}.

At leading order in κ, the Hamiltonian constraint implies that the T-component of the metric vanishes: $h^{\mathrm{T}} = 0 + \mathrm{O}(\kappa)$. At next order, the Hamiltonian constraint fixes the T-component of the metric via

$$\left(-\mathcal{D}^{ij} h^{\mathrm{T}}_{ij} + \kappa\, Q\right) \Psi[h,\phi] = 0 \ , \tag{4.4}$$

where $\mathcal{D}^{ij}$ is given in (4.131) and Q is given in (4.74). This sets h^{T}_{ij} to a non-trivial $\mathrm{O}(\kappa)$ value.

To analyze these constraints, we first integrate the Hamiltonian constraint over a Cauchy slice Σ to obtain a simpler constraint, which takes the form

$$\left(-H_\partial + \int_\Sigma d^d x \sqrt{\gamma}\, N \mathcal{H}_{\mathrm{bulk}}\right) \Psi[h,\phi] = 0 \ , \tag{4.5}$$

where

$$H_\partial \equiv \frac{1}{2\kappa}\int_{\partial\Sigma} d^{d-1}\Omega\, J^i n_i \ . \tag{4.6}$$

Here the ADM current J^i, which is integrated over the boundary $\partial\Sigma$ after contracting with the normal n_i, is linear in the metric fluctuation and defined in (4.39). It depends only on the T-component of the metric as shown in (4.66) and gives the ADM energy H_∂. In (4.5), N is the lapse function; $\mathcal{H}_{\mathrm{bulk}}$ is quadratic in the canonical variables and its precise definition is given in equation (4.89). It can be viewed as the "bulk energy density" involving the transverse-traceless gravitons and the matter. Thus, the integrated Hamiltonian constraint gives a quantum version of the familiar statement that the energy is a boundary term in canonical gravity.

Since the integrated Hamiltonian constraint is so simple, we can explicitly obtain wavefunctionals that solve it. The solutions take the form of a "dressed" Fock space that we construct as follows. First, we obtain wavefunctionals of h^{TT} and ϕ that form an ordinary free-field Fock space in AdS and are eigenstates of the free-field Hamiltonian. We choose a basis for these wavefunctionals that we denote by

$$\psi_{\mathrm{F}}^{E,\{a\}}[h^{\mathrm{TT}}, \phi] \ .$$

The superscript E indicates the energy of the state in the Fock space, and the superscript $\{a\}$ is an additional label for degenerate energy eigenstates.

These Fock space wavefunctionals can be promoted to a solution of the integrated constraint by additionally specifying that they are eigenstates of the integral of the boundary metric that appears on the left of equation (4.5):

$$\psi_{\mathrm{I}}^{E,\{a\}}[H_{\partial}, h^{\mathrm{TT}}, \phi] = \psi_{\mathrm{F}}^{E,\{a\}}[h^{\mathrm{TT}}, \phi] \otimes |H_{\partial} = E\rangle \ . \tag{4.7}$$

The constraints (4.4) and (4.3) constitute an infinite number of constraints — one at each point of the Cauchy slice. So the solution to the integrated Hamiltonian constraint obtained above needs to be improved further to obtain a solution to these constraints. We present an explicit leading order solution to the pointwise constraints in section 4.5.2 and Appendix B.4. In addition, we give a simple discussion of a procedure that makes it clear that each solution of the integrated constraint (4.7) can be uniquely uplifted to a solution of the pointwise constraint (4.1):

$$\psi_{\mathrm{I}}^{E,\{a\}}[H_{\partial}, h^{\mathrm{TT}}, \phi] \to \Psi^{E,\{a\}}[h, \phi] \ . \tag{4.8}$$

This argument is enough to ensure that once the dependence of the wavefunctional on h^{TT} and ϕ in the auxiliary Fock space is chosen, there is no further freedom to specify its dependence on h^{T} and h^{L}. The integrated constraint fixes the detailed form of h^{T} at the boundary and, although the solution to the pointwise constraints that we find is both new and interesting, we do not require the explicit form of the dependence of the wavefunctional on h^{T} and h^{L} in the bulk for obtaining our main result.

We then define a natural inner product on the space of solutions (see section 4.5.3) and show that it is compatible with the structure of the constraints. This allows us to meaningfully compute correlation functions of observables using these wavefunctionals.

The above analysis of the constraints allows us to obtain a striking result. We show that any two pure or mixed states in a theory of gravity that agree on the boundary of AdS for an infinitesimal interval of time must agree everywhere in the bulk. To demonstrate this result we consider a general density matrix that depends on two metric perturbations, h_{ij} and $\tilde{h}_{ij}$ and two matter perturbations, ϕ and $\tilde{\phi}$. We write it in the form

$$\rho[h,\phi,\tilde{h},\tilde{\phi}] = \sum_{E,E',\{a\},\{a'\}} c(E,E',\{a\},\{a'\})\rho^{E,E',\{a\},\{a'\}}[h,\phi,\tilde{h},\tilde{\phi}] \ , \qquad (4.9)$$

where $c(E,E',\{a\},\{a'\})$ is a list of coefficients and a basis of density matrices

$$\rho^{E,E',\{a\},\{a'\}}[h,\phi,\tilde{h},\tilde{\phi}] \equiv \Psi^{E',\{a'\}}[\tilde{h},\tilde{\phi}]\Psi^{E,\{a\}}[h,\phi]^* \ , \qquad (4.10)$$

is obtained by combining the solutions to the constraints obtained above.

We consider a simple class of gauge invariant operators that are supported only on the boundary, and therefore automatically commute with the constraints (4.1). One such operator is H_∂ displayed in (4.6), whereas other operators — which we denote by $\mathcal{O}(t,\Omega)$ — correspond to the boundary limit of fluctuations of the dynamical fields, including the transverse-traceless graviton and matter fields. We first show that if two density matrices ρ_1 and ρ_2 yield the same correlators of the following combination of such operators

$$\langle H_\partial^n\, \mathcal{O}(t_1,\Omega_1)\ldots\mathcal{O}(t_q,\Omega_q)H_\partial^m\rangle_{\rho_1} = \langle H_\partial^n\, \mathcal{O}(t_1,\Omega_1)\ldots\mathcal{O}(t_q,\Omega_q)H_\partial^m\rangle_{\rho_2} \ , \qquad (4.11)$$

then the respective coefficients $c_1(E,E',\{a\},\{a'\})$ and $c_2(E,E',\{a\},\{a'\})$ must satisfy the following identity at each individual value of E and E'

$$\sum_{\{a\},\{a'\}} \Big[c_1(E,E',\{a\},\{a'\}) - c_2(E,E',\{a\},\{a'\})\Big]\langle\mathcal{O}(t_1,\Omega_1)\ldots\mathcal{O}(t_q,\Omega_q)\rangle_{\rho^{E,E',\{a\},\{a'\}}} = 0 \ . \qquad (4.12)$$

We only demand that the equations above hold at O(1) and not at O(κ) so that we can study them reliably within our perturbative setup. In particular, this means that n, m, q are limited to O(1) integers as well and cannot scale with an inverse power of κ and the passage from (4.11) to (4.12) can be performed reliably provided that the energy of the state in (4.9) does not scale with $\log\frac{1}{\kappa}$ in AdS units. We show that there is no non-trivial solution to these equations if the t_i above are allowed to range in the infinitesimal interval $[0,\epsilon]$. Therefore if two pure or mixed states agree on the boundary for even an infinitesimal time interval then they must be the same.

This last result that we obtain is central to the notion of holography since it tells us that, in a theory of gravity, the state in the bulk is completely determined by boundary data in an infinitesimal time interval. Here we see that this surprising aspect of gravity follows directly from the constraints of the theory.

4.3 Preliminaries

In this section, we set the stage for our analysis, establish some notation, and review the constraints that must be satisfied by physical states in any theory of gravity.

4.3.1 Action and boundary conditions

We will study gravity with a negative cosmological constant in $d+1$ dimensions, as described by the action

$$S = \frac{1}{2\kappa^2} \int dt d^d x \sqrt{-\hat{g}} \, (\hat{R} - 2\Lambda) + S_{\text{GHY}} + S_{\text{matter}}, \tag{4.13}$$

where $\kappa = \sqrt{8\pi G}$, $\hat{R}$ is the $d+1$-dimensional Ricci scalar, S_{GHY} is the Gibbons-Hawking-York boundary term and Λ is a cosmological constant. We will use hats to differentiate spacetime quantities with Cauchy slice quantities. The specific details of the matter sector will not be important in the subsequent analysis although we will use scalar fields as an example for illustration.

We are interested in spacetimes that are asymptotically AdS. Note that in both the classical and the quantum theory it is necessary to fix asymptotic boundary conditions on the metric. The metric is then allowed to fluctuate in the bulk. We introduce a coordinate r so that the conformal boundary is attained as $r \to \infty$. We then demand that near this boundary

$$ds^2 \underset{r\to\infty}{\longrightarrow} \ell^2 \left(-(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-1}^2 \right) \tag{4.14}$$

where the AdS length $\ell = 1$. Note that, in these units, κ is a dimensionless number and we will assume that $\kappa \ll 1$ which is simply the assumption that the Planck length is much smaller than the cosmological length.

This means that we allow for the standard normalizable boundary conditions for fluctuations of the metric and matter fields following [113, 114, 49], demanding that the metric and matter fluctuations have appropriate falloffs near the boundary.[2]

4.3.2 Canonical formalism

In the canonical formalism for gravity described by ADM [82], the line element is written using a $d+1$ split

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \ , \tag{4.15}$$

where N is called the lapse function, and N^i is called the shift vector. The metric on a Cauchy slice Σ, at a fixed value of t, is g_{ij} where $i, j, \ldots$ run only over the spatial coordinates.

We can rewrite the action as

$$S = \frac{1}{2\kappa^2} \int dt d^d x \, N\sqrt{g} \left(K_{ij} K_{kl} g^{ik} g^{jl} - K^2 + R - 2\Lambda \right) + S_{\text{GHY}} + S_{\text{matter}} \ , \tag{4.16}$$

using the extrinsic curvature of the slice of constant t, given by

$$K_{ij} = \frac{1}{2N} \left(-\dot{g}_{ij} + D_j N_i + D_i N_j \right), \tag{4.17}$$

where D_i is the covariant derivative with respect to g_{ij}, $K = g^{ij} K_{ij}$ and R is the Ricci scalar on the slice.

The canonical momentum is defined as

$$\pi^{ij} = \frac{\delta S}{\delta \dot{g}_{ij}} = -\frac{1}{2\kappa^2} \sqrt{g} \left(g^{il} g^{jk} K_{lk} - g^{ij} K \right) \ . \tag{4.18}$$

The conjugate momenta for the lapse and shift vanish identically leading to the primary constraints [118]

$$\pi_N = \frac{\delta S}{\delta \dot{N}} = 0, \qquad \pi_{N_i} = \frac{\delta S}{\delta \dot{N}_i} = 0 \ . \tag{4.19}$$

[2] It is of interest to consider other kinds of boundary conditions [115, 116]. However, if the boundary conditions allow energy to escape from AdS, then one-loop effects generically generate a mass for the graviton in the bulk [117] leading to a theory that might have qualitatively different properties from standard theories of gravity.

The Hamiltonian can be written in the form

$$H = H_0 + H_\partial \ , \tag{4.20}$$

where

$$H_0 = \int_\Sigma d^d x \sqrt{g} \left(N\mathcal{H} + N^i \mathcal{H}_i \right) \ , \tag{4.21}$$

and $\mathcal{H}$ and $\mathcal{H}_i$ are given by

$$\mathcal{H} = 2\kappa^2 g^{-1} \left(g_{ik} g_{jl} \pi^{kl} \pi^{ij} - \frac{1}{d-1} (g_{ij}\pi^{ij})^2 \right) - \frac{1}{2\kappa^2}(R - 2\Lambda) + \mathcal{H}^{\text{matter}}, \tag{4.22}$$

$$\mathcal{H}_i = -2 g_{ij} D_k \frac{\pi^{jk}}{\sqrt{g}} + \mathcal{H}_i^{\text{matter}}, \tag{4.23}$$

where $\mathcal{H}^{\text{matter}}$ is the matter Hamiltonian density, $\mathcal{H}_i^{\text{matter}}$ is the matter momentum density and H_∂ is a boundary contribution [119] whose explicit form we give below in (4.38).

The matter Hamiltonian is obtained in a standard way using canonical quantization. Let us illustrate this in the example of a scalar field of mass m, described by the action

$$S_{\text{matter}} = -\frac{1}{2} \int dt d^d x \sqrt{g}\, N \left((\partial\phi)^2 + m^2 \phi^2 \right) \ . \tag{4.24}$$

the conjugate momentum is $\pi = \sqrt{g} N^{-1} (\partial_t \phi - N^i \partial_i \phi)$ and the Hamiltonian and momentum density are

$$\mathcal{H}^{\text{matter}} = \frac{1}{2} g^{-1} \pi^2 + \frac{1}{2} \left(g^{ij} \partial_i \phi \partial_j \phi + m^2 \phi^2 \right) \ , \qquad \mathcal{H}_i^{\text{matter}} = \frac{1}{\sqrt{g}}\, \pi\, \partial_i \phi \ . \tag{4.25}$$

We obtain secondary constraints by demanding that the primary constraints are preserved by time evolution. These secondary constraints are nontrivial and are called the Hamiltonian and momentum constraints. They can be described as follows. Let f be any function that dies off smoothly as $r \to \infty$ and let

$$\mathcal{H}(f) \equiv \int_\Sigma d^d x\, \mathcal{H} f, \qquad \mathcal{H}_i(f) \equiv \int_\Sigma d^d x\, \mathcal{H}_i f \ . \tag{4.26}$$

Then the Hamiltonian and momentum constraints are

$$\mathcal{H}(f) = 0, \qquad \mathcal{H}_i(f) = 0 \ . \tag{4.27}$$

Note that (4.27) are equivalent to imposing $\mathcal{H} = 0$ and $\mathcal{H}_i = 0$ at all points except for the conformal boundary.

The exclusion of the boundary can be understood using a simple physical argument. The constraints (4.27) express the diffeomorphism invariance of the theory. But, as is standard in gauge theories, only small diffeomorphisms — those diffeomorphisms that vanish smoothly at the conformally boundary — are redundancies in the description. Large diffeomorphisms — those diffeomorphisms that act nontrivially at the conformal boundary — generate physical transformations and should not be viewed as trivial.

4.3.3 Quantum theory

So far our description has been classical. In the quantum theory, the states are given by wavefunctionals

$$\Psi[g, \phi] \ .$$

Note that, to lighten the notation, we do not display the indices on g and on other tensors when they appear in an argument of the wavefunctional. Here, ϕ is used as a collective variable for the matter fields in the theory. The wavefunctional returns a complex number upon being given a configuration of the metric and matter fields on the entire Cauchy slice.

The conjugate momenta act on these wavefunctionals via

$$\pi_{ij}\Psi[g, \phi] = -i\frac{\delta}{\delta g_{ij}}\Psi[g, \phi], \qquad \pi\Psi[g, \phi] = -i\frac{\delta}{\delta \phi}\Psi[g, \phi] \ . \tag{4.28}$$

In the quantum theory, we demand that all valid wavefunctionals are annihilated by the constraints. The primary constraints tell us that the wavefunctional is independent of N and N_i since they imply that

$$\frac{\delta}{\delta N}\Psi[g, \phi] = 0, \qquad \frac{\delta}{\delta N^i}\Psi[g, \phi] = 0 \ . \tag{4.29}$$

In the quantum theory, the information about how the d-geometries are glued together into a spacetime geometry must be extracted from the canonical momentum and not from the values of N or N^i. If one takes the classical limit in the quantum theory, then the expectation value of the momentum operator can be related to the classical extrinsic curvature via (4.18).

Finally, the wavefunctional must be annihilated by the Hamiltonian and momentum constraints

$$\mathcal{H}(f)\Psi[g, \phi] = 0, \qquad \mathcal{H}_i(f)\Psi[g, \phi] = 0 \ . \tag{4.30}$$

These constraints can be understood as imposing the gauge invariance of the wavefunctional in the quantum theory. As usual, we do not impose invariance under large gauge transformations which may act non-trivially on the state. For mixed states, the corresponding condition is that the density matrix must commute with the constraints.

A valid observable in the theory, denoted $\mathcal{O}$, is a Hermitian operator that commutes with the constraints

$$[\mathcal{O}, \mathcal{H}(f)] = 0, \qquad [\mathcal{O}, \mathcal{H}_i(f)] = 0 \ . \tag{4.31}$$

A simple set of gauge-invariant observables are just given by the boundary limits of bulk operators. Such observables manifestly satisfy (4.31) because $\mathcal{H}(f)$ and $\mathcal{H}_i(f)$ vanish near the boundary. Such observables may depend on the boundary coordinates including the boundary time and, in the discussion below, we display this dependence as $\mathcal{O}(t, \Omega)$. We discuss these observables further in section 4.6.

4.4 Perturbative expansion

In this section, we will expand the constraints in the perturbative regime about the AdS background. We start by introducing the perturbative variables and then proceed to the perturbative expansion. All the derivations described in this section are checked using xAct [120] and xPert [121] in a Mathematica notebook [122].

4.4.1 Perturbative setup

Metric fluctuation. In perturbation theory, we expand the metric as

$$g_{ij} = \gamma_{ij} + \kappa h_{ij} \ , \tag{4.32}$$

where $\kappa = \sqrt{8\pi G}$ and the background metric, γ_{ij}, corresponds to the metric on a constant time slice of global AdS_{d+1}.

$$\gamma_{ij} dx^i dx^j = \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-1}^2 \ , \tag{4.33}$$

Equation (4.32) should be taken as the definition of the perturbative variable h_{ij}. Note that, for now, this equation is just an exact change of variables although below we will perform a perturbative expansion in κ. We will find it convenient to represent states as wavefunctionals of this new variable using the notation

$$\Psi[h, \phi] \ .$$

Momentum operator. It is also convenient, in perturbation theory, to work in terms of the momentum operator

$$\Pi^{ij} = \frac{\kappa}{\sqrt{\gamma}} \pi^{ij} \ . \tag{4.34}$$

In the wavefunctional representation, the action of this operator is just

$$\Pi^{ij} = -\frac{i}{\sqrt{\gamma}} \frac{\delta}{\delta h_{ij}} \ , \tag{4.35}$$

and so this operator is canonically conjugate to h_{ij} up to a factor of $\frac{1}{\sqrt{\gamma}}$ that is included so that Π^{ij} transforms like a tensor field on the background.

Derivatives and indices. We will use ∇_i to denote the covariant derivative associated to the background metric γ_{ij}. This should be distinguished from D_i which is the covariant derivative associated to the full metric g_{ij}. Furthermore, for the rest of this chapter, we will raise and lower indices using only the background metric γ_{ij}. We remind the reader that indices are summed only over the spatial coordinates and if time appears in a formula, it is displayed separately.

Shift and lapse. The primary constraints imply that neither N nor N^i enter in any wavefunctional or observable. Nevertheless, in our analysis it will be convenient to fix the background value of N to be

$$N^2 = 1 + r^2. \tag{4.36}$$

Since N is not an observable, the reader can just take equation (4.36) to specify a certain function of the coordinates that will be useful in the analysis.

Background properties. In our computations, it will be useful to use the following identities satisfied by the background quantities:

$$
\begin{aligned}
R_{ijk\ell} &= \gamma_{i\ell}\gamma_{jk} - \gamma_{ik}\gamma_{j\ell}, & R_{ij} &= -(d-1)\gamma_{ij} \ , \\
R &= -d(d-1), & \nabla_i \nabla_j N &= \gamma_{ij} N,
\end{aligned}
\tag{4.37}
$$

and the cosmological constant is $\Lambda = -d(d-1)/2$. We are using conventions where $\ell_{\text{AdS}} = 1$.

Boundary Hamiltonian.

In terms of the notation introduced above, the boundary contribution to the Hamiltonian in (4.20) takes on a simple form. This can be viewed as an AdS version of the ADM energy. It is given as

$$
H_\partial = \frac{1}{2\kappa} \int_{\partial\Sigma} d^{d-1}\Omega \, n_i J^i \ , \tag{4.38}
$$

where

$$
J_i \equiv N \nabla^j (h_{ij} - h\gamma_{ij}) - \nabla^j N (h_{ij} - h\gamma_{ij}) \tag{4.39}
$$

will be called the ADM current. We show in Appendix B.3 that this agrees with various prescriptions for the gravitational energy in AdS. Here $d^{d-1}\Omega$ denotes the appropriate measure for boundary integration and n_i denotes the normal to the boundary.[3] Note that, in the coordinates (4.33) the area of a sphere at large r grows like $\mathrm{O}\big(r^{d-1}\big)$ which precisely compensates the large r falloff of J^i. Also note that as a consequence of (4.27), the bulk contribution to the energy of any state vanishes. The nonzero contribution to the energy comes only from the boundary term (4.38).

4.4.2 ADM decomposition

In order to better understand the Hamiltonian and momentum constraints given in (4.22) and (4.23), it is convenient to use the ADM decomposition of symmetric tensors [82]. ADM originally introduced this decomposition about flat space, and here we present the generalization to an AdS background. We refer the reader to [123] for related discussion.

[3]For concreteness, we can take $d^{d-1}\Omega$ to be the volume form of the unit sphere and $n_i = r^{d-1}\mathbf{n}_i$ where $\mathbf{n}_i$ is the unit normal to the boundary.

We decompose the metric perturbation as

$$h_{ij} = h_{ij}^{\mathrm{TT}} + h_{ij}^{\mathrm{T}} + h_{ij}^{\mathrm{L}} \ , \tag{4.40}$$

and the three terms in the sum are called the transverse-traceless component, the T-component and the longitudinal component respectively. We will perform precisely analogous decompositions for other tensor fields below and, in each case, the three components will be labeled by "TT", "T" and "L" as above.

The transverse-traceless component obeys

$$\nabla^i h_{ij}^{\mathrm{TT}} = 0, \qquad \gamma^{ij} h_{ij}^{\mathrm{TT}} = 0 \ . \tag{4.41}$$

The T-component of the metric is also transverse

$$\nabla^i h_{ij}^{\mathrm{T}} = 0 \ , \tag{4.42}$$

but only captures information about the trace of the transverse part of the decomposition. The longitudinal component is of the form

$$h_{ij}^{\mathrm{L}} = \nabla_i \epsilon_j + \nabla_j \epsilon_i \ , \tag{4.43}$$

in terms of an arbitrary vector field ϵ_i that vanishes at the conformal boundary.

Given any tensor field h_{ij}, the decomposition (4.40) is unique and can be obtained by solving a set of elliptic partial differential equations as we now describe. The transversality conditions (4.42) and (4.41) imply that ϵ_i is obtained as the solution to

$$\nabla^i \nabla_i \epsilon_j + \nabla^i \nabla_j \epsilon_i = \nabla^i h_{ij} \ , \tag{4.44}$$

which has a unique solution for ϵ_i subject to our boundary conditions and thereby yields h_{ij}^{L}. Note that the Killing vectors of the background cannot be added to a solution of the equation above to obtain another solution since they do not vanish asymptotically.

We denote the trace of the transverse part of the metric by

$$h^{\mathrm{T}} = \gamma^{ij} \left(h_{ij} - h_{ij}^{\mathrm{L}} \right) = \gamma^{ij} h_{ij}^{\mathrm{T}} \ . \tag{4.45}$$

We want the T-component of the metric to depend linearly on the metric, correspond to a single degree of freedom, and vanish when h^{T} vanishes. This is achieved by introducing an auxiliary scalar field χ and writing

$$h_{ij}^{\mathrm{T}} = \frac{1}{d} h^{\mathrm{T}} \gamma_{ij} - \frac{1}{d-1} \left[\nabla_i \nabla_j - \frac{1}{d} \gamma_{ij} \Delta \right] \chi \ , \tag{4.46}$$

where $\Delta \equiv \nabla^i \nabla_i$. The condition (4.42) implies that χ must obey

$$(\Delta - d)\,\chi = h^{\mathrm{T}} \ . \tag{4.47}$$

Once the longitudinal and T-component have been determined as above, the h^{TT} component of the metric is what remains: $h^{\mathrm{TT}}_{ij} = h_{ij} - h^{\mathrm{L}}_{ij} - h^{\mathrm{T}}_{ij}$. Note that, by construction, the conditions (4.41) are met.

It is also clear that the degrees of freedom on both sides of equation (4.40) match. The propagating modes of the graviton are contained in h^{TT}_{ij} and represent $(d+1)(d-2)/2$ degrees of freedom. There are d degrees of freedom in h^{L}_{ij} corresponding to the components of ϵ_i and 1 degree of freedom in h^{T}_{ij}. This gives a total of $d(d+1)/2$ as appropriate for a symmetric tensor.

The terms in the decomposition (4.40) are orthogonal when contracted and integrated over the Cauchy slice. For instance,

$$\int_\Sigma d^d x \sqrt{\gamma}\, h^{\mathrm{TT},ij} h^{\mathrm{L}}_{ij} = -2 \int_\Sigma d^d x \sqrt{\gamma}\, \nabla_i h^{\mathrm{TT},ij} \epsilon_j = 0 \ , \tag{4.48}$$

where we have integrated by parts and utilized (4.41). A similar argument shows that the integral of a T-component with the longitudinal component vanishes. We also find that

$$\int_\Sigma d^d x \sqrt{\gamma}\, h^{\mathrm{TT},ij} h^{\mathrm{T}}_{ij} = \frac{1}{1-d} \int_\Sigma d^d x \sqrt{\gamma}\, h^{\mathrm{TT},ij} \nabla_i \nabla_j \chi = 0 \ , \tag{4.49}$$

where in the first step we used the fact that $h^{\mathrm{TT},ij}$ is traceless and in the second step we integrated by parts and used the property (4.41).

We now turn to the canonical momenta. Note that (4.35) tells us that Π^{ij} is an operator-valued field. Nevertheless we can perform a decomposition similar to (4.40). We write

$$\Pi^{ij} = \Pi^{ij}_{\mathrm{TT}} + \Pi^{ij}_{\mathrm{T}} + \Pi^{ij}_{\mathrm{L}} \ . \tag{4.50}$$

The canonical generator [82] that induces an infinitesimal shift in the metric fluctuation, $h_{ij} \to h_{ij} + \zeta_{ij}$, is simply

$$G = i \int_\Sigma d^d x \sqrt{\gamma}\, \Pi^{ij} \zeta_{ij} \ . \tag{4.51}$$

Using the orthogonality of the components demonstrated above, it is clear that the canonical generator diagonalizes so that

$$G = i \int_\Sigma d^d x \sqrt{\gamma} \left(\Pi^{ij}_{\mathrm{TT}} \zeta^{\mathrm{TT}}_{ij} + \Pi^{ij}_{\mathrm{T}} \zeta^{\mathrm{T}}_{ij} + \Pi^{ij}_{\mathrm{L}} \zeta^{\mathrm{L}}_{ij} \right) , \tag{4.52}$$

which implies that

$$\Pi^{ij}_{\rm TT} = -\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta h^{\rm TT}_{ij}}\ , \qquad \Pi^{ij}_{\rm T} = -\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta h^{\rm T}_{ij}}\ , \qquad \Pi^{ij}_{\rm L} = -\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta h^{\rm L}_{ij}}\ . \tag{4.53}$$

4.4.3 Expansion of the constraints

In this section, we present the perturbative expansion of the constraints. A similar analysis was performed in [81] about Minkowski space.

Momentum constraint

Let us start by considering the momentum constraint (4.23). We consider successive approximations to the constraint which we write as

$$\sqrt{g}\,\mathcal{H}_i = \sqrt{\gamma}\,\mathcal{H}^{(n)}_i + {\rm O}\big(\kappa^{n-1}\big)\,, \qquad n = 0, 1, 2, \dots \tag{4.54}$$

by which we mean that $\mathcal{H}^{(n)}_i$ captures all terms in the expansion of the left hand side up to terms of order κ^{n-2}.

The zeroth order term in the momentum constraint vanishes trivially,

$$\mathcal{H}^{(0)}_i = 0\ . \tag{4.55}$$

First order. At leading order, the momentum constraint takes the form

$$\mathcal{H}^{(1)}_i = -\frac{2}{\kappa}\gamma_{ij}\nabla_k\Pi^{jk}\ . \tag{4.56}$$

The momentum constraint simply tells us that the wavefunctional is independent of $h^{\rm L}_{ij}$ to leading order. This can be seen as follows. Consider the infinitesimal gauge transformation $x^i \to x^i + \xi^i$. Then we see that

$$\begin{aligned}\Psi[h_{ij} + \nabla_i\epsilon_j + \nabla_j\epsilon_i, \phi] &= \Psi[h,\phi] + \int_\Sigma d^dx\,(\nabla_i\epsilon_j + \nabla_j\epsilon_i)(x)\frac{\delta}{\delta h_{ij}(x)}\Psi[h,\phi] \\ &= \Psi[h,\phi] - 2i\int_\Sigma d^dx\sqrt{\gamma}\,\epsilon_j(x)\nabla_i\Pi^{ij}(x)\Psi[h,\phi] = \Psi[h,\phi]\end{aligned} \tag{4.57}$$

at leading order in ϵ_i, where we have used the leading-order momentum constraint in the last equality.

Alternately, this can also be seen from the decomposition (4.50). The leading order momentum constraint tells us that

$$-2\gamma_{ij}\nabla_k\Pi_{\mathrm{L}}^{jk}\,\Psi[h,\phi] = 0 + \mathrm{O}(\kappa)\,. \tag{4.58}$$

which is equivalent to $\Pi_{\mathrm{L}}^{jk}\Psi[h,\phi] = 0 + \mathrm{O}(\kappa)$.

Second order. At next order, we have

$$\mathcal{H}_i^{(2)} = (\nabla_i h_{jk} - 2\nabla_k h_{ij})\Pi^{jk} - 2h_{ij}\nabla_k\Pi^{jk} - \frac{2}{\kappa}\gamma_{ij}\nabla_k\Pi^{jk} + \mathcal{H}_i^{\mathrm{matter}}\ . \tag{4.59}$$

The first order constraint implies that $h_{ij}\nabla_k\Pi^{jk} = \mathrm{O}(\kappa)$ which is subleading. We can then rewrite the constraint as

$$\frac{2}{\kappa}\gamma_{ij}\nabla_k\Pi_{\mathrm{L}}^{jk} = Q_i \tag{4.60}$$

where we have defined

$$Q_i \equiv (\nabla_i h_{jk} - 2\nabla_k h_{ij})\Pi^{jk} + \mathcal{H}_i^{\mathrm{matter}}\ , \tag{4.61}$$

and $\mathcal{H}_i^{\mathrm{matter}}$ is the contribution of the matter to momentum constraint. For a free scalar field, we have from (4.25)

$$\mathcal{H}_i^{\mathrm{matter}} = \frac{1}{\sqrt{\gamma}}\pi\,\partial_i\phi\ . \tag{4.62}$$

This shows that the second order momentum constraint determines the $\mathrm{O}(\kappa)$ part of Π_{L}^{jk} in terms of $\mathrm{O}(1)$ quantities.

Hamiltonian constraint

We now consider the perturbative expansion of the Hamiltonian constraint. We consider successive approximations

$$\sqrt{g}\,\mathcal{H} = \sqrt{\gamma}\,\mathcal{H}^{(n)} + \mathrm{O}\big(\kappa^{n-1}\big)\,, \tag{4.63}$$

by which we mean that $\mathcal{H}^{(n)}$ includes all the terms from the Hamiltonian constraint up to terms of order κ^{n-2}.

At zeroth order, we simply have

$$\mathcal{H}^{(0)} = -\frac{1}{2\kappa^2}(R - 2\Lambda)\ . \tag{4.64}$$

Plugging in the values from (4.37), we see that this term vanishes identically: $\mathcal{H}^{(0)} = 0$.

First order. At first order, we obtain

$$N\mathcal{H}^{(1)} = -\frac{1}{2\kappa}N\left(\nabla^i\nabla^j h_{ij} - \nabla_i\nabla^i h + (d-1)h\right) = -\frac{1}{2\kappa}\nabla^i J_i \,, \qquad (4.65)$$

which we have written as a total derivative in terms of the ADM current (4.39). Note the factor of N that we have inserted on the LHS of (4.65). It is only with this factor that the expression turns into a total derivative, and this fact will play an important role in the analysis below.

In the decomposition (4.40), it can be seen that this expression (4.65) involves only h^{T}_{ij} and not h^{TT}_{ij} or h^{L}_{ij}. It is clear that h^{TT}_{ij} disappears because of the transverse-traceless condition. The longitudinal component also disappears from this expression. This can be checked explicitly from (4.43) by evaluating $\mathcal{H}^{(1)}$ on (4.43) and commuting the covariant derivatives and using the background identities (4.37). This can also be understood from the fact that, at first order, the longitudinal component corresponds to an infinitesimal spatial diffeomorphism. Hence, it doesn't change the Ricci scalar which is constant according to (4.37). This implies that $\mathcal{H}^{(1)}$ doesn't depend on h^{L}_{ij}. The end result is that

$$N\mathcal{H}^{(1)} = \frac{1}{2\kappa}N\left(-\nabla_i\nabla^i h^{\mathrm{T}} + (d-1)h^{\mathrm{T}}\right). \qquad (4.66)$$

where we denote $h^{\mathrm{T}} = \gamma^{ij}h^{\mathrm{T}}_{ij}$. Since h^{T}_{ij} has only one degree of freedom, which can be taken to be h^{T}, the first order Hamiltonian constraint implies that

$$h^{\mathrm{T}}_{ij} = 0 + \mathrm{O}(\kappa)\,. \qquad (4.67)$$

In the sections below, we will work out aspects of the $\mathrm{O}(\kappa)$ correction to this equation, which will play a central role in our analysis.

Second order. At second order, we have contributions from the term quadratic in Π_{ij} and the matter stress tensor:

$$N\sqrt{\gamma}\,\mathcal{H}^{(2)} = 2N\sqrt{\gamma}\left(\Pi^{ij}\Pi_{ij} - \frac{1}{d-1}\Pi^2\right) - \frac{1}{2}N\left[\sqrt{g}(R-2\Lambda)\right]^{(2)} + N\sqrt{\gamma}\,\mathcal{H}^{\mathrm{matter}}\,, \qquad (4.68)$$

and we should expand the term in the brackets to second order in κ. The expansion is performed in the accompanying Mathematica script [122]. It leads to many terms which we can organize as

$$
\begin{aligned}
-N\left[\sqrt{g}(R-2\Lambda)\right]^{(2)} \quad = \quad & \frac{1}{4}\sqrt{\gamma}N\left(-h^{ij}(\Delta_N+2)h_{ij}+h(\Delta_N-d)h\right) \\
& +\frac{1}{2}\sqrt{\gamma}N\left(2h^{ij}\nabla_i\nabla^k h_{jk}+\nabla_i h^{ij}\nabla^k h_{jk}+2\nabla_i h\nabla_j h^{ij}+h\nabla_i\nabla_j h^{ij}\right) \\
& +\frac{1}{2}\sqrt{\gamma}\nabla_i L^i-\frac{1}{2\kappa}\nabla_i J^i\ ,
\end{aligned}
\tag{4.69}
$$

where we have introduced a Laplace-type operator

$$
\Delta_N h_{ij}=N^{-1}\nabla_k(N\nabla^k h_{ij})\ . \tag{4.70}
$$

The total derivative involves a current

$$
L_i\equiv -N\nabla^j L_{ij}+L_{ij}\nabla^j N+\frac{1}{2}N\left(h_{jk}\nabla_i h^{jk}-h\nabla_i h\right)\ , \tag{4.71}
$$

where we have defined

$$
L_{ij}\equiv 2hh_{ij}-h_{ik}h_j{}^k+\gamma_{ij}h_{k\ell}h^{k\ell}-\frac{1}{2}\gamma_{ij}h^2\ . \tag{4.72}
$$

There is also a contribution from the ADM current J^i evaluated on the $\mathrm{O}(\kappa)$ part of h_{ij}, the order one part being zero by the first order constraint. Finally, the second order Hamiltonian constraint takes the form

$$
N\mathcal{H}^{(2)}=NQ-\frac{1}{2\kappa}\nabla_i J^i\ , \tag{4.73}
$$

where we have defined

$$
\begin{aligned}
Q\equiv\ & 2\left(\Pi^{ij}\Pi_{ij}-\frac{1}{d-1}\Pi^2\right)+\frac{1}{8}\left(-h^{ij}(\Delta_N+2)h_{ij}+h(\Delta_N-d)h\right)+\frac{1}{4}\nabla_i L^i \\
& +\frac{1}{4}\left(2h^{ij}\nabla_i\nabla^k h_{jk}+\nabla_i h^{ij}\nabla^k h_{jk}+2\nabla_i h\nabla_j h^{ij}+h\nabla_i\nabla_j h^{ij}\right)+N\mathcal{H}^{\mathrm{matter}}\ .
\end{aligned}
\tag{4.74}
$$

4.4.4 Integrated constraint

We will find it very useful to also consider the integrated second order Hamiltonian constraint

$$
H_0^{(2)}=\int_\Sigma d^d x\sqrt{\gamma}\,N\mathcal{H}^{(2)}\ . \tag{4.75}
$$

It is important to perform the integral with the measure that defines the canonical Hamiltonian, i.e. with a factor of N as shown above. In this section, we show that the complicated expression obtained in (4.73) and (4.74) simplifies greatly upon integration.

To show this, we will use the ADM decomposition (4.40) and (4.50). We will also use the fact that, as shown above, the first order constraints will set $h^{\mathrm{T}}_{ij} = \mathrm{O}(\kappa)$ and $\Pi^{ij}_{\mathrm{L}} = \mathrm{O}(\kappa)$. So we drop terms where h^{T}_{ij} and Π^{ij}_{L} multiply another O(1) quantity since this allows us to avoid writing a number of unnecessary terms that will eventually not be relevant for our analysis.

First, the integrated constraint becomes independent of $h^{\mathrm{L}}_{ij} = \nabla_i\epsilon_j + \nabla_j\epsilon_i$. This is trivial at first order because h^{L}_{ij} corresponds to an infinitesimal diffeomorphism. At second order, the cancellation is non-trivial and quite remarkable[4]. It follows from the fact that we can write the constraint as

$$\begin{aligned} N\mathcal{H}^{(2)} &= 2N\left(\Pi^{ij}\Pi_{ij} - \frac{1}{d-1}\Pi^2\right) - \frac{1}{8}N\, h^{ij}_{\mathrm{TT}}(\Delta_N + 2)h^{\mathrm{TT}}_{ij} + N\mathcal{H}^{\mathrm{matter}} \\ &\quad + \frac{1}{2}\nabla_i M^i + \frac{1}{4}\nabla_i L^i[h^{\mathrm{TT}}] - \frac{1}{2\kappa}\nabla_i J^i + \mathrm{O}\big(h^{\mathrm{T}}\big)\,, \end{aligned} \tag{4.77}$$

where the $\mathrm{O}\big(h^{\mathrm{T}}\big)$ term is subleading as explained above, and where the de-

[4]This can be understood as follows. The Ricci scalar is constant on the background, so it is invariant under the diffeomorphism $x_i \to x_i + \kappa\epsilon_i$. As a result, the second order Hamiltonian constraint is also invariant under that diffeomorphism, as the variation of $\sqrt{\gamma}$ and N can be ignored because we assume $\mathcal{H}^{(0)} = \mathcal{H}^{(1)} = 0$. This diffeomorphism modifies the metric according to $h_{ij} \to h_{ij} + \nabla_i\epsilon_i + \nabla_j\epsilon_i + \kappa h^{(2)}_{ij} + O(\kappa^2)$. At linearized order, this generates an arbitrary h^{L}_{ij} and shows that the first order constraint is independent of the longitudinal metric. At second order, we also generate a subleading term whose explicit expression is $h^{(2)}_{ij} = \nabla_i\epsilon_k\nabla_j\epsilon^k - \epsilon_i\epsilon_j + \gamma_{ij}\epsilon_k\epsilon^k + \mathcal{L}_\epsilon h^{\mathrm{TT}}_{ij}$. By applying the above diffeomorphism to the constraint at $h^{\mathrm{L}} = 0$, we obtain

$$N\mathcal{H}^{(2)}\Big|_{h^{\mathrm{L}}=0} = N\mathcal{H}^{(2)} - \frac{1}{2}\nabla_i J^i[h^{(2)}]\,, \tag{4.76}$$

where we used the fact that since $\mathcal{H}^{(2)}$ captures terms up to $\mathrm{O}\big(\kappa^0\big)$ the subleading term $h^{(2)}$ can only affect it through those terms that have an explicit factor of κ^{-1}. This shows the dependence on h^{L} in $N\mathcal{H}^{(2)}$ is indeed captured by a total divergence.

pendence in ϵ_i is fully captured by the divergence of the following current

$$M^i = N\left(-\nabla_j M^{ij} + h^{\mathrm{TT}}_{jk}\nabla^j\nabla^k\epsilon^i - 2\nabla^i h^{\mathrm{TT}}_{jk}\nabla^j\epsilon^k - h^{\mathrm{TT},ij}\nabla_j\nabla_k\epsilon^k + 2d h^{\mathrm{TT},ij}\epsilon_j\right)$$
$$+\nabla_j N\left(M^{ij} + h^{\mathrm{TT},ij}\nabla_k\epsilon^k - h^{\mathrm{TT},ik}\nabla_k\epsilon^j + 2\epsilon_k\nabla^j h^{\mathrm{TT},ik}\right), \tag{4.78}$$

where we have defined

$$M^{ij} \equiv \epsilon^i\epsilon^j + \epsilon^k\nabla^j\nabla^i\epsilon_k + \gamma^{ij}\left((d-2)\epsilon_k\epsilon^k - \epsilon_k\Delta\epsilon^k\right). \tag{4.79}$$

Above, the symbol $L^i[h^{\mathrm{TT}}]$ means that (4.71) is evaluated only on h^{TT}_{ij} and this evaluation reduces to

$$L^i[h^{\mathrm{TT}}] = N\, h^{\mathrm{TT}}_{jk}\nabla^k h^{ij}_{\mathrm{TT}} - \frac{3}{2}N\, h^{jk}_{\mathrm{TT}}\nabla_i h^{\mathrm{TT}}_{jk} + \nabla^i N\, h^{\mathrm{TT}}_{jk}h^{jk}_{\mathrm{TT}} - \nabla^j N\, h^{ik}_{\mathrm{TT}}h^{\mathrm{TT}}_{jk}\ . \tag{4.80}$$

The validity of this rewriting is checked in the associated Mathematica notebook [122].

From equation (4.77) it is clear that the integration of $N\mathcal{H}^{(2)}$ over the entire Cauchy slice leads to boundary terms that involve $M^i, L^i[h^{\mathrm{TT}}]$ and J^i. The terms involving M^i and $L^i[h^{\mathrm{TT}}]$ are quadratic in the metric fluctuation and since we have imposed normalizable boundary conditions their decay at large r is faster than the growth of the area of the sphere. Therefore the boundary contribution from these terms vanishes. On the other hand, the boundary term involving J^i, upon integration over the boundary, gives the ADM energy H_∂.

Let's now consider the kinetic piece

$$2\int_\Sigma d^dx\sqrt{\gamma}\,N\left(\Pi^{ij}\Pi_{ij} - \frac{1}{d-1}\Pi^2\right)\ . \tag{4.81}$$

To analyze the term quadratic in Π^{ij}_{T} it is convenient to write the decomposition of section 4.4.2 as

$$\Pi_{\mathrm{T},ij} = \frac{1}{d-1}\left(\gamma_{ij}\Pi_{\mathrm{T}} - \alpha_{ij}\right) \tag{4.82}$$

where

$$\alpha_{ij} = N\nabla_i\nabla_j\alpha + \nabla_i N\nabla_j\alpha + \nabla_j N\nabla_i\alpha \tag{4.83}$$

in terms of a scalar operator-valued field α that satisfies the analogue of (4.47) for Π_{T}:

$$(\Delta - d)(N\alpha) = \Pi_{\mathrm{T}}\ . \tag{4.84}$$

From the expression (4.82), we see that the term quadratic in $\Pi_{\rm T}^{ij}$ in the kinetic piece can be written

$$\int_\Sigma d^dx\sqrt{\gamma}\,N\left(\Pi_{{\rm T},ij}\Pi_{\rm T}^{ij}-\frac{1}{d-1}\Pi_{\rm T}^2\right)=-\frac{1}{d-1}\int_\Sigma d^dx\sqrt{\gamma}\,N\alpha_{ij}\Pi_{\rm T}^{ij}\ . \quad (4.85)$$

Using (4.83), we can write

$$N\alpha_{ij}=\nabla_i\alpha_j+\nabla_j\alpha_i, \qquad \alpha_i\equiv\frac{1}{2}N^2\nabla_i\alpha\ , \quad (4.86)$$

and we finally obtain

$$\int_\Sigma d^dx\sqrt{\gamma}\,N\left(\Pi_{\rm T}^{ij}\Pi_{{\rm T},ij}-\frac{1}{d-1}\Pi_{\rm T}^2\right)=-\frac{2}{d-1}\int_\Sigma d^dx\sqrt{\gamma}\,\nabla_i\left(\Pi_{\rm T}^{ij}\alpha_j\right), \quad (4.87)$$

which becomes a boundary term. Since $\Pi_{\rm T}$ vanishes at the boundary, the boundary term vanishes.[5] In a similar way, we can show that the cross terms involving $\Pi_{\rm TT}^{ij}$ and $\Pi_{\rm T}^{ij}$ vanish. Recall that $\Pi_{\rm L}^{ij}$ does not appear since it vanishes at O(1) in perturbation theory by the first order momentum constraint. This shows that $\Pi_{\rm T}^{ij}$ disappears from the integrated constraint.

Finally, the integrated Hamiltonian constraint takes the form

$$H_0^{(2)}=-H_\partial+\int_\Sigma d^dx\sqrt{\gamma}\,N\mathcal{H}_{\rm bulk} \quad (4.88)$$

where

$$\mathcal{H}_{\rm bulk}=2\,\Pi_{{\rm TT},ij}\Pi_{\rm TT}^{ij}-\frac{1}{8}\,h^{{\rm TT}ij}(\Delta_N+2)h_{ij}^{\rm TT}+\mathcal{H}^{\rm matter} \quad (4.89)$$

and the explicit expression of H_∂ is given in (4.38). The constraint $H_0^{(2)}=0$ can be understood as the equality of the ADM energy H_∂ with a bulk energy defined as the second term of (4.88). Here we see that this relationship follows naturally from the Hamiltonian constraint.

4.5 Solving the constraints

We now describe how the constraints discussed in the previous section can be solved to reveal a remarkable structure of correlations in gravitational wavefunctionals.

[5] More precisely, we only need $\Pi_{\rm T}<{\rm O}\big(r^{(d-4)/2}\big)$ for the boundary term to vanish.

The analysis of section 4.4 immediately yields solutions to the first order constraints. We find from the first order Hamiltonian constraint that

$$\mathcal{H}^{(1)}\Psi[h,\phi] = 0 \Rightarrow h^{\mathrm{T}}\Psi[h,\phi] = 0 + \mathrm{O}(\kappa)\,. \tag{4.90}$$

This equation should be interpreted as telling us that the support of the wavefunctional is negligible when the value of h^{T} is parametrically larger than $\mathrm{O}(\kappa)$. The first order momentum constraint tells us that

$$\mathcal{H}_i^{(1)}\Psi[h,\phi] = 0 \Rightarrow \frac{\delta\Psi[h,\phi]}{\delta h_{ij}^{\mathrm{L}}} = 0 + \mathrm{O}(\kappa)\,. \tag{4.91}$$

The interesting features in the solutions appear at the next order in perturbation theory, and this is what we will focus on.

4.5.1 Integrated Hamiltonian constraint

We first describe how to solve the integrated Hamiltonian constraint (4.88). Here we look for wavefunctionals $\psi_{\mathrm{I}}^{E,\{a\}}[H_\partial, h^{\mathrm{TT}}, \phi]$ with a specified dependence on h^{TT}, ϕ and H_∂. The reason it is possible to restrict to only these variables is that, as explained in section 4.4.4, the other degrees of freedom all drop out of the integrated Hamiltonian constraint. Note that H_∂ corresponds to a single degree of freedom from h_{ij}^{T} as can be seen from (4.66) and (4.38). In section 4.5.2, we describe how the remaining dependence on h_{ij}^{T} and h_{ij}^{L} can be fixed in the full wavefunctional.

We remind the reader that a factor of κ^{-1} is implicit in the definition of H_∂, which can be seen in (4.38). Therefore, in perturbation theory, it is natural to think in terms of κH_∂. The first order solutions to $H_0^{(2)} = 0$ all have a degenerate value of $\kappa H_\partial = 0 + \mathrm{O}(\kappa)$ by virtue of equation (4.90). To work out the solution at $\mathrm{O}(\kappa)$ is a standard problem in degenerate perturbation theory. We need to look for solutions that diagonalize the "perturbation" in (4.88), which is the bulk term.

$$\int_\Sigma d^dx\sqrt{\gamma}N\,\mathcal{H}_{\mathrm{bulk}}\,\psi_{\mathrm{I}}^{E,\{a\}}[H_\partial, h^{\mathrm{TT}}, \phi] = E\,\psi_{\mathrm{I}}^{E,\{a\}}[H_\partial, h^{\mathrm{TT}}, \phi]\;. \tag{4.92}$$

Here the eigenvalue of the integrated bulk term is E and the superscript $\{a\}$ simply reminds us that the eigenvalues of the bulk Hamiltonian operator are degenerate and the wavefunctional is not completely specified by only a value of E. Then, the integrated constraint implies that

$$H_\partial \, \psi_{\mathrm{I}}^{E,\{a\}}[H_\partial, h^{\mathrm{TT}}, \phi] = E \, \psi_{\mathrm{I}}^{E,\{a\}}[H_\partial, h^{\mathrm{TT}}, \phi] \; . \tag{4.93}$$

Since the integral $\mathcal{H}_{\mathrm{bulk}}$ depends only on the propagating degrees of freedom h_{ij}^{TT} and ϕ, as explained in section 4.4.4, it is useful to introduce an auxiliary Hilbert space of wavefunctionals that depend only on h_{ij}^{TT} and ϕ. We will see that these states form a Fock space.

In this auxiliary space, equation (4.92) simply becomes

$$\int_\Sigma d^d x \sqrt{\gamma} N \, \mathcal{H}_{\mathrm{bulk}} \, \psi_{\mathrm{F}}^{E,\{a\}}[h^{\mathrm{TT}}, \phi] = E \, \psi_{\mathrm{F}}^{E,\{a\}}[h^{\mathrm{TT}}, \phi] \; . \tag{4.94}$$

The equation above is the same as (4.92) except that the wavefunctional has no dependence of H_∂.

We show below that this can be solved by taking a factorized basis of wavefunctionals that depend, respectively, on only the transverse-traceless metric fluctuation and the matter fluctuation.

$$\psi_{\mathrm{F}}^{E,\{a\}}[h^{\mathrm{TT}}, \phi] = \psi_{\mathrm{g}}[h^{\mathrm{TT}}] \psi_{\mathrm{m}}[\phi] \; , \tag{4.95}$$

where

$$\int_\Sigma d^d x \sqrt{\gamma} N \left[2\Pi_{\mathrm{TT}}^{ij} \Pi_{\mathrm{TT},ij} - \frac{1}{8} \, h^{\mathrm{TT}ij} (\Delta_N + 2) h_{ij}^{\mathrm{TT}} \right] \psi_{\mathrm{g}}[h^{\mathrm{TT}}] = E_{\mathrm{g}} \, \psi_{\mathrm{g}}[h^{\mathrm{TT}}] \; , \tag{4.96}$$

$$\int_\Sigma d^d x \sqrt{\gamma} N \, \mathcal{H}^{\mathrm{matter}} \psi_{\mathrm{m}}[\phi] = E_{\mathrm{m}} \, \psi_{\mathrm{m}}[\phi] \; , \tag{4.97}$$

with $E = E_{\mathrm{g}} + E_{\mathrm{m}}$. Solutions to (4.96) and (4.97) are also degenerate although we have suppressed additional labels on the right hand side of (4.95) to lighten the notation.

In the subsections below, we describe, in some detail, the solutions to (4.96) and (4.97). The eigenvalues E in (4.92) are obtained after introducing a normal ordering prescription to regulate $\mathcal{H}_{\mathrm{bulk}}$. We specify this prescription below.

Here we note that once these solutions are found, the solution to the integrated Hamiltonian constraint is simply

$$\psi_{\mathrm{I}}^{E,\{a\}}[H_\partial, h^{\mathrm{TT}}, \phi] = \psi_{\mathrm{F}}^{E,\{a\}}[h^{\mathrm{TT}}, \phi] \otimes |H_\partial = E\rangle. \tag{4.98}$$

Our choice of notation above emphasizes that the spectrum of H_∂, which is a single-degree of freedom, is discrete.

Any linear combination of solutions of the form (4.98) is also a solution. The solution (4.98) will be very important for our analysis since it shows how the Hamiltonian constraint, at second order, correlates the energy of the dynamical degrees of freedom in the wavefunctional to the value of H_∂, which is given by an integrated component of the metric.

We will see later that the constraints fully determine the physical state $\Psi^{E,\{a\}}[h,\phi]$. The full wavefunctional is obtained by dressing the Fock state $\psi_{\rm F}^{E,\{a\}}[h^{\rm TT},\phi]$ with the appropriate $h_{ij}^{\rm T}$ and $h_{ij}^{\rm L}$ dependence, as will be explained in section 4.5.2.

Graviton wavefunctionals

We will describe here the solutions of (4.96). From the integrand appearing in equation (4.96), it is natural to consider a basis of transverse-traceless eigenfunctions $h_{ij}^{(n)}$ satisfying

$$-N^2(\Delta_N+2)h_{ij}^{(n)} = \omega_n^2 h_{ij}^{(n)}. \tag{4.99}$$

As shown in Appendix B.2.1, this eigenvalue problem is equivalent to the standard quantization of the graviton in global AdS_{d+1}. We will take this basis to be normalized with respect to the inner product

$$\frac{1}{2}\int_\Sigma d^dx\sqrt{\gamma}\,N^{-1}\,h_{ij}^{(m)}h^{(n)ij} = \delta_{mn}\ . \tag{4.100}$$

We then use the decomposition

$$h_{ij}^{\rm TT} = \sum_n c_n h_{ij}^{(n)}, \tag{4.101}$$

and the variables c_n can be written, using the orthogonality condition above as

$$c_n = \frac{1}{2}\int_\Sigma d^dx\sqrt{\gamma}\,N^{-1}h_{ij}^{\rm TT}h^{(n)ij}\ . \tag{4.102}$$

Using the chain rule we see that

$$\Pi_{\rm TT}^{ij}\,\psi_{\rm g}[h^{\rm TT}] = -\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta h_{ij}^{\rm TT}}\psi_{\rm g}[h^{\rm TT}] = -\frac{i}{2N}\sum_n\frac{\partial\psi_{\rm g}[h^{\rm TT}]}{\partial c_n}h^{(n)ij}\ , \tag{4.103}$$

so that we have

$$2\int_\Sigma d^dx\sqrt{\gamma}N\,\Pi^{ij}_{\rm TT}\Pi_{{\rm TT},ij}\psi_{\rm g}[h^{\rm TT}] = -\frac{1}{2}\int_\Sigma d^dx\sqrt{\gamma}N^{-1}\sum_{n,m}\frac{\partial^2\psi_{\rm g}[h^{\rm TT}]}{\partial c_n\partial c_m}h^{(n)ij}h^{(m)}_{ij}$$
$$= -\sum_n \frac{\partial^2\psi_{\rm g}[h^{\rm TT}]}{\partial c_n^2}\ , \tag{4.104}$$

where we have again used the orthogonality relation (4.100).

Then equation (4.96) reduces to

$$\sum_n\left(-\frac{\partial^2}{\partial c_n^2}+\frac{1}{4}c_n^2\omega_n^2\right)\psi_{\rm g}[h^{\rm TT}] = E_{\rm g}\,\psi_{\rm g}[h^{\rm TT}]\ . \tag{4.105}$$

We define the raising and lowering operators

$$A_n^\dagger = \frac{1}{\sqrt{\omega_n}}\left(\frac{\partial}{\partial c_n}-\frac{1}{2}\omega_n c_n\right),\qquad A_n = -\frac{1}{\sqrt{\omega_n}}\left(\frac{\partial}{\partial c_n}+\frac{1}{2}\omega_n c_n\right),\qquad [A_n, A_m^\dagger] = \delta_{mn}\ . \tag{4.106}$$

We also assume that $\mathcal{H}_{\rm bulk}$ should be normal ordered so that all annihilation operators, A_n are placed to the right of creation operators $A_n^\dagger$. With this simplification the constraint becomes

$$\sum_n \omega_n A_n^\dagger A_n\,\psi_{\rm g}[h^{\rm TT}] = E_{\rm grav}\,\psi_{\rm g}[h^{\rm TT}]\ . \tag{4.107}$$

Our normal ordering prescription ensures that the energy vanishes for the vacuum, which is defined as

$$A_n\,\psi_0[h^{\rm TT}] = 0\ ,\quad \text{for all } n\ . \tag{4.108}$$

This is the vacuum wavefunctional for the transverse-traceless gravitons. It has the expression

$$\psi_0[h^{\rm TT}] = \mathcal{N}\prod_n \exp\left(-\frac{1}{4}\omega_n c_n^2\right) = \mathcal{N}\exp\left(-\frac{1}{8}\int_\Sigma d^dx\sqrt{\gamma}\,h^{{\rm TT}ij}\sqrt{-(\Delta_N+2)}\,h^{\rm TT}_{ij}\right) \tag{4.109}$$

up to a normalization constant $\mathcal{N}$ that we specify below. In the flat space limit, this reproduces the results of [81] obtained using a similar method, or in [124] from a Euclidean path integral.

The space of solutions is then a Fock space spanned by states of the form

$$\psi_{\rm g}[h^{\rm TT}] = \frac{1}{\prod_i \sqrt{d_i!}} (A^\dagger_{n_1})^{d_1} (A^\dagger_{n_2})^{d_2} \dots \psi_0[h^{\rm TT}] \ , \tag{4.110}$$

with energy

$$E_{\rm g} = \sum_i d_i \, \omega_{n_i} \ . \tag{4.111}$$

We have written the wavefunctionals that appear in equation (4.110) in terms of the action of operators on the vacuum wavefunctionals. But they can also be written, as usual, in terms of Hermite polynomials. Note that the validity of perturbation theory requires that, in the Fock space, (4.110) we restrict attention to states where $\omega_{n_i} \ll \frac{1}{\kappa}$.

The natural measure on this space of wavefunctionals is simply

$$Dh^{\rm TT} = \prod_n dc_n \ , \tag{4.112}$$

and we choose the normalization constant $\mathcal{N}$ so that with respect to this measure the wavefunctionals that appear in (4.110) are unit normalized

$$(\psi_{\rm g}, \psi_{\rm g}) \equiv \int Dh^{\rm TT} \psi_{\rm g}[h^{\rm TT}] \psi_{\rm g}[h^{\rm TT}]^* = 1 \ . \tag{4.113}$$

Of course, wavefunctionals that differ by even a single value of d_i in equation (4.110) are orthogonal.

Matter wavefunctionals

The matter part of the wavefunctional can be obtained in a similar way as for the transverse-traceless gravitons. To illustrate this, we will consider a minimally coupled massive scalar field. From (4.25), it follows that the canonical Hamiltonian is

$$\mathcal{H}^{\rm matter} = \frac{1}{2} \int_\Sigma d^d x \sqrt{\gamma} N \left(\gamma^{-1} \pi^2 - \phi (\Delta_N - m^2) \phi \right) \ ,$$

where we have imposed a normalizable boundary condition at infinity for the scalar field to remove a boundary term. The operator Δ_N appearing here is the same as in (4.70).

We consider eigenfunctions $\phi^{(n)}$ satisfying

$$-N^2(\Delta_N - m^2)\phi^{(n)} = \tilde{\omega}_n^2\phi^{(n)} \ , \tag{4.114}$$

normalized so that

$$\int_\Sigma d^dx\sqrt{\gamma}N^{-1}\,\phi^{(m)}\phi^{(n)} = \delta_{mn} \ . \tag{4.115}$$

Using that

$$\pi(x) = -i\frac{\delta}{\delta\phi(x)} \ , \tag{4.116}$$

we can perform the same analysis as for the graviton. We obtain a Fock space constructed from the frequencies $\tilde{\omega}_n$.

We can check that the Wheeler-DeWitt analysis reproduces the correct frequencies by considering the equation of motion in the full spacetime

$$(\hat{\Box} - m^2)\phi = 0 \ , \tag{4.117}$$

which becomes on the Cauchy slice,

$$-N^{-2}\partial_t^2\phi + (\Delta_N - m^2)\phi = 0 \ . \tag{4.118}$$

In the same manner as for the graviton, this shows that $\tilde{\omega}_n$ as defined in (4.114) are indeed the frequencies obtained from (4.117).

In global AdS_{d+1} with normalizable boundary conditions, the resulting spectrum is [125]

$$\tilde{\omega}_n = \Delta + \ell + 2n, \qquad n \in \mathbb{Z}_{\geq 0} \ . \tag{4.119}$$

where ℓ labels a spherical harmonic of S^{d-1} with eigenvalue $\ell(\ell + d - 2)$ and the conformal dimension is

$$\Delta = \frac{1}{2}(d + \sqrt{d^2 + 4m^2}) \ . \tag{4.120}$$

In precise analogy with the analysis above, we expand the matter field as

$$\phi = \sum_n \tilde{c}_n\phi^{(n)}. \tag{4.121}$$

The equation (4.97) then reduces to

$$\frac{1}{2}\sum_n\left(-\frac{\partial^2}{\partial\tilde{c}_n^2} + \tilde{c}_n^2\omega_n^2\right)\psi_{\mathrm{m}}[\phi] = E_{\mathrm{m}}\psi_{\mathrm{m}}[\phi] \ . \tag{4.122}$$

We then define

$$\tilde{A}_n^\dagger = \frac{1}{\sqrt{2\tilde{\omega}_n}}\left(\frac{\partial}{\partial\tilde{c}_n} - \tilde{\omega}_n\tilde{c}_n\right) , \qquad \tilde{A}_n = -\frac{1}{\sqrt{2\tilde{\omega}_n}}\left(\frac{\partial}{\partial\tilde{c}_n} + \tilde{\omega}_n\tilde{c}_n\right) , \qquad [\tilde{A}_n, \tilde{A}_m^\dagger] = \delta_{mn} , \tag{4.123}$$

and the vacuum wavefunctional, which is annihilated by all the $\tilde{A}_n$ operators is given by

$$\psi_0[\phi] = \widetilde{\mathcal{N}}\prod_n \exp\left(-\frac{1}{2}\tilde{\omega}_n\tilde{c}_n^2\right) = \widetilde{\mathcal{N}}\exp\left(-\frac{1}{2}\int_\Sigma d^dx\sqrt{\gamma}\,\phi\sqrt{-(\Delta_N - m^2)}\,\phi\right) , \tag{4.124}$$

where $\widetilde{\mathcal{N}}$ is a normalization constant. Once again, excited states can be obtained by acting with creation operators:

$$\psi_{\rm m}[\phi] = \frac{1}{\prod\sqrt{d_i!}}(A_{n_1}^\dagger)^{d_1}(A_{n_2}^\dagger)^{d_2}\ldots\psi_0[\phi] \ . \tag{4.125}$$

As above, we normal order the matter contribution to $\mathcal{H}_{\rm bulk}$ so that the annihilation operators A_n are placed to the right of the creation operators $A_n^\dagger$. With this convention, the energy is given by

$$E_{\rm m} = \sum_i d_i\,\tilde{\omega}_{n_i} \ . \tag{4.126}$$

We remind the reader that, as in the case of graviton wavefunctionals, within perturbation theory, we are restricted to states of the form (4.124) where $\tilde{\omega}_{n_i} \ll \frac{1}{\kappa}$. The natural measure on this space of wavefunctionals is simply

$$D\phi = \prod_n d\tilde{c}_n \ , \tag{4.127}$$

and we choose the normalization constant so that the wavefunctionals are unit normalized

$$(\psi_{\rm m}, \psi_{\rm m}) \equiv \int D\phi\,\psi_{\rm m}[\phi]\psi_{\rm m}[\phi]^* = 1 \ . \tag{4.128}$$

As above, wavefunctionals that differ by even a single value of d_i in the expression (4.124) are orthogonal.

The analysis of matter wavefunctionals completes our analysis of the auxiliary Fock space. These wavefunctionals can be combined with the transverse-traceless graviton wavefunctionals obtained above as displayed in (4.95). The resulting wavefunctional enters the solution of the integrated Hamiltonian constraint displayed in (4.98).

4.5.2 Pointwise constraints

In the previous section, we have described how to solve the integrated Hamiltonian constraint. However, the Hamiltonian and momentum constraints, displayed in (4.27), actually comprise an infinite set of constraints — one for each spacetime point. In this section we will present the leading order solution to these pointwise constraints. We will also describe a procedure to obtain higher order solutions.

We show in section 4.6 that the main result of this chapter — which is that wavefunctionals that coincide near the boundary must also coincide in the bulk — does not require the detailed form of the dependence of the wavefunctionals on h^{L} and h^{T} in the bulk. For us, it is only important that the pointwise constraints can be used to uniquely lift a solution of the integrated Hamiltonian constraint displayed in (4.98) to a solution of the full constraints (4.27). So, in the bulk of this section, we focus on a procedure that makes it evident that the pointwise constraints can be used to perturbatively fix the dependence of the wavefunctional on the pointwise values of h^{T}_{ij} and h^{L}_{ij}. In the solution (4.98), it was only the dependence on H_∂ — which is the integral of a particular component of h^{T}_{ij} — that was fixed. Therefore our procedure leads to the following uplift.

$$\psi_{\mathrm{I}}^{E,\{a\}}[H_\partial, h^{\mathrm{TT}}, \phi] \xrightarrow{\text{pointwise constraints}} \Psi^{E,\{a\}}[h, \phi] \ . \tag{4.129}$$

In section 4.5.2 we then provide an indirect argument that leads to the same conclusion: namely that the uplift (4.129) can be performed uniquely. The details, and checks of the explicit solution itself are presented in Appendix B.4.

Rewriting the pointwise constraints

We start by putting the pointwise Hamiltonian and momentum constraint in a convenient form. In this section, we often display the dependence of the wavefunctional on the individual components of the ADM decomposition of the metric fluctuation and momenta using notation like $\Psi[h^{\mathrm{TT}}, h^{\mathrm{T}}, h^{\mathrm{L}}, \phi]$.

Hamiltonian constraint. First, consider the second order Hamiltonian constraint. From expression (4.73) the Hamiltonian constraint is equivalent to

$$\mathcal{D}^{ij} h_{ij}(x) \Psi[h^{\mathrm{TT}}, h^{\mathrm{T}}, h^{\mathrm{L}}, \phi] = \kappa\, Q(x) \Psi[h^{\mathrm{TT}}, h^{\mathrm{T}}, h^{\mathrm{L}}, \phi] \ , \tag{4.130}$$

where we have defined the differential operator

$$\mathcal{D}^{ij} \equiv \frac{1}{2}\left(\nabla^i\nabla^j - \gamma^{ij}\nabla_k\nabla^k + (d-1)\gamma^{ij}\right) \tag{4.131}$$

and Q is defined in (4.74).

As explained near (4.66), the LHS of (4.130) only depends on h^{T} since the operator $\mathcal{D}^{ij}$ annihilates the longitudinal and the transverse-traceless components. So we can also write the equation above as

$$\mathcal{D}^{ij}h^{\mathrm{T}}_{ij}(x)\Psi[h^{\mathrm{TT}}, h^{\mathrm{T}}, h^{\mathrm{L}}, \phi] = \kappa\, Q(x)\Psi[h^{\mathrm{TT}}, h^{\mathrm{T}}, h^{\mathrm{L}}, \phi]\ , \tag{4.132}$$

which can be rewritten as

$$h^{\mathrm{T}}_{ij}(x)\Psi[h^{\mathrm{TT}}, h^{\mathrm{T}}, h^{\mathrm{L}}, \phi] = \kappa\int_\Sigma d^dx'\sqrt{\gamma}'\, G_{ij}(x,x')Q(x')\Psi[h^{\mathrm{TT}}, h^{\mathrm{T}}, h^{\mathrm{L}}, \phi]\ , \tag{4.133}$$

where the Green's function G_{ij} satisfies $\mathcal{D}^{ij}G_{ij}(x,x') = \frac{1}{\sqrt{\gamma}}\delta(x,x')$ with boundary conditions that it vanishes as x' approaches the boundary. We emphasize that (4.133) is just an exact rewriting of (4.132) and not really a solution.

The equation (4.133) may seem complicated. However, we can develop a perturbative procedure to solve (4.133) as follows. The idea, as indicated originally by ADM [82] and then elaborated in [126, 81, 127] is to think of the momentum Π^{ij}_{T} as a notion of "local" time. Therefore we can think of the pointwise constraint (4.132) as telling us how initial data on a slice "evolve" as we change time locally but keep the endpoints of the Cauchy slice fixed.

Thus, we must view Π^{ij}_{T} to be the "position" variable while h^{T}_{ij} is the conjugate momentum. This idea can be implemented by performing a partial Fourier transform of the wavefunctional

$$\Psi[h^{\mathrm{TT}}, \Pi_{\mathrm{T}}, h^{\mathrm{L}}, \phi] = \int Dh^{\mathrm{T}}\, e^{-i\int_\Sigma d^dx\sqrt{\gamma}\,\Pi^{ij}_{\mathrm{T}}h^{\mathrm{T}}_{ij}}\Psi[h^{\mathrm{TT}}, h^{\mathrm{T}}, h^{\mathrm{L}}, \phi] \tag{4.134}$$

and we will slightly abuse notation by also denoting this wavefunctional by Ψ. This allows us to rewrite the constraint (4.133) as

$$\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta\Pi^{ij}_{\mathrm{T}}(x)}\Psi[h^{\mathrm{TT}}, \Pi_{\mathrm{T}}, h^{\mathrm{L}}, \phi] = \kappa\int_\Sigma d^dx'\,\sqrt{\gamma}'G_{ij}(x,x')Q(x')\Psi[h^{\mathrm{TT}}, \Pi_{\mathrm{T}}, h^{\mathrm{L}}, \phi]\,. \tag{4.135}$$

Momentum constraint. We now rewrite the momentum constraint using a similar procedure. We do not display all intermediate steps since the procedure used is almost identical to the procedure used above.

We start with the form of the constraint as shown in equation (4.60). Then we note that it can be written in the form

$$-\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta h^{\mathrm{L}}_{ij}(x)}\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi] = \frac{\kappa}{2}\int_{\Sigma} d^dx'\,\sqrt{\gamma}'\tilde{G}^{ijk}(x,x')Q_k(x')\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi]\,. \tag{4.136}$$

Here the Green's function $\widetilde{G}^i$ is the solution to

$$\nabla_i\widetilde{G}^{ijk}(x,x') = \frac{1}{\sqrt{\gamma}'}\delta^d(x,x')\gamma^{jk} \tag{4.137}$$

with boundary conditions so that $\widetilde{G}^{ijk}(x,x')$ vanishes as x is taken to the boundary.

Note that in (4.135) the operator Q still involves both Π^{ij}_{T} and also h^{T}_{ij} which should be interpreted as $\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta\Pi^{ij}_T}$ while acting on Ψ. Similarly in (4.136), the right hand side Q_k still involves h^{L}_{ij}. So it may appear that we have not achieved much by recasting the pointwise constraints in the form (4.135) and (4.136). Nevertheless we can take advantage of the factor of κ that appears in (4.135) to develop an iterative procedure to solve this equation.

Leading order solutions

We start by considering the wavefunctionals described in section 4.4 that have a specified dependence on h^{TT}_{ij} and the matter field ϕ and are eigenfunctions of the energy with eigenvalue E. We then specify that for the constant function $\Pi^{ij}_{\mathrm{T}}(x) = 0$ and $h^{\mathrm{L}}_{ij}(x) = 0$ we have

$$\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi]\big|_{\Pi^{ij}_{\mathrm{T}}=0,h^{\mathrm{L}}_{ij}=0} = \psi^{E,\{a\}}_{\mathrm{F}}[h^{\mathrm{TT}},\phi]\,. \tag{4.138}$$

We then truncate (4.135) and (4.136) by dropping occurrences of h^{T}_{ij} and Π^{ij}_{L}

$$Q^{(0)}(x) \equiv Q(x)\Big|_{h^{\mathrm{T}}_{ij}=0,\Pi^{ij}_{\mathrm{L}}=0}; \qquad Q^{(0)}_i(x) \equiv Q_i(x)\Big|_{h^{\mathrm{T}}_{ij}=0,\Pi^{ij}_{\mathrm{L}}=0}\,. \tag{4.139}$$

As h^{T}_{ij} and Π^{ij}_{L} are $\mathrm{O}(\kappa)$ by the first order constraints, this corresponds to restricting to the leading order.

Then the leading order wavefunctional solution satisfies

$$\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta\Pi_{\mathrm{T}}^{ij}(x)}\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi] = \kappa\int_{\Sigma} d^dx'\sqrt{\gamma'}\,G_{ij}(x,x')Q^{(0)}(x')\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi],$$
$$-\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta h_{ij}^{\mathrm{L}}(x)}\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi] = \frac{\kappa}{2}\int_{\Sigma} d^dx'\sqrt{\gamma'}\,\tilde{G}^{ijk}(x,x')Q_k^{(0)}(x')\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi]\ . \tag{4.140}$$

Note that, for consistency, we must adopt the same normal ordering prescription for $Q^{(0)}$ and $Q_k^{(0)}$ that was adopted in section 4.5.1. This normal ordering prescription leads to the subtraction of a position-dependent constant in (4.140).

These leading order equations can be solved by performing a change of variable for Π_{ij}^{T} and h_{ij}^{L}. It proves convenient to define a "time" variable $\mathbf{t}$ by the equation

$$\Pi_{ij}^{\mathrm{T}} = D_{ij}\mathbf{t}\ . \tag{4.141}$$

This is the generalization to AdS of the time variable used for example in [82, 81]. Note that this is related to the variable α appearing in (4.84) by $\mathbf{t} = -\frac{2}{d-1}N\alpha$.

Differentiating with respect to $\mathbf{t}$ instead of Π_{T} simplifies the Hamiltonian constraint to

$$\left[-\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta\mathbf{t}(x)} - \kappa\,Q^{(0)}(x)\right]\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi] = 0\ . \tag{4.142}$$

Similarly, using ϵ^i instead of h_{ij}^{L} allows to write the momentum constraint as

$$\left[-\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta\epsilon_i(x)} - \kappa\,Q_i^{(0)}(x)\right]\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi] = 0\ . \tag{4.143}$$

These equations are derived in Appendix B.4. We can look for a solution of the form

$$\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi] = \exp(i\kappa\mathcal{S})\,\psi_{\mathrm{F}}^{E,\{a\}}[h^{\mathrm{TT}},\phi] + \mathrm{O}\big(\kappa^2\big)\,, \tag{4.144}$$

where the exponent $\mathcal{S}$ must satisfy

$$\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\mathbf{t}(x)} = Q^{(0)}(x), \qquad \frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\epsilon^i(x)} = -Q_i^{(0)}(x)\ . \tag{4.145}$$

Remarkably, the solution can be found as it takes the simple form

$$\mathcal{S} = \int d^dx\sqrt{\gamma}\left(-\frac{2}{3}\,\mathbf{t}\left(\Pi^{ij}_{\mathrm{T}}\Pi^{\mathrm{T}}_{ij} - \frac{1}{d-1}\Pi^2_{\mathrm{T}}\right) + 2\,\mathbf{t}\,\Pi^{ij}_{\mathrm{TT}}\Pi^{ij}_{\mathrm{T}} + Q^{(0)}_0\mathbf{t} - \epsilon^i\mathcal{H}^{\mathrm{matter}}_i\right). \tag{4.146}$$

It is proven in Appendix B.4 that this is indeed the solution. This relies on a non-trivial permutation symmetry in the terms of $\mathcal{S}$ that are cubic and quadratic in $\mathbf{t}$. We can confirm that the approximation used in (4.139) is valid since we can explicitly check on the solution that $h^{\mathrm{T}}\Psi, (h^{\mathrm{T}})^2\Psi, \Pi_{\mathrm{L}}\Psi$ and $(\Pi_{\mathrm{L}})^2\Psi$ are all subleading in κ.

By inverting the Fourier transform (4.134) we can also obtain wavefunctionals in the original metric representation

$$\Psi^{E,\{a\}}[h,\phi] = \int D\Pi_{\mathrm{T}}\, e^{i\int d^dx\sqrt{\gamma}\,\Pi^{ij}_{\mathrm{T}}h^{\mathrm{T}}_{ij}}\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi]\ . \tag{4.147}$$

We can see that the dependence on h^{T}_{ij} is captured by an integral that is qualitatively similar to the Airy function.

An iterative solution algorithm

We can obtain solutions to the pointwise constraints at higher order by using an iterative procedure. At $\mathrm{O}(\kappa^2)$ one must also account for the terms that involve $h^{\mathrm{T}}_{ij} = \frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta\Pi^{ij}_{\mathrm{T}}}$ and $\Pi^{ij}_{\mathrm{L}} = -\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta h^{\mathrm{L}}_{ij}}$ on the right hand side of (4.135) and (4.136). But it is clear that to obtain the solution to $\mathrm{O}(\kappa^2)$ one only needs to account for the action of these terms on the $\mathrm{O}(\kappa)$ solution obtained through (4.140). In fact if one expands the wavefunctional in a power series in κ then this pattern continues at higher order in perturbation theory: at each order in perturbation theory, these operators act on the lower-order terms and produce a "source term" on the right hand side of the first order differential equation (4.135) and (4.136).

Note that at higher orders it is not enough to keep only the terms involving h^{T} in Q but it is also necessary to include the other higher-order terms from the expansion of the Hamiltonian constraint (4.63). But provided this is done, the procedure above can be extended to higher order.

It is clear that to leading order in κ the dependence on Π^{ij}_{T} as one approaches the boundary continuously goes over to the dependence obtained in the solutions of 4.4.4. However, the solutions obtained there were very simple because Π^{ij}_{T} drops out from the integrated constraint as described in

(4.87). At a general bulk point this does not happen and therefore (4.135) leads in general to a complicated set of coupled differential equations.

An indirect argument implying a bijection between solutions to the pointwise and integrated constraints

The subsection above proposed an iterative algorithm to uniquely uplift a solution of the integrated constraint to a solution of the full pointwise constraints and an explicit solution to leading order. However, it is possible to argue indirectly, even without the help of the explicit solution or the algorithm above, that there is a one-to-one map between solutions of the integrated constraint and solutions of the full pointwise constraints.

This is because it is possible to obtain a description of the low-energy Hilbert space of gravity coupled to matter by other means. One common procedure adopted is simply to fix the gauge, which allows an identification of the independent degrees of freedom. As expected, these degrees of freedom correspond to the transverse-traceless graviton and the matter fields. Another equivalent procedure is to examine the set of all classical solutions of the theory and then quantize them. Both procedures can be seen to lead to precisely the Fock space described in section 4.5.1. The solutions that we have described here are also in one-to-one correspondence with this Fock space. This implies that there are no additional solutions that we have missed, and nor does our procedure yield any spurious solutions.

4.5.3 Inner product

To complete the definition of the canonical theory, we need to give the definition of the inner product. The inner product has been the subject of some discussion in the literature [80]. Here we will propose a specific definition of the inner product at leading order in perturbation theory and demonstrate its consistency.

Consider two solutions of the constraints that we denote by Ψ_1 and Ψ_2. We propose that the inner product between these two solutions obtained above is defined as

$$(\Psi_2, \Psi_1) = \int Dh^{\mathrm{TT}} D\phi \, \Psi_1[h^{\mathrm{TT}}, \Pi_{\mathrm{T}}, h^{\mathrm{L}}, \phi] \, \Psi_2[h^{\mathrm{TT}}, \Pi_{\mathrm{T}}, h^{\mathrm{L}}, \phi]^* \,, \qquad (4.148)$$

where $*$ refers to complex conjugation. Note that the integral is only over the propagating degrees of freedom $h^{\rm TT}$ and ϕ and is performed at fixed values of $\Pi_{\rm T}^{ij}$ and $h_{ij}^{\rm L}$.

To see that this definition makes sense, we must show that the inner product doesn't depend on the value of $h_{ij}^{\rm L}$ and $\Pi_{\rm T}^{ij}$ at which the wavefunctionals are evaluated. At leading order in κ this follows directly from the "evolution" equations obeyed by the wavefunctionals in these variables. In particular, by conjugating equation (4.140), we find that

$$-\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta\Pi_{\rm T}^{ij}(x)}\Psi_2[h^{\rm TT},\Pi_{\rm T},h^{\rm L},\phi]^* = \kappa\int_\Sigma d^dx'\sqrt{\gamma}'\,G_{ij}(x,x')Q^{(0)*}(x')\Psi_2[h^{\rm TT},\Pi_{\rm T},h^{\rm L},\phi]^*\,. \tag{4.149}$$

Note that in the basis used above, $Q^{(0)}(x)$ is not a real operator due to the presence of cross terms in its definition that mix, for instance, $\Pi_{\rm TT}^{ij}$ and $\Pi_{\rm T}^{ij}$. But since $\Pi_{\rm T}^{ij}$ is realized, in the basis used above, as $-i\frac{\delta}{\delta\Pi_{\rm TT}^{ij}}$ complex conjugation of this operator introduces a negative sign. Nevertheless, by integrating by parts, and using the identities

$$\begin{aligned}\int Dh^{\rm TT}D\phi\,\Psi_1\Big(\frac{\delta}{\delta h_{ij}^{\rm TT}}\Psi_2^*\Big) &= -\int Dh^{\rm TT}D\phi\,\Big(\frac{\delta}{\delta h_{ij}^{\rm TT}}\Psi_1\Big)\Psi_2^*\,,\\ \int Dh^{\rm TT}D\phi\,\Psi_1\Big(\frac{\delta^2}{\delta h_{ij}^{\rm TT}\delta h_{kl}^{\rm TT}}\Psi_2^*\Big) &= \int Dh^{\rm TT}D\phi\,\Big(\frac{\delta^2}{\delta h_{ij}^{\rm TT}\delta h_{kl}^{\rm TT}}\Psi_1\Big)\Psi_2^*\,,\end{aligned} \tag{4.150}$$

we find that

$$\int Dh^{\rm TT}D\phi\,\Psi_1 Q^{(0)*}(x')\Psi_2^* = \int Dh^{\rm TT}D\phi\,\Big(Q^{(0)}(x')\Psi_1\Big)\Psi_2^*\,. \tag{4.151}$$

In the sequence of equations above, we have suppressed the arguments of the wavefunctionals for clarity.

Now, using the evolution equation we find that

$$\begin{aligned}&\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta\Pi_{\rm T}^{ij}(x)}(\Psi_2,\Psi_1)\\ &= \int_\Sigma d^dx'\int Dh^{\rm TT}D\phi\,\Big(Q^{(0)}(x')G_{ij}(x,x')\Psi_1\Psi_2^* - \Psi_1 G_{ij}(x,x')Q^{(0)*}(x')\Psi_2^*\Big) = 0\,.\end{aligned} \tag{4.152}$$

Similarly, the second order momentum constraint (4.60) equates $-\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta h_{ij}^{\rm L}}$ with a self-adjoint operator, which ensures that

$$-\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta h_{ij}^{\rm L}}(\Psi_1,\Psi_2) = 0\,, \tag{4.153}$$

and the inner product is independent of h^{L}_{ij}.

This inner product reproduces the Fock space inner product if we use the natural measure

$$Dh^{\mathrm{TT}} = \prod_n dc_n, \qquad D\phi = \prod_n d\tilde{c}_n \ . \tag{4.154}$$

Then, using the above normalization, we find the simple result

$$\left(\Psi^{E,\{a\}}, \Psi^{E',\{a'\}}\right) = \delta_{E,E'}\delta_{\{a\},\{a'\}} \ . \tag{4.155}$$

4.6 Holography of information

In previous sections we have analyzed the form of the Hamiltonian and momentum constraints. We have shown that these constraints force a certain component of the metric fluctuation to have specific correlations with the excitations of the matter fields and transverse-traceless gravitons. We will now show that these correlations are sufficient to completely identify a state in the bulk from boundary correlators.

More precisely, we will establish the result.

Result. *If two pure or mixed states of the theory coincide at the boundary of AdS for an infinitesimal interval of time then they must coincide everywhere in the bulk.*

An intuitive way to think of our strategy to establish this result is as follows. At the boundary, we have available to us the boundary values of the metric and other matter fields. Let us first consider pure states. Then the correlations that we have analyzed at length in section 4.5 allow us to determine the energy of a state from the measurement of a certain component of the metric at the boundary. The value of this component is suppressed by a factor of κ but our analysis is already sufficient to reveal its nontrivial value.

A determination of the energy is not sufficient to determine the state. Since a pure state must be a superposition of energy eigenstates, the determination of the energy still leaves us with an ambiguity of relative phases between different energy eigenstates and also an ambiguity associated with degeneracies in energy eigenstates.

To resolve this ambiguity, we exploit the fact that energy eigenstates are necessarily delocalized states. This is true just by virtue of the Heisenberg uncertainty principle. We demonstrate that the ambiguity associated with degeneracy and the ambiguity associated with the phases of eigenstates can be resolved by additional measurements of the metric and matter fields near the boundary in an infinitesimal time interval. These latter measurements are not suppressed by κ and involve just the O(1) fluctuations of the transverse-traceless metric component and the matter fields. The end result is that correlations of the energy and other observables near the boundary suffice to completely fix the form of the bulk state.

The extension of our result to mixed states is straightforward. A basis of density matrices is obtained by combining a wavefunctional corresponding to one energy eigenstate with the conjugate of a wavefunctional corresponding to another energy eigenstate. Let us denote such a basis by $\rho^{E,E',\{a\},\{a'\}}[h, \phi, \tilde{h}, \tilde{\phi}]$ where E, E' are the energy eigenvalues of the wavefunctionals and $\{a\}, \{a'\}$ are additional labels necessary since energy eigenstates can be degenerate and, as usual, the density matrix has double the arguments of the wavefunctional. Any density matrix can be written as a linear combination of such elementary density matrices with certain coefficients. Two density matrices can only yield the same values for all moments of the energy if these coefficients satisfy certain strong constraints. As in the case of pure states, measurements of the energy are insufficient to fix these coefficients. However, we show that correlators of additional dynamical fields uniquely fix these coefficients.

We now present a precise mathematical argument that realizes the intuition above. In preparation for this argument, we first discuss the set of boundary observables and also the set of valid mixed states in the theory before turning to the proof in section 4.6.3

4.6.1 Boundary observables

Let us briefly recapitulate the set of boundary observables. Recall that, as explained below equation (4.27) boundary observables are automatically gauge invariant. The constraints only impose the invariance of observables under small gauge transformations, and since such transformations die off near the boundary, the constraints commute with boundary observables.

One special boundary observable that will be required is the ADM Hamiltonian, H_∂, given in equation (4.38). In addition, we will require the boundary values of the metric and also the matter fields in the theory. In order to adopt a compact notation, we denote such local boundary operators collectively by

$$\mathcal{O}(t, \Omega); \qquad \Omega \in S^{d-1}.$$

Note that these observables are naturally defined by a value of the boundary time, t, and also a position on the boundary sphere.

For instance, consider the scalar field that we have discussed above with mass m. Then a gauge-invariant boundary observable is obtained through

$$\mathcal{O}(t, \Omega) = \lim_{r \to \infty} r^{\Delta} \phi(r, t, \Omega) \ , \tag{4.156}$$

where we are using the coordinate system in (4.14) and Δ is defined in (4.120). In our notation, we assume that unlike H_∂ (defined in equation (4.38)), no explicit factors of $\frac{1}{\kappa}$ are inserted while taking the boundary limits of bulk operators. The reader should keep this distinction between H_∂ and the observables $\mathcal{O}(t, \Omega)$ in mind for the analysis below.

We pause to address a subtlety associated with the limit described in equation (4.156). In order to take the limit, the operator on the right hand side of equation (4.156), which is a bulk operator, must be first made gauge invariant in the sense of equation (4.31). It can be seen that there is no unique way to dress the bulk operator in order to make it gauge invariant.

A simple way to understand this lack of uniqueness is as follows. Until now, we have not invoked a specific gauge. But another way of obtaining approximately local bulk operator is simply to choose a gauge. To every such gauge-fixed operator, there exists a gauge-invariant representation of the operator that satisfies the constraints (4.31). But different choices of gauge lead to different operators. This is why the symbol $\phi(r, t, \Omega)$ does not have a unique meaning unless its precise dressing is specified.

This lack of uniqueness changes some correlators at $\mathrm{O}(\kappa)$ [128]. Nevertheless, this issue is not important for our analysis because we will use the operators shown in (4.156) only within specific correlators. We will only need the fact that when we take the limit to the boundary, the final operator commutes with the constraints and its correlators with other local boundary operators at $\mathrm{O}(1)$ are independent of how the operators was dressed in the intermediate step. The precise property used is stated precisely in equation (4.164) below and also holds for gauge-fixed operators.

We have displayed a scalar field in (4.156) but a similar limit can be taken for observables that contain the metric or other dynamical fields in the theory.

In the case of observables that depend on the metric, the only element of the ADM decomposition that is relevant at O(1) in such an observable is $h^{\rm TT}$. It is easiest to see this in the mixed representation of (4.134), which can also be used for observables. Then the first order Hamiltonian and momentum constraints tell us that such an observable must be independent of $h^{\rm L}$ and $\Pi_{\rm T}$ at O(1). Therefore, at O(1) such observables can only depend on $h^{\rm TT}$ and $\Pi_{\rm TT}$. To lighten the notation, in the analysis below, $\mathcal{O}(t,\Omega)$ can stand for an insertion of either the metric or the insertion of a matter field.

4.6.2 Mixed states

In the previous sections, we have focused on pure states in the theory. It is a short step to generalize this discussion to mixed states, and we do so now.

In section 4.5 we have obtained wavefunctionals that are annihilated by the constraints. A basis of density matrices is obtained by combining them:

$$\rho^{E,E',\{a\},\{a'\}}[h,\phi,\tilde{h},\tilde{\phi}] = \Psi^{(a')}_{E'}[h_{ij},\phi]\Psi^{(a)}_{E}[\tilde{h}_{ij},\tilde{\phi}]^* \ , \tag{4.157}$$

where the wavefunctionals are normalized with respect to the inner product (4.148). Note that the density matrix depends on <u>two</u> metric configurations, which we have denoted above by h_{ij} and $\tilde{h}_{ij}$, and two matter-field configurations, denoted above by ϕ and $\tilde{\phi}$.

A general density matrix is a linear combination of elements of (4.157):

$$\rho[h,\phi,\tilde{h},\tilde{\phi}] = \sum_{E,E',\{a\},\{a'\}} c(E,E',\{a\},\{a'\})\rho^{E,E',\{a\},\{a'\}}[h,\phi,\tilde{h},\tilde{\phi}] \ .$$

As usual, these density matrices satisfy the constraints that

$$c(E',E,\{a'\},\{a\}) = c(E,E',\{a\},\{a'\})^* \ . \tag{4.158}$$

Moreover, the eigenvalues of the density matrix must be positive and we additionally have

$$\sum_{E,\{a\}} c(E,E,\{a\},\{a\}) = 1 \ . \tag{4.159}$$

We denote expectation values of an operator A in a density matrix using the notation $\langle A\rangle_\rho$. These expectation values are computed through

$$\langle A\rangle_\rho = \sum_{E,E',\{a\},\{a'\}} c(E,E',\{a\},\{a'\}) \left(\Psi^{E',\{a'\}}, A\,\Psi^{E,\{a\}}\right) \tag{4.160}$$

where the inner product is as defined in (4.148).

4.6.3 Proof of the main result

We are now in a position to prove the result above.

Let ρ_1 and ρ_2 be two density matrices of the form (4.157) with coefficients $c_1(E, E', \{a\}, \{a'\})$ and $c_2(E, E', \{a\}, \{a'\})$ respectively. We will now show that if the we have equality of the expectation values

$$\langle H_\partial^n \, \mathcal{O}(t_1, \Omega_1) \ldots \mathcal{O}(t_q, \Omega_q) H_\partial^m \rangle_{\rho_1} = \langle H_\partial^n \, \mathcal{O}(t_1, \Omega_1) \ldots \mathcal{O}(t_q, \Omega_q) H_\partial^m \rangle_{\rho_2} \tag{4.161}$$

for arbitrary values of n, m, q and and for any $t_i \in [0, \epsilon]$, we then have $\rho_1 = \rho_2$.

First note that equation (4.161) implies that

$$\sum_{\substack{E,E' \\ \{a\},\{a'\}}} \Big[c_1(E, E', \{a\}, \{a'\}) - c_2(E, E', \{a\}, \{a'\}) \Big] E^n E'^m \, \langle \mathcal{O}(t_1, \Omega_1) \ldots \mathcal{O}(t_q, \Omega_q) \rangle_{\rho^{E,E',\{a\},\{a'\}}} = 0 \, . \tag{4.162}$$

Since this is true for arbitrary values of n, m it must be true that for each individual value of E, E'

$$\sum_{\{a\},\{a'\}} \Big[c_1(E, E', \{a\}, \{a'\}) - c_2(E, E', \{a\}, \{a'\}) \Big] \langle \mathcal{O}(t_1, \Omega_1) \ldots \mathcal{O}(t_q, \Omega_q) \rangle_{\rho^{E,E',\{a\},\{a'\}}} = 0 \, , \tag{4.163}$$

where the important difference with the previous equation is that (4.163) does not involve any sum over E, E'.

We now note that the correlators that appear in (4.163) can be evaluated in the auxiliary Fock space introduced in section 4.5.1. That is,

$$\langle \mathcal{O}(t_1, \Omega_1) \ldots \mathcal{O}(t_q, \Omega_q) \rangle_{\rho^{E,E',\{a\},\{a'\}}} = \Big(\psi_{\mathrm{F}}^{E',\{a'\}}, \mathcal{O}(t_1, \Omega_1) \ldots \mathcal{O}(t_q, \Omega_q) \psi_{\mathrm{F}}^{E,\{a\}} \Big) + \mathrm{O}(\kappa) \, . \tag{4.164}$$

Note that the correlator on the left hand side does not include H_∂ and it only includes operators of the form (4.156). The equation above then follows from the discussion of section 4.6.1. Computing an ordinary matter correlator with the full wavefunctional is the same at O(1) as computing the same correlator in the Fock space.

To complete the proof, we will use the analytic properties of the correlators that appear on the RHS of (4.164). By inserting a complete set of energy eigenstates in the auxiliary Fock space, we find that

$$\left(\psi_{\rm F}^{E',\{a'\}}, \mathcal{O}(t_1,\Omega_1)\ldots\mathcal{O}(t_q,\Omega_q)\psi_{\rm F}^{E,\{a\}}\right) = e^{i(E't_1 - Et_q)} \sum_{E_j,\{a_j\}} e^{i\sum_{i=1}^{q-1} E_i(t_{i+1}-t_i)}$$
$$\times \left(\psi_{\rm F}^{E',\{a'\}}, \mathcal{O}(0,\Omega_1)\psi_{\rm F}^{E_1,\{a_1\}}\right)\left(\psi_{\rm F}^{E_1,\{a_1\}}, \mathcal{O}(0,\Omega_2)\psi_{\rm F}^{E_2,\{a_2\}}\right)\left(\psi_{\rm F}^{E_2,\{a_2\}}, \mathcal{O}(0,\Omega_2)\psi_{\rm F}^{E_3,\{a_3\}}\right)$$
$$\times \ldots \times \left(\psi_{\rm F}^{E_{q-1},\{a_{q-1}\}}, \mathcal{O}(0,\Omega_q)\psi_{\rm F}^{E,\{a\}}\right). \tag{4.165}$$

We emphasize that the entire identity above is simply in the auxiliary Fock space, and we have used completeness and the transformation properties of the operators under time translations only in the Fock space. This correlator is clearly analytic when the variables

$$z_1 = t_1; \quad z_2 = t_2 - t_1; \quad \ldots \quad ; z_q = t_q - t_{q-1} \tag{4.166}$$

are continued in the upper half plane. This follows just from the positivity of energy in the auxiliary Fock space. Note that in the correlator above the term in the exponent involving $E't_1 - Et_q$ is outside the sum over energies and when the variables z_i are extended in the upper half plane each term in the exponential inside the sum picks up a factor that decays exponentially with energy. Hence, if the correlator vanishes when $t_i \in [0,\epsilon]$ it must also vanish for $t_i \in [0,\pi]$ by the edge of the wedge theorem [129, 53].

But, in the Fock space, the individual creation and annihilation operators can be obtained by integrating $\mathcal{O}(t_i)$ in a band of size π. This follows from the discrete frequencies for the excitations found in section 4.5.1 and 4.5.1. So the algebra of operators for all $t_i \in [0,\pi]$ provides a complete basis for the algebra of all operators in the Fock space. Therefore the correlator in equation (4.163) vanishes for all $t_i \in [0,\pi]$ if and only if $c_1 = c_2$.

This proves our assertion.

Comments on the proof

We would like to comment on some subtle aspects of the proof above.

1. Note that the correlator (4.161) involves high powers of H_∂. Nevertheless, our perturbative solution can be used to reliably compute these correlators. This can be seen by rewriting the expression for the integrated constraint after Fourier transforming the wavefunctional as was

done in section 4.5.2. The constraint then takes the form

$$\frac{i}{2\kappa}\int_{\partial\Sigma} d^{d-1}\Omega\, n^i(N\nabla^j - \nabla^j N)(\delta_i^k\delta_j^\ell - \gamma^{k\ell}\gamma_{ij})\frac{1}{\sqrt{\gamma}}\frac{\delta}{\delta\Pi_{\mathrm{T}}^{k\ell}}\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi]$$
$$= \int_\Sigma d^dx\sqrt{\gamma}N\,\mathcal{H}_{\mathrm{bulk}}\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi] + \mathrm{O}\big(h_{ij}^{\mathrm{T}}\big) + \mathrm{O}(\kappa) \tag{4.167}$$

where we have explicitly also displayed the $\mathrm{O}\big(h_{ij}^{\mathrm{T}}\big)$ and higher-order terms that were dropped in the analysis of section 4.4.4. Now one of the key simplifications that we found in section 4.4.4 was that Π_{T}^{ij} drops out of the integrated expression for $\mathcal{H}_{\mathrm{bulk}}$. Consequently we were able to examine wavefunctionals that satisfied

$$\int_\Sigma d^dx\sqrt{\gamma}N\,\mathcal{H}_{\mathrm{bulk}}\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi] = E\,\Psi[h^{\mathrm{TT}},\Pi_{\mathrm{T}},h^{\mathrm{L}},\phi]\ . \tag{4.168}$$

We can then move to new variables

$$\widetilde{\Pi}_{\mathrm{T}}^{ij} = \kappa\,\Pi_{\mathrm{T}}^{ij}, \qquad \widetilde{h}_{ij}^{\mathrm{T}} = \frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta\widetilde{\Pi}_{\mathrm{T}}^{ij}} = \frac{1}{\kappa}h_{ij}^{\mathrm{T}}\ , \tag{4.169}$$

so that the equation above takes the form

$$\frac{i}{2}\int_{\partial\Sigma} d^{d-1}\Omega\, n^i(N\nabla^j - \nabla^j N)(\delta_i^k\delta_j^\ell - \gamma^{k\ell}\gamma_{ij})\frac{1}{\sqrt{\gamma}}\frac{\delta}{\delta\widetilde{\Pi}_{\mathrm{T}}^{k\ell}}\Psi[h^{\mathrm{TT}},\widetilde{\Pi}_{\mathrm{T}},h^{\mathrm{L}},\phi]$$
$$= E\,\Psi[h^{\mathrm{TT}},\widetilde{\Pi}_{\mathrm{T}},h^{\mathrm{L}},\phi] + \kappa\,\mathrm{O}\Big(\widetilde{h}_{ij}^{\mathrm{T}}\Big) + \mathrm{O}(\kappa)\,. \tag{4.170}$$

Note that the factor of κ has disappeared on the LHS above, and an additional factor of κ has appeared in front of the functional derivatives with respect to $\widetilde{\Pi}_{\mathrm{T}}^{ij}$ on the second line of the RHS. This entire equation clearly has a smooth limit as $\kappa \to 0$ and this allows us to conclude that repeated applications of H_∂ produce a simple result:

$$\left[\frac{i}{2}\int_{\partial\Sigma} d^{d-1}\Omega\, n^i(N\nabla^j - \nabla^j N)(\delta_i^k\delta_j^\ell - \gamma^{k\ell}\gamma_{ij})\frac{1}{\sqrt{\gamma}}\frac{\delta}{\delta\widetilde{\Pi}_{\mathrm{T}}^{k\ell}}\right]^n\Psi[h^{\mathrm{TT}},\widetilde{\Pi}_{\mathrm{T}},h^{\mathrm{L}},\phi]$$
$$= E^n\,\Psi[h^{\mathrm{TT}},\widetilde{\Pi}_{\mathrm{T}},h^{\mathrm{L}},\phi] + \mathrm{O}(\kappa)\,. \tag{4.171}$$

This is precisely what we have used above.

2. Our perturbative analysis in section 4.4 and section 4.5 assumes that the states under consideration do not have energies that scale parametrically with $\mathrm{O}\left(\frac{1}{\kappa}\right)$ so that there are no factors of κ that we need to keep track of except for the ones that appear explicitly in perturbation theory. But the proof above requires somewhat more stringent conditions on the energies. This can be seen by examining the passage from equation (4.162) to equation (4.163). If we denote the number of energy levels below a given energy E by $D(E)$ then this passage is valid provided we can take n, m in (4.161) to satisfy $n, m > D(E)$. Since we are limited to using $n, m < \mathrm{O}\left(\frac{1}{\kappa}\right)$, the proof above holds provided the states that enter (4.161) satisfy $D(E) < \mathrm{O}\left(\frac{1}{\kappa}\right)$ in AdS units. We remind the reader that $\log(D(E))$ can grow no faster than E on thermodynamic grounds and the linear bound is saturated by the Hagedorn behaviour of string-theory.

   This limitation should not be surprising. When $D(E) = \mathrm{O}\left(\frac{1}{\kappa}\right)$, the expected difference in the value of <u>any</u> observable between two typical state is suppressed by a factor of $\mathrm{O}(\sqrt{\kappa})$. (See [130] or section 2.4 of [2]) Therefore even if one were to consider all correlators, including bulk correlators, it would still be necessary to measure these correlators to highers order in κ in order to differentiate two typical states.

   We emphasize that this limitation does not mean that the result above fails to hold for high-energy states. The arguments of [5] arrive at the same result with no such limitation. So our observation simply implies that we need to refine our proof for high-energy states.

3. We note that the proof above can also be rewritten using the projector on the vacuum as was done in [5] or by replacing powers of H_∂ with projectors onto eigenstates of H_∂. Indeed, from a physical perspective, projective measurements are more natural than correlators as was explained in [10]. We have provided a proof using the correlators of (4.161) only to keep our argument simple and explicit.

4. In the proof above we have utilized a small time band in order to make the assertion below (4.166) rigorous. We expect that it should be possible to trade this infinitesimal time band for an infinitesimal "thickness" in the bulk. If so, the result above can also be stated as "if two states coincide <u>near</u> the boundary at a single instant of time, they must coincide everywhere in the bulk." However, to make this

rigorous requires some delicate analysis since, in an intermediate step, it will be necessary to construct bulk operators that commute with the constraints.

5. The products of operators that appear in (4.161) are not necessarily Hermitian. However, the expectation value we need can always be obtained by combining the expectation values of Hermitian observables. We first write each product of operators, A in (4.161) as $A = X + iY$ where X and Y are Hermitian. We then have $\langle A \rangle = \langle X \rangle + i\langle Y \rangle$. For further discussion of a "physical protocol" that can be used to extract information about the state, by combining a boundary unitary operation with a measurement of the energy, we refer the reader to [10].

6. The proof above takes advantage of the infrared cutoff provided by global AdS. Since the spectrum of energies is discrete, a finite set of powers of H_∂ in (4.161) are sufficient to make the passage to (4.163). This means that the method of proof presented here must be refined before it can be applied to asymptotically flat space where there is no infrared cutoff.

   We note that the result one should aim for in asymptotically flat space is clear. In [5] it was shown, using operator-theoretic techniques, that two states of massless particles that coincide in a small retarded-time band near the past boundary of $\mathcal{I}^+$ (or a small advanced-time band near the future boundary of $\mathcal{I}^-$) must be identical. We expect that a refinement of the techniques developed here, to account for the infrared subtleties of flat space, will lead to the same result.

4.7 Discussion

Summary of results. In this chapter, we have explicitly shown that a careful analysis of the solutions of the gravitational constraints leads to a perturbative proof of the principle of holography of information: any wave-functional that satisfies the gravitational constraints in AdS is determined uniquely by its boundary values over an infinitesimal interval of time. As we reviewed in section 4.3, these constraints can be obtained from the straightforward canonical quantization of gravity. In the canonical formalism, states of the theory are represented as wavefunctionals of the metric and matter degrees of freedom. The requirement that these wavefunctionals yield the same amplitude for configurations that are related by diffeomorphisms of a spatial slice leads to the momentum constraint; requiring the same amplitude for configurations related by diffeomorphisms that move points in time leads to the Hamiltonian constraint, which is also called the Wheeler-DeWitt equation. The precise form of these constraints can be found in equations (4.22) and (4.23).

In section 4.4, we expanded these constraints up to second order in the metric fluctuation. An important tool introduced in section 4.4 was the ADM decomposition presented in equation (4.40). This decomposition has previously been used in flat space; our results show that when suitably generalized it is also a very useful decomposition in curved space.

In section 4.5, we analyzed solutions to the perturbative Hamiltonian constraint. We first considered the equation obtained by integrating the Hamiltonian constraint over an entire Cauchy slice. This procedure greatly simplifies the constraint. We were able to obtain explicit solutions to the integrated constraint: these solutions are just dressed versions of wavefunctionals in an auxiliary Fock space that describe the matter excitations and the transverse-traceless metric excitations. We also showed how the pointwise Hamiltonian constraint can be solved through an iterative procedure.

In section 4.6, we showed that these wavefunctionals obey the remarkable property that their boundary values for an infinitesimal interval of time determine their behavior everywhere in the bulk. This result follows from the solutions that we obtained in sections 4.4 and 4.5. It sheds light, in a precise and explicit setting, on how and why gravitational theories are holographic.

Natural extensions. It is instructive to see what our analysis gives in the case of AdS_3. There are no nontrivial propagating gravitons in AdS_3 but it is still meaningful to define a boundary Hamiltonian that measures the total energy of the state. So we see that the present formalism can be applied to AdS_3. It would be interesting to go further and recast the Brown-Henneaux analysis [131] in the language of wavefunctionals.

This work was focused on global AdS_{d+1} but the analysis can also be performed for subregions of AdS. In particular, it appears straightforward to extend our analysis to the the Rindler wedge of a spherical region [132] and also perhaps to more general entanglement wedges. This promises to shed light on subregion duality and entanglement wedge reconstruction and we hope to return to this problem in the near future[6].

Future work. The analysis in this chapter and the later ones are perturbative in nature. In [5], it was shown that with weak assumptions on the Hilbert space and the nature of boundary observables, theories of gravity must be holographic even nonperturbatively. The analysis of [5] used operator algebra techniques. It would be very interesting if the perturbative analysis of this chapter could be generalized to show that, even nonperturbatively, solutions of the WDW equation that coincide on the boundary must coincide everywhere in the bulk. Although the nonperturbative WDW equation may seem formidable, the results of [5] suggest that obtaining such a result might be possible.

In this chapter, we have been agnostic to the matter content of the theory and its interactions. However, it is well known, from the AdS/CFT literature, that not all low-energy effective theories can be consistently extended to obtain a UV-complete theory of quantum gravity in AdS. It would be very interesting to understand whether and how these constraints enter possible extensions of our analysis.

The results of our work again illustrate the dramatic difference between the storage of quantum information in quantum gravity compared to quantum field theories. In ordinary quantum field theories, it is possible to find states that differ inside a bounded region but are identical outside that region; such states localize information in the interior of some region. The existence of such states corresponds to the "split property" of ordinary quantum field theories where the Hilbert space factorizes into a factor associated with the interior of the region and another factor associated with the exterior. In classical theories of gravity, configurations that differ inside a ball but coincide outside it can be constructed. For this reason, it has often been assumed that split states should also exist in quantum gravity. But our results show that this seemingly innocuous assumption is false.

It is described in [2] how this incorrect assumption plays a key role, both in Hawking's formulation of the information paradox and also in its various refinements (see also [76, 42]). More interestingly, the idea that black hole radiation should obey a "Page curve" also relies implicitly on the same incorrect assumption of factorization. By focusing on this assumption, it was

[6]There has been recent work on using the WDW equation for analyzing the Hilbert space and the holography of information in dS spacetime [133, 134].

recently shown in [135] that the paradigm of "islands" [6] that has been used to derive this Page curve is applicable only to theories of massive gravity and does not apply to standard theories with long-range gravity.

This chapter shows how the impossibility of localizing information in a bounded region in gravity is directly related to the structure of valid wave-functionals in the theory. We hope that a study of the solutions that we have found will help to shed further light on this remarkable property of quantum gravity.

Chapter 5

Asymptotic Symmetries in Higher Dimensional Flat Space

5.1 Introduction

In this chapter we investigate the symmetries of asymptotically flat spacetimes near Null Infinity[1]. Ever since the work of BMS, it has been known that the asymptotic symmetry group of flat spacetime in four dimensions is larger than the Poincare group and is known as the BMS group. This contains additional symmetries known as *supertranslations*[2] along with the usual Poincare transformations. These asymptotic symmetries have been extensively studied in four dimensions [18, 140] and there has been some progress in understanding them in the linearized gravitational theory in higher even dimensions as well [22, 26][3]. However, there are crucial differences between the situations in four and higher dimensions. In four dimensions, supertranslations and radiative degrees of freedom are described by the same field which constitutes the free data. In higher dimensions, these are accounted for by two different fields, occuring at two different fallofs in the asymptotic expansion. While

[1]The interested reader can refer to [21, 136, 137, 138, 139] for a discussion of asymptotic symmetries near timelike and spacelike infinities.

[2]Supertranslations are "angle dependent translations", where the angle is defined with respect to the sphere at null infinity (also known as the Celestial sphere). We will define this concretely in the main text.

[3]The behaviour of solutions in odd dimensions differs greatly from that of even dimensions and shows non-integer fall offs. In this work we only consider even dimensional spacetimes.

in four dimensions the theory is effectively free as one goes to null infinity and therefore a linearized analysis is often sufficient, in higher dimensions, certain non-linearities persist at null infinity[4]. This leads to a tension with the boundary conditions needed for defining the symplectic stucture. As we discuss in the main text, this tension also exists while studying the linearized theory about a non-trivial background.

In this work, we resolve this issue by first redefining the fields that contain the radiative data. This redefinition does not affect the News tensor characterizing the radiation. Using this redefinition we find the Symplectic structure of the resulting phase space and impose boundary conditions that are consistent with the action of supertranslations. This allows us to compute the Noether charge corresponding to supertranslations. We also find this charge by an explicit evaluation of the equations of motion of the full non-linear theory after first expressing it as an integral of the Bondi mass. We show that these two expressions for the Noether charge agree with each other.

The Noether charge corresponding to supertranslation symmetry gives a description of the classically conserved quantities. In the quantum regime, these classical conservation laws manifest as the Ward identities, which help us understand the Weinberg soft theorems. The connections between asymptotic symmetries and soft theorems are well established in the literature for four dimensions and in higher dimensional linearized theory [18, 22, 26]. In this chapter, we show that this connection persists in the non-linear theory by demonstrating that it retains the same structure as in the linearized theory when described in the redefined variables.

Along with soft theorems, asymptotic symmetries have also been important for understanding the *memory effect*. Such a connection has been well understood in four dimensions and in the linearized theory in higher-dimensions [18, 73, 141, 78]. Together, asymptotic symmetries, soft theorems and the memory effect complete the *Infrared Triangle*. The triangle represents a set of mathematical operations which relate the three seemingly different concepts. In this chapter we explain how one can derive the memory effect in the non-linear theory which then leads us to a generalization of the Infrared triangle in higher-dimensions. This provides the first example of the Infrared triangle in the full non-linear theory in higher-dimensions.

[4]The non-linear action of the supertranslations on the free data was also pointed out in [26].

We end with some comments on the implication of our results in the quantum theory in section 5.5.1.

5.2 Supertranslations in higher dimensions

5.2.1 Metric Conventions and Gauge Choice

Although our main aim is to analyze the implications of allowing supertranslations in higher dimensional gravity, for concreteness we work in six dimensions. However, many of the qualitative features should not change for higher even dimensions. In contrast to earlier works [22, 26] we work in the full non-linear theory. As we go along, we highlight the key differences between asymptotic symmetries in linearized gravity and our results. We denote the retarded Bondi cooridnates by (u, r, z^a) with the small Latin indices denoting the coordinates on the Celestial sphere $\mathbb{S}^4$ and the Greek indices denote the spacetime coordinates, $\mu \in (u, r, z^a)$. Although our analysis is restricted to six spacetime dimensions, several results are independent of the dimensionality and we will highlight them explicitly. We work in the unit of $c = \hbar = 8\pi G_N = 1$ which is the same as [26].

In the Bondi gauge the (asymptotically) flat metric can be written as [12, 13][5],

$$ds^2 = e^{2\beta} M du^2 - 2e^{2\beta} du dr + g_{ab}(dz^a - U^a du)(dz^b - U^b du). \tag{5.1}$$

All quantities in the metric can be a function of (u, r, z). The indices of U^a are raised and lowered using g_{ab}, i.e, $U_a = g_{ab}U^b$ and $U^2 = g^{ab}U_a U_b$, where the inverse g^{ab} is defined as $g^{ac}g_{cb} = \delta^a_b$. The inverse metric is then written in the following form,

$$g_{\mu\nu} = \begin{pmatrix} Me^{2\beta} + U^2 & -e^{2\beta} & -U_a \\ -e^{2\beta} & 0 & 0 \\ -U_b & 0 & g_{ab} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & -e^{-2\beta} & 0 \\ -e^{-2\beta} & -Me^{-2\beta} & -U^a e^{-2\beta} \\ 0 & -U^b e^{-2\beta} & g^{ab} \end{pmatrix}. \tag{5.2}$$

The Bondi gauge condition also places a constraint on the determinant of the metric g_{ab},

$$\det\left(\frac{g_{ab}}{r^2}\right) = \det(\gamma_{ab}) \tag{5.3}$$

[5]Note the difference in notation from chapter 3 .

where γ_{ab} is the metric on the 4-sphere. As shown in appendix C.1, the determinant condition helps us fix the traces of $g_{ab}^{(n)}$. As we are interested in the analysis near null infinity, we will be mostly interested in the large$-r$ limit of the metric components. Depending on how the large$-r$ limit is taken we can either end up near $\mathcal{I}^+$ or $\mathcal{I}^-$ (the future/past null infinity) but the basic construction remains the same in either case, and therefore we shall stick to working near $\mathcal{I}^+$.

We will need the fall off conditions near $\mathcal{I}^+$ which are given as,

$$
\begin{aligned}
&\beta = \sum_{n=2}^{\infty} \frac{\beta^{(n)}(u,z)}{r^n}, \qquad M = \sum_{n=0}^{\infty} \frac{M^{(n)}(u,z)}{r^n}, \qquad U_a = \sum_{n=0}^{\infty} \frac{U_a^{(n)}(u,z)}{r^n}, \\
&g_{ab} = r^2\gamma_{ab} + \sum_{n=-1}^{\infty} \frac{g_{ab}^{(n)}}{r^n}.
\end{aligned}
\tag{5.4}
$$

These fall off conditions have been motivated in the literature, see eg. [22, 26, 18]. Further, these conditions imply that the components of the Ricci tensor fall off as

$$
\begin{aligned}
R_{uu} &= O(r^{-4}), \qquad R_{ur} = O(r^{-5}), \qquad R_{ua} = O(r^{-4}), \\
R_{rr} &= O(r^{-6}), \qquad R_{ra} = O(r^{-5}), \qquad R_{ab} = O(r^{-4}).
\end{aligned}
\tag{5.5}
$$

The fall off conditions are chosen to ensure the finiteness of energy flux and other physical observables when we couple matter fields with gravity. Note that without including external matter fields we do not need to specify any fall off condition for the Ricci Tensor (as they become zero to arbitrary orders in r when evaluated on-shell), however the addition of matter fields necessitates the fall off conditions mentioned above and therefore also constrain the fall off conditions of the matter stress tensor which are equivalent to the fall off for $R_{\mu\nu}$.

For convenience we denote some of the important components of $g_{ab}(u,z)^{(n)}$ using the following notations

$$
g_{ab}^{(-1)} \equiv C_{ab}, \quad g_{ab}^{(0)} \equiv D_{ab}, \quad g_{ab}^{(1)} \equiv E_{ab}, \quad g_{ab}^{(2)} \equiv F_{ab}. \tag{5.6}
$$

The indices of $U_a^{(n)}$ and $g_{ab}^{(n)}$ are lowered and raised using the metric γ_{ab}. The determinant condition eq.(5.3) fixes the traces of $g_{ab}^{(n)}$, for example,

$$C_a^a = 0 \ , \tag{5.7a}$$

$$D_a^a = \frac{1}{2} C^{ab} C_{ab} \ , \tag{5.7b}$$

$$E_a^a = C \cdot D - \frac{1}{3} C^3 \ , \tag{5.7c}$$

$$F_a^a = C^{ab} E_{ab} + \frac{1}{2} D^{ab} D_{ab} - C^{am} C_{bm} D_a^b + \frac{1}{4} C_b^a C_c^b C_d^c C_a^d \ . \tag{5.7d}$$

A detailed derivation of this is given in appendix C.1. It is easily seen that in the linearized limit (which means that non-linear terms involving $g_{ab}^{(n \geq -1)}$ are neglected), the traces of D_{ab}, E_{ab}, F_{ab} just become 0. In fact in the linearized limit, the trace of every $g_{ab}^{(n)} = 0 \ \forall \ n \geq -1$ as shown in [26]. One can repeat the above anlaysis at $\mathcal{I}^-$. The interested reader is encouraged to look at [22] for more details.

For the sake of convenience, we decompactify the celestial sphere. Thus γ_{ab} is simply the flat Euclidean metric δ_{ab}. As the leading order term $M^{(0)}$ in the expansion of g_{uu} is related to the curvature scalar of γ_{ab}, it vanishes in the case of celestial plane, i.e, $M^{(0)} = 0$ when $\gamma_{ab} = \delta_{ab}$. We will often be referring to $\gamma_{ab} = \delta_{ab}$ as the *flat sphere*. This choice of coordinates has been adopted in many papers including [142, 143, 144, 145] and the explicit coordinate transformation from the round sphere to the flat sphere is provided in the references above. The simplification will not alter any of the results, as at the end of the computation we could always "covariantize" the result and revert back to γ_{ab} as the sphere metric[6].

5.2.2 Free data

In the previous section we parametrized the space of asymptotically flat spacetimes in the Bondi gauge. In this section we analyze the radiative phase space of the theory at future null infinity. In $D = 4$, the radiative phase space

[6] We note that the flat metric on the celestial sphere has already been used in [142] to compute the conserved charges in electromagnetism in higher dimensions, which we independently reproduce from a calculation using the symplectic form in appendix C.4. This has also been used in the context of gravity in [145] and we refer the reader to that paper for further details on these coordinates. It can also be seen from our computations that we get the expected results in linearized gravity by working with the flat sphere metric.

of General Relativity is coordinatized by the shear $C_{ab}^{(4D)}(u,z)$. It is a linear space of the unconstrained free data such that the sub-leading components of the metric (in the $1/r$ expansion) can be determined in terms of $C_{ab}^{(4D)}$ and the boundary conditions at $\mathcal{I}^+_-$. As we show below, in $D=6$, the free data for Einstein's equations at $\mathcal{I}^+$ is parametrized by C_{ab} and D_{ab}. All the fields appearing in eq.(5.1) except for $M^{(3)}$ can be written in terms of C_{ab} and D_{ab} (see appendix C.2). As we shall see later, $M^{(3)}$ will be identified as the analgoue of the Bondi mass (see sec.5.3.3) and its u-derivative is determined by the constraint equation. The differences between the $D=4$ and $D=6$ radiative data are summarized in table 5.1.

From the leading order non-trivial equation for R_{ab}, i.e, at $O(r^0)$ we get,

$$\partial_u C_{ab} = 0. \tag{5.8}$$

This implies that $C_{ab}(z)$ is u-independent and hence has no radiative information. In six dimensions the radiative degree comes with a $1/r^2$ fall off and hence is expected to be contained in D_{ab}. The subsequent equations explicitly show that $D_{ab}(u,z)$ is unconstrained. Thus D_{ab} is analogous to the shear field in four dimensions and the corresponding News tensor $\partial_u D_{ab}$ captures radiative content.

We note that as $\partial_u C_{ab} = 0$, a field redefinition of $D_{ab} \to D_{ab} + \mathbf{func}(C_{ab})$ does not change the News tensor. The freedom in defining D_{ab} will play a central role in our analysis. However, as we prove in section 5.2.3, this freedom is fixed and **func** has a specific form such that certain fall-off conditions are well defined (these fall-offs refer to the behavior of the radiative field near $\mathcal{I}^+_-$ and are discussed in section 5.2.3).

We also note that the radiative degrees of freedom parametrized by (trace-free) D_{ab} are exactly equal to nine, which is the number of graviton polarizations in $D=6$.

We finally note that in four dimensions, if we restrict ourselves to the so-called Christoudoulou-Klainermann (CK) spacetimes then the leading component of the magnetic part of the Weyl tensor vanishes at $\mathcal{I}^\pm$ [146]. In the present case, the vanishing magnetic charge condition takes the form

$$\partial_a U_b^{(0)} - \partial_b U_a^{(0)} = 0. \tag{5.9}$$

It can be checked that this condition sets the leading order term in the magnetic part of the Weyl tensor to zero[7]. As $H^1(\mathbb{S}^4, \mathbf{R}) = 0$, $U_a^{(0)}$ is an

[7]The relationship between the above constraint and the magnetic part of the Weyl tensor can be found in [147], also see [148].

exact form on $\mathbb{S}^4$ if the magnetic charge vanishes (a similar argument also holds in $\mathbb{R}^4$). Using (C.13), we thus see that C_{ab} is parametrized in terms of a single scalar potential[8].

$$C_{ab} = -2\big(\partial_a\partial_b\psi\big)^{\text{tf}} \equiv -2\Big(\partial_a\partial_b\psi - \frac{1}{4}\delta_{ab}\partial^2\psi\Big). \tag{5.10}$$

Here X^{tf} denotes the tracefree part of X defined as $X^{\text{tf}}_{ab} = X_{ab} - \frac{1}{4}\delta_{ab}\text{Tr}(X)$; and $\partial^2 \equiv \partial^a\partial_a$. Here the factor of -2 in the equation for the potential is chosen for convenience and will be explained in section 5.2.3. We note that the magnetic constraint in (5.9) can be trivially satisfied by choosing $C_{ab} = 0$, but as was shown in the linearized case [26], this condition is not preserved under supertranslations (see eq.(5.14)).

5.2.3 Supertranslation Generators

In this section, we analyze the symmetries that preserve the space of all $D = 6$ (and in general all even) dimensional asymptotically flat geometries[9]. If we restrict the analysis to linearized gravity around flat spacetime, then it has been shown in [22, 26] that the asymptotic symmetries include infinite dimensional supertranslations. The complete group of such symmetries is a Cartesian product of BMS group on $\mathcal{I}^+$, $\mathcal{I}^-$. The diagonal subgroup obtained by the anti-podal identification of supertranslation generators at $\mathcal{I}^+$ and $\mathcal{I}^-$ and is conjectured to be the symmetry group of tree-level S-matrix with massless external particles (which are not gravitons) such that the corresponding Ward identity is equivalent to the Weinberg soft graviton theorem.

Although this is a promising data point for showing existence of an IR triangle in higher (even) dimensions, there is a caveat which has not been analysed in the literature so far[10]. As seen in the literature [22, 26] and also discussed previously, the action of supertranslation even in linearized gravity generates an infinity of flat spacetimes parametrized by $C_{ab} \neq 0$. Hence in order to show that supertranslations at $\mathcal{I}^\pm$ preserve the space of linearized

[8]This counting can even be done in the linearized theory and was mentioned in [26], but we find the reasoning below (5.9) to be more robust than the one in [26]. We would like to thank Ankit Aggarwal for discussions about this issue.

[9]We thank Prahar Mitra for several discussions regarding this.

[10]Henceforth, we shall always mean higher even dimensions, whenever we refer to any results in higher dimensions.

asymptotically flat geometries, we need to include linearisation around flat space-times for which $C_{ab} \neq 0$. To the best of our understanding, analysis of supertranslation symmetries and the subsequent conservation laws have not been studied previously and we fill this gap in our work.

We start by reviewing the derivation of BMS asymptotic symmetries as given in [22, 26]. They are generated by vector fields which are asymptotically Killing. I.e, ξ^α is an asymptotic symmetry if,

$$\mathcal{L}_\xi g_{rr} = 0, \qquad \mathcal{L}_\xi g_{ra} = 0, \qquad g^{ab}\mathcal{L}_\xi g_{ab} = 0. \tag{5.11}$$

We note that this definition of asymptotic symmetry is agnostic to the dimension of spacetime. These equations tell us that the gauge conditions should be preserved under a diffeomorphism. They can be solved exactly and the solutions for ξ^μ is given as [22],

$$\xi^u = f, \qquad \xi^a = -\partial_b f \int_r^\infty e^{2\beta} g^{ab} dr', \qquad \xi^r = \frac{r}{4}\Big(U^a\partial_a f - \partial_a \xi^a\Big), \tag{5.12}$$

which can be summarized as,

$$\xi^\mu = f\partial_u + \frac{r}{4}\Big(U^a\partial_a f - \partial_a\xi^a\Big)\partial_r - \partial_b f \int_r^\infty e^{2\beta} g^{ab} dr' \partial_a. \tag{5.13}$$

These are the well known BMS vector fields which include supertranslations parametrized by an arbitrary function $f(z^a)$[11]. We now derive the action of the supertranslations on the radiative data at $\mathcal{I}^+$. By evaluating $\mathcal{L}_\xi g_{ab}$ (which we shall often denote as $\delta_f g_{ab}$ when we are referring to supertranslations) to $O(r^{-1})$ we can see that,

$$\lim_{r\to\infty} r\mathcal{L}_\xi g_{ab} = \delta_f C_{ab} = -2\Big(\partial_a\partial_b f - \frac{1}{4}\delta_{ab}\partial^2 f\Big). \tag{5.14}$$

where ξ is the ST vector field defined in (5.13). It can be verified immediately that the transformation of the potential ψ in (5.10) is as follows[12]

$$\delta_f \psi = f \tag{5.15}$$

[11]If we relax the boundary conditions and demand that the asymptotic symmetries are generated by vector fields which are volume preserving as opposed to asymptotically Killing, then we will be led to the generalisation of the so-called Generalised BMS group which is the semi-direct product ST $\otimes$ Diff(S^4). We expect the conservation laws associated to Diff(S^4) to be equivalent to sub-leading soft graviton theorem. However the detailed analysis of this question is outside the scope of this chapter.

[12]This explains our reason for introducing -2 in the definition of the scalar potential in (5.10). Without which we would have to rescale ψ in an appropriate manner.

We can now compute the action of ST on D_{ab},

$$\lim_{r\to\infty} r^2\mathcal{L}_\xi g_{ab} = \delta_f D_{ab} = f\partial_u D_{ab} + \frac{1}{4}\delta_{ab}\left[-\frac{4}{3}\partial_c C^{cd}\partial_d f - C^{cd}\partial_c\partial_d f\right] + \frac{1}{4}C_{ab}\partial^2 f - \partial_c C_{ab}\partial^c f - \frac{1}{2}\big(C_{bc}\partial_a\partial^c f + C_{ac}\partial_b\partial^c f\big) + \frac{1}{2}\big[\partial_a C_{bc}\partial^c f + \partial_b C_{ac}\partial^c f\big] + \frac{1}{6}\big[\partial^c C_{bc}\partial_a f + \partial^c C_{ac}\partial_b f\big]. \tag{5.16}$$

This result is completely general and is valid in the non-linear theory. From the perspective of linearized gravity, it is an extension of the result in [26] when the background spacetime has non-zero C_{ab}[13].

We thus note that the action of ST on $D_{ab}(u, z)$ (Graviton mode) generates a constant (u-independent) term even in *linearized gravity around a flat metric with non-zero* C_{ab}[14]. However, both the saddle point analysis in appendix C.6 and the finiteness of symplectic structure (see section 5.3.1) at $\mathcal{I}^+$ impose stronger fall off conditions on the Graviton mode. The finiteness of the symplectic structure implies that as $|u| \to \infty$

$$\lim_{|u|\to\pm\infty} \text{Graviton}(u, z) = O\Big(\frac{1}{|u|^{1+0_+}}\Big) \tag{5.17}$$

This fall-off condition appears to be in tension with (5.16). We now show that it is possible to redefine D_{ab} to $\tilde{D}_{ab}$ such that (1) the corresponding News tensor $\partial_u D_{ab}$ remains unchanged and (2) action of ST preserves the fall-off condition of $\tilde{D}_{ab}$. Let us define $\tilde{D}_{ab}$ as,

$$\tilde{D}_{ab} = D_{ab} - \frac{1}{4}C^c_a C_{bc} - \frac{1}{16}\delta_{ab}C^{cd}C_{cd}. \tag{5.18}$$

Under ST it transforms as,

$$\delta_f \tilde{D}_{ab} = f\partial_u \tilde{D}_{ab}. \tag{5.19}$$

It can also be verified that $\tilde{D}_{ab}$ is trace-free, i.e, $\tilde{D}^a_a = 0$. Implications of this field redefinition in linearized gravity are now clear. When combined with the

[13]From the perspective of soft theorems, our analysis can be understood as a derivation of the leading order multi-soft graviton theorem from supertranslation Ward identity in linearized gravity.

[14]An analysis about a flat metric with non-zero C_{ab} means that the background is $r^2\delta_{ab} + rC_{ab}$ instead of just $r^2\delta_{ab}$

results of [22], we see that the Gravitational scattering in the background of any flat spacetime has ST symmetries whose associated conservation laws are equivalent to classical and quantum soft graviton theorem [149]. The complete implications of this field redefinition in non-linear theory will emerge in the following sections. For now, we have the following statement. In the non-linear theory, supertranslations map an asymptotically flat spacetime to a distinct asymptotically flat spacetime via the following action,

$$\delta_f C_{ab} = -2\Big(\partial_a\partial_b f - \frac{1}{4}\delta_{ab}\partial^2 f\Big), \tag{5.20a}$$

$$\delta_f \tilde{D}_{ab} = f\partial_u \tilde{D}_{ab}. \tag{5.20b}$$

Having described the generators of ST, we move onto computing the charge corresponding to this symmetry in the next section.

5.3 Supertranslation Charge

In this section, we evaluate the Noether charge corresponding to supertranslations. We first compute the symplectic form of the theory at $\mathcal{I}^+$ and then compute the Noether charge using that. Finally, we also compute the Bondi mass and explain how we obtain the same Noether charge as computed before.

5.3.1 Symplectic Form

In order to compute the Noether charge we use the covariant phase formalism and first analyze the symplectic form of the theory[15]. This will also help us in the identification of the canonically conjugate variables of the theory.

The Symplectic form can be constructed on any Cauchy slice but since our interest lies in the theory at future null infinity, we will perform our analysis on $\mathcal{I}^+ \cup i^+$. The advantage of doing this is that most interaction terms die off near that region. Since we are working with massless fields, we can neglect the contribution from i^+ and just work by defining the data on the null slice $\mathcal{I}^+$.

[15]For a detailed discussion of this formalism, we refer the reader to [150, 151, 152]. This was first analyzed near null infinity in [15, 14].

We have already established in the previous sections that C_{ab} and $\tilde{D}_{ab}$ constitute the free data and it would therefore make sense to construct the symplectic form with these two variables.

The symplectic form in higher dimensions at $\mathcal{I}^+$ has been evaluated in the linearized limit about $C_{ab} = 0$ in [26]. We follow the same strategy and evaluate this in the non-linear theory for the Einstein-Hilbert action at $\mathcal{I}^+$. In order to do this we first construct the symplectic current and later, integrate over the current to obtain the symplectic form. Following [151], the symplectic current for the Einstein-Hilbert action is given as[16],

$$J^\alpha = \frac{1}{2}\delta\Gamma^\alpha_{\mu\nu} \wedge \left[\delta g^{\mu\nu} + \frac{1}{2}g^{\mu\nu}\delta \log g\right] - \frac{1}{2}\delta\Gamma^\nu_{\mu\nu} \wedge \left[\delta g^{\alpha\mu} + \frac{1}{2}g^{\alpha\mu}\delta \log g\right] \quad (5.21)$$

Here δ represents the exterior derivative on phase space and $\wedge$ denotes the wedge product. The basic idea of computing this symplectic form at $\mathcal{I}^+$ follows a similar strategy as QED (see appendix C.4) or the linearized gravitational case, which is already done in [26]. We explain the technicalities of evaluating this in the non-linear gravitational case in full detail in appendix C.5 and summarize the conceptual points here.

The symplectic form is defined as an integral over the symplectic current,

$$\Omega \equiv \int d\Sigma_\alpha J^\alpha \quad (5.22)$$

with $d\Sigma_\alpha$ denoting the measure of integration on the chosen Cauchy slice. For us, this is evaluated on $\mathcal{I}^+$. In order to reach $\mathcal{I}^+$, we first choose a constant-t cauchy slice and then take the limit $t \to \infty$ by holding $u =$ constant. Note that in our notations $t = u + r$. Since we start with a constant-t Cauchy slice, the component of the current that contributes to the symplectic form is $J^t = J^u + J^r$,

$$\Omega^t = \lim_{r\to\infty} \int_{\mathcal{I}^+} r^4 du d^4z \; J^t = \lim_{r\to\infty} \int_{\mathcal{I}^+} r^4 du d^4z \; (J^u + J^r). \quad (5.23)$$

It is simpler to evaluate J^u first and that is given as (refer to (C.49) for more details),

$$-J^u = \frac{1}{4r^4}\delta C^{ab} \wedge \delta(D_{ab} - C^a_c C^{bc}) \quad (5.24)$$

[16]A detailed derivation of this expression can be found in section 4.2 of [153]. There is an improved version of this result given in [154] which takes care of boundary terms on a spatial slice.

For J^r, we get a leading order term at $O(r^{-3})$ which we call J^r_{div}. This term can potentially diverge since the measure of the integral in (5.23) at $\mathcal{I}^+$ is r^4 and we are working in the large-r limit. However, as shown in (C.5.2), it ends up giving a finite contribution to the symplectic form[17]. Thus the contribution from J^r is given as,

$$J^r = J^r_{div} + J^r_{fin} \tag{5.25}$$

with

$$-2J^r_{div} = -\frac{1}{2r^4}\partial_u\Big[(t-u)\delta C^{ab}\wedge\delta D_{ab}\Big] - \frac{1}{2r^4}\delta C^{ab}\wedge\delta D_{ab}\ , \tag{5.26a}$$

$$\begin{aligned}-2J^r_{fin} = &-\frac{1}{2}\delta D^{ab}\wedge\delta\partial_u D_{ab} - \frac{1}{2}\delta C^{ab}\wedge\delta\partial_u E_{ab}\\ &-\frac{1}{2}\delta C^{ab}\wedge\delta\Big[2\partial_a U^{(1)}_b - U^{(0)c}\big(2\partial_a C_{bc} - \partial_c C_{ab}\big) - \frac{1}{3}\partial_b\big[U^{(1)}_a + C^c_a U^{(0)}_c\big]\\ &\qquad + \frac{1}{3}\partial_m\partial_a(D^m_b - C^m_c C^c_b)\Big].\end{aligned} \tag{5.26b}$$

The equations can be simplified further by expressing it in terms of the free data C_{ab} and $\tilde{D}_{ab}$ (see appendix C.5 for details). We then combine J^r and J^u to obtain J^t which we integrate to get the symplectic form (5.23). In order to ensure a finite contribution from the first term in (5.26a) to the symplectic form (in the large$-r$ limit), we see that the fall off condition on $\tilde{D}_{ab}$ with large$-u$ should be[18]

$$\lim_{u\to\pm\infty}\tilde{D}_{ab} = O\Big(\frac{1}{|u|^{1+0_+}}\Big). \tag{5.27}$$

For further details we refer the reader to the appendix C.5.2. We also note that the same fall-off is expected from the saddle point analysis, as explained in appendix C.6.

Leaving the details to appendix C.5.5, we state the final result for the the symplectic form here. We find that the symplectic form Ω^t can be uniquely spit into two different terms

[17]We emphasize that this apparent large-r divergence is also seen in QED (C.26) and in the linearized limit as well [26]. As shown here, this divergence disappears upon appropriately choosing the boundary conditions.

[18]A similar analysis also motivates the fall off in the linearized theory, as shown in [26], and can also be seen in QED (C.28).

1. **Finite and Integrable piece:** This is the term that leads to the correct Noether charge and also defines the radiative phase space at $\mathcal{I}^+$,

$$\Omega^{rad} = \int_{\mathcal{I}^+} dud^4zJ^t_I = -\frac{1}{2}\int_{\mathcal{I}^+} d^4zdu\Big[-\frac{1}{2}\delta\tilde{D}_{ab}\wedge\delta\partial_u\tilde{D}^{ab}+\frac{1}{2}\delta C^{ab}\wedge\delta\partial_u E_{ab}+\frac{1}{9}\delta C^{ab}\wedge\delta\partial_a\partial^c\tilde{D}_{bc}\Big]. \tag{5.28}$$

It contains the radiative degree of freedom $\tilde{D}_{ab}$ and defines its Poisson brackets near $\mathcal{I}^+$. We thus obtain a generalization of the Ashtekar-Streubel symplectic structure at $\mathcal{I}^+$ [15] in the higher dimensional non-linear theory. We also note that the expression above can be obtained from [26] by replacing D_{ab} with $\tilde{D}_{ab}$, which is the radiative data in the non-linear theory. Therefore upon using the EOM for E_{ab} (see eq.(5.44)), the canonically conjugate variable of $\tilde{D}_{ab}$ can be identified using this structure. As shown in section 5.3.2, we obtain the Noether charge of supertranslation using Ω^{rad}. Later (in section 5.3.3), we also compare it with the expression obtained from the Bondi mass. The divergent piece as discussed below does not affect the charge as it cannot be expressed as a total variation.

2. **Divergent and Non-Integrable piece:** The other piece of Ω^t is given as

$$\begin{aligned}\Omega^{div} &= \frac{1}{2}\int_{\mathcal{I}^+} dud^4zJ^t_{NI} \\ &= -\int_{\mathcal{I}^+} dud^4z\ \delta C^{ab}\wedge\delta\Big[-\frac{1}{2}C^c_aC_{bc}-\frac{1}{9}\partial_a\Big(C^{cd}\partial_cC_{bd}+\frac{4}{3}C^c_b\partial^dC_{cd}-\frac{1}{16}C^{cd}\partial_bC_{cd}\Big) \\ &\qquad +\frac{1}{3}\partial^cC_{cd}\partial_aC^d_b-\frac{1}{6}\partial^cC_{ab}\partial^dC_{cd}\Big].\end{aligned} \tag{5.29}$$

Note that this contains no radiative degree of freedom $\tilde{D}_{ab}$, and as $\partial_uC_{ab}=0$, the integrand is independent of u. Therefore, this gives a divergent term when integrated with u. However, this does not make the symplectic structure ambiguous at $\mathcal{I}^+$. The ambiguity is settled by noting that this term will not contribute to the Noether charge, as it does not lead to a total variation. As seen from the computation in [26], such a term does not arise while computing the symplectic form in the linearized theory about the $C_{ab}=0$ background. It will be interesting to see the implication of such a term and we leave that for a future work.

5.3.2 Noether Charge

The Noether charge[19] can be computed using

$$\delta Q_f \equiv \Omega^{rad}(\delta_f, \delta) \tag{5.30}$$

Here δ_f represents the ST variations[20]. The following calculation is almost similar to that of [26] with D_{ab} replaced by $\tilde{D}_{ab}$. However in addition to the soft charge we also obtain the hard charge as shown below.

We first work out the soft charge (the term linear in $\tilde{D}$). Using the formula (5.30), the soft charge obtained from the symplectic form becomes,

$$Q_f^{soft} = \frac{1}{12}\int_{\mathcal{I}^+} d^4z du\ f\partial^2\partial^{ab}\tilde{D}_{ab} + \int_{\mathcal{I}^+} d^4z du\ f(\partial^{ab} - \frac{1}{4}\delta^{ab}\partial^2)\partial_u E_{ab} \tag{5.31}$$

where $\partial^{abc\cdots} \equiv \partial^a\partial^b\partial^c\cdots$. From the equation of motion of E_{ab} (see eq.(5.44)) it is easy to see that the term containing $\partial_u E_{ab}$ does not contribute,

$$\Big(\partial^{ab} - \frac{1}{4}\delta^{ab}\partial^2\Big)\partial_u E_{ab} = \partial^{ab}\big(C^c_{(a}\partial_u\tilde{D}_{b)c}\big) - \frac{1}{4}\partial^2\big(C^{ca}\partial_u\tilde{D}_{ca}\big) \tag{5.32}$$

Using this and the fall condition (5.27), it is clear that such a term does not contribute to the charge, as upon integrating (5.32) w.r.t u, we get zero. The other term in (5.31) is similar to the expression obtained in [26], with the identification $D_{ab} \to \tilde{D}_{ab}$. The only other difference is that we are working with the flat sphere δ_{ab}, and hence we do not see the contribution from the curvature term of $\mathbb{S}^4$ that arose in [26]. Therefore, the soft charge reduces to

$$Q_f^{soft} = \frac{1}{12}\int_{\mathcal{I}^+} dud^4z\ f(z)\partial^2\partial^{ab}\tilde{D}_{ab}. \tag{5.33}$$

The total charge is obtained by adding this to the contribution from the matter and hard gravitons (also known as the *hard charge*), which is given as,

$$Q_f = Q_f^{soft} + Q_f^{hard} = Q_f^{soft} + \int_{\mathcal{I}^+} dud^4z\ f(z)\Big(T^{M(4)}_{uu} + \frac{1}{4}N_{ab}N^{ab}\Big) \tag{5.34}$$

[19]Charges corresponding to gauge symmetries require a study of the generalized Noether theorem, which is reviewed in [150].

[20]For a computation of the charge in QED, refer to appendix C.4.3 and also [155, 156].

where we explicitly state the contribution from the matter and the graviton part. The graviton hard charge is obtained from the following term in the symplectic form (5.28) and by using the variation (5.19)

$$\int d^4z du \; \delta\tilde{D}_{ab} \wedge \delta\partial_u \tilde{D}^{ab}.$$

We can also obtain the same expression for the charge by using the Bondi mass, as shown in the next subsection (see section 5.3.3). We see that our final result (5.34) is a neat generalization of [22, 26] to the non-linear theory with D_{ab} replaced by $\tilde{D}_{ab}$.

In the following section we demonstrate how we get the total charge (5.34) using the Bondi mass.

5.3.3 Bondi Mass

We first give a brief idea of the Bondi mass in the six dimensional non-linear theory, explain the notations, and then demonstrate how it generates the expression for the supertranslation charge (5.34).

The *Bondi mass aspect*, $M^{(3)}(u,z)$ is denoted by the notation $m_B(u,z)$. At $\mathcal{I}^+$ this gives us a definition of the angular density of energy. For Kerr-like spacetimes the Bondi mass aspect is proportional to the mass of the object in the bulk. We obtain the *total Bondi mass* by integrating the Bondi mass aspect over the Celestial sphere (or plane). And further, the total Bondi mass in the limit $u \to -\infty$ gives us the ADM mass, which is identified as the Hamiltonian of the theory.

We can study the evolution of the Bondi mass aspect along $\mathcal{I}^+$ using the $uu-$component of the Einstein equation (we use the same conventions as that of [26]),

$$\begin{aligned}
&-2\partial_u m_B - \frac{1}{4}\partial_u D^a_b \partial_u D^b_a - \partial_u \partial_a U^{(2)a} + \partial_a\big(C^{ab}\partial_u U^{(1)}_b\big) - 3U^{(0)a}\partial_u U^{(1)}_a \\
&= T^{M(4)}_{uu} + \frac{1}{2}\partial^2(U^{(0)2} + M^{(2)})
\end{aligned} \tag{5.35}$$

Here we have also included a matter source term as denoted by $T^{M(4)}_{uu}$. Using the EOM for $M^{(2)}$ and U_a's (as given in appendix C.2), we can express $\partial_u m_B$ entirely in terms of the free data C_{ab} and $\tilde{D}_{ab}$. Further, it is simple to see that in the linearized limit, equation (5.35) reduces to the results in [22, 26]. In order to get the total Bondi mass, we need to integrate $\partial_u m_B$ over z^a. Since we are not considering massive particles in our system, there is no non-trivial information at $u \to +\infty$ and hence $m_B(+\infty, z) = 0$. Using this information and eq.(5.35), the total Bondi mass is given as,

$$\int d^4z \ m_B(u,z) = \frac{1}{2}\int_u^\infty dud^4z \ T^{M(4)}_{uu} + \frac{1}{4}\partial_u \tilde{D}^{ab}\partial_u \tilde{D}_{ab}. \tag{5.36}$$

The ADM mass can be obtained by taking the $u \to -\infty$ limit in the equation above,

$$\lim_{u\to-\infty}\int d^4z \ m_B(u,z) = \frac{1}{2}\int_{-\infty}^\infty dud^4z \ T^{M(4)}_{uu} + \frac{1}{4}\partial_u \tilde{D}^{ab}\partial_u \tilde{D}_{ab}. \tag{5.37}$$

Having explained the basic properties of $m_B(u,z)$, we now describe its relevance as a candidate for the supertranslation charge. This is motivated as a valid candidate through the results in the linearized theory [22, 26], and is given as

$$\mathcal{Q}_f \equiv \lim_{u\to-\infty} 2\int f(z) m_B(u,z) d^4z \tag{5.38}$$

where we have an extra factor of 2 since we set $8\pi G_N = 1$[21] and $f(z)$ is the function parametrizing supertranslations (see section 5.2). We can recover the ADM mass (5.37) by setting $f(z) = \text{constant}$.

Substituting the constraint equation (5.35) in (5.38) and after integrating by parts, we obtain $\mathcal{Q}_f$ in terms of the radiative data, which is expressed as a summation of two separate pieces. Comparing with equation (5.34), these are identified as the *soft* and *hard* piece

$$\mathcal{Q}_f = \mathcal{Q}_f^{soft} + \mathcal{Q}_f^{hard} \tag{5.39}$$

where,

$$\mathcal{Q}_f^{soft} = \frac{1}{12}\int_{\mathcal{I}^+} d^4z du \ f(z)\partial^2\partial^{ab}\tilde{D}_{ab} \tag{5.40a}$$

[21] The expression without working with $8\pi G_N = 1$ is given as [22]

$$\mathcal{Q}_f \equiv \lim_{u\to-\infty} \frac{1}{4\pi G_N}\int f(z) m_B(u,z) d^4z$$

$$\mathcal{Q}_f^{hard} = \int_{\mathcal{I}^+} d^4z du \; f(z) T_{uu}^{(4)}. \tag{5.40b}$$

The hard charge contains the stress tensor of matter and hard gravitons, which is given as

$$T_{uu}^{(4)} = T_{uu}^{M(4)} + \frac{1}{4}\partial_u \tilde{D}^{ab}\partial_u \tilde{D}_{ab}. \tag{5.41}$$

Therefore we see that the expression derived in (5.39) and the one derived using the symplectic form (see eq.(5.34)) are the same. As noted before, the charge derived in this section, generalizes the results in [22, 26] to the non-linear theory (and also to the linearized theory about a $C_{ab} \neq 0$ background). From the discussion above, we see that it is possible to formally obtain the result for the charge in the non-linear theory by using the results in [22, 26] and replacing D_{ab} with $\tilde{D}_{ab}$. This is another way of deriving the redefined Graviton mode $\tilde{D}_{ab}$. Therefore, combining with the results of [22], we expect to get a similar structure for the Ward identity and correspondingly, the leading soft-theorem in the non-linear theory. We elaborate on this point and discuss some more implications of the Ward identity in section 5.5.

We note that in the linearized theory about $C_{ab} = 0$, the total charge Q_f has also been derived from the electric part of Weyl tensor in [26] and it will be interesting to compute the same in the non-linear theory and verify that we get the same result as (5.39), which we leave for a future work.

This completes our analysis of the Supertranslation charge and it also sheds light on the phase space of the theory. Along with the description of asymptotic symmetries and soft theorems, there is another important ingredient alluding to the infrared properties of the theory. This is known as the *memory effect* and together with the soft theorems and asymptotic symmetries, it completes the IR triangle. The triangle represents a set of mathematical operations which connects the three corners. We now move onto the analysis of the memory effect and describe how it is generalized from the linearized results and also, how it is related with the discussions in the sections above.

5.4 Memory and IR Triangle

In [79], the authors have defined the linearized memory in (even) higher dimensions. In 6-dimensions, it is proportional to $D_{ab}(u, z)$. The existence of memory follows directly from the equation of motion and has been proved in [157]. The fact that memory only depends on the scattering data and not on details of the interaction is the statement of classical soft graviton theorem. However the results of [157] are valid for linear as well as non-linear memory (also known as null memory [158]). In this section, we will show that the non-linear memory can be obtained directly from $\tilde{D}_{ab}$ (defined in (5.18)) such that in the linearized theory around Minkowski vacuum, it reduces to the linear memory derived in [79].

A definition of the memory effect convenient for our purpose is the following. Memory effect is a measure of how the distance between two detectors near $\mathcal{I}^+$ changes upon the passage of gravitational radiation. These detectors move in a time like trajectory with the tangent vector $k = \partial_u$. The relative separation of the detectors is usually computed using the Geodesic deviation equation,

$$\frac{d^2 s^a}{du^2} = R^a{}_{uub} s^b \tag{5.42}$$

where s^a is the relative transverse displacement between the two detectors and $R^a{}_{uub}$ is the Riemann curvature. This equation gives us s^a as a function of u and we will solve this equation perturbatively in G_N, i.e, assume that the s^b on the RHS of (5.42) is at a fixed value u and then study the separation by recursively solving (5.42). For our purpose, the leading order solution to this recursion equation will suffice. We note that the detectors can make a measurement between retarded times u_i and u_f and therefore, the memory is defined as $s^a(u_f) - s^a(u_i) \equiv \Delta s^a$. This represents the change in separation between the detectors in the retarded time interval $u_f - u_i \equiv \Delta u$. To account for gravitational radiation at late and early times, memory is usually defined in the limit $u_i \to -\infty$ and $u_f \to +\infty$.

In order to study Δs^a via (5.42), we first compute the Riemann tensor $R^a{}_{uub}$. Note that the indices of $R^a{}_{uub}$ are raised and lowered using $g_{\mu\nu}$ but the indices of C_{ab}, $\tilde{D}_{ab}$ and E_{ab} are raised and lowered using δ_{ab}. In the large-r limit $R^a{}_{uub}$ is given as,

$$R^a{}_{uub} = \frac{1}{2r^2}\partial_u^2 \tilde{D}^a_b + \frac{1}{2r^3}\left(\partial_u^2 E^a_b - \frac{2}{3}\partial_u \partial^{(a}_c \tilde{D}^c_{b)} - C^{ac}\partial_u^2 \tilde{D}_{bc} + \frac{1}{6}\delta^a_b \partial_u \partial^{cd}\tilde{D}_{cd}\right) + O(r^{-4}). \tag{5.43}$$

This can be expressed in terms of the radiative data by using the EOM for E_{ab},

$$\partial_u E_{ab} = C^c_{(a}\partial_u \tilde{D}_{b)c} + \frac{2}{3}\partial^c_{(b}\tilde{D}_{a)c} - \frac{1}{2}\partial^2 \tilde{D}_{ab} - \frac{1}{6}\delta_{ab}\partial^{cd}\tilde{D}_{cd}. \tag{5.44}$$

Using this we find,

$$R^a{}_{uub} = \frac{1}{2r^2}\partial_u^2 \tilde{D}^a_b - \frac{1}{4r^3}\left(\partial^2 \partial_u \tilde{D}^a_b + C^{ca}\partial_u^2 \tilde{D}_{bc} - C^c_b \partial_u^2 \tilde{D}^a_c\right) + O(r^{-4}) \quad (5.45)$$

Integrating this twice with u gives us the memory Δs^a (see (5.42)) to leading order in r,

$$\Delta s^a = -\frac{1}{4r^3}\int_{-\infty}^{\infty} du\ \partial^2 \tilde{D}^a{}_b\ s^b_i \quad (5.46)$$

where we have used the boundary condition $\tilde{D}_{ab}(u \to \pm\infty) \to 0$ (see (5.27)) and $s^b_i \equiv s^b(u_i)$ with $u_i \to -\infty$. The boundary condition sets the term at $O(1/r^2)$ and the non-linear term at $O(1/r^3)$ to zero.

This generalizes the definition of the linear memory to the non-linear theory in six dimensions. Thus we get a contribution of the six dimensional memory at the Coulombic order, in contrast with the four dimensional case (see table 5.1 for a summary the differences between the important physical quantities in four and six dimensions). In appendix C.3, we show how one can recover a similar form of the answer from the computation in [79] in the linearized regime.

From (5.46) we see that the memory in the non-linear theory is measured via an integration over the radiative degree of freedom $\tilde{D}_{ab}$. This is the precisely the same quantity which appears in the soft charge (see (5.40a)) and eventually, in the soft theorems as well (we expand upon how one gets to the soft theorems using the conservation laws in section 5.5). Upon comparing with the results in the linearized theory [22, 26, 79], we see how our results are generalized to the non-linear theory via the replacement of $D_{ab} \to \tilde{D}_{ab}$. This establishes the connection between the three corners of the IR in the full non-linear theory. Therefore, we see how the IR triangle is generalized to the full non-linear theory of gravity in higher-dimensions.

5.5 Discussion and summary

We summarize our main results and state some important implications for the quantum theory in this section.

5.5.1 Possible Implications for the Quantum theory

As discussed in section 5.2, the free data of the theory is constituted by C_{ab} and $\tilde{D}_{ab}$, where the mode C_{ab} is u-independent and $\tilde{D}_{ab}$ is identified as the shear. The u-independent mode labels the vacuum of the theory. It has been well known [18] that for a theory having an infinite dimensional BMS symmetry, there are multiple vacuua states possible. As given in equation (5.14), the generators of supertranslations modify the value of C_{ab} so it is identified as the Goldstone mode and since the theory has infinite dimensional BMS symmetry it has multiple vacuua.

A table summarizing the comparison with 4D is given in table 5.1.

Quantity	**4D**	**6D**
Goldstone Mode	$\int N_{ab}^{(4D)} du$	C_{ab}
Radiative Mode	$C_{ab}^{(4D)}$	$\tilde{D}_{ab}$
Memory	$C_{ab}^{(4D)}$	$\tilde{D}_{ab}$
Soft Mode	$\int N_{ab}^{(4D)} du$	$\int \tilde{D}_{ab} du$

Table 5.1: Here we summarize the importance of the components in the r-expansion of g_{ab} and show how they contrast with the 4D counterpart. Note that the quantities given are only up to a proportionality, and focuses on the u-dependence.

We quote our first main result in this language: The true graviton degree of freedom gets redefined upon working about a specific vacuum (labeled by C_{ab}). From the structure of (5.18) we see that it is specifically the zero-mode of D_{ab} which gets redefined.

We now describe an important application of our results in the context of the S-matrix[22]. As described above, there exists multiple soft vacuua in flat space and thus the initial and the final states in a generic S-matrix can be built on different soft vacuua. Therefore, there is a possibility of a vacuum-vacuum transition in a generic scattering process. There is a detailed calculation demonstrating this effect in a four dimensional scattering process [159]. The main reason behind this transition is the *Ward Identity*, which, for a scattering process would imply that the total supertranslation charge

[22]We would end up with a similar implication even for the QED S-matrix in higher dimensions.

(see eq.(5.39)) is conserved during scattering,

$$\begin{aligned}&\langle out|[\hat{Q}_f, S]|in\rangle = \langle out|[\hat{Q}_f^{hard} + \hat{Q}_f^{soft}, S]|in\rangle = 0\\ &\implies (Q_{f,+}^{soft} - Q_{f,-}^{soft})\,\langle out|S|in\rangle = -(Q_{f,+}^{hard} - Q_{f,-}^{hard})\,\langle out|S|in\rangle\end{aligned} \tag{5.47}$$

where we use the notation $\hat{Q}|out\rangle = Q_+|out\rangle$ and $\hat{Q}|in\rangle = Q_-|in\rangle$. We can consider an $|in\rangle$ state built on an eigenstate of $\hat{Q}_f^{soft}$ with eigen charge $Q_{f,-}^{soft}$. For any general scattering with a non-zero S-matrix, $\langle out|S|in\rangle \neq 0$ we have $Q_{f,+}^{hard} \neq Q_{f,-}^{hard}$. Therefore for a scattering process which conserves the supertranslation charge (satisfies (5.47)), we must have $Q_{f,+}^{soft} \neq Q_{f,-}^{soft}$. This indicates that there is a vacuum to vacuum transition in any general scattering which conserves the supertranslation charge.

It is well known that the S-matrix in a theory of gravity in 4-dimensions suffers from IR-divergences [160]. However the physical S-matrix (which is free of IR divergences) can be obtained by dressing the original S-matrix using the KF (Kulish-Faddeev) prescription [161]. Therefore, the IR finite S-matrix in 4-dimensions is given by the KF dressed S-matrix. This leaves us with a puzzle in higher dimensions. As the bare S-matrix in higher dimensions is already IR finite, it is not apriori clear from this perspective whether one should be dressing the S-matrix or not. However the need for dressing can arises from an attempt to define gauge invariant observables in gravity [128]. It is proven in four dimensional spacetimes [159, 162] that the S-matrix which conserves the supertranslation charge is the KF dressed S-matrix and it is reasonable to expect that a similar proof should hold in higher dimensions.

5.5.2 Summary

In this chapter we discuss supertranslations in (even) higher dimensions, specifically focusing on the six-dimensional case. We first specify the free data of the theory, which are given by the first two subleading coefficients in the large$-r$ expansion of g_{ab}. The graviton is defined by a combination of this free data (C_{ab} and D_{ab}), which we call $\tilde{D}_{ab}$. This redefined field $\tilde{D}_{ab}$ contains the radiative data. The redefinition does not effect the News tensor ($\partial_u \tilde{D}_{ab}$), but is necessary for the graviton to have the correct asymptotic fall offs and further, a finite symplectic form in the full non-linear theory. We emphasize that the redefinition is also required even if one is studying the linearized theory about a non-trivial Minkowski background ($C_{ab} \neq 0$). In the quantum theory, this implies that the graviton gets redefined depending on the vacuum one is working with. We discuss the symmetries of supertranslations in terms of the redefined variable $\tilde{D}_{ab}$.

We compute the supertranslation charge using the covariant phase space formalism. For this, we first evaluate the Symplectic form of the theory. We find that the symplectic form is uniquely split into two parts: one which is finite and characterizes the radiation of the system, and the other which is divergent (when integrated along $\mathcal{I}^+$). As the name suggests, the entire radiation content of the theory is contained in the finite & radiative part. The split is unambiguous and the ambiguity is fixed by noting that only the radiative part leads to the correct Noether charge. In fact, upon trying to construct the charge from the divergent piece, we notice that it cannot be expressed as a total variation. Therefore, the radiative symplectic form at $\mathcal{I}^+$ is uniquely defined and it also helps us understand the canonically conjugate variables in the theory. As we point out, the symplectic form in the non-linear theory is a simple generalization of the result in [26], and therefore the Noether charge can be obtained by following similar steps. We also compute the Bondi mass and evaluate the supertranslation charge using that. We find an exact matching between the two expressions – the one via the radiative symplectic form and the one via the Bondi mass. The Noether charge in the full non-linear theory is a simple generalization of the result in the linearized theory about $C_{ab} = 0$. Upon combining with the results in [22] it is easy to see that we end up getting a similar Ward identity and therefore, the same structure for the Weinberg Soft theorem. In appendix C.4, we show how the gravitational case generalizes from the electromagnetic case which helps us understand these issues better.

Finally, we move onto a discussion of the memory effect and the IR triangle in the six dimensional non-linear theory. We find that the generalization works in a similar way as that of the Noether charge, where we simply have to replace D_{ab} (in the linearized answers) with $\tilde{D}_{ab}$, to get the result in the non-linear theory. Therefore, this gives a very neat generalization of the IR triangle in higher-dimensions in the non-linear theory. As per our knowledge, this is the first example of the IR triangle in the non-linear theory in higher dimensions. In appendix C.3 we also compare the final form of our answer with [79] in the linearized limit and show that it is gauge invariant. Finally, in section 5.5.1 we mention the important implications of our analysis in the quantum theory.

5.5.3 Future Directions

One of the major motivations for this analysis was to understand the principle of holography of information [83, 5, 10, 2, 9] in higher dimensional flat spacetime. This would be a direct extension of [5] and would require a proper understanding of the Hilbert space in higher dimensions. In this chapter, we have taken the first step of identifying the right phase space. By following the arguments in the previous chapter we can see that the principle of holography of information can be easily generalized to 6-dimensions.

The main focus of this chapter has been the study of supertranslations in higher dimensions. A similar study for superrotations in higher dimensions has been done in the paper [24]. Our analysis has been restricted to even dimensional spacetimes and we would like to explore the case of odd dimensions [163] in the future.

Chapter 6

Conclusion

One of our main results is that in a theory of quantum gravity in flat spacetime or global AdS, observers located near the boundary are able to extract information about the bulk via a physical protocol. This result builds on older work [4, 5]. We assumed that observers near the boundary could measure the energy of the state and modify the state by acting with a unitary operators. These operators that the observers use in the protocol are obtained via low energy Hermitian operators near the boundary region. Using these tools and assuming the validity of the Born rule, we show how the observers can determine the bulk state.

In chapter 4 of this thesis we show that perturbative solutions of the gravitational constraints leads to a proof of the principle of holography of information. In the wavefunctional language it states that for any wavefunctional that satisfies the constraint equations in gravity (in AdS) are uniquely determined by their boundary values over an infinitesimal interval of time.

In the final chapter of the thesis we analyze the symmetries of non-linear general relativity in six dimensional spacetime and identify the correct variables that capture the radiative data. These then allow us to construct the symplectic form, the supertranslation charge and also make a connection with the memory effect in non-linear general relativity. Although we have explicitly worked in six-dimensional flat spacetime, all our results can be extended to higher even dimensions.

We refer the interested reader to delve into the final sections of each chapter for a detailed summary.

Future Directions

While some aspects of the principle of holography of information have been explored there are still many interesting properties that are left to uncover. For example, it was only recently shown how the principle of holography of information generalizes to dS spacetime and also sheds light on the Hilbert space at late times [134, 133]. It would be interesting to study the consequences of the states found in this Hilbert space for cosmology and other phenomenological implications. It is still not understood how massive particles are to be taken into account for establishing the principle of holography of information in flat spacetime and that needs more investigation. Similarly, it would be interesting to understand how one can generalize the story in flat spacetime to all dimensions, including the odd-dimensional case. For this one needs a proper understanding of the phase space of general relativity near null infinity in odd dimensions.

Appendix A

Details on Constructing the Protocol in AdS and Flat Space

A.1 State-operator map

In this appendix, we would like to describe, in some more detail, how states of the form

$$|X\rangle = X|0\rangle, \tag{A.1}$$

form a basis for the entire Hilbert space. Here X is a simple Hermitian operator near the boundary. We first review two quick formal arguments — one purely from a bulk perspective, and another assuming AdS/CFT. Then we describe this construction in more detail and explain how at large N, the low-energy Fock space can be generated explicitly in the form above.

Formal arguments

From a bulk perspective, we have a number of weakly interacting fields at low-energies. These fields include the low-energy gravitons but may also include stringy excitations. At low-energies, it is possible to frame all dynamics with the Hilbert space formed by

$$\mathcal{H} = \text{span of}\{X(\tau)|0\rangle\}, \tag{A.2}$$

where the $X(\tau)$ are simple Hermitian operators at an arbitrary point of time i.e. τ can range from $(-\infty, \infty)$. This is true since time-evolution manifestly evolves this set back to itself. But then standard analyticity arguments (see

Appendix A of [5]) tell us that Hermitian operators from a smaller time-band $[0, \epsilon]$ generate a space of states that is dense in the Hilbert space above.

A second argument, which assumes AdS/CFT, is as follows. The set of bulk operators near the boundary are dual to boundary operators by the extrapolate dictionary [164]. Now, it is clear, from the state-operator map in the conformal field theory that the set of states obtained by applying boundary Hermitian operators on a time-slice to the vacuum form a complete basis for the Hilbert space. If we restrict to simple Hermitian operators — boundary generalized free-fields and low-order polynomials in these generalized free-fields — we obtain the low-energy Hilbert space in the bulk. We use a time-band in this thesis, rather than a time-slice, which removes the need for using operators with time-derivatives in the state-operator map.

Explicit constructions

We now demonstrate how this construction works explicitly in low-energy bulk effective field theory. Consider a weakly interacting bulk field of dimension Δ, which we denoted by ϕ in the text, with boundary value O. In global AdS, this operator can be expanded as

$$O(t, \Omega) = \sum_{n,\ell} \sqrt{G_{n,\ell}} a_{n,\ell} e^{-i(\Delta+2n+\ell)t} Y_\ell(\Omega) + \text{h.c}, \tag{A.3}$$

where the operators obey $[a_{n,\ell}, a^\dagger_{n',\ell'}] = \delta_{nn'}\delta_{\ell\ell'}$ and the function $G_{n,\ell}$ is explicitly given by

$$G_{n,\ell} = \frac{4\pi^d \Gamma(\Delta+n+\ell)\Gamma(\Delta+n+1-d/2)}{\Gamma(d/2)\Gamma(n+1)\Gamma(\Delta)\Gamma(\Delta+1-d/2)\Gamma(d/2+n+\ell)}. \tag{A.4}$$

The normalized low-energy single-particle states are simply created by

$$|n, \ell\rangle = a^\dagger_{n,\ell}|0\rangle. \tag{A.5}$$

Now, it is obvious that one way to create such states through a Hermitian operator localized near the boundary is to consider

$$|n, \ell\rangle = \frac{1}{\sqrt{G_{n,\ell}}} \int_0^\pi O(t, \Omega) e^{-i(\Delta+2n+\ell)t} Y^*_\ell(\Omega) d^{d-1}\Omega \frac{dt}{\pi} |0\rangle, \tag{A.6}$$

but this is <u>not</u> what our observers need since it involves an integral over the light-crossing time of AdS. The claim above is that we <u>can also generate</u> the state by smearing the bulk field in the time-band $[0, \epsilon]$.

A simple algorithm to numerically approximate the state is as follows. Consider the window function

$$w(t) = \theta(t)\theta(\epsilon - t)e^{\frac{1}{t(t-\epsilon)}}, \tag{A.7}$$

which is a real function that vanishes smoothly at the end-points of $[0, \epsilon]$. Now, consider the states

$$|h_m\rangle = \int_0^\epsilon \left[O(t,\Omega)Y_\ell^*(\Omega)e^{imt}w(t)d^{d-1}\Omega dt\right]|0\rangle = \sum_{n=0}^{\infty} h_{n,m}\sqrt{G_{n,\ell}}|n,\ell\rangle, \tag{A.8}$$

where

$$h_{n,m} = \int_0^\epsilon e^{i(\Delta+2n+m+\ell)t}w(t)dt. \tag{A.9}$$

We take states from the basis above with values of m ranging between $-m_{\max}$ to $m_{\max}$ and look for coefficients, c_m, that minimize

$$\Delta_{m_{\max}} = \left| |n,\ell\rangle - \sum_{m=-m_{\max}}^{m_{\max}} c_m |h_m\rangle \right|^2 .$$

The arguments above tell us that this error term converges to 0 as the value of $m_{\max}$ is increased. The original state is thus approximated better and better by a sum of states dual to operators in a small time band.

An extremely simple Mathematica code to generate a numerical approximation to the state $|n,\ell\rangle$ can be found at [165] . Note that, in this code, apart from putting a cutoff on $m_{\max}$, it is also necessary to truncate the sum over n in (A.8) to some finite value N_{cut}. Figure A.1 shows a plot of $\Delta_{m_{\max}}$ vs $m_{\max}$ for two possible choices of n, ℓ for a massless bulk field in $d = 4$, with $N_{\text{cut}} = 100$. As is evident, this error term consistently tends to 0.

Once we have constructed single-particle states, multi-particle states are easy to construct. In the weakly coupled limit, the structure of the Hilbert space is that of a Fock space. So multi-particle states can be constructed just by multiplying operators of the form above, and then acting on the vacuum after smearing them appropriately.

The careful reader may notice a subtlety. In the analysis above, we have used the back-reaction of bulk excitations on the metric, which appears only during interactions, and yet we are content with using the Fock space approximation here. The reason is that any effect of interactions is only at

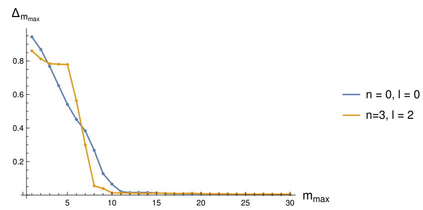


Figure A.1: The figure displays the error $\Delta_{m_{\text{max}}}$ in approximating a state as a function of m_{max} for two low-energy states. It is evident that this declines monotonically as m_{max} is increased and becomes very small.

subleading order in $\frac{1}{N}$, and therefore it does not affect the observers ability to distinguish the bulk state at leading order at all. In particular, say that the observers wish to find the operator dual to a state $|X\rangle$ and they use an approximation so that

$$X_{\text{approx}}|0\rangle = |X\rangle + \frac{1}{N}|X_{\text{err}}\rangle \tag{A.10}$$

The reader can check by examining the procedure described in section 2.3 that this error makes only a $\frac{1}{N}$ difference in all measurements conducted by the observer. Since the accuracy of the task set for the observers is set by $\delta \gg \frac{1}{N}$ this error is unimportant.

For this reason, one can also use the free boundary-bulk smearing function given in [4]; map the boundary operators above to bulk fields on the slice $t = \frac{\epsilon}{2}$, and with radial coordinate $r \in [\cot \frac{\epsilon}{2}, \infty)$ and use these smeared bulk fields to generate the state.

In this chapter, for simplicity, we have considered the case where all interactions are controlled by $\frac{1}{N}$. But, our protocol would work even if there was a hierarchy of interactions. If interactions between quantum fields were controlled by a parameter $\lambda \gg \frac{1}{N}$ but with $\lambda \ll 1$, then we would need to correct the operators above perturbatively in λ. This calculation can still be done reliably within the framework of QFT in curved spacetime.

A.2 Orthogonalizing unitaries

In section 2.3, we explained that if the observers encountered a state with $|\langle 0|g\rangle|^2 \neq 0$, they could act with a preliminary unitary $\mathfrak{U}^z$ with the property that $\langle 0|\mathfrak{U}^z|g\rangle = 0$. They could then apply the protocol of section 2.3 to $\mathfrak{U}^z|g\rangle$, and back-calculate $|g\rangle$. We now explain how to construct the unitary $\mathfrak{U}^z$.

Although, we present the construction slightly formally below, the basic idea behind the construction of this unitary is quite simple. Using a single propagating field, and by smearing the field and the conjugate momentum over a small region of spacetime, the observers can construct one simple-harmonic degree of freedom. The unitary they need to find acts only on this simple-harmonic degree of freedom. By rotating the states of this degree of freedom appropriately, it is always possible to make the state $\mathfrak{U}^z|g\rangle$ orthogonal to $|0\rangle$.

Let $O(t,\Omega)$ be the boundary value of a bulk field as defined in Appendix A.1. Then by a suitable choice of two functions it is possible to define operators

$$\mathcal{X} = \int dt d^{d-1}\Omega O(t,\Omega) f_1(t,\Omega); \qquad \mathcal{P} = \int dt d^{d-1}\Omega O(t,\Omega) f_2(t,\Omega), \quad \text{(A.11)}$$

so that these operators obey the Heisenberg algebra

$$[\mathcal{X},\mathcal{P}] = i. \qquad \text{(A.12)}$$

Here the functions f_1, f_2 have limited support in the time band $t \in [0,\epsilon]$ and also on the boundary sphere. There is no unique choice of f_1 and f_2 and so we do not write down their forms explicitly. The relation above is valid in the large-N approximation where the commutator of a bulk field with itself is well-approximated by a c-number. By the same reasoning as in Appendix A.1, it is acceptable to use this approximation while constructing the orthogonalizing unitary.

We can also define the operators

$$\mathcal{A} = \frac{1}{\sqrt{2}}(\mathcal{X} + i\mathcal{P}); \qquad \mathcal{A}^\dagger = \frac{1}{\sqrt{2}}(\mathcal{X} - i\mathcal{P}). \qquad \text{(A.13)}$$

The Hilbert space of the full theory furnishes a representation of the algebra corresponding to this simple harmonic degrees of freedom i.e. the algebra that can be expanded in terms of the $\mathcal{X}$ and $\mathcal{P}$ operators. One of the useful elements of this algebra is the projector onto the zero-eigenspace of the number operator

$$\mathcal{P}_0^{\text{sho}} = \int_0^{2\pi} e^{i\theta(\mathcal{A}^\dagger\mathcal{A})}\frac{d\theta}{2\pi}. \qquad \text{(A.14)}$$

Note that this projector projects onto an infinite dimensional subspace in the full theory. Using this projector we can also construct projectors onto other number eigenspaces.

$$\mathcal{P}_n^{\rm sho} = \frac{1}{n!}(\mathcal{A}^\dagger)^n \int_0^{2\pi} e^{i\theta(\mathcal{A}^\dagger \mathcal{A})} \frac{d\theta}{2\pi} (\mathcal{A})^n. \tag{A.15}$$

These operators furnish a partition of the identity so that $\sum \mathcal{P}_n^{\rm sho} = 1$. One can also construct more general "transition operators"

$$\mathcal{T}_{m,n}^{\rm sho} = \frac{1}{\sqrt{m!n!}}(\mathcal{A}^\dagger)^m \int_0^{2\pi} e^{i\theta(\mathcal{A}^\dagger \mathcal{A})} \frac{d\theta}{2\pi} (\mathcal{A})^n. \tag{A.16}$$

Note that $\mathcal{P}_n^{\rm sho} = \mathcal{T}_{n,n}^{\rm sho}$, and also that the operators $\mathcal{T}_{m,n}^{\rm sho}$ furnish a complete basis for all elements of the algebra.

Now consider a unitary $\mathfrak{U}^z$ that is within this algebra. Since the transition operators form a basis, we can expand

$$\mathfrak{U}^z = \sum_{n,m=0}^{\infty} u_{nm} \mathcal{T}_{n,m}^{\rm sho}, \tag{A.17}$$

where u_{nm} is just a matrix of c-numbers. The condition that $\mathfrak{U}^z$ is unitary is the same as the condition that the c-number matrix u_{nm} is unitary. So we have

$$\langle 0|\mathfrak{U}^z|g\rangle = \sum_{n,m=0}^{\infty} u_{mn} \langle 0|\mathcal{T}_{n,m}^{\rm sho}|g\rangle, \tag{A.18}$$

Once again $t_{nm} = \langle 0|\mathcal{T}_{n,m}^{\rm sho}|g\rangle$ is just some matrix of c-numbers. So the observers need to find a unitary matrix that satisfies a single linear constraint.

$$\sum_{n,m=0}^{\infty} u_{mn} t_{nm} = 0. \tag{A.19}$$

The problem (A.19) is a linear-algebra problem of finding a unitary matrix with the property that when multiplied with another matrix and traced, it yields zero. It is not difficult to see that such a unitary matrix can always be found for any choice of t_{nm}. But for the sake of completeness, we now give a proof.

We will give a proof in steps. First consider the two-dimensional version of this problem, where we are given an arbitrary 2×2 matrix t_{nm} and need to find a 2×2 unitary matrix, u_{nm}, so that $\mathrm{Tr}(ut) = 0$. We express the unitary matrix in terms of a unit-vector on the sphere, $\hat{n}$, and an angle θ as $u_{nm} = \cos(\frac{\theta}{2})\delta_{nm} + i(\vec{\sigma} \cdot \hat{n})_{nm}\sin(\frac{\theta}{2})$. Then we first choose $\hat{n}$ to be orthogonal to the vector $\mathrm{Re}\,(\mathrm{Tr}(\vec{\sigma}t))$. Then we are left with the single equation $\mathrm{Tr}(t)\cos(\frac{\theta}{2}) - \mathrm{Im}\,(\mathrm{Tr}((\vec{\sigma} \cdot \hat{n})\, t))\sin(\frac{\theta}{2})$ which can be solved by choosing θ appropriately.

Now, consider the D-dimensional problem. If we collect the columns of t into a set of vectors $\vec{t}_1 \ldots \vec{t}_D$ we need to find a set of orthonormal basis vectors $\vec{v}_1, \ldots \vec{v}_D$ that satisfy $\sum \langle \vec{v}_i, \vec{t}_i \rangle = 0$, with the usual conjugate-bilinear inner-product. The unitary matrix is then obtained by taking $\vec{v}_i$ to be the columns of $u^\dagger$.

Such an orthonormal basis can be found as follows. We choose $\vec{v}_D$ to be orthogonal to $\vec{t}_D$. Then we choose $\vec{v}_{D-1}$ to be orthogonal to $\vec{v}_D$ and $\vec{t}_{D-1}$. We similarly choose $\vec{v}_P$ to be orthogonal to $\vec{t}_P$ and $\vec{v}_{P+1}, \vec{v}_{P+2} \ldots \vec{v}_D$ for all vectors until $P = 3$. Then $\vec{v}_1$ and $\vec{v}_2$ are confined to a two-dimensional space, since they must be orthogonal to $\vec{v}_3 \ldots \vec{v}_D$, and further they must satisfy $\langle \vec{v}_1, \vec{t}_1 \rangle + \langle \vec{v}_2, \vec{t}_2 \rangle = 0$. But this is just the two-dimensional problem that we solved above.

We now turn to the infinite dimensional case (A.19). On physical grounds, we expect that $\langle 0|\mathcal{T}^{\mathrm{sho}}_{n,m}|g\rangle \ll 1$ when n, m are much larger than the "occupancy" in the two states i.e. when $n, m \gg \langle 0|\mathcal{A}^\dagger\mathcal{A}|0\rangle$ and $n, m \gg \langle g|\mathcal{A}^\dagger\mathcal{A}|g\rangle$. So we consider a unitary that does not act on states with occupancy beyond some D i.e. $u_{ij} = \delta_{ij}$ for $i, j \geq D$ and $u_{ij} = 0$ if $i \geq D, j < D$ or $i < D, j \geq D$. Then we just need to deform the finite dimensional solution above so that $\sum_{n,m=0}^{D-1} u_{mn}t_{nm} = -\sum_{i \geq D} t_{ii}$. This can clearly be done provided the right hand side is small enough, which can be achieved by taking D to be large enough.

A.3 Soft Structure of Vacuua

The correlation functions that we measure are of the form $\langle\psi|U_i^\dagger P_0 U_j|\psi\rangle$ where P_0 is defined in eq.(3.13), the states $|\psi\rangle$ are generically of the form eq.(3.14) and the unitaries U are of the form eq.(3.17). The following discussion only refers to states built out of a single flavoured field and the extension to multiple flavours is similar. Since the states and the unitaries are constructed out of hard operators only (i.e, do not include operators like the zero mode of the shear), we show that the soft structure of the vacuum and the projector become unimportant[1]. From the decomposition of the vacuum and the projector as given in (3.11) and (3.13), a generic correlation function of interest takes the form

$$\langle\psi|U_i^\dagger P_0 U_j|\psi\rangle = \langle 0|Z^\dagger U_i^\dagger P_0 U_j Z|0\rangle = \int DsDs'Ds'' g_{\{s\}} g_{\{s''\}} \langle\{s\}|Z^\dagger U_i^\dagger|\{s'\}\rangle \langle\{s'\}|U_j Z|\{s''\}\rangle \tag{A.20}$$

with Z representing the operators on the RHS in the definition of the state (3.14) and $Ds \equiv \prod_{lm} ds_{lm}$. To avoid cluttering of notation we suppress the l, m indices but it is trivial to reinstate them. We now use the fact that [75]

$$\langle\{s\}|O|\{s'\}\rangle = \langle 0|O|0\rangle\, \delta(\{s\} - \{s'\}) \tag{A.21}$$

for any hard operator O. This can be intuitively understood by noting that the operator O, a hard operator, does not induce a vacuum transition as it does not contain the zero mode of the shear. The above result can be also proven explicitly by using the condition $\langle 0|0\rangle = 1$. Using this result we can simplify (A.20) to get

$$\begin{aligned}\langle\psi|U_i^\dagger P_0 U_j|\psi\rangle &= \int DsDs'Ds'' g^*_{\{s\}} g_{\{s''\}} \langle 0|Z^\dagger U_i^\dagger|0\rangle \langle 0|U_j Z|0\rangle\, \delta(\{s\} - \{s'\})\delta(\{s'\} - \{s''\}) \\ &= \langle 0|Z^\dagger U_i^\dagger|0\rangle \langle 0|U_j Z|0\rangle \int |g_{\{s\}}|^2\, Ds = \langle\psi|U_i^\dagger|0\rangle \langle 0|U_j|\psi\rangle\end{aligned} \tag{A.22}$$

This is an expected identity when there is no vacuum degeneracy. For this identity to hold in the presence of a degeneracy it is necessary that we are not considering expectation values of soft operators (like the zero mode of the shear).

A.4 Saddle point approximation for scalar field

In this appendix we work out the 2-point Wightman function for a free massless scalar field at null infinity in $(d+2)$-dimensional Minkowski spacetime. The free field correlation functions suffice at leading order in large$-r$ as the interaction terms do not survive at this order. A similar derivation can be performed for gauge and gravitational fields as well [18].

[1] We thank Tuneer Chakraborty and Priyadarshi Paul for several useful discussions on this and clarifying many doubts.

The mode expansion for the scalar field in the bulk is given as

$$\phi(t,r,\Omega) = \int \frac{d^{d+1}q}{(2\pi)^{d+1}} \frac{1}{2\omega_q} \big[a_q e^{iq\cdot x} + a_q^\dagger e^{-iq\cdot x}\big]. \tag{A.23}$$

The mode expansion at null infinity can be obtained by taking the large$-r$ limit of this expression. It it convenient to decompose $e^{iq\cdot x} = e^{-i\omega u} e^{-i\omega r(1-\cos\theta)}$, where we use $|\vec{q}| = q^0 = \omega$ and θ denotes the angle between $\vec{q}$ and $\vec{r}$. In the limit $r \to \infty$ the only saddle that contributes is $\theta = 0$ (the saddle at 2π is prevented by the Riemann-Lebesgue lemma). This allows us to approximate $1 - \cos\theta \approx \frac{\theta^2}{2}$ and therefore $\lim_{r\to\infty} e^{iq\cdot x} = e^{-i\omega u} e^{-i\omega r\theta^2/2}$.

Using this we perform the $\int d^{d+1}q$ integral (where the integral over the remaining $(d-1)$ variables is denoted by S_{d-1}.),

$$\int \frac{d^{d+1}q}{2\omega_q} a_q e^{iq\cdot x} = -\frac{S_{d-1} i}{2r} \int d\omega \omega^{d-2} a(\omega\hat{\Omega}) e^{-i\omega u} \tag{A.24}$$

Thus the expansion of the scalar field at $\mathcal{I}^+$ using the saddle point approximation becomes

$$\phi(u,\Omega) = -\frac{iS_{d-1}}{2(2\pi)^{d+1}} \int_{-\infty}^{\infty} d\omega \omega^{d-2} \big[a(\omega\hat{\Omega}) e^{-i\omega u} - a^\dagger(\omega\hat{\Omega}) e^{i\omega u}\big] \tag{A.25}$$

For computing the 2-point Wightman function of the fields $\langle s|\phi(u,\Omega)\phi(u',\Omega')|s\rangle$, it is necessary to compute the expectation values of the ladder operators $a, a^\dagger$. This can be obtained by computing the commutator of the ladder operators

$$[a(\omega\hat{\Omega}), a^\dagger(\omega'\hat{\Omega}')] = \frac{2}{\omega^{d-1}} (2\pi)^d \delta(\omega - \omega') \delta^d(\Omega, \Omega'). \tag{A.26}$$

Since $a|0\rangle = 0$ a simple computation then gives us the 2-point Wightman function of the scalar field

$$\langle 0|\phi(u,\Omega)\phi(u',\Omega')|0\rangle = \frac{S_{d-1}^2 i^{2-d} \Gamma(d-2)}{(2\pi)^{2d+2}} \times \frac{\delta^d(\Omega,\Omega')}{(u - u' - i\epsilon)^{d-2}}, \tag{A.27}$$

where we have introduced a factor of $i\epsilon$, with $\epsilon > 0$, for the integral to converge. In $d = 2$ this gives a Logarithmic dependence and it is therefore more convenient to evaluate $\langle 0|\pi(u,\Omega)\phi(u',\Omega')|0\rangle = \langle 0|\partial_u\phi(u,\Omega)\phi(u',\Omega')|0\rangle$,

$$\langle 0|\pi(u,\Omega)\phi(u',\Omega')|0\rangle_{d=2} = -i\frac{\delta^2(\Omega,\Omega')}{4\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(u-u'-i\epsilon)} = -\frac{1}{4\pi} \frac{\delta^2(\Omega,\Omega')}{u - u' - i\epsilon} \tag{A.28}$$

where we use $S_1 = 2\pi$.

A.5 Explicit reconstruction of g_n's

In this appendix we explain how one can explicitly reconstruct the functions g_n's. The kind of correlation functions that we encounter lead to equations of the following kind for $g_n(\vec{u},\vec{\Omega})$

$$\int d\vec{u}d\vec{\Omega}d\vec{u}'d\vec{\Omega}'\ f_n(\vec{u}',\vec{\Omega}')g_n(\vec{u},\vec{\Omega})\frac{\delta(\Omega_1,\Omega_1')}{u_1-u_1'-i\epsilon}\cdots\frac{\delta(\Omega_n,\Omega_n')}{u_n-u_n'-i\epsilon}=C_n. \quad (A.29)$$

Here $f_n(\vec{u},\vec{\Omega}')$ is a smearing function localized near $\mathcal{I}^+_-$ which is up to the observers to tune and C_n is the result of the measurement they perform. The structures $\frac{\delta(\Omega,\Omega')}{u-u'-i\epsilon}$ arise from the 2-point functions computed in appendix C.6. The integration of the delta functions gives

$$\int d\vec{u}d\vec{\Omega}d\vec{u}'\ f_n(\vec{u}',\vec{\Omega})g_n(\vec{u},\vec{\Omega})\frac{1}{u_1-u_1'-i\epsilon}\cdots\frac{1}{u_n-u_n'-i\epsilon}=C_n\,. \quad (A.30)$$

From this equation we see that the spherical dependence of g_n's is fixed as the functions f_n's can be tuned arbitrarily by the observers. In order to fix the u dependence, let us consider the integrand of $\vec{u}'$ on the LHS

$$\mathcal{I}\equiv\int d\vec{u}g_n(\vec{u},\vec{\Omega})\frac{1}{u_1-u_1'-i\epsilon}\cdots\frac{1}{u_n-u_n'-i\epsilon} \quad (A.31)$$

and expand the integrands as $u_i'\to-\infty$ (since these arise from fields that are localized near $\mathcal{I}^+_-$) which gives

$$\mathcal{I}=\sum_{n_1,\cdots,n_n=0}^{\infty}\frac{(-1)^{n\mathrm{mod}2}}{u_1'^{n_1+1}\cdots u_n'^{n_n+1}}\int g_n(\vec{u},\vec{\Omega})u_1^{n_1}\cdots u_n^{n_n}d\vec{u}\,. \quad (A.32)$$

Plugging this integrand back into C_n we obtain

$$\int d\vec{u}'d\Omega f(\vec{u}',\vec{\Omega})\sum_{n_1,\cdots,n_n=0}^{\infty}\frac{(-1)^{n\mathrm{mod}2}}{u_1'^{n_1+1}\cdots u_n'^{n_n+1}}\int g_n(\vec{u},\vec{\Omega})u_1^{n_1}\cdots u_n^{n_n}d\vec{u}=C_n\,. \quad (A.33)$$

Since the observers are allowed to tune the function $f_n(\vec{u}',\vec{\Omega})$ arbitrarily, they can obtain all possible moments of the distribution $g_n(\vec{u},\vec{\Omega})$ and therefore reconstruct the function $g_n(\vec{u},\vec{\Omega})$ completely. We have included a mathematica notebook with this submission that demonstrates how this works in a numerical example for a 1-dimensional function.

A.6 Special cases of states

In this appendix we explain how we can determine the state when we have only even or only odd excitations. The states we consider are of the form

$$|\psi_{even}\rangle = \sum_{n=1}^{\infty} \prod_{j=1}^{2n} \int : \phi(u_j, \Omega_j) : g_{2n}(\vec{u}, \vec{\Omega}) d\vec{u} d\vec{\Omega} \, |0\rangle \tag{A.34a}$$

$$|\psi_{odd}\rangle = \sum_{n=1}^{\infty} \prod_{j=1}^{2n-1} \int : \phi(u_j, \Omega_j) : g_{2n-1}(\vec{u}, \vec{\Omega}) d\vec{u} d\vec{\Omega} \, |0\rangle \, . \tag{A.34b}$$

The main reason why these cases are special is because we cannot determine the value of $\sin\theta_j$ (defined in eq.(3.22)) in the same way as we did in section 3.3.5. This is because that method requires the presence of both even and odd n. We demonstrate below how that can be extended to include these special cases.

The multiple flavoured case works in a similar way as to that of the single flavour case and therefore we shall only demonstrate the former.

A.6.1 All even

Consider a state of the form in eq.(A.34a). In this case, the first non-zero $g_n = g_2$. Therefore we first measure the value of $\langle\psi|U_2^\dagger P_0 U_2|\psi\rangle$, which allows us to determine the value of g_2 up to a phase. However, since the overall phase of the state $|\psi\rangle$ is meaningless, we can fix the phase of g_2 to be $\theta_2 = 0$.

This completely fixes the value of g_2. Using this it is simple to determine the value of $\cos\theta_{2j}$ for all j. This can be performed in a similar manner as in eq.(3.23), where we now have U_2 instead of U_1, i.e, measure $\langle\psi|U_2^\dagger U_{2j}^\dagger P_0 U_{2j} U_2|\psi\rangle$ at $\mathcal{O}(f^2)$, which gives

$$\cos\theta_{2j} = \frac{\langle\psi|U_{2j}^\dagger U_2^\dagger P_0 U_2 U_{2j}|\psi\rangle - \langle\psi|U_2^\dagger P_0 U_2|\psi\rangle - \langle\psi|U_{2j}^\dagger P_0 U_{2j}|\psi\rangle}{2\sqrt{\langle\psi|U_2^\dagger P_0 U_2|\psi\rangle}\sqrt{\langle\psi|U_{2j}^\dagger P_0 U_{2j}|\psi\rangle}} \, . \tag{A.35}$$

We have therefore determined the value of $\cos\theta_{2j}$ for all j. In this measurement we also encounter the value of $\langle\psi|U_{2j}^\dagger P_0 U_{2j}|\psi\rangle$ which allows us to measure the value of g_{2j+2} after knowing the value of g_{2j}.

Thus the only thing that we are now left with is to find the value of $\sin\theta_{2j}$, where we only need to fix its sign. This is something that can be done in two steps. We first need the value of any $\sin\theta_{2j}$ other than $j = 1$ (which is already fixed to be 0). By trial and error we can easily find some $2j_0$ for which $\sin\theta_{2j_0} \neq 0$ (since we can measure $\cos\theta_{2j_0}$). Then using the procedure as described around eq.(3.3.5), we fix the value of $\sin\theta_{2j_0}$ and then go on to fix the other signs using an analogous equation to eq.(3.24).

We explain how to carry this out for $j_0 = 2$ but in case $\sin\theta_4 = 0$, we can easily extend this for any other j_0. For $j_0 = 2$, we just need to measure $\langle\psi|U_2^\dagger P_0 U_2|\psi\rangle$ to $\mathcal{O}(f^3)$. This fixes the sign of $\sin\theta_4$, although it is not enough to determine the value of g_4. However, since we can measure $\langle\psi|U_4^\dagger P_0 U_4|\psi\rangle$, we can determine g_4.

Once we have fixed $\sin\theta_4$, we can measure $\langle\psi|U_4^\dagger U_{2j}^\dagger P_0 U_{2j} U_4|\psi\rangle$ at $\mathcal{O}(f^2)$ in order to determine the value of $\sin\theta_{2j}$ completely. Therefore we have demonstrated how one can fix the functions g_{2j} including the phase factors θ_{2j} completely, in a sieve-like manner.

A.6.2 All odd

This is very similar to the previous case, therefore we only list the important steps.

1. We first fix the value of g_1 by measuring $\langle\psi|U_1^\dagger P_0 U_1|\psi\rangle$ and then using the fact that the overall phase of the state $|\psi\rangle$ is meaningless, we fix $\theta_1 = 0$.

2. Then we measure the value of $\langle\psi|U_1^\dagger U_{2j-1}^\dagger P_0 U_{2j-1} U_1|\psi\rangle$ at $\mathcal{O}(f^2)$ to fix the value of $\cos\theta_{2j-1}$.

3. Measuring $\langle\psi|U_{2j-1}^\dagger P_0 U_{2j-1}|\psi\rangle$ at $\mathcal{O}(f^2)$ yields the value of g_{2j-1} up to the phase factor $e^{i\theta_{2j-1}}$. This step has to be performed in a sieve-like procedure; the value of $\cos\theta_{2j-1}$ is fixed from the previous step and therefore we are left with fixing the sign of $\sin\theta_{2j-1}$ only.

4. In order to fix the signs of all $\sin\theta_{2j-1}$ we first fix the value of any particular $\sin\theta_{2j-1} \neq 0$ and then performing a measurement of the kind in eq.(3.24), allows to fix everything else. For the case when $\sin\theta_3 \neq 0$ (which is an example, that can be extended to any $2j - 1$) we have to measure $\langle\psi|U_2^\dagger U_1^\dagger P_0 U_1 U_2|\psi\rangle$ at $\mathcal{O}(f^3)$ in order to fix the sign of $\sin\theta_3$.

5. By measuring $\langle\psi|U_3^\dagger U_{2j-1}^\dagger P_0 U_{2j-1} U_3|\psi\rangle$ we can fix the signs of all $\sin\theta_{2j-1}$.

Appendix B

Split States in QED, Gravitational Energy, and Point-Wise Constraints

B.1 Split states in QED

In this appendix, we show that ordinary gauge theories localize information much like ordinary quantum field theories, and very differently from gravity. To illustrate this, we will solve the constraint of a $U(1)$ gauge theory coupled to matter and construct explicit wavefunctionals that are identical outside a bounded region but differ inside. Such states are called "split states" and the argument provided in the main text of the paper shows that split states do not exist in theories of quantum gravity. A useful reference for the analysis of wavefunctionals in QED and ordinary quantum field theories is [166]. An analysis of the canonical quantization of QED can also be found in Appendix B of [44]. We caution the reader that some of the conventions below differ from those of [44] by terms involving N and the determinant of the spatial metric.

B.1.1 Action and constraints

We work about the fixed global AdS background

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\Omega_{d-1}^2 \ , \tag{B.1}$$

where N is the same as (4.36). We emphasize that in this Appendix, we are not considering a theory with dynamical gravity and so the metric (B.1) is exact. We continue to use the $d+1$ notation of the main text for covariant derivatives.

The action of QED takes the form,

$$S = -\frac{1}{4}\int dt d^d x \sqrt{\gamma} N \, \hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + S_{\text{matter}} \ . \tag{B.2}$$

Note that we have included the interactions of the gauge field and the matter in the term denoted as S_{matter} above. The details of this action will not be important except for a few features that we mention below. But for the purpose of illustration, we consider a charged scalar field with the action,

$$S_{\text{matter}} = -\frac{1}{2}\int d^{d+1}x \sqrt{\gamma} N \, (\mathcal{D}_\mu \phi)^* \mathcal{D}^\mu \phi \ , \tag{B.3}$$

where $\mathcal{D}_\mu = \partial_\mu + iA_\mu$ is the gauge covariant derivative with the coupling constant set to 1.

As there is no kinetic term for the A_0 field, we immediately obtain a primary constraint

$$\Pi^t_{\text{em}} = 0 \ , \tag{B.4}$$

whereas the canonical momentum for the spatial part of the gauge field is

$$\Pi^i_{\text{em}} = \frac{1}{\sqrt{\gamma}}\frac{\delta S}{\delta \dot{A}_i} = -N\hat{F}^{ti} \tag{B.5}$$

which is just the electric field. This is similar to the primary constraints (4.19) in gravity. Imposing that this constraint is preserved under time evolution leads to a secondary constraint. This is the pointwise Gauss law

$$\nabla_i \Pi^i_{\text{em}} = \rho \ , \tag{B.6}$$

where ρ is the charge density of the matter. The left hand side of (B.6) is reminiscent of the momentum constraint in gravity since it is linear in the canonical momentum. However, the momentum constraint in gravity couples the metric and its canonical momentum whereas we see that (B.6) has no such nonlinear terms. This will allow us to present a general solution to this constraint.

For the action (B.3), the momentum conjugate to the scalar field is

$$\Pi_\phi = \frac{1}{\sqrt{\gamma}}\frac{\delta S}{\delta \dot{\phi}} = \frac{1}{2N}(\dot{\phi}^* - iA_t\phi^*); \qquad \Pi_{\phi^*} = \frac{1}{\sqrt{\gamma}}\frac{\delta S}{\delta \dot{\phi}^*} = \frac{1}{2N}(\dot{\phi} + iA_t\phi) \; , \tag{B.7}$$

and in terms of the canonical variables, we have

$$\rho = i(\phi\, \Pi_\phi - \phi^*\, \Pi_{\phi^*}) \; . \tag{B.8}$$

But the details of the matter sector will be unimportant in the analysis below and we will only use the fact that, in the classical canonical theory, the Poisson bracket between the charge density at two distinct points, x and x', on the same spatial slice vanishes:

$$\{\rho(x), \rho(x')\}_{\mathrm{PB}} = 0 \; . \tag{B.9}$$

In the quantum theory, the states of the theory are described by wavefunctionals of the gauge field and matter fields. The primary constraint (B.4) tells us that wavefunctionals, and observables that commute with the constraints, are independent of A_0. Therefore these wavefunctionals $\psi[A,\phi]$ depend on only the spatial components of the gauge field. The momentum operator is realized as

$$\Pi^i_{\mathrm{em}} = -\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta A_i} \; , \tag{B.10}$$

The secondary constraint then implies that

$$\left[\nabla_i \Pi^i_{\mathrm{em}} - \rho\right] \psi[A,\phi] = 0 \; . \tag{B.11}$$

Since we will exclusively consider wavefunctionals that satisfy the constraints the Poisson brackets (B.9) are directly promoted to commutators in the quantum theory. Therefore we have

$$[\rho(x), \rho(x')] = 0 \; , \tag{B.12}$$

at any two points x, x' on the same spatial slice. This property will be utilized below.

B.1.2 Solution to the constraints

Since the constraints in electromagnetism are simple, it is possible to write down an exact solution to the constraints. As in the main text, it is convenient to decompose the gauge field into a longitudinal and a transverse part

$$A_i = A_i^{\mathrm{L}} + A_i^{\mathrm{T}} \tag{B.13}$$

which satisfy

$$\nabla^i A_i^{\mathrm{T}} = 0, \qquad A_i^{\mathrm{L}} = \nabla_i \chi \ . \tag{B.14}$$

for some χ that vanishes asymptotically. The momentum can be similarly decomposed as

$$\Pi^i_{\mathrm{em}} = \Pi^i_{\mathrm{em,T}} + \Pi^i_{\mathrm{em,L}} \tag{B.15}$$

and by a simple extension of the argument near equation (4.53) we find that

$$\Pi^i_{\mathrm{em,T}} = -\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta A_i^{\mathrm{T}}}; \qquad \Pi^i_{\mathrm{em,L}} = -\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta A_i^{\mathrm{L}}} \ . \tag{B.16}$$

The constraint (B.11) correlates the part of the wavefunctional that depends on A^{L} with the part that depends on the charge density, leaving the part that depends on A^{T} unconstrained. A solution to the constraints is given by

$$\psi[A,\phi] = \exp\left[i\int_\Sigma d^dx\sqrt{\gamma}\int_\Sigma d^dx'\sqrt{\gamma'}\, A_i^{\mathrm{L}}(x)\nabla^i G(x,x')\rho(x')\right]\psi_A[A^{\mathrm{T}}]\psi_\phi[\phi] \tag{B.17}$$

where ψ_ϕ and ψ_A are arbitrary functionals and the Green's function $G(x,x')$ satisfies

$$\nabla_i\nabla^i G(x,x') = \frac{1}{\sqrt{\gamma}}\delta^{(d)}(x,x') \ . \tag{B.18}$$

Since the spatial slice is just Euclidean AdS$_d$, the Green's function can be written as [167, 168]

$$G(x,x') = \frac{2^{-\tilde\Delta}}{\tilde\Delta}\frac{\Gamma(\tilde\Delta)}{\pi^{\frac{\tilde\Delta}{2}}\Gamma(\frac{\tilde\Delta}{2})}\xi^{\tilde\Delta}\,{}_2F_1\big(\tfrac{\tilde\Delta}{2},\tfrac{\tilde\Delta}{2}+1;\tfrac{\tilde\Delta}{2}+1;\xi^2\big) \tag{B.19}$$

where $\tilde\Delta = d-1$ and $\xi(x,x') = (\cosh d(x,x'))^{-1}$ and $d(x,x')$ is the geodesic distance between x and x'. In our coordinates, we have explicitly

$$\xi(x,x') = \sqrt{1+r^2}\sqrt{1+(r')^2} + rr'\,e\cdot e' \ , \tag{B.20}$$

where e, e' are unit vectors in $\mathbb{R}^d$ parameterizing S^{d-1}.

Note that (B.17) is not a factorized solution since the ρ in the exponent of the right hand side acts as an operator on ψ_ϕ and this forces correlations between the matter fields and the longitudinal part of the gauge field.

B.1.3 Split states in QED

Although the solution obtained above is not factorized, it is still possible to find split states. A simple example is obtained by taking two wavefunctionals $\psi_\phi^{(1)}[\phi]$ and $\psi_\phi^{(2)}[\phi]$ that are both eigenfunctions of the charge operator ρ:

$$\rho\, \psi_\phi^{(1)}[\phi] = \rho_1 \psi_\phi^{(1)}[\phi], \qquad \rho\, \psi_\phi^{(2)}[\phi] = \rho_2 \psi_\phi^{(2)}[\phi] \ . \tag{B.21}$$

Consider the case where the eigenfunctions ρ_1 and ρ_2 are both spherically symmetric, vanish outside a ball of finite radius B_R centered at $r = 0$ but differ inside the ball. The fact that states of the form (B.21) exist relies crucially on the fact that the charge density can be specified independently at each point in space by (B.12) and also on the fact that for ordinary matter fields it is possible to construct split wavefunctionals that agree outside a bounded region but differ inside [53].

If we impose the condition that

$$\int_{B_R} d^d x \sqrt{\gamma}\, \rho_1 = \int_{B_R} d^d x \sqrt{\gamma}\, \rho_2 \ , \tag{B.22}$$

then we see that the wavefunctionals

$$\psi^{(1)}[A, \phi] = \exp\left[i \int_\Sigma d^d x \sqrt{\gamma} \int_\Sigma d^d x' \sqrt{\gamma'}\, A_i^{\mathrm{L}}(x) \nabla^i G(x, x') \rho(x') \right] \psi_A[A^{\mathrm{T}}] \psi_\phi^{(1)}[\phi] \tag{B.23}$$

and

$$\psi^{(2)}[A, \phi] = \exp\left[i \int_\Sigma d^d x \sqrt{\gamma} \int_\Sigma d^d x' \sqrt{\gamma'}\, A_i^{\mathrm{L}}(x) \nabla^i G(x, x') \rho(x') \right] \psi_A[A^{\mathrm{T}}] \psi_\phi^{(2)}[\phi] \tag{B.24}$$

solve the constraints for an arbitrary choice of $\psi_A[A^{\mathrm{T}}]$, are identical outside B_R but differ inside. Note that we have used the fact that the electric field produced by ρ_1 and ρ_2, which enters in the exponents above, agrees outside B_R by spherical symmetry and equality of the total charge but differs inside.

Another example of a split state is obtained by simply taking two wavefunctionals $\psi^{(1)}[A^{\rm L}, A^{\rm T}]$ and $\psi^{(2)}[A^{\rm L}, A^{\rm T}]$ that are eigenstates of $\Pi_{\rm em,L}$ with different eigenvalues

$$\nabla_i \Pi^i_{\rm em,L}\, \psi^{(1)}[A^{\rm L}, A^{\rm T}] = \rho_1\, \psi^{(1)}[A^{\rm L}, A^{\rm T}]; \qquad \nabla_i \Pi^i_{\rm em,L}\, \psi^{(2)}[A^{\rm L}, A^{\rm T}] = \rho_2\, \psi^{(2)}[A^{\rm L}, A^{\rm T}]\,. \tag{B.25}$$

Unlike the example above, ρ_1 and ρ_2 do not need to be spherically symmetric in this case but we again demand that they differ inside a ball B_R but agree outside. We can then simply choose two matter wavefunctionals that satisfy (B.21) and we see that the wavefunctionals

$$\psi^{(1)}[A^{\rm L}, A^{\rm T}]\psi^{(1)}[\phi] \qquad \text{and} \qquad \psi^{(2)}[A^{\rm L}, A^{\rm T}]\psi^{(2)}[\phi] \tag{B.26}$$

differ inside B_R but agree outside.

B.1.4 Difference between QED and gravity

From a technical perspective what allows us to construct split states in QED is the relation (B.12). Unlike the charge density, the energy density cannot be independently specified at each spacetime point. This is because the commutator of the stress tensor with itself leads to the so-called Schwinger terms [169]. For example, in a lattice regularization, the stress tensor at one lattice point does not commute with the stress tensor at adjacent lattice points.

The significance of this difference can be seen by considering the global AdS vacuum. Here the specification of the total energy completely fixes the state in the bulk and so it is clear that once the integral of the stress tensor has been specified and set to vanish, there is no freedom to specify it arbitrarily in different parts of space. In contrast, specifying the integral of the charge density leaves an infinite ambiguity in the local charge density.

There is a more physical way to understand the difference between gravity and nongravitational gauge theories. In gravity, the "charge" is the energy but, by the Heisenberg uncertainty principle, an excitation with a fixed total energy must be delocalized. There is no similar principle for excitations of the electric charges or other gauge charges. This is why it is possible to find split states in ordinary gauge theories, which localize information much like other local quantum field theories, but impossible to find split states in gravity.

B.2 Graviton modes in global AdS

We verify here that the eigenvalue problem (4.99) coming from the Wheeler-DeWitt equation corresponds to graviton modes in AdS_{d+1}. We then provide an explicit solution and compute the frequencies ω_n in global AdS_4.

B.2.1 Graviton eigenvalue problem

To relate graviton modes to the analysis of section 4.5.1, we should write the linearized Einstein equation in global AdS_{d+1} in terms of d-dimensional quantities on the slice Σ. We use hats for spacetime quantities to distinguish them from slice quantities. The background metric is taken to be

$$ds^2 = \hat{\gamma}_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} dx^i dx^j \tag{B.27}$$

and the perturbation is

$$\hat{h}_{\mu\nu} dx^\mu dx^\nu = \hat{h}_{tt} dt^2 + 2\hat{h}_{ti} dt dx^i + h_{ij} dx^i dx^j \ , \tag{B.28}$$

which we take to be transverse and traceless

$$\hat{\nabla}_\mu \hat{h}^{\mu\nu} = 0 \ , \qquad \hat{h}_\mu{}^\mu = 0 \ . \tag{B.29}$$

This is known as the generalized de Donder gauge.

The linearized equation of motion can be obtained by expanding the Einstein-Hilbert action to quadratic order [170]

$$S = \frac{1}{16\pi G} \int dt d^d x \sqrt{-\hat{g}} \, (\hat{R} - 2\Lambda) \ . \tag{B.30}$$

This leads to the linearized equation

$$(\hat{\Box} + 2)\hat{h}_{\mu\nu} = 0 \ . \tag{B.31}$$

To write this in terms of slice quantities, we use that the non-zero Christoffel symbols of the background are

$$\hat{\Gamma}^i_{tt} = N\partial_i N, \qquad \hat{\Gamma}^t_{it} = N^{-1}\partial_i N, \qquad \hat{\Gamma}^k_{ij} = \Gamma^k_{ij} \ , \tag{B.32}$$

and a tedious but straightforward computation gives

$$(\hat{\Box} + 2)\hat{h}_{ij} = -N^{-2}\partial_t^2 h_{ij} + (\Delta_N + 2) h_{ij} \ , \tag{B.33}$$

where the Laplace-type operator Δ_N defined in (4.70) appears. The equations $(\hat{\Box}+2)\hat{h}_{ti}=0$ can be used to fix the components $\hat{h}_{ti}$ and one can check that $(\hat{\Box}+2)\hat{h}_{tt}=0$ is then automatically satisfied.

The frequencies ω_n of the graviton modes can be defined by the eigenvalue equation $i\partial_t h_{ij}^{(n)} = \omega_n h_{ij}^{(n)}$, and we see that (B.33) indeed reduces to (4.99).

B.2.2 Graviton spectrum in AdS_4

For completeness, we give here a derivation of the graviton frequencies ω_n in the case of global AdS_4. The background metric is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ . \tag{B.34}$$

An efficient method to obtain the graviton spectrum in AdS_4 is to make use of the Teukolsky equation [171]. We start by defining a Newman-Penrose tetrad [172] which here takes the form

$$l = \frac{1}{1+r^2}\partial_t + \partial_r \ , \qquad n = \frac{1}{2}(\partial_t - (1+r^2)\partial_r) \ , \tag{B.35}$$

$$m = \frac{1}{\sqrt{2}r}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \ , \qquad \bar{m} = \frac{1}{\sqrt{2}r}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right) \ , \tag{B.36}$$

and satisfies

$$g_{\mu\nu} = -l_\mu n_\nu - l_\nu n_\mu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu \ . \tag{B.37}$$

It consists of null vectors which are all orthogonal to each other except for $l\cdot n = -1$ and $m \cdot \bar{m} = 1$.

The Teukolsky equation can be written for any type D spacetime using the Newman-Penrose formalism. For global AdS_4, it takes the form

$$\begin{aligned} 0 = {} & \frac{r^2}{1+r^2}\partial_t^2\Psi_\eta - r^2(1+r^2)\partial_r^2\Psi_\eta - 2(1+2\eta)\partial_r\Psi_\eta + \eta\frac{4r}{1+r^2}\partial_t\Psi_\eta - \frac{1}{\sin^2\theta}\partial_\phi^2\Psi_\eta \\ & - \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta\Psi_\eta) - 4i\eta\frac{\cos\theta}{\sin^2\theta}\partial_\phi\Psi_\eta - 2\left(3r^2(3+2\eta) + 2 + \eta - \frac{2}{\sin^2\theta}\right)\Psi_\eta \end{aligned} \tag{B.38}$$

where $\eta = \pm 1$ corresponds to the two polarizations.[1]

[1] This equation can also be obtained by taking the $M = a = 0$ limit of the Kerr-AdS analysis of [173].

We can consider a separated ansatz

$$\Psi_\eta(t,r,\theta,\phi) = e^{-i\omega t + im\phi} R_\eta(r) S_\eta(\theta) \ , \tag{B.39}$$

and the master equation reduces to two coupled ODEs. The equation for $S(\theta)$ can be written using the variable $x = \cos\theta$ as

$$\partial_x((1-x^2)\partial_x S) + \left(\lambda + s - \frac{(m+sx)^2}{1-x^2}\right) S = 0 \tag{B.40}$$

and corresponds to spin-weighted spherical harmonics of spin s [174, 175]. It is well-known that the corresponding eigenvalues are

$$\lambda = \ell(\ell+1) - s(s+1) \ , \qquad \ell = |s|, |s|+1, |s|+2, \dots \tag{B.41}$$

with azimuthal number degeneracies $m = -\ell, -\ell+1, \dots, \ell-1, \ell$. The eigenvalue λ enters in the radial equation which takes the form

$$\begin{aligned} 0 \ = \ & R''(r) + \frac{2(1+2\eta)(1+2r^2)}{r(1+r^2)} R'(r) \\ & + \frac{1}{r^2(1+r^2)^2} \left((1+r^2)(4+2\eta+6r^2(3+2\eta) - \ell(\ell+1)) + \omega^2 r^2 + 4i\eta\omega r \right) R(r) \ . \end{aligned} \tag{B.42}$$

For each polarization $\eta = \pm 1$, the solutions are given in terms of hypergeometric functions. Imposing regularity at the origin $r = 0$ selects one of the two solutions. Imposing normalizability at $r = \infty$ makes the spectrum discrete, with frequencies

$$\omega_{\ell,n} = \ell + n + 1 \ , \qquad n \in \mathbb{Z}_{\geq 0} \ , \tag{B.43}$$

for each polarization. For fixed ℓ and n, the degeneracy of $\omega_{\ell,n}$ is $2(2\ell+1)$ coming from the two polarizations and the $2\ell+1$ values of the azimuthal quantum number m.

B.3 Gravitational energy in AdS

In this appendix, we compare the boundary Hamiltonian (4.38) with various expressions for the gravitational energy in AdS.

B.3.1 Hawking-Horowitz prescription

A formula of the gravitational energy in AdS was obtained in [176]. For linearized perturbation, this takes the form

$$H_\partial^{\mathrm{HH}} = \frac{1}{2\kappa}\int_{\partial\Sigma} d^{d-1}\Omega\, N n^i \nabla^j (h_{ij} - h\gamma_{ij})\ . \tag{B.44}$$

This was derived assuming in the gauge $h_{ij}|_{\partial\Sigma} = 0$. It is easy to see that (4.38) reduces to (B.44) under this gauge condition.

B.3.2 Holographic energy

In the context of AdS/CFT, a notion of holographic energy was defined in [49]. To compare, we will write (4.38) in Fefferman-Graham (FG) gauge. We define a new radial coordinate ρ in which the global AdS_{d+1} metric (B.34) takes the form

$$\hat{\gamma}_{\mu\nu} dx^\mu dx^\nu = -\frac{(4\rho^2+1)^2}{16\rho^2} dt^2 + \frac{d\rho^2}{\rho^2} + \frac{(4\rho^2-1)^2}{16\rho^2} d\Omega_{d-1} \tag{B.45}$$

and we assume that the perturbation satisfies $\hat{h}_{\rho\mu} = 0$. The prescription of [49] was written in terms of $\hat{h}_{00}$ but [49] also showed that the trace of the perturbation $\hat{\gamma}^{\mu\nu}\hat{h}_{\mu\nu}$ was fixed in terms of a dimension-dependent number. Therefore, in order to compare our expression with [49] it is permissible to replace $\gamma^{ij}h_{ij}$ with $\hat{h}_{00}$ up to a constant that only shifts the zero-point of the energy.

With this substitution, the expression (4.38) for the energy gives

$$H_\partial = \frac{1}{2\kappa}\lim_{\rho\to\infty}\rho^{d-2}\int d^{d-1}\Omega\,(-\rho\,\partial_\rho \hat{h}_{00} + 2\hat{h}_{00}). \tag{B.46}$$

In a large ρ expansion, a normalizable perturbation behaves as

$$\hat{h}_{00}(t,\rho,\Omega) = \rho^{2-d}\hat{h}_{00}^{(d-2)}(t,\Omega) + \ldots \tag{B.47}$$

up to subleading terms. This gives

$$H_\partial = \frac{d}{2\kappa}\int d^{d-1}\Omega\, h_{00}^{(d-2)}, \tag{B.48}$$

which matches the holographic energy of [49].

B.3.3 Iyer-Wald energy

In the covariant phase space formalism [177], the energy is given by the boundary integral of the $(d-1)$-form

$$\boldsymbol{\chi}_\xi(\hat{h}) = \frac{1}{2\kappa}\boldsymbol{\varepsilon}_{\mu\nu}\left(\hat{h}^{\mu\rho}\hat{\nabla}_\rho\xi^\nu - \frac{1}{2}\hat{h}_\rho{}^\rho\hat{\nabla}^\mu\xi^\nu + \xi^\rho\hat{\nabla}^\nu\hat{h}^\mu{}_\rho - \xi^\nu\hat{\nabla}_\rho\hat{h}^{\mu\rho} + \xi^\nu\hat{\nabla}^\mu\hat{h}_\rho{}^\rho\right), \tag{B.49}$$

where $\xi = \partial_t$ and using the notation of [178]. Evaluating this on the slice Σ gives

$$\boldsymbol{\chi}_\xi(\hat{h}) = \frac{1}{2\kappa}\boldsymbol{\varepsilon}_{it}\left(\hat{h}^{i\rho}\hat{\nabla}_\rho\xi^t - \hat{h}^{t\rho}\hat{\nabla}_\rho\xi^i - \frac{1}{2}\hat{h}_\rho{}^\rho(\hat{\nabla}^i\xi^t - \hat{\nabla}^t\xi^i) + \hat{\nabla}^t\hat{h}^i{}_t - \hat{\nabla}^i\hat{h}^t{}_t - \hat{\nabla}_\rho\hat{h}^{i\rho} + \hat{\nabla}^i\hat{h}_\rho{}^\rho\right), \tag{B.50}$$

This can be simplified using

$$\hat{\nabla}_t\xi^t = \hat{\Gamma}^t_{tt} = 0, \qquad \hat{\nabla}_i\xi^i = \hat{\Gamma}^i_{it} = 0, \tag{B.51}$$

$$\hat{\nabla}_t\xi^i = \hat{\Gamma}^i_{tt} = \frac{1}{2}\gamma^{ij}\partial_j(N^2), \qquad \hat{\nabla}_i\xi^t = \hat{\Gamma}^t_{it} = \frac{1}{2N^2}\partial_i(N^2)\ , \tag{B.52}$$

and we obtain

$$\boldsymbol{\chi}_\xi(\hat{h}) = \frac{1}{2\kappa}\boldsymbol{\varepsilon}_i\left(\nabla_j N(h^{ij} - h\gamma^{ij}) - N\nabla_j(h^{ij} - h\gamma^{ij})\right), \tag{B.53}$$

using the relation between the volume forms $\hat{\boldsymbol{\varepsilon}}_{it} = N\boldsymbol{\varepsilon}_i$. This shows that the Iyer-Wald energy matches (4.38):

$$H_\partial = \int_{\partial\Sigma}\boldsymbol{\chi}_\xi(\hat{h})\ . \tag{B.54}$$

In fact, the integrated Hamiltonian constraint (4.88) can be viewed as the quantization of a classical equation which can be expressed in this formalism. For linearized AdS spacetimes, this was detailed in [178, 179]. The result is that the tt component of Einstein's equation gives an identity

$$\int_{\partial\Sigma}\boldsymbol{\chi}_\xi(h) = \int_\Sigma\left(\boldsymbol{\omega}_{\text{grav}}(h, \mathcal{L}_\xi h) + \boldsymbol{\omega}_\phi(\phi, \mathcal{L}_\xi\phi)\right), \tag{B.55}$$

for linearized on-shell perturbations. The LHS is the the energy H_∂ as shown above. The RHS involves the symplectic forms $\boldsymbol{\omega}_{\text{grav}}$ and $\boldsymbol{\omega}_\phi$ associated to gravity and matter and corresponds to a "bulk energy" known as the Hollands-Wald canonical energy [180].

B.4 Leading order solutions

We derive here the leading order solutions to the pointwise Hamiltonian and momentum constraints presented in equation (4.144) and equation (4.146).

B.4.1 Hamiltonian constraint

As explained in section 4.5.2, the second order Hamiltonian constraint takes the form

$$\mathcal{D}^{ij} h^{\mathrm{T}}_{ij} = \kappa\, Q^{(0)}, \tag{B.56}$$

where $\mathcal{D}_{ij}$ is defined in (4.131) and $Q^{(0)}$ is the truncation to leading order of Q which can be written

$$Q^{(0)} = 2\left(\Pi^{ij}_{\mathrm{T}}\Pi^{\mathrm{T}}_{ij} - \frac{1}{d-1}\Pi^2_{\mathrm{T}}\right) + 4\Pi^{\mathrm{TT}}_{ij}\Pi^{ij}_{\mathrm{T}} + Q^{(0)}_0 \tag{B.57}$$

where we have isolated the part with no Π^{T}:

$$\begin{aligned} Q^{(0)}_0 = {} & 2\Pi^{ij}_{\mathrm{TT}}\Pi^{\mathrm{TT}}_{ij} - \frac{1}{8}h^{ij}(\Delta_N + 2)h_{ij} + \frac{1}{4}\left(2h^{ij}\nabla_i\nabla^k h_{jk} + \nabla_i h^{ij}\nabla^k h_{jk}\right) \\ & + \frac{1}{4}N^{-1}\nabla_i L^i + \mathcal{H}^{\mathrm{matter}}\ . \end{aligned} \tag{B.58}$$

It is convenient to define the time variable $\mathbf{t}$ by the equation

$$\Pi^{\mathrm{T}}_{ij} = \mathcal{D}_{ij}\mathbf{t}\ , \tag{B.59}$$

which is explicitly

$$\Pi^{\mathrm{T}}_{ij} = \frac{1}{2}\left(\nabla^i\nabla^j\mathbf{t} - \gamma^{ij}\nabla_k\nabla^k\mathbf{t} + (d-1)\gamma^{ij}\mathbf{t}\right)\ . \tag{B.60}$$

This is the generalization to AdS of the time variable used in [82, 81]. Taking the trace we see that

$$\gamma^{ij}\Pi^{\mathrm{T}}_{ij} = \frac{d-1}{2}(-\Delta + d)\mathbf{t}\ , \tag{B.61}$$

so the relation with α in (4.84) is:

$$\mathbf{t} = -\frac{2}{d-1}N\alpha\ . \tag{B.62}$$

An identity that will prove important is

$$N\alpha_{ij} = \nabla_i \alpha_j + \nabla_j \alpha_i \tag{B.63}$$

where we have

$$\alpha_i = \frac{1}{2} N^2 \nabla_i \alpha = -\frac{1}{(d-1)} (N \nabla_i \mathbf{t} - \nabla_i N \mathbf{t}) \ . \tag{B.64}$$

This allows us to solve the Hamiltonian constraint at leading order. We have

$$\begin{aligned} \delta\Psi &= \int d^d x \, \frac{\delta \Psi}{\delta \Pi^{ij}_{\mathrm{T}}(x)} \, \delta \Pi^{ij}_{\mathrm{T}}(x) \\ &= i \int d^d x \sqrt{\gamma} \, \mathcal{D}^{ij} h^{\mathrm{T}}_{ij} \Psi \, \delta \mathbf{t} \end{aligned} \tag{B.65}$$

using (B.59) and integration by parts. Hence,

$$-\frac{i}{\sqrt{\gamma}} \frac{\delta \Psi}{\delta \mathbf{t}} = \mathcal{D}^{ij} h^{\mathrm{T}}_{ij} \ , \tag{B.66}$$

and the constraint can be written

$$\left(-\frac{i}{\sqrt{\gamma}} \frac{\delta}{\delta \mathbf{t}} - \kappa \, Q^{(0)} \right) \Psi = 0 \ . \tag{B.67}$$

We can write the solution in the form

$$\Psi[\mathbf{t}, h^{\mathrm{TT}}, h^{\mathrm{L}}, \phi] = \exp\left(i \kappa \mathcal{P}\right) \Psi_0[h^{\mathrm{TT}}, h^{\mathrm{L}}, \phi] + \mathrm{O}\big(\kappa^2\big) \,, \tag{B.68}$$

where $\mathcal{P}$ needs to satisfy

$$\frac{1}{\sqrt{\gamma}} \frac{\delta \mathcal{P}}{\delta \mathbf{t}(x)} = Q^{(0)} \ . \tag{B.69}$$

The solution can be found and takes a remarkably simple form:

$$\mathcal{P} = \int d^d x \sqrt{\gamma} \left(-\frac{2}{3} \, \mathbf{t} \left(\Pi^{ij}_{\mathrm{T}} \Pi^{\mathrm{T}}_{ij} - \frac{1}{d-1} \Pi^2_{\mathrm{T}} \right) + 2 \, \mathbf{t} \, \Pi^{ij}_{\mathrm{TT}} \Pi^{ij}_{\mathrm{T}} + \mathbf{t} \, Q^{(0)}_0 \right) . \tag{B.70}$$

The first term is cubic in $\mathbf{t}$ and the second term is quadratic in $\mathbf{t}$. We will now check that differentiating these terms with respect to $\mathbf{t}$ gives (B.57).

Cubic term. Let's consider the cubic term, allowing each entries to be different:

$$\mathcal{P}^{(3)}[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3] = \frac{2}{3}\int d^dx\sqrt{\gamma}\,\mathbf{t}_1\left(\Pi_{\mathrm{T}}^{ij}[\mathbf{t_2}]\Pi_{ij}^{\mathrm{T}}[\mathbf{t_3}] - \frac{1}{d-1}\Pi_{\mathrm{T}}[\mathbf{t_2}]\Pi_{\mathrm{T}}[\mathbf{t_3}]\right), \tag{B.71}$$

We want to show that

$$\frac{1}{\sqrt{\gamma}}\frac{\delta}{\delta\mathbf{t}}\mathcal{P}^{(3)}[\mathbf{t},\mathbf{t},\mathbf{t}] = 2\left(\Pi_{\mathrm{T}}^{ij}\Pi_{ij}^{\mathrm{T}} - \frac{1}{d-1}\Pi_{\mathrm{T}}^2\right). \tag{B.72}$$

Since the derivative with respect to $\mathbf{t}_1$ gives one third of the RHS, we just need $\mathcal{P}^{(3)}[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3]$ to be invariant under permutation of its arguments. As $\mathcal{P}^{(3)}[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3]$ is manifestly invariant under $\mathbf{t_2} \leftrightarrow \mathbf{t_3}$, we just need to show that it's also invariant under $\mathbf{t_1} \leftrightarrow \mathbf{t_3}$.

First, we note that the combination gives

$$\Pi_{\mathrm{T}}^{ij}\Pi_{ij}^{\mathrm{T}} - \frac{1}{d-1}\Pi_{\mathrm{T}}^2 = -\frac{1}{d-1}\Pi_{ij}^{\mathrm{T}}\alpha^{ij}\ , \tag{B.73}$$

so that we have

$$\mathcal{P}^{(3)}[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3] = -\frac{2}{3(d-1)}\int d^dx\sqrt{\gamma}\,\mathbf{t}_1\,\Pi_{\mathrm{T}}^{ij}[\mathbf{t_2}]\alpha_{ij}[\mathbf{t_3}]\ . \tag{B.74}$$

We now use the identity (B.63) and integration by parts:

$$\begin{aligned}
\mathcal{P}^{(3)}[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3] &= -\frac{4}{3(d-1)}\int d^dx\sqrt{\gamma}\,N^{-1}\,\mathbf{t}_1\,\Pi_{\mathrm{T}}^{ij}[\mathbf{t_2}]\nabla_i\alpha_j[\mathbf{t}_3] \\
&= \frac{4}{3(d-1)^2}\int d^dx\sqrt{\gamma}\,N^{-1}\,\mathbf{t}_1\,\Pi_{\mathrm{T}}^{ij}[\mathbf{t_2}]\nabla_i(N\nabla_j\mathbf{t}_3 - \nabla_jN\mathbf{t}_3) \\
&= \frac{4}{3(d-1)^2}\int d^dx\sqrt{\gamma}\,\Pi_{\mathrm{T}}^{ij}[\mathbf{t_2}]\left(-\nabla_i\mathbf{t}_1\nabla_j\mathbf{t_3} - \gamma_{ij}\mathbf{t}_1\mathbf{t}_3\right).
\end{aligned} \tag{B.75}$$

This is manifestly symmetric under $\mathbf{t_1} \leftrightarrow \mathbf{t_3}$. Hence, $\mathcal{P}^{(3)}[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3]$ is invariant under permutation of its arguments and (B.72) is satisfied.

Quadratic term. Similarly, we introduce the quantity

$$\mathcal{P}^{(2)}[\mathbf{t}_1, \mathbf{t}_2] = 2\int d^dx\sqrt{\gamma}\,\mathbf{t}_1\,\Pi_{\mathrm{TT}}^{ij}\Pi_{\mathrm{T}}^{ij}[\mathbf{t}_2]\ . \tag{B.76}$$

We want to show that

$$\frac{1}{\sqrt{\gamma}}\frac{\delta}{\delta\mathbf{t}}\mathcal{P}^{(2)}[\mathbf{t},\mathbf{t}] = 4\,\Pi^{ij}_{\mathrm{TT}}\Pi^{\mathrm{T}}_{ij}\ . \tag{B.77}$$

This would follow if $\mathcal{P}^{(2)}[\mathbf{t}_1,\mathbf{t}_2]$ is invariant under $\mathbf{t}_1 \leftrightarrow \mathbf{t}_2$. We can write

$$\begin{aligned}\mathcal{P}^{(2)}[\mathbf{t}_1,\mathbf{t}_2] &= 2\int d^dx\sqrt{\gamma}\,\mathbf{t}_1\,\Pi^{ij}_{\mathrm{TT}}\mathcal{D}_{ij}\mathbf{t}_2,\\ &= -\int d^dx\sqrt{\gamma}\,\Pi^{ij}_{\mathrm{TT}}(\nabla^i\,\mathbf{t}_1\nabla^j\mathbf{t}_2 - \gamma^{ij}\nabla_k\,\mathbf{t}_1\nabla^k\mathbf{t}_2)\ ,\end{aligned} \tag{B.78}$$

which is manifestly invariant under $\mathbf{t}_1 \leftrightarrow \mathbf{t}_2$.

This shows that (B.70) is indeed the solution.

B.4.2 Momentum constraint

The leading order constraint is

$$\frac{2}{\kappa}\gamma_{ij}\nabla_k\Pi^{jk}_{\mathrm{L}} = Q^{(0)}_i\ . \tag{B.79}$$

What plays the role of $\mathbf{t}$ here is the vector ϵ_i defined as

$$h^{\mathrm{L}}_{ij} = \nabla_i\epsilon_j + \nabla_j\epsilon_i\ . \tag{B.80}$$

We then have

$$\begin{aligned}\delta\Psi &= \int d^dx\frac{\delta\Psi}{\delta h^{\mathrm{L}}_{ij}(x)}\delta h^{\mathrm{L}}_{ij}(x)\\ &= -2i\int d^dx\sqrt{\gamma}\,\nabla_i\Pi^{ij}_{\mathrm{L}}\Psi\,\delta\epsilon_j\ ,\end{aligned} \tag{B.81}$$

which allows to write the momentum constraint as

$$\left(-\frac{i}{\sqrt{\gamma}}\frac{\delta}{\delta\epsilon_i} - \kappa\,Q^{(0)}_i\right)\Psi[h,\phi] = 0\ , \tag{B.82}$$

where $Q^{(0)}_i$ is the leading order truncation of Q_i which takes the form

$$Q^{(0)}_i = (\nabla_i h_{jk} - 2\nabla_k h_{ij})(\Pi^{jk}_{\mathrm{T}} + \Pi^{jk}_{\mathrm{TT}}) + \mathcal{H}^{\mathrm{matter}}_i\ . \tag{B.83}$$

This can be written in terms of ϵ^i as

$$Q_i^{(0)} = \big(2(R_{k\ell ij}\epsilon^\ell - \nabla_k\nabla_j\epsilon_i) + \nabla_i h^{\rm TT}_{jk} - 2\nabla_k h^{\rm TT}_{ij}\big)\,(\Pi^{jk}_{\rm T} + \Pi^{jk}_{\rm TT}) + \mathcal{H}_i^{\rm matter}\,. \tag{B.84}$$

To properly define this operator, we should adopt the same normal ordering prescription as in section 4.5.1.

As above, we can write the solution as

$$\Psi[h^{\rm TT}, h^{\rm T}, h^{\rm L}, \phi] = \exp\left(i\kappa\mathcal{R}\right)\Psi_0[h^{\rm TT}, h^{\rm T}, \phi] + {\rm O}\big(\kappa^2\big)\,, \tag{B.85}$$

where Ψ_0 is an arbitrary functional. We need to have

$$\frac{1}{\sqrt{\gamma}}\frac{\delta}{\delta\epsilon^i}\mathcal{R} = -\,Q_i^{(0)}\;. \tag{B.86}$$

The solution can be found explicitly to be

$$\mathcal{R} = -\int d^dx\sqrt{\gamma}\left(R_{ijk\ell}\epsilon^i\epsilon^\ell - \epsilon^i\nabla_k\nabla_j\epsilon_i + \epsilon^i\nabla_i h^{\rm TT}_{jk} - 2\epsilon^i\nabla_k h^{\rm TT}_{ij})(\Pi^{jk}_{\rm T} + \Pi^{jk}_{\rm TT}) + \epsilon^i\mathcal{H}_i^{\rm matter}\right). \tag{B.87}$$

As above, we can prove that this is a solution by showing that the term quadratic in ϵ^i is symmetric in its two entries. This follows from integration by parts.

B.4.3 Solutions to both constraints

We have presented above general perturbative solutions of the Hamiltonian and momentum constraint independently. Here, we will give solutions to both constraints.

The solutions found above must be compatible with each other. This implies that the "interaction" part of $\mathcal{P}$ and $\mathcal{R}$, involving products of $\mathbf{t}$ and ϵ_i must be exactly the same. We will use below the subscript "int" to denote this part. It is a rather non-trivial consistency check to verify this.

From the solution of the Hamiltonian constraint, we have

$$\mathcal{P}_{\rm int} = \frac{1}{2}\int d^dx\sqrt{\gamma}\,\nabla_i M^i\mathbf{t} \tag{B.88}$$

using the results of section 4.4.4. In particular, M^i is defined in (4.78). From the momentum constraint, this term is

$$\begin{aligned}\mathcal{R}_{\text{int}} &= -\int d^dx\sqrt{\gamma}\left(2(R_{k\ell ij}\epsilon^\ell - \nabla_k\nabla_j\epsilon_i) + \nabla_i h^{\text{TT}}_{jk} - 2\nabla_k h^{\text{TT}}_{ij}\right)\Pi^{jk}_{\text{T}} \\ &= \int d^dx\sqrt{\gamma}\, D^{jk}\left(2(R_{k\ell ij}\epsilon^\ell - \nabla_k\nabla_j\epsilon_i) + \nabla_i h^{\text{TT}}_{jk} - 2\nabla_k h^{\text{TT}}_{ij}\right)\mathbf{t}\end{aligned} \tag{B.89}$$

using integration by parts. Consistency of our solutions then requires that $\mathcal{P}_{\text{int}} = \mathcal{R}_{\text{int}}$ which is explicitly

$$D^{jk}\left(R_{ijk\ell}\epsilon^i\epsilon^\ell + \nabla_k\epsilon^i\nabla_j\epsilon_i + \epsilon^i\nabla_i h^{\text{TT}}_{jk} - 2\epsilon^i\nabla_k h^{\text{TT}}_{ij}\right) = \frac{1}{2}\nabla_i M^i \;. \tag{B.90}$$

This is a rather non-trivial identity since the LHS comes from the expansion of the momentum constraint while the RHS comes from the expansion of the Hamiltonian constraint. We have checked that this identity indeed holds, see the associated Mathematica notebook [122].

Finally, we can write the leading order solution to both constraints as

$$\Psi[\Pi^{\text{T}}, h^{\text{TT}}, h^{\text{L}}, \phi] = \exp(i\kappa\mathcal{S})\psi[h^{\text{TT}}, \phi] + \text{O}\big(\kappa^2\big)\,, \tag{B.91}$$

where ψ is an arbitrary functional and

$$\mathcal{S} = \int d^dx\sqrt{\gamma}\left(-\frac{2}{3}\,\mathbf{t}\left(\Pi^{ij}_{\text{T}}\Pi^{\text{T}}_{ij} - \frac{1}{d-1}\Pi^2_{\text{T}}\right) + 2\,\mathbf{t}\,\Pi^{ij}_{\text{TT}}\Pi^{ij}_{\text{T}} + Q^{(0)}_0\mathbf{t} - \epsilon^i\mathcal{H}^{\text{matter}}_i\right), \tag{B.92}$$

with $Q^{(0)}_0$ given in (B.58).

Appendix C

Derivations of physical quantities at null infinity in 6-dimensions

C.1 Metric Inverse and Traces

In this appendix we expand upon the computation of the metric inverse and the trace of various components in the metric. In the linearized theory this computation is fairly trivial but in the non-linear theory, it becomes slightly convoluted. The inverse metric g^{ab} corresponding to the metric (5.2), is given as

$$g^{ab} g_{bc} = \delta^a_c. \tag{C.1}$$

We will expand the LHS order by order in r and then evaluate the r-expansion of the metric g^{ab}. Since g_{bc}'s leading order term is r^2, we expect that the leading order term of g^{ab} will be $1/r^2$. Therefore, let us take the expansion of g^{ab} as,

$$g^{ab} = \sum_{n=2}^{\infty} \frac{g^{(n)ab}}{r^n} \tag{C.2}$$

and contract this with

$$g_{bc} = r^2 \delta_{bc} + r C_{bc} + D_{bc} + \frac{E_{bc}}{r} + \frac{F_{bc}}{r^2} + \cdots \tag{C.3}$$

Noting that the indices of $g^{(n)}_{ab}$ are lowered and raised with γ_{ab} we get the values of $g^{(n)}_{ab}$

$$g^{(2)ab} = \delta^{ab}, \tag{C.4}$$
$$g^{(3)ab} = -C^{ab}, \tag{C.5}$$
$$g^{(4)ab} = C^a_c C^{bc} - D^{ab}, \tag{C.6}$$
$$g^{(5)ab} = C^{ac} D^b_c + D^{ac} C^b_c - C^a_m C^{mn} C^b_n - E^{ab}, \tag{C.7}$$
$$\begin{aligned} g^{(6)ab} = &-F^{ab} + C^{ac}E^b_c + C^{bc}E^a_c + D^{ac}D^b_c + C^a_m C^{mn} C^c_n C^b_c \\ &- (D^b_c C^a_m C^{cm} + D^c_m C^{am} C^b_c + D^a_c C^b_m C^{cm}). \end{aligned} \tag{C.8}$$

Trace of $g^{(n)}_{ab}$

We will now demonstrate how the Bondi gauge condition (5.3) fixes the traces of $g^{(n)}_{ab}$. The condition we have is $\det(\frac{g_{AB}}{r^2}) = \det(\delta_{ab})$. This condition results in,

$$\begin{aligned} \det\left(\frac{g_{ab}}{r^2}\right) &= \det\left(\delta_{ab} + \frac{C_{ab}}{r} + \frac{D_{ab}}{r^2} + \frac{E_{ab}}{r^3} + \frac{F_{ab}}{r^4}\right) \\ &= \det(\delta_{ab}) \times \exp \mathrm{Tr} \log\left(\delta^c_b + \frac{C^c_b}{r} + \frac{D^c_b}{r^2} + \frac{E^c_b}{r^3} + \frac{F^c_b}{r^3}\right) \end{aligned} \tag{C.9}$$

From the gauge condition (5.3) we see that we need,

$$\mathrm{Tr} \log\left(\delta^c_b + \frac{C^c_b}{r} + \frac{D^c_b}{r^2} + \frac{E^c_b}{r^3} + \frac{F^c_b}{r^3}\right) = 0. \tag{C.10}$$

In order to simplify the term inside the Tr we note that,

$$\begin{aligned} &\log\left(\delta^c_b + \frac{C^c_b}{r} + \frac{D^c_b}{r^2} + \frac{E^c_b}{r^3} + \frac{F^c_b}{r^4}\right) \\ &= \frac{C^c_b}{r} + \frac{D^c_b - \frac{1}{2}C^m_b C^c_m}{r^2} + \frac{E^c_b - C^m_b D^c_m + \frac{1}{3}C^m_b C^n_m C^c_n}{r^3} \\ &\quad + \frac{F^c_b - C^{ab}E_{ab} - \frac{1}{2}D^{ab}D_{ab} + C^{am}C_{bm}D^b_a - \frac{1}{4}C^a_b C^b_c C^c_d C^d_a}{r^4} + O(1/r^5) \end{aligned} \tag{C.11}$$

Taking the trace of this equation and using (C.10) then gives us (5.7). Although here we have shown how we can fix the traces of $g^{(n)}_{ab}$ for $n \leq 2$, this procedure is applicable $\forall\ n$.

C.2 Equations of motion

We eventually want to express everything in terms of free data, which, as argued in sec. 5.2 are C_{ab} and $\tilde{D}_{ab}$. Therefore, using the Einstein equations we represent everything else in terms of those variables. We solve the Einstein equations in flat spacetime $R_{\mu\nu} = 0$ (where the specific fall off conditions are mentioned in (5.5)) and determine the components of the metric in terms of the free data. The ones which are important for us in this analysis are R_{rr}, R_{ra}, R_{ur}, R_{ab}, R_{uu}. We solve them order by order in r in the large$-r$ limit.

From the component R_{ur} we have the following:

$$M^{(1)} = 0, \tag{C.12a}$$

$$-M^{(2)} = \frac{1}{2}\partial^a U_a^{(1)} + \partial^2\beta^{(2)} + U^{(0)2} \ . \tag{C.12b}$$

The next order equation does not yield us any non-trivial equation for $M^{(3)}$. For that one, we will have to work with the R_{uu} equation which is like the Hamiltonian constraint.

From the R_{ra} component we obtain,

$$U_a^{(0)} = -\frac{1}{6}\partial^b C_{ab}, \tag{C.13a}$$

$$3U_a^{(1)} = -\partial_b D_a^b + C_{ab}U^{(0)b} + \frac{1}{2}\partial_b(C^{bm}C_{am}) + 6\partial_a\beta^{(2)} + \frac{1}{8}\partial_a(C^{bc}C_{bc}). \tag{C.13b}$$

We can also explicitly write down the value of $U^{(2)a}$ but we will not be needing that for any specific calculation. All we need is the basic structure of $\partial_u U^{(2)a}$ for the computation of the Bondi mass evolution equation and we shall quote that here

$$2\partial_u U^{(2)a} = -\frac{3}{2}\partial_u\partial_b E^{ab} + \partial_u\mathcal{U}(C,D) \tag{C.14}$$

where $\mathcal{U}(C,D)$ is a bi-linear in C_{ab} and D_{ab}. Using the fall off condition (5.27) and the equation of motion for E_{ab} (see equation (5.44)), it is simple to see that $\int_{-\infty}^{\infty} du\ \partial_u\partial_a U^{(2)a} = 0$ and hence this does not contribute to the integrals in (5.38).

From the R_{rr} equation we obtain,

$$\beta^{(2)} = -\frac{1}{64}C^{ab}C_{ab},$$
$$\beta^{(3)} = \frac{1}{48}(C^{ab}C_{bc}C^{c}_{a} - 2C^{ab}D_{ab}), \tag{C.15a}$$
$$\beta^{(4)} = \frac{1}{64}\Big[5C^{a}_{b}C^{m}_{a}D^{b}_{m} - 2C^{a}_{b}C^{b}_{c}C^{c}_{d}C^{d}_{a} - 3C^{ab}E_{ab} - D^{ab}D_{ab}\Big].$$

In the linearized theory about $C_{ab} = 0$ it is clear that $\beta = 0$ [22, 26]. The equation of motion for R_{ab} and R_{uu} are discussed in the main text. To summarize them, the leading order non-trivial equation for R_{ab} implies that $\partial_u C_{ab} = 0$ and the next non-trivial equation (at $O(1/r^2)$) gives us the value of $\partial_u E_{ab}$ (see equation (5.44)). There does not exist an equation of motion for $\partial_u D_{ab}$ implying that is the free data of the theory. The leading non-trivial EOM for R_{uu} gives the time evolution of the Bondi mass (see (5.35)).

C.3 Memory in linearized gravity

We can derive the memory in the linearized theory by using the formulas in [79]. Even though the results in there are derived in the Harmonic gauge, the final answer is shown to match with ours (in the linearized limit), which reflects the fact that memory is gauge invariant (refer to section 5.4). Note that the results in [79] are derived on the compact celestial sphere $\mathbb{S}^4$, whose curvature contributes to the final answer. To avoid a confusion with notations, we use the $\gamma_{\mathbb{S}}^{AB}$ and $\mathcal{D}_{\mathbb{S}}$ to denote the metric and the derivative on the compact sphere. The notations for the important metric fluctuations in [79] are related to ours as,

$$\boldsymbol{h}^{(0)}_{AB} \equiv D_{ab}, \qquad \boldsymbol{h}^{(1)}_{AB} \equiv E_{ab}$$

and the relative transverse displacement in the linearized case is denoted as s_{lin}. Therefore, using equation 3.4 and 3.5 [79] of we get,

$$\Delta s^{A}_{lin} = \frac{\gamma_{\mathbb{S}}^{AC}}{2r^3}\Delta \boldsymbol{h}^{(1)}_{CB} s^{B}_{lin,i} \tag{C.16}$$

Next we consider their equation 4.17 which gives (here the factor of -4 appears due to the curvature of the sphere),

$$\partial_u \boldsymbol{h}^{(1)}_{BC} = -\frac{1}{2}(D^2_{\mathbb{S}} - 4)\boldsymbol{h}^{(0)}_{BC} \tag{C.17}$$

Substituting this in (C.16), we get,

$$\Delta s^A_{lin} = -\frac{1}{4r^3}(D^2_{\mathbb{S}} - 4)\int du\ \boldsymbol{h}^{(0)A}_{B} s^B_{lin,i} \tag{C.18}$$

Since $\boldsymbol{h}^{(0)AB} \equiv D^{ab}$ this expression can be obtained from (5.46) in the linearized limit and upon taking care of the curvature of the sphere.

C.4 Lessons from QED Symplectic Form

We demonstrate a detailed computation of the symplectic form in source-free QED as a toy model, and also describe how there are similar issues which crop up in gravity. Note that the behavior of linearized gravity is exactly similar to QED and hence we should expect the QED analysis to behave similarly to the linearized GR analysis done in [26], with the small difference that we are working with $\gamma_{ab} = \delta_{ab}$. We encourage the reader to look at [155, 156] for a more mathematically robust treatment of the symplectic form in QED.

Before going to the computation of the symplectic form, let us mention the quantities that parametrize the phase space. The free data here is $A^{(0)}_i$ and $A^{(1)}_i$ and these are similar to C_{ab} and D_{ab} in gravity, as discussed in section 5.2. This can be shown by analyzing the Maxwell Equations. The notation for the fields in the large r expansion here is the same as the one in gravity,

$$A_\mu(u,r,z) = \sum_n \frac{A^{(n)}_\mu(u,z)}{r^n}. \tag{C.19}$$

We shall be working the radial gauge $A_r = 0$, which is analogous to the Bondi gauge in gravity [181]. The other subsidiary conditions in this gauge choice are $A^{(0)}_u = A^{(1)}_u = 0$.

From the equations of motion we can show that

$$\partial_u A^{(0)}_i = 0 \tag{C.20}$$

which is analogous to (5.8). There does not exist an equation of motion for $A^{(1)}_i$ and that in general can be u-dependent, hence along with it being the free data, it is used to parametrize electromagnetic radiation. This is analogous to $\tilde{D}_{ab}$ in gravity.

We now proceed with the computation of the symplectic form. To do this, we first evaluate the symplectic current (whose general expression is given in [151]). Like we did for the case of gravity, the component of the symplectic form that we are interested in $\mathcal{J}^t$ (where $t = u + r$)

$$\mathcal{J}^t = \mathcal{J}^u + \mathcal{J}^r = -(\mathcal{J}_r + \mathcal{J}_u). \tag{C.21}$$

The general form of $\mathcal{J}_\alpha$ is given as [151],

$$\mathcal{J}_\alpha = g^{\mu\nu}\delta A_\nu \wedge \delta F_{\mu\alpha}. \tag{C.22}$$

Thus we have to now compute the value of $\mathcal{J}_r$ and $\mathcal{J}_u$, which are given as,

$$\mathcal{J}_r = g^{\mu\nu}\delta A_\nu \wedge \delta F_{\mu r} = \frac{\delta^{ij}}{r^4}\delta A_j^{(0)} \wedge \delta A_i^{(1)}. \tag{C.23}$$

and

$$\begin{aligned}\mathcal{J}_u &= g^{\mu\nu}\delta A_\nu \wedge \delta F_{\mu u} = \frac{1}{r^2}\delta A^i \wedge \delta F_{iu} \\ &= -\frac{1}{r^3}\delta A^{(0)i} \wedge \delta\partial_u A_i^{(1)} + \frac{1}{r^4}\Big[-\delta A^{(1)i} \wedge \delta\partial_u A_i^{(1)} + \delta A^{(0)i} \wedge \delta F_{iu}^{(2)}\Big]\end{aligned} \tag{C.24}$$

Where to get the $1/r^4$ term we have used $A_u^{(0)} = A_u^{(1)} = \partial_u A_i^{(0)} = 0$. The first two are part of the gauge condition and the last one follows from an equation of motion (C.20). The currents can be simplified using the equation of motion which are given below. We specifically need the $\nabla_\mu F^{\mu r} = 0$ equation, and that is given as,

$$\partial_u F_{ru} + \frac{1}{r^2}\partial^i F_{iu} = 0 \implies \partial^i F_{iu}^{(2)} = \partial_u F_{ur}^{(4)}. \tag{C.25}$$

Where the second equation is one of the terms in the r-expansion of the first one.

C.4.1 Understanding $\mathcal{J}^t_{div}$

Next, we look at the $1/r^3$ term appearing in $\mathcal{J}_u$. This is analogous to the term J^r_{div} appearing in gravity (see (C.42)). Naively this would lead to a divergent term in the expression for the symplectic form as the volume of spacetime comes with a determinant factor of $\sqrt{-g} = r^4$ which in the $r \to \infty$ limit will give a divergence with this. Therefore, we need to be a bit careful in order to handle this. This term in the current is labeled as $\mathcal{J}^t_{div} \equiv \mathcal{J}^u_{div}$ and is given as,

$$\mathcal{J}^{div}_u = -\frac{1}{r^3}\delta A^{(0)i} \wedge \delta\partial_u A^{(1)}_i \tag{C.26}$$

In order to simplify this we use the fact that we are working with constant t-slices and we define $u = t - r$. By using this, we interpret the $1/r^3$ in the expression as r/r^4 and the r on the numerator here will be written as $r = t - u$, where we hold u constant as we take $r \to \infty$[1]. Therefore,

$$\begin{aligned}\mathcal{J}^{div}_u &= -\frac{1}{r^3}\delta A^{(0)i} \wedge \delta\partial_u A^{(1)}_i = \frac{u-t}{r^4}\delta A^{(0)i} \wedge \delta\partial_u A^{(1)}_i \\ &= \frac{1}{r^4}\partial_u\Big[(u-t)\delta A^{(0)i} \wedge \delta A^{(1)}_i\Big] - \frac{1}{r^4}\delta A^{(0)i} \wedge \delta A^{(1)}_i.\end{aligned} \tag{C.27}$$

Notice that while computing the symplectic form we are eventually interested in $\int_{-\infty}^{\infty} \mathcal{J}^{div}_u$ and with the fall off

$$\lim_{u\to\pm\infty} A^{(1)}_i \sim O(\frac{1}{|u|^{2+0_+}}) \tag{C.28}$$

we see that the first term in equation (C.27) vanishes (by repeating a similar analysis as that of appendix C.6 we also get this fall off using the saddle point approximation in QED). Therefore, we get,

$$\int_{-\infty}^{\infty} du\ \mathcal{J}^{div}_u = -\frac{1}{r^4}\int_{-\infty}^{\infty} du\ \delta A^{(0)i} \wedge \delta A^{(1)}_i \tag{C.29}$$

With this in place, we simplify the full symplectic form, i.e, the combination of (C.23) and (C.24).

C.4.2 Simplifying $\mathcal{J}^t$

From the computation above, we see that upon combining (C.23) and (C.29) we get,

$$\int_{-\infty}^{\infty} du\ \mathcal{J}^{div}_u + \mathcal{J}_r = 0. \tag{C.30}$$

[1] These limits are in general hard to make sense of, and we do not provide a rigorous mathematical argument to prove this here. For a more robust mathematical discussion of the symplectic form in QED, we refer the reader to [155].

In order to further simplify, we shall assume that we have no magnetic charges present (where the analogous statement in gravity is (5.9)) and therefore the curl of $A_i^{(0)}$ is zero. Which means that $\epsilon^{ijk}\partial_j A_k^{(0)} = 0$ and this leads to the condition,

$$A_i^{(0)} = \partial_i \Phi \tag{C.31}$$

where $\Phi(z)$ is some function on $\mathbb{R}^4$ and is like the "potential for the vector potential A_i". Like in the case of gravity, there is a freedom of adding a constant term in this expression, but that is not going to affect any of our analysis and hence we set that to zero. Therefore, in terms of Φ, the symplectic form becomes,

$$\Omega^t_{QED}(\delta, \delta') = \int_{-\infty}^{\infty} du r^4 \mathcal{J}_u = \int_{-\infty}^{\infty} du - \delta A^{(1)i} \wedge \delta' \partial_u A_i^{(1)} - \delta\Phi \wedge \delta' \partial_u \partial^i F_{ur}^{(4)}. \tag{C.32}$$

Where we have used the EOM in (C.25) to get the second term. From this equation we can read of the poisson brackets of the free theory, and these are stated in [181]. The first term in this like the kinetic term for the degree of freedom capturing radiation $A_i^{(1)}$. The second term gives us the conjugate to the soft mode. This is in contrast with the 4D case, as in there both of these come from the gauge field component at the same order $A_i^{(0)(4D)}$.

C.4.3 Charge

The charge is constructed using the variation under large gauge transformation. By working in the radial gauge we have the following variations,

$$\delta_\epsilon A_i^{(0)} = \partial_i \epsilon, \qquad \delta_\epsilon A_i^{(1)} = 0. \tag{C.33}$$

And therefore the charge is going to become,

$$\delta Q_\epsilon = \Omega^t(\delta_\epsilon, \delta) \implies Q_\epsilon = \int d^4 z \ \epsilon(z) F_{ur}^{(4)}\big|_{\mathcal{I}^-_+} \tag{C.34}$$

where we are neglecting the contribution at $\mathcal{I}^+_+$ due to the absence of massive particles. This is the same expression for the charge in QED as given in equation 3.2 of [142].

C.5 Details of the Gravity Symplectic form

In this section we give a detailed description of the derivation of the symplectic form in gravity. Our starting point will be the Witten-Crnkovic current as given in (5.21). As stated before, from the determinant condition of the Bondi gauge (5.3) we have $\delta g = 0$ and hence the equation for the symplectic current simplifies to,

$$J^\alpha = \frac{1}{2}\delta\Gamma^\alpha_{\mu\nu} \wedge \delta g^{\mu\nu} - \frac{1}{2}\delta\Gamma^\nu_{\mu\nu} \wedge \delta g^{\alpha\mu} \tag{C.35}$$

The components of interest in here will be J^t, which in turn just reduces to $J^r + J^u$. Therefore, we proceed onto evaluating these two. Since we are in 6-dimensions, the symplectic form constructed out of the current J^t is integrated with the measure $r^4\sqrt{\gamma} = r^4$ (when $\gamma_{ab} = \delta_{ab}$). Therefore we shall expand J^t in inverse powers of r and ignore the terms which are $O(1/r^5)$ and higher orders.

C.5.1 J^r

This is given as,

$$2J^r = \delta\Gamma^r_{\mu\nu} \wedge \delta g^{\mu\nu} - \delta\Gamma^\nu_{\mu\nu} \wedge \delta g^{r\mu} \tag{C.36}$$

By directly expanding this we get,

$$\begin{aligned} -2J^r = \delta g^{ru} \wedge (2\Gamma^r_{ur} - \Gamma^\mu_{u\mu}) + \delta g^{rr} \wedge \delta(\Gamma^r_{rr} - \Gamma^\mu_{\mu r}) \\ + \delta g^{ra} \wedge \delta(2\Gamma^r_{ra} - \Gamma^\mu_{\mu a}) + \delta g^{ab} \wedge \delta\Gamma^r_{ab} \end{aligned} \tag{C.37}$$

We see that the terms in the first line are of higher order, i.e,

$$\delta g^{ru} \wedge (2\Gamma^r_{ur} - \Gamma^\mu_{u\mu}) + \delta g^{rr} \wedge \delta(\Gamma^r_{rr} - \Gamma^\mu_{\mu r}) = O(1/r^5) \tag{C.38}$$

And thus, for getting the terms at the required order we have,

$$-2J^r = \delta g^{ra} \wedge \delta(2\Gamma^r_{ra} - \Gamma^\mu_{\mu a}) + \delta g^{ab} \wedge \delta\Gamma^r_{ab} \tag{C.39}$$

These terms are simple to compute and we list the final forms of the individual terms in here,

1.

$$\delta g^{ra} \wedge \delta(2\Gamma^r_{ra} - \Gamma^\mu_{\mu a}) = \frac{1}{r^4}\Big[2\delta(U^{(1)a} - C^{ab}U^{(0)}_b) \wedge \delta U^{(0)}_a + \delta U^{(0)}_a \wedge \delta(3U^{(1)}_a - C_{ac}U^{(0)c})\Big] \tag{C.40}$$

2.

$$\delta g^{ab}\wedge\delta\Gamma^{r}_{ab}=-\frac{1}{r^3}\delta C^{ab}\wedge\delta\Gamma^{r(0)}_{ab}+\frac{1}{r^4}\Big[-\delta C^{ab}\wedge\delta\Gamma^{(1)r}_{ab}+\delta(C^{a}_{c}C^{bc}-D^{ab})\wedge\delta\Gamma^{(0)r}_{ab}\Big] \tag{C.41}$$

Here $\Gamma^{(n)r}_{ab}$ denotes the coefficient of $1/r^n$ of Γ^r_{ab}. Therefore we see that the terms in J^r contain terms at $O(1/r^3)$ and $O(1/r^4)$. As we had mentioned before, in order to get the symplectic form, we need to integrate this with the measure which contains r^4 and therefore the term in J^r containing $1/r^3$ has to be handled with caution. We shall treat these two separately below. This is treatment is analogous to the treatment of EM in C.4.1. We will call the $1/r^3$ term J^r_{div} and the $1/r^4$ term as J^r_{fin}.

C.5.2 J^r_{div}

Here we have,

$$-2J^r_{div}=-\frac{1}{r^3}\delta C^{ab}\wedge\delta\Gamma^{r(0)}_{ab}=-\frac{1}{r^3}\Big[\delta C^{ab}\wedge\delta\partial_a U^{(0)}_b+\frac{1}{2}\delta C^{ab}\wedge\delta\partial_u D_{ab}\Big]. \tag{C.42}$$

The first term in this is actually zero when we look at its contribution in the symplectic form. And this can be seen by integrating on the flat sphere ($\gamma_{ab}=\delta_{ab}$) and using the equation of motion for $U^{(0)}_a$ (C.13),

$$\begin{aligned}\int d^4z\ \delta C^{ab}\wedge\delta\partial_a U^{(0)}_b &\sim \int d^4z\ \delta C^{ab}\wedge\delta\partial_a\partial^c C_{bc}\\ &=\int d^4z\ \partial_a\Big[\delta C^{ab}\wedge\delta\partial^c C_{bc}\Big]-\int d^4z\ \delta\partial_a C^{ab}\wedge\delta\partial^c C_{bc}=0.\end{aligned} \tag{C.43}$$

Here the first term is a total derivative on the flat sphere and hence is zero, and the second term is a wedge product of the same object and hence zero as well.

Let us look at the other term in (C.42). This term is similar to that of (C.26) and thus can be handled in the same way. We shall also be using (5.18) and (5.8) to write $\partial_u D_{ab}=\partial_u\tilde{D}_{ab}$. This replacement is done because, as motivated before, $\tilde{D}_{ab}$ is the radiative data in the full non-linear theory. It will be demonstrated below as to how this is also necessary for defining a finite symplectic form and is also implied by the Saddle point approximation (see appendix C.6). Therefore, the final form of (C.42) becomes,

$$-2J^r_{div} = -\frac{1}{2r^3}\delta C^{ab} \wedge \delta\partial_u \tilde{D}_{ab} \tag{C.44}$$

Like we did in the case of QED, we would like to think of the $1/r^3$ term as r/r^4 which is further thought of as $(t-u)/r^4$ where we would like to hold t fixed as we take $r \to \infty$. Thus we get,

$$-2J^r_{div} = -\frac{1}{2r^4}\partial_u\Big[(t-u)\delta C^{ab} \wedge \delta\tilde{D}_{ab}\Big] - \frac{1}{2r^4}\delta C^{ab} \wedge \delta\tilde{D}_{ab}. \tag{C.45}$$

We will describe how the finiteness of the symplectic form as computed from this expression fixes the fall off for $\tilde{D}_{ab}$.

C.5.3 J^r_{fin}

We now study the other part of J^r, which we had called J^r_{fin}. That is read off from (C.39)-(C.41),

$$\begin{aligned} 2J^r_{fin} = \frac{1}{r^4}\Big[&2\delta(U^{(1)a} - C^{ab}U^{(0)}_b) \wedge \delta U^{(0)}_a + \delta U^{(0)}_a \wedge \delta(3U^{(1)}_a - C_{ac}U^{(0)c}) \\ &- \delta C^{ab} \wedge \delta\Gamma^{(1)r}_{ab} + \delta(C^a_c C^{bc} - D^{ab}) \wedge \delta\Gamma^{(0)r}_{ab}\Big]. \end{aligned} \tag{C.46}$$

With a bit of work and integrating out total derivatives on the flat sphere, this becomes,

$$\begin{aligned} -2J^r_{fin} = &-\frac{1}{2}\delta D^{ab} \wedge \delta\partial_u D_{ab} - \frac{1}{2}\delta C^{ab} \wedge \delta\partial_u E_{ab} \\ &- \frac{1}{2}\delta C^{ab} \wedge \delta\Big[2\partial_a U^{(1)}_b - U^{(0)c}\big(2\partial_a C_{bc} - \partial_c C_{ab}\big) - \frac{1}{3}\partial_b\big[U^{(1)}_a + C^c_a U^{(0)}_c\big] \\ &+ \frac{1}{3}\partial_m\partial_a(D^m_b - C^m_c C^c_b)\Big]. \end{aligned} \tag{C.47}$$

This equation can be simplified further and everything can be expressed in terms of C_{ab} alone, which we will do after combining the results of J^u and J^r_{div}.

C.5.4 J^u

From the general equation for the symplectic current in (5.21) we have,

$$2J^u = \delta\Gamma^u_{\mu\nu} \wedge \delta g^{\mu\nu} - \delta\Gamma^\nu_{\mu\nu} \wedge \delta g^{u\mu}. \tag{C.48}$$

This is relatively easy to compute and can be written as,

$$2J^u = \frac{1}{2r^4}\delta C^{ab} \wedge \delta(C^a_c C^{bc} - D_{ab}) = -\frac{1}{2r^4}\delta C_{ab} \wedge \delta\tilde{D}^{ab} + \frac{3}{8r^4}\delta C_{ab} \wedge \delta(C^a_c C^{bc}) \tag{C.49}$$

Where we have used $C^a_a = 0$.

C.5.5 J^t

From the definition $t = u + r$, J^t is given as $J^u + J^r$ and therefore combining (C.45), (C.47) and (C.49), we have,

$$\begin{aligned}2J^t = &-\frac{1}{2r^4}\delta D^{ab} \wedge \delta\partial_u D_{ab} - \frac{1}{2r^4}\delta C^{ab} \wedge \delta\partial_u E_{ab} - \frac{1}{2r^4}\partial_u\Big[(t-u)\delta C^{ab} \wedge \tilde{D}_{ab}\Big] \\ &- \frac{1}{2}\delta C^{ab} \wedge \delta\Big[2\partial_a U^{(1)}_b - U^{(0)c}\big(2\partial_a C_{bc} - \partial_c C_{ab}\big) - \frac{1}{3}\partial_b\big[U^{(1)}_a + C^c_a U^{(0)}_c\big] \\ &+ \frac{1}{3}\partial_m\partial_a(D^m_b - C^m_c C^c_b) + \frac{1}{4}C^c_a C_{bc}\Big].\end{aligned} \tag{C.50}$$

From the equations of motion we have in sec.(C.2) we can write J^t completely in terms of C_{ab} and $\tilde{D}_{ab}$. And after doing that, we will find three kinds of terms in the expansion. One of which is a remnant of J^r_{div} and in the equation above that appears with a $\partial_u\Big[\cdots\Big]$. And in the other two terms, one of them contains the terms dependent on $\tilde{D}_{ab}$ and the other is independent of $\tilde{D}_{ab}$. These three parts are called J^t_{div}, J^t_{fin} and J^t_{NI} respectively. Here J^t_{NI} stands for "non-integrable" and we will explain the meaning of that in more detail in the upcoming section.

J^t_{div}

Here,

$$-2J^t_{div} = -\frac{1}{2r^4}\partial_u\Big[(t-u)\delta C^{ab} \wedge \delta\tilde{D}_{ab}\Big] \tag{C.51}$$

This term will give a contribution to the symplectic form which looks like,

$$\Omega^t_{div} \sim \int du\, \partial_u \Big[(t-u)\delta C^{ab}\wedge\delta\tilde{D}_{ab}\Big] = \Big[(t-u)\delta C^{ab}\wedge\delta\tilde{D}_{ab}\Big]_{u\to\infty} - \Big[(t-u)\delta C^{ab}\wedge\delta\tilde{D}_{ab}\Big]_{u\to-\infty}. \tag{C.52}$$

The way in which we regulate this is to choose the fall off for $\tilde{D}_{ab}$ as,

$$\lim_{u\to\pm\infty} \tilde{D}_{ab} \sim O\Big(\frac{1}{|u|^{2+0_+}}\Big), \tag{C.53}$$

which is exactly similar to the way in which $A_i^{(1)}$ falls off in EM which is discussed around (C.28), and is the same fall-off as derived using the saddle point approximation in appendix C.6. With this in place, we see that the contribution to

$$\Omega^t_{div} = 0. \tag{C.54}$$

Henceforth we shall be making the following replacement in the symplectic current, as it does not contribute to the form as motivated by the falloff (C.53),

$$\frac{1}{r^4}\delta C^{ab} \wedge \delta\partial_u D_{ab} = \frac{1}{r^4}\delta C^{ab} \wedge \delta\partial_u \tilde{D}_{ab} \to 0. \tag{C.55}$$

J^t_I

After substituting the EOM in (C.50) we collect the terms dependent on $\tilde{D}_{ab}$ and that is equal to,

$$-2J^t_I = -\frac{1}{2}\delta\tilde{D}_{ab} \wedge \delta\partial_u\tilde{D}^{ab} + \frac{1}{2}\delta C^{ab} \wedge \delta\partial_u E_{ab} + \frac{1}{9}\delta C^{ab} \wedge \delta\partial_a\partial^c\tilde{D}_{bc}. \tag{C.56}$$

In order to get this we have used the fall off (C.53) to replace,

$$\delta D^{ab} \wedge \delta\partial_u D_{ab} = \delta\tilde{D}^{ab} \wedge \delta\partial_u\tilde{D}_{ab} \tag{C.57}$$

which is possible because of the replacement (C.55). We are also including the $\partial_u E_{ab}$ term as it is a function of $\tilde{D}_{ab}$ (see (5.44)). It is interesting to note that the contribution to the symplectic form due to this term is the same as the linearized case (about $C = 0$ vacuum as in [26]), but with D_{ab} replaced by $\tilde{D}_{ab}$.

Therefore, the integrable part of the symplectic form becomes,

$$\Omega^t_I = \int_{\mathcal{I}^+} d^4z du J^t_I = -\frac{1}{2}\int_{\mathcal{I}^+} d^4z du -\frac{1}{2}\delta\tilde{D}_{ab}\wedge\delta\partial_u\tilde{D}^{ab} + \frac{1}{2}\delta C^{ab}\wedge\delta\partial_u E_{ab} + \frac{1}{9}\delta C^{ab}\wedge\delta\partial_a\partial^c\tilde{D}_{bc}. \tag{C.58}$$

It has been shown in section 5.3.1 that this leads to the correct soft charge, which is obtained by convoluting m_B with a function $f(z)$ (see (5.38)).

J^t_{NI}

Let us consider the final piece of the symplectic current (C.50) now, which solely depends on C_{ab}. Given the form of (C.50), a tedious but straightforward computation leads us to,

$$-2J^t_{NI} = \delta C^{ab}\wedge\delta\Big[-\frac{1}{2}C^c_aC_{bc} - \frac{1}{9}\partial_a\Big(C^{cd}\partial_cC_{bd} + \frac{4}{3}C^c_b\partial^dC_{cd} - \frac{1}{16}C^{cd}\partial_bC_{cd}\Big) + \frac{1}{3}\partial^cC_{cd}\partial_aC^d_b - \frac{1}{6}\partial^cC_{ab}\partial^dC_{cd}\Big]. \tag{C.59}$$

C.6 Saddle Point analysis

We perform the saddle point approximation for the Graviton to get an idea of the fall-off of the field at $\mathcal{I}^+_\pm$. Since we are only interested in the fall-offs, we will be cavalier about the numerical constants as they will not be necessary for the final answer. We will follow the treatment in [18][2].

The mode expansion of the Gravitational perturbation in Cartesian coordinates is given as

$$h_{\mu\nu}(u,\vec{r}) = \sum_\alpha \int \frac{d^5q}{(2\pi)^5 2|\vec{q}|}\Big[\epsilon^{(\alpha)}_{\mu\nu}a_\alpha(|\vec{q}|,\hat{q})e^{iq\cdot x} + h.c.\Big] \tag{C.60}$$

Here α is the polarization index, $\epsilon^{(\alpha)}_{\mu\nu}$ represents the polarization vector and a_α represents the mode functions. The factor $e^{iq\cdot x}$ is given as,

$$e^{iq\cdot x} = e^{-iqt+i\vec{q}\cdot\vec{x}} = e^{-iqu-iqr(1-\cos\theta)}, \tag{C.61}$$

where θ is the angle between $\vec{x}$ and $\vec{q}$ and we often use $|\vec{q}| \equiv q$ when there is no conflict of notation. With this expansion in place, we evaluate the integral in the $r \to \infty$ limit. Along with the $e^{iq\cdot x}$ term there is another contribution of θ that comes from the measure of the integral, which is proportional to $\sin^3\theta$

[2] Exercise 4 in [18] treats the 4D electromagnetic case, but the formalism is very similar.

(since it is a Five Dimensional integral). Thus the necessary contribution of the integral becomes,

$$h_{\mu\nu}(u,\vec{r}) \sim \sum_{\alpha} \int dq d\theta \; q^3 \sin^3\theta \big[\epsilon^{(\alpha)}_{\mu\nu} a_\alpha e^{-iqu - iqr(1-\cos\theta)} + h.c.\big] \tag{C.62}$$

Now we consider the saddle point approximation of the integral in the $r \to \infty$ limit. Here the saddle we pick is $\theta = 0$, where the other one is forbidden by the Riemann-Lebesgue lemma. The leading order saddle point answer will give us a 0, hence we have to go the subleading order, which means that we consider the following expansions,

$$\lim_{r\to\infty} \sin^3\theta e^{-iqr(1-\cos\theta)} \approx \lim_{r\to\infty} \theta^3 e^{-\frac{iqr\theta^2}{2}} \tag{C.63}$$

After which the integral is proportional to,

$$\lim_{r\to\infty} h_{\mu\nu} \sim \sum_{\alpha} \int dq d\theta \; \theta^3 q^3 \big[\epsilon^{(\alpha)}_{\mu\nu} a_\alpha e^{-iqu - \frac{iqr\theta^2}{2}} + h.c.\big] \tag{C.64}$$

Since we only want the r and q dependence of the θ integral we get,

$$\lim_{r\to\infty} h_{\mu\nu} \sim \frac{1}{r^2} \sum_{\alpha} \int dq \frac{q^3}{q^2} \Big[\epsilon^{(\alpha)}_{\mu\nu} a_\alpha e^{-iqu} k_1 + h.c.\Big] = \frac{1}{r^2} \sum_{\alpha} \int dq q \Big[\epsilon^{(\alpha)}_{\mu\nu} a_\alpha e^{-iqu} k_1 + h.c.\Big]. \tag{C.65}$$

where k_1 is an unimportant constant used to represent the numerical value for the integral over θ. In order to get the behavior of the Graviton field at $\mathcal{I}^+_\pm$ we now need to take the limit $|u| \to \infty$. Again, via saddle point approximation, this will now enforce the limit $q \to 0$ on the integrand of RHS. Using the normalization of the soft factor as given in [149] (see equations 2.10 and 2.29 of the reference mentioned) we get the behavior of the mode functions at $q \to 0$ in six Dimensions

$$\lim_{q\to 0} \epsilon^{(\alpha)}_{\mu\nu} a_\alpha(q,\hat{q}) \propto q^0 k_2(\hat{q}) \tag{C.66}$$

where $k_2(\hat{q})$ is an unimportant function depending on the direction of $\hat{q}$. This leads us to the asymptotic fall off at $\mathcal{I}^+_\pm$of $h_{\mu\nu}$

$$\lim_{u\to\pm\infty} \lim_{r\to\infty} h_{\mu\nu} \sim \frac{1}{r^2} \lim_{q\to 0} \int dq q e^{iqu} k_1 k_2(\hat{q}) = \frac{1}{r^2} \times O\Big(\frac{1}{u^2}\Big) \tag{C.67}$$

where the final step can be seen by performing a simple change of variable $qu = l$ for example. Since the physical Graviton mode is given as $\tilde{D}_{ab}$ (see eq. (5.18)) we can relate $h_{\mu\nu}$ to D_{ab} by a simple coordinate transformation

$$\tilde{D}_{ab} = \lim_{r\to\infty} \frac{\partial x^\mu}{\partial x^a}\frac{\partial x^\nu}{\partial x^b} h_{\mu\nu} \propto \lim_{r\to\infty} r^2 h_{\mu\nu} \tag{C.68}$$

and thus we arrive at the fall-off for the graviton field near $\mathcal{I}^+_\pm$ (see (5.27))

$$\lim_{u\to\pm\infty} \tilde{D}_{ab} = O\Big(\frac{1}{u^2}\Big). \tag{C.69}$$

A similar analysis can also be performed for the Photon in six dimensions and we end up with a similar fall-off condition for the radiative mode in QED.

Bibliography

[1] Juan Martin Maldacena. “The Large N limit of superconformal field theories and supergravity”. In: Adv. Theor. Math. Phys. 2 (1998), pp. 231–252. DOI: 10.1023/A:1026654312961. arXiv: hep-th/9711200.

[2] Suvrat Raju. “Lessons from the Information Paradox”. In: (Dec. 2020). arXiv: 2012.05770 [hep-th].

[3] Edward Witten. “Anti-de Sitter space and holography”. In: Adv. Theor. Math. Phys. 2 (1998), pp. 253–291. DOI: 10.4310/ATMP.1998.v2.n2.a2. arXiv: hep-th/9802150.

[4] Souvik Banerjee et al. “A toy model of black hole complementarity”. In: JHEP 05 (2016), p. 004. DOI: 10.1007/JHEP05(2016)004. arXiv: 1603.02812 [hep-th].

[5] Alok Laddha et al. “The Holographic Nature of Null Infinity”. In: SciPost Phys. 10 (2021), p. 041. DOI: 10.21468/SciPostPhys.10.2.041. arXiv: 2002.02448 [hep-th].

[6] Ahmed Almheiri et al. “The entropy of Hawking radiation”. In: (June 2020). arXiv: 2006.06872 [hep-th].

[7] S.W. Hawking. “Particle Creation by Black Holes”. In: Commun. Math. Phys. 43 (1975). Ed. by G.W. Gibbons and S.W. Hawking. [Erratum: Commun.Math.Phys. 46, 206 (1976)], pp. 199–220. DOI: 10.1007/BF02345020.

[8] Suvrat Raju. “Failure of the split property in gravity and the information paradox”. In: Class. Quant. Grav. 39.6 (2022), p. 064002. DOI: 10.1088/1361-6382/ac482b. arXiv: 2110.05470 [hep-th].

[9] Chandramouli Chowdhury et al. “Holography from the Wheeler-DeWitt equation”. In: (July 2021). arXiv: 2107.14802 [hep-th].

[10] Chandramouli Chowdhury, Olga Papadoulaki, and Suvrat Raju. "A physical protocol for observers near the boundary to obtain bulk information in quantum gravity". In: SciPost Phys. 10.5 (2021), p. 106. DOI: 10.21468/SciPostPhys.10.5.106. arXiv: 2008.01740 [hep-th].

[11] Chandramouli Chowdhury and Olga Papadoulaki. "Recovering information in an asymptotically flat spacetime in quantum gravity". In: Class. Quant. Grav. 39.24 (2022), p. 245012. DOI: 10.1088/1361-6382/aca192. arXiv: 2203.07449 [hep-th].

[12] H. Bondi, M.G.J. van der Burg, and A.W.K. Metzner. "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems". In: Proc. Roy. Soc. Lond. A 269 (1962), pp. 21–52. DOI: 10.1098/rspa.1962.0161.

[13] R. K. Sachs. "Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times". In: Proc. Roy. Soc. Lond. A 270 (1962), pp. 103–126. DOI: 10.1098/rspa.1962.0206.

[14] A. Ashtekar. "Asymptotic Quantization of the Gravitational Field". In: Phys. Rev. Lett. 46 (1981), pp. 573–576. DOI: 10.1103/PhysRevLett.46.573.

[15] A. Ashtekar and M. Streubel. "Symplectic Geometry of Radiative Modes and Conserved Quantities at Null Infinity". In: Proc. Roy. Soc. Lond. A376 (1981), pp. 585–607. DOI: 10.1098/rspa.1981.0109.

[16] Andrew Strominger. "On BMS Invariance of Gravitational Scattering". In: JHEP 07 (2014), p. 152. DOI: 10.1007/JHEP07(2014)152. arXiv: 1312.2229 [hep-th].

[17] Temple He et al. "BMS supertranslations and Weinberg?s soft graviton theorem". In: JHEP 05 (2015), p. 151. DOI: 10.1007/JHEP05(2015)151. arXiv: 1401.7026 [hep-th].

[18] Andrew Strominger. "Lectures on the Infrared Structure of Gravity and Gauge Theory". In: (Mar. 2017). arXiv: 1703.05448 [hep-th].

[19] Glenn Barnich and Cedric Troessaert. "Aspects of the BMS/CFT correspondence". In: JHEP 05 (2010), p. 062. DOI: 10.1007/JHEP05(2010)062. arXiv: 1001.1541 [hep-th].

[20] Daniel Kapec et al. "Semiclassical Virasoro symmetry of the quantum gravity $\mathcal{S}$-matrix". In: JHEP 08 (2014), p. 058. DOI: 10.1007/JHEP08(2014)058. arXiv: 1406.3312 [hep-th].

[21] Miguel Campiglia and Alok Laddha. "Asymptotic symmetries of gravity and soft theorems for massive particles". In: JHEP 12 (2015), p. 094. DOI: 10.1007/JHEP12(2015)094. arXiv: 1509.01406 [hep-th].

[22] Daniel Kapec et al. "Higher-dimensional supertranslations and Weinberg's soft graviton theorem". In: Ann. Math. Sci. Appl. 02 (2017), pp. 69–94. DOI: 10.4310/AMSA.2017.v2.n1.a2. arXiv: 1502.07644 [gr-qc].

[23] Chandramouli Chowdhury, Ruchira Mishra, and Siddharth G. Prabhu. "The Asymptotic Structure of Gravity in Higher Even Dimensions". In: (Jan. 2022). arXiv: 2201.07813 [hep-th].

[24] Chandramouli Chowdhury, Anupam A. H., and Arpan Kundu. "Generalized BMS algebra in higher even dimensions". In: Phys. Rev. D 106.12 (2022), p. 126025. DOI: 10.1103/PhysRevD.106.126025. arXiv: 2209.06839 [hep-th].

[25] Federico Capone et al. "Phase Space Renormalization and Finite BMS Charges in Six Dimensions". In: (Apr. 2023). arXiv: 2304.09330 [hep-th].

[26] Ankit Aggarwal. "Supertranslations in Higher Dimensions Revisited". In: Phys. Rev. D 99.2 (2019), p. 026015. DOI: 10.1103/PhysRevD.99.026015. arXiv: 1811.00093 [hep-th].

[27] Donald Marolf. "Unitarity and Holography in Gravitational Physics". In: Phys. Rev. D 79 (2009), p. 044010. DOI: 10.1103/PhysRevD.79.044010. arXiv: 0808.2842 [gr-qc].

[28] Donald Marolf. "Asymptotic flatness, little string theory, and holography". In: JHEP 03 (2007), p. 122. DOI: 10.1088/1126-6708/2007/03/122. arXiv: hep-th/0612012.

[29] Donald Marolf. "Holography without strings?" In: Class. Quant. Grav. 31 (2014), p. 015008. DOI: 10.1088/0264-9381/31/1/015008. arXiv: 1308.1977 [hep-th].

[30] Ted Jacobson and Phuc Nguyen. "Diffeomorphism invariance and the black hole information paradox". In: Phys. Rev. D 100.4 (2019), p. 046002. DOI: 10.1103/PhysRevD.100.046002. arXiv: 1904.04434 [gr-qc].

[31] Samir D. Mathur. "The Information paradox: A Pedagogical introduction". In: Class. Quant. Grav. 26 (2009). Ed. by A.M. Uranga, p. 224001. DOI: 10.1088/0264-9381/26/22/224001. arXiv: 0909.1038 [hep-th].

[32] Ahmed Almheiri et al. "Black Holes: Complementarity or Firewalls?" In: JHEP 02 (2013), p. 062. DOI: 10.1007/JHEP02(2013)062. arXiv: 1207.3123 [hep-th].

[33] Kyriakos Papadodimas and Suvrat Raju. "An Infalling Observer in AdS/CFT". In: JHEP 10 (2013), p. 212. DOI: 10.1007/JHEP10(2013)212. arXiv: 1211.6767 [hep-th].

[34] Erik Verlinde and Herman Verlinde. "Black Hole Entanglement and Quantum Error Correction". In: JHEP 10 (2013), p. 107. DOI: 10.1007/JHEP10(2013)107. arXiv: 1211.6913 [hep-th].

[35] Yasunori Nomura, Jaime Varela, and Sean J. Weinberg. "Complementarity Endures: No Firewall for an Infalling Observer". In: JHEP 03 (2013), p. 059. DOI: 10.1007/JHEP03(2013)059. arXiv: 1207.6626 [hep-th].

[36] Juan Maldacena and Leonard Susskind. "Cool horizons for entangled black holes". In: Fortsch. Phys. 61 (2013), pp. 781–811. DOI: 10.1002/prop.201300020. arXiv: 1306.0533 [hep-th].

[37] Geoffrey Penington. "Entanglement Wedge Reconstruction and the Information Paradox". In: JHEP 09 (2020), p. 002. DOI: 10.1007/JHEP09(2020)002. arXiv: 1905.08255 [hep-th].

[38] Ahmed Almheiri, Raghu Mahajan, and Juan Maldacena. "Islands outside the horizon". In: (Oct. 2019). arXiv: 1910.11077 [hep-th].

[39] Ahmed Almheiri et al. "The Page curve of Hawking radiation from semiclassical geometry". In: JHEP 03 (2020), p. 149. DOI: 10.1007/JHEP03(2020)149. arXiv: 1908.10996 [hep-th].

[40] Ahmed Almheiri et al. "Replica Wormholes and the Entropy of Hawking Radiation". In: JHEP 05 (2020), p. 013. DOI: 10.1007/JHEP05(2020)013. arXiv: 1911.12333 [hep-th].

[41] Geoff Penington et al. "Replica wormholes and the black hole interior". In: (Nov. 2019). arXiv: 1911.11977 [hep-th].

[42] Suvrat Raju. "A Toy Model of the Information Paradox in Empty Space". In: SciPost Phys. 6.6 (2019), p. 073. DOI: 10.21468/SciPostPhys.6.6.073. arXiv: 1809.10154 [hep-th].

[43] Sudip Ghosh and Suvrat Raju. "Quantum information measures for restricted sets of observables". In: Phys. Rev. D 98.4 (2018), p. 046005. DOI: 10.1103/PhysRevD.98.046005. arXiv: 1712.09365 [hep-th].

[44] Kyriakos Papadodimas and Suvrat Raju. "State-Dependent Bulk-Boundary Maps and Black Hole Complementarity". In: Phys. Rev. D 89.8 (2014), p. 086010. DOI: 10.1103/PhysRevD.89.086010. arXiv: 1310.6335 [hep-th].

[45] Ahmed Almheiri, Xi Dong, and Daniel Harlow. "Bulk Locality and Quantum Error Correction in AdS/CFT". In: JHEP 04 (2015), p. 163. DOI: 10.1007/JHEP04(2015)163. arXiv: 1411.7041 [hep-th].

[46] Thomas Faulkner, Aitor Lewkowycz, and Juan Maldacena. "Quantum corrections to holographic entanglement entropy". In: JHEP 11 (2013), p. 074. DOI: 10.1007/JHEP11(2013)074. arXiv: 1307.2892 [hep-th].

[47] Daniel L. Jafferis et al. "Relative entropy equals bulk relative entropy". In: JHEP 06 (2016), p. 004. DOI: 10.1007/JHEP06(2016)004. arXiv: 1512.06431 [hep-th].

[48] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. "Gauge theory correlators from noncritical string theory". In: Phys. Lett. B 428 (1998), pp. 105–114. DOI: 10.1016/S0370-2693(98)00377-3. arXiv: hep-th/9802109.

[49] Sebastian de Haro, Sergey N. Solodukhin, and Kostas Skenderis. "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence". In: Commun. Math. Phys. 217 (2001), pp. 595–622. DOI: 10.1007/s002200100381. arXiv: hep-th/0002230.

[50] Vijay Balasubramanian and Per Kraus. "A Stress tensor for Anti-de Sitter gravity". In: Commun. Math. Phys. 208 (1999), pp. 413–428. DOI: 10.1007/s002200050764. arXiv: hep-th/9902121.

[51] Nima Arkani-Hamed et al. "The String landscape, black holes and gravity as the weakest force". In: JHEP 0706 (2007), p. 060. DOI: 10.1088/1126-6708/2007/06/060. arXiv: hep-th/0601001.

[52] Daniel Harlow and Hirosi Ooguri. "Symmetries in quantum field theory and quantum gravity". In: (Oct. 2018). arXiv: 1810.05338 [hep-th].

[53] R. Haag. Local quantum physics: Fields, particles, algebras. Springer-Verlag, 1992.

[54] Edward Witten. "APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory". In: Rev. Mod. Phys. 90.4 (2018), p. 045003. DOI: 10.1103/RevModPhys.90.045003. arXiv: 1803.04993 [hep-th].

[55] Horacio Casini, Marina Huerta, and Jose Alejandro Rosabal. "Remarks on entanglement entropy for gauge fields". In: Phys. Rev. D 89.8 (2014), p. 085012. DOI: 10.1103/PhysRevD.89.085012. arXiv: 1312.1183 [hep-th].

[56] Horacio Casini and Marina Huerta. "Entanglement entropy for a Maxwell field: Numerical calculation on a two dimensional lattice". In: Phys. Rev. D 90.10 (2014), p. 105013. DOI: 10.1103/PhysRevD.90.105013. arXiv: 1406.2991 [hep-th].

[57] Ronak M Soni and Sandip P. Trivedi. "Aspects of Entanglement Entropy for Gauge Theories". In: JHEP 01 (2016), p. 136. DOI: 10.1007/JHEP01(2016)136. arXiv: 1510.07455 [hep-th].

[58] Sudip Ghosh, Ronak M Soni, and Sandip P. Trivedi. "On The Entanglement Entropy For Gauge Theories". In: JHEP 09 (2015), p. 069. DOI: 10.1007/JHEP09(2015)069. arXiv: 1501.02593 [hep-th].

[59] William Donnelly and Steven B. Giddings. "How is quantum information localized in gravity?" In: Phys. Rev. D96.8 (2017), p. 086013. DOI: 10.1103/PhysRevD.96.086013. arXiv: 1706.03104 [hep-th].

[60] Sumit R. Das et al. "Bulk Entanglement Entropy and Matrices". In: (Apr. 2020). arXiv: 2004.00613 [hep-th].

[61] Xi Dong et al. "Effective entropy of quantum fields coupled with gravity". In: (July 2020). arXiv: 2007.02987 [hep-th].

[62] Shinsei Ryu and Tadashi Takayanagi. "Aspects of Holographic Entanglement Entropy". In: JHEP 08 (2006), p. 045. DOI: 10.1088/1126-6708/2006/08/045. arXiv: hep-th/0605073.

[63] Shinsei Ryu and Tadashi Takayanagi. "Holographic derivation of entanglement entropy from AdS/CFT". In: Phys. Rev. Lett. 96 (2006), p. 181602. DOI: 10.1103/PhysRevLett.96.181602. arXiv: hep-th/0603001.

[64] Veronika E. Hubeny, Mukund Rangamani, and Tadashi Takayanagi. "A Covariant holographic entanglement entropy proposal". In: JHEP 07 (2007), p. 062. DOI: 10.1088/1126-6708/2007/07/062. arXiv: 0705.0016 [hep-th].

[65] Steven B. Giddings. "Gravitational dressing, soft charges, and perturbative gravitational splitting". In: Phys. Rev. D 100.12 (2019), p. 126001. DOI: 10.1103/PhysRevD.100.126001. arXiv: 1903.06160 [hep-th].

[66] Xian O. Camanho et al. "Causality Constraints on Corrections to the Graviton Three-Point Coupling". In: JHEP 02 (2016), p. 020. DOI: 10.1007/JHEP02(2016)020. arXiv: 1407.5597 [hep-th].

[67] Subham Dutta Chowdhury et al. "Classifying and constraining local four photon and four graviton S-matrices". In: JHEP 02 (2020), p. 114. DOI: 10.1007/JHEP02(2020)114. arXiv: 1910.14392 [hep-th].

[68] Daniel Harlow. "Aspects of the Papadodimas-Raju Proposal for the Black Hole Interior". In: JHEP 11 (2014), p. 055. DOI: 10.1007/JHEP11(2014)055. arXiv: 1405.1995 [hep-th].

[69] Kyriakos Papadodimas and Suvrat Raju. "Remarks on the necessity and implications of state-dependence in the black hole interior". In: Phys. Rev. D 93.8 (2016), p. 084049. DOI: 10.1103/PhysRevD.93.084049. arXiv: 1503.08825 [hep-th].

[70] Kyriakos Papadodimas. "A class of non-equilibrium states and the black hole interior". In: (Aug. 2017). arXiv: 1708.06328 [hep-th].

[71] Raphael Bousso et al. "Asymptotic Charges Cannot Be Measured in Finite Time". In: Phys. Rev. D 97.4 (2018), p. 046014. DOI: 10.1103/PhysRevD.97.046014. arXiv: 1709.08632 [hep-th].

[72] Donald Marolf. "Comments on Microcausality, Chaos, and Gravitational Observables". In: Class. Quant. Grav. 32.24 (2015), p. 245003. DOI: 10.1088/0264-9381/32/24/245003. arXiv: 1508.00939 [gr-qc].

[73] Andrew Strominger and Alexander Zhiboedov. "Gravitational Memory, BMS Supertranslations and Soft Theorems". In: JHEP 01 (2016), p. 086. DOI: 10.1007/JHEP01(2016)086. arXiv: 1411.5745 [hep-th].

[74] A. Ashtekar. Asymptotic Quantization: based on 1984 Naples Lectures. Bibliopolis, 1987.

[75] Abhay Ashtekar, Miguel Campiglia, and Alok Laddha. "Null infinity, the BMS group and infrared issues". In: Gen. Rel. Grav. 50.11 (2018), pp. 140–163. DOI: 10.1007/s10714-018-2464-3. arXiv: 1808.07093 [gr-qc].

[76] Tuneer Chakraborty, Joydeep Chakravarty, and Priyadarshi Paul. "Monogamy paradox: A toy model in flat space". In: (July 2021). arXiv: 2107.06919 [hep-th].

[77] Geoffrey Compère and Jiang Long. "Vacua of the gravitational field". In: JHEP 07 (2016), p. 137. DOI: 10.1007/JHEP07(2016)137. arXiv: 1601.04958 [hep-th].

[78] Gautam Satishchandran and Robert M. Wald. "Asymptotic behavior of massless fields and the memory effect". In: Phys. Rev. D 99.8 (2019), p. 084007. DOI: 10.1103/PhysRevD.99.084007. arXiv: 1901.05942 [gr-qc].

[79] Monica Pate, Ana-Maria Raclariu, and Andrew Strominger. "Gravitational Memory in Higher Dimensions". In: JHEP 06 (2018), p. 138. DOI: 10.1007/JHEP06(2018)138. arXiv: 1712.01204 [hep-th].

[80] Bryce S. DeWitt. "Quantum Theory of Gravity. 1. The Canonical Theory". In: Phys. Rev. 160 (1967). Ed. by Li-Zhi Fang and R. Ruffini, pp. 1113–1148. DOI: 10.1103/PhysRev.160.1113.

[81] K. Kuchar. "Ground state functional of the linearized gravitational field". In: J. Math. Phys. 11 (1970), pp. 3322–3334. DOI: 10.1063/1.1665133.

[82] Richard L. Arnowitt, Stanley Deser, and Charles W. Misner. "The Dynamics of general relativity". In: Gen. Rel. Grav. 40 (2008), pp. 1997–2027. DOI: 10.1007/s10714-008-0661-1. arXiv: gr-qc/0405109.

[83] Suvrat Raju. "Is Holography Implicit in Canonical Gravity?" In: Int. J. Mod. Phys. D28.14 (2019), p. 1944011. DOI: 10.1142/S0218271819440115. arXiv: 1903.11073 [gr-qc].

[84] Ted Jacobson. “Boundary unitarity and the black hole information paradox”. In: Int. J. Mod. Phys. D22 (2013), p. 1342002. DOI: 10.1142/S0218271813420029. arXiv: 1212.6944 [hep-th].

[85] Laurent Freidel. “Reconstructing AdS/CFT”. In: (Apr. 2008). arXiv: 0804.0632 [hep-th].

[86] Francesco Cianfrani and Jerzy Kowalski-Glikman. “Wheeler-DeWitt equation and AdS/CFT correspondence”. In: Phys. Lett. B 725 (2013), pp. 463–467. DOI: 10.1016/j.physletb.2013.07.034. arXiv: 1306.0353 [hep-th].

[87] Lauren McGough, Márk Mezei, and Herman Verlinde. “Moving the CFT into the bulk with $T\overline{T}$”. In: JHEP 04 (2018), p. 010. DOI: 10.1007/JHEP04(2018)010. arXiv: 1611.03470 [hep-th].

[88] Pawel Caputa et al. “Geometrizing $T\overline{T}$”. In: JHEP 03 (2021), p. 140. DOI: 10.1007/JHEP03(2021)140. arXiv: 2011.04664 [hep-th].

[89] Jorrit Kruthoff and Onkar Parrikar. “On the flow of states under $T\overline{T}$”. In: (June 2020). arXiv: 2006.03054 [hep-th].

[90] Evan A. Coleman et al. “$T\overline{T}$ -deformed actions and (1,1) supersymmetry”. In: JHEP 10 (2019), p. 080. DOI: 10.1007/JHEP10(2019)080. arXiv: 1906.05439 [hep-th].

[91] William Donnelly and Vasudev Shyam. “Entanglement entropy and $T\overline{T}$ deformation”. In: Phys. Rev. Lett. 121.13 (2018), p. 131602. DOI: 10.1103/PhysRevLett.121.131602. arXiv: 1806.07444 [hep-th].

[92] Andrew J. Tolley. “$T\overline{T}$ deformations, massive gravity and non-critical strings”. In: JHEP 06 (2020), p. 050. DOI: 10.1007/JHEP06(2020)050. arXiv: 1911.06142 [hep-th].

[93] Aurora Ireland and Vasudev Shyam. “$T\overline{T}$ deformed YM_2 on general backgrounds from an integral transformation”. In: JHEP 07 (2020), p. 058. DOI: 10.1007/JHEP07(2020)058. arXiv: 1912.04686 [hep-th].

[94] Shinji Hirano and Masaki Shigemori. “Random boundary geometry and gravity dual of $T\overline{T}$ deformation”. In: JHEP 11 (2020), p. 108. DOI: 10.1007/JHEP11(2020)108. arXiv: 2003.06300 [hep-th].

[95] Alexandre Belin, Aitor Lewkowycz, and Gabor Sarosi. “Gravitational path integral from the T^2 deformation”. In: JHEP 09 (2020), p. 156. DOI: 10.1007/JHEP09(2020)156. arXiv: 2006.01835 [hep-th].

[96] Pawel Caputa, Shouvik Datta, and Vasudev Shyam. "Sphere partition functions \& cut-off AdS". In: JHEP 05 (2019), p. 112. DOI: 10.1007/JHEP05(2019)112. arXiv: 1902.10893 [hep-th].

[97] Masakatsu Kenmoku et al. "Analytic solutions of the Wheeler-DeWitt equation in spherically symmetric space-time". In: Phys. Rev. D 59 (1999), p. 124004. DOI: 10.1103/PhysRevD.59.124004. arXiv: gr-qc/9810042.

[98] P. Hajicek. "Spherically Symmetric Systems of Fields and Black Holes. 2. Apparent Horizon in Canonical Formalism". In: Phys. Rev. D 30 (1984), p. 1178. DOI: 10.1103/PhysRevD.30.1178.

[99] W. Fischler et al. "Quantum Mechanics of the Googolplexus". In: Nucl. Phys. B 327 (1989), pp. 157–177. DOI: 10.1016/0550-3213(89)90290-3.

[100] Willy Fischler, Daniel Morgan, and Joseph Polchinski. "Quantum Nucleation of False Vacuum Bubbles". In: Phys. Rev. D 41 (1990), p. 2638. DOI: 10.1103/PhysRevD.41.2638.

[101] J. J. Halliwell and S. W. Hawking. "The Origin of Structure in the Universe". In: Phys. Rev. D 31 (1985). Ed. by Li-Zhi Fang and R. Ruffini, p. 1777. DOI: 10.1103/PhysRevD.31.1777.

[102] Takayuki Hori. "Exact solutions to the Wheeler-DeWitt equation of two-dimensional dilaton gravity". In: Prog. Theor. Phys. 90 (1993), pp. 743–745. DOI: 10.1143/PTP.90.743. arXiv: hep-th/9303049.

[103] D. Louis-Martinez, J. Gegenberg, and G. Kunstatter. "Exact Dirac quantization of all 2-D dilaton gravity theories". In: Phys. Lett. B 321 (1994), pp. 193–198. DOI: 10.1016/0370-2693(94)90463-4. arXiv: gr-qc/9309018.

[104] J. Gegenberg, G. Kunstatter, and D. Louis-Martinez. "Observables for two-dimensional black holes". In: Phys. Rev. D 51 (1995), pp. 1781–1786. DOI: 10.1103/PhysRevD.51.1781. arXiv: gr-qc/9408015.

[105] Juan Maldacena, Gustavo J. Turiaci, and Zhenbin Yang. "Two dimensional Nearly de Sitter gravity". In: JHEP 01 (2021), p. 139. DOI: 10.1007/JHEP01(2021)139. arXiv: 1904.01911 [hep-th].

[106] Panagiotis Betzios and Olga Papadoulaki. "Liouville theory and Matrix models: A Wheeler DeWitt perspective". In: JHEP 09 (2020), p. 125. DOI: 10.1007/JHEP09(2020)125. arXiv: 2004.00002 [hep-th].

[107] A. Ashtekar. “New Variables for Classical and Quantum Gravity”. In: Phys. Rev. Lett. 57 (1986), pp. 2244–2247. DOI: 10.1103/PhysRevLett.57.2244.

[108] Carlo Rovelli. Quantum gravity. Cambridge university press, 2004.

[109] Jonathan J. Halliwell. “Introductory Lectures on Quantum Cosmology”. In: 7th Jerusalem Winter School for Theoretical Physics: Quantum Cosmology and B 1989. arXiv: 0909.2566 [gr-qc].

[110] Sydney Coleman et al. Quantum Cosmology And Baby Universes: Proceedings Of 7th Jerus Vol. 7. World Scientific, 1991.

[111] William Donnelly and Steven B. Giddings. “Gravitational splitting at first order: Quantum information localization in gravity”. In: Phys. Rev. D 98.8 (2018), p. 086006. DOI: 10.1103/PhysRevD.98.086006. arXiv: 1805.11095 [hep-th].

[112] William Donnelly and Steven B. Giddings. “Observables, gravitational dressing, and obstructions to locality and subsystems”. In: Phys. Rev. D94.10 (2016), p. 104038. DOI: 10.1103/PhysRevD.94.104038. arXiv: 1607.01025 [hep-th].

[113] Marc Henneaux and Claudio Teitelboim. “Asymptotically anti-de Sitter spaces”. In: Communications in Mathematical Physics 98.3 (1985), pp. 391–424. DOI: cmp/1103942446. URL: https://doi.org/.

[114] Abhay Ashtekar and Saurya Das. “Asymptotically Anti-de Sitter spacetimes: Conserved quantities”. In: Class. Quant. Grav. 17 (2000), pp. L17–L30. DOI: 10.1088/0264-9381/17/2/101. arXiv: hep-th/9911230.

[115] Laurent Freidel, Marc Geiller, and Daniele Pranzetti. “Edge modes of gravity. Part I. Corner potentials and charges”. In: JHEP 11 (2020), p. 026. DOI: 10.1007/JHEP11(2020)026. arXiv: 2006.12527 [hep-th].

[116] Laurent Freidel, Marc Geiller, and Daniele Pranzetti. “Edge modes of gravity. Part II. Corner metric and Lorentz charges”. In: JHEP 11 (2020), p. 027. DOI: 10.1007/JHEP11(2020)027. arXiv: 2007.03563 [hep-th].

[117] Ofer Aharony et al. “Defect conformal field theory and locally localized gravity”. In: JHEP 07 (2003), p. 030. DOI: 10.1088/1126-6708/2003/07/030. arXiv: hep-th/0303249.

[118] Paul Adrien Maurice Dirac. Lectures on Quantum mechanics. Original published by Belfer Graduate School of Science in 1964. Dover, 2001.

[119] Tullio Regge and Claudio Teitelboim. "Role of Surface Integrals in the Hamiltonian Formulation of General Relativity". In: Annals Phys. 88 (1974), p. 286. DOI: 10.1016/0003-4916(74)90404-7.

[120] José M. Martín-García. "xAct: Efficient tensor computer algebra for the Wolfram Language". In: (2002).

[121] David Brizuela, Jose M. Martin-Garcia, and Guillermo A. Mena Marugan. "xPert: Computer algebra for metric perturbation theory". In: Gen. Rel. Grav. 41 (2009), pp. 2415–2431. DOI: 10.1007/s10714-009-0773-2. arXiv: 0807.0824 [gr-qc].

[122] Github repository: Wheeler-DeWitt holography. https://github.com/victorgodet/wdw-holography. 2021.

[123] James W York Jr. "Conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial-value problem of general relativity". In: Journal of Mathematical Physics 14.4 (1973), pp. 456–464.

[124] J. B. Hartle. "Ground State Wave Function Of Linearized Gravity". In: Phys. Rev. D 29 (1984), pp. 2730–2737. DOI: 10.1103/PhysRevD.29.2730.

[125] Ofer Aharony et al. "Large N field theories, string theory and gravity". In: Phys. Rept. 323 (2000), pp. 183–386. DOI: 10.1016/S0370-1573(99)00083-6. arXiv: hep-th/9905111.

[126] James W. York Jr. "Role of conformal three geometry in the dynamics of gravitation". In: Phys. Rev. Lett. 28 (1972), pp. 1082–1085. DOI: 10.1103/PhysRevLett.28.1082.

[127] Karel Kuchar. "The problem of time in canonical quantization". In: Conceptual Problems of Quantum Gravity. Ed. by A. Ashtekar and J. Stachel. Birkhauser, 1991.

[128] William Donnelly and Steven B. Giddings. "Diffeomorphism-invariant observables and their nonlocal algebra". In: Phys. Rev. D93.2 (2016). [Erratum: Phys. Rev.D94,no.2,029903(2016)], p. 024030. DOI: 10.1103/PhysRevD.94.029903, 10.1103/PhysRevD.93.024030. arXiv: 1507.07921 [hep-th].

[129] Raymond F Streater and Arthur S Wightman. PCT, spin and statistics, and all that. Princeton University Press, 2016.

[130] Seth Lloyd. "Black Holes, Demons and the Loss of Coherence: How complex systems get information, and what they do with it". PhD thesis. Rockefeller University, 1988.

[131] J. David Brown and M. Henneaux. "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity". In: Commun. Math. Phys. 104 (1986), pp. 207–226. DOI: 10.1007/BF01211590.

[132] Horacio Casini, Marina Huerta, and Robert C. Myers. "Towards a derivation of holographic entanglement entropy". In: JHEP 1105 (2011), p. 036. DOI: 10.1007/JHEP05(2011)036. arXiv: 1102.0440 [hep-th].

[133] Tuneer Chakraborty et al. "The Hilbert space of de Sitter quantum gravity". In: (Mar. 2023). arXiv: 2303.16315 [hep-th].

[134] Tuneer Chakraborty et al. "Holography of information in de Sitter space". In: (Mar. 2023). arXiv: 2303.16316 [hep-th].

[135] Hao Geng et al. "Inconsistency of Islands in Theories with Long-Range Gravity". In: (July 2021). arXiv: 2107.03390 [hep-th].

[136] Miguel Campiglia and Rodrigo Eyheralde. "Asymptotic $U(1)$ charges at spatial infinity". In: JHEP 11 (2017), p. 168. DOI: 10.1007/JHEP11(2017)168. arXiv: 1703.07884 [hep-th].

[137] Miguel Campiglia and Alok Laddha. "Asymptotic charges in massless QED revisited: A view from Spatial Infinity". In: JHEP 05 (2019), p. 207. DOI: 10.1007/JHEP05(2019)207. arXiv: 1810.04619 [hep-th].

[138] Kartik Prabhu. "Conservation of asymptotic charges from past to future null infinity: Maxwell fields". In: JHEP 10 (2018), p. 113. DOI: 10.1007/JHEP10(2018)113. arXiv: 1808.07863 [gr-qc].

[139] Kartik Prabhu. "Conservation of asymptotic charges from past to future null infinity: Supermomentum in general relativity". In: JHEP 03 (2019), p. 148. DOI: 10.1007/JHEP03(2019)148. arXiv: 1902.08200 [gr-qc].

[140] Evangelos Melas. "On the representation theory of the Bondi–Metzner–Sachs group and its variants in three space–time dimensions". In: Journal of Mathematical Physics 58.7 (July 2017), p. 071705. ISSN: 1089-7658. DOI: 10.1063/1.4993198. URL: http://dx.doi.org/10.1063/1.4993198.

[141] Stefan Hollands, Akihiro Ishibashi, and Robert M. Wald. "BMS Supertranslations and Memory in Four and Higher Dimensions". In: Class. Quant. Grav. 34.15 (2017), p. 155005. DOI: 10.1088/1361-6382/aa777a. arXiv: 1612.03290 [gr-qc].

[142] Temple He and Prahar Mitra. "Asymptotic symmetries and Weinberg's soft photon theorem in Mink_{d+2}". In: JHEP 10 (2019), p. 213. DOI: 10.1007/JHEP10(2019)213. arXiv: 1903.02608 [hep-th].

[143] Thomas T. Dumitrescu et al. "Infinite-Dimensional Fermionic Symmetry in Supersymmetric Gauge Theories". In: (Nov. 2015). arXiv: 1511.07429 [hep-th].

[144] Temple He and Prahar Mitra. "Asymptotic symmetries in (d + 2)-dimensional gauge theories". In: JHEP 10 (2019), p. 277. DOI: 10.1007/JHEP10(2019)277. arXiv: 1903.03607 [hep-th].

[145] Daniel Kapec and Prahar Mitra. "A d-Dimensional Stress Tensor for Mink_{d+2} Gravity". In: JHEP 05 (2018), p. 186. DOI: 10.1007/JHEP05(2018)186. arXiv: 1711.04371 [hep-th].

[146] D. Christodoulou and S. Klainerman. The Global nonlinear stability of the Minkowski space. Princeton: Princeton University Press, 1993.

[147] Sigbjørn Hervik, Marcello Ortaggio, and Lode Wylleman. "Minimal tensors and purely electric or magnetic spacetimes of arbitrary dimension". In: Class. Quant. Grav. 30 (2013), p. 165014. DOI: 10.1088/0264-9381/30/16/165014. arXiv: 1203.3563 [gr-qc].

[148] Miguel Campiglia and Alok Laddha. "Sub-subleading soft gravitons: New symmetries of quantum gravity?" In: Phys. Lett. B 764 (2017), pp. 218–221. DOI: 10.1016/j.physletb.2016.11.046. arXiv: 1605.09094 [gr-qc].

[149] Alok Laddha and Ashoke Sen. "Classical proof of the classical soft graviton theorem in D>4". In: Phys. Rev. D 101.8 (2020), p. 084011. DOI: 10.1103/PhysRevD.101.084011. arXiv: 1906.08288 [gr-qc].

[150] Geoffrey Compère and Adrien Fiorucci. "Advanced Lectures on General Relativity". In: (Jan. 2018). arXiv: 1801.07064 [hep-th].

[151] Cedomir Crnkovic and Edward Witten. “Covariant Description of Canonical Formalism in Geometrical Theories”. In: (Sept. 1986).

[152] Temple He and Prahar Mitra. “Covariant Phase Space and Soft Factorization in Non-Abelian Gauge Theories”. In: JHEP 03 (2021), p. 015. DOI: 10.1007/JHEP03(2021)015. arXiv: 2009.14334 [hep-th].

[153] Lars Grant et al. “Minisuperspace quantization of ’Bubbling AdS’ and free fermion droplets”. In: JHEP 08 (2005), p. 025. DOI: 10.1088/1126-6708/2005/08/025. arXiv: hep-th/0505079.

[154] Daniel Harlow and Jie-Qiang Wu. “Covariant phase space with boundaries”. In: JHEP 10 (2020), p. 146. DOI: 10.1007/JHEP10(2020)146. arXiv: 1906.08616 [hep-th].

[155] Laurent Freidel, Florian Hopfmüller, and Aldo Riello. “Asymptotic Renormalization in Flat Space: Symplectic Potential and Charges of Electromagnetism”. In: JHEP 10 (2019), p. 126. DOI: 10.1007/JHEP10(2019)126. arXiv: 1904.04384 [hep-th].

[156] Marc Henneaux and Cédric Troessaert. “Asymptotic structure of electromagnetism in higher spacetime dimensions”. In: Physical Review D 99.12 (June 2019). ISSN: 2470-0029. DOI: 10.1103/physrevd.99.125006. URL: http://dx.doi.org/10.1103/PhysRevD.99.125006.

[157] Alok Laddha and Ashoke Sen. “Gravity Waves from Soft Theorem in General Dimensions”. In: JHEP 09 (2018), p. 105. DOI: 10.1007/JHEP09(2018)105. arXiv: 1801.07719 [hep-th].

[158] Lydia Bieri and David Garfinkle. “Perturbative and gauge invariant treatment of gravitational wave memory”. In: Physical Review D 89.8 (Apr. 2014). ISSN: 1550-2368. DOI: 10.1103/physrevd.89.084039. URL: http://dx.doi.org/10.1103/PhysRevD.89.084039.

[159] Sangmin Choi and Ratindranath Akhoury. “BMS Supertranslation Symmetry Implies Faddeev-Kulish Amplitudes”. In: JHEP 02 (2018), p. 171. DOI: 10.1007/JHEP02(2018)171. arXiv: 1712.04551 [hep-th].

[160] Steven Weinberg. “Infrared photons and gravitons”. In: Phys. Rev. 140 (1965), B516–B524. DOI: 10.1103/PhysRev.140.B516.

[161] P. P. Kulish and L. D. Faddeev. “Asymptotic conditions and infrared divergences in quantum electrodynamics”. In: Theor. Math. Phys. 4 (1970), p. 745. DOI: 10.1007/BF01066485.

[162] Sayali Atul Bhatkar. "Ward identity for loop corrected soft photon theorem for massless QED coupled to gravity". In: (2019). arXiv: 1912.10229 [hep-th].

[163] Oscar Fuentealba et al. "The Bondi-Metzner-Sachs group in five space-time dimensions". In: (Nov. 2021). arXiv: 2111.09664 [hep-th].

[164] Tom Banks et al. "AdS dynamics from conformal field theory". In: (Aug. 1998). arXiv: hep-th/9808016.

[165] Chandramouli Chowdhury, Olga Papadoulaki, and Suvrat Raju. A numerical script to verify

[166] Brian Hatfield. Quantum field theory of point particles and strings. Perseus Books, 1991.

[167] Steven B. Giddings. "Flat space scattering and bulk locality in the AdS / CFT correspondence". In: Phys. Rev. D 61 (2000), p. 106008. DOI: 10.1103/PhysRevD.61.106008. arXiv: hep-th/9907129.

[168] Eric D'Hoker and Daniel Z. Freedman. "Supersymmetric gauge theories and the AdS / CFT correspondence". In: Theoretical Advanced Study Institute in Elem Jan. 2002. arXiv: hep-th/0201253.

[169] Stanley Deser and D. Boulware. "Stress-Tensor Commutators and Schwinger Terms". In: J. Math. Phys. 8 (1967), p. 1468. DOI: 10.1063/1.1705368.

[170] S.M. Christensen and M.J. Duff. "Quantizing Gravity with a Cosmological Constant". In: Nucl. Phys. B 170 (1980), pp. 480–506. DOI: 10.1016/0550-3213(80)90423-X.

[171] Saul A. Teukolsky. "Perturbations of a rotating black hole. 1. Fundamental equations for gravitational electromagnetic and neutrino field perturbations". In: Astrophys. J. 185 (1973), pp. 635–647. DOI: 10.1086/152444.

[172] Ezra Newman and Roger Penrose. "An Approach to gravitational radiation by a method of spin coefficients". In: J. Math. Phys. 3 (1962), pp. 566–578. DOI: 10.1063/1.1724257.

[173] Oscar J. C. Dias, Jorge E. Santos, and Maren Stein. "Kerr-AdS and its Near-horizon Geometry: Perturbations and the Kerr/CFT Correspondence". In: JHEP 10 (2012), p. 182. DOI: 10.1007/JHEP10(2012)182. arXiv: 1208.3322 [hep-th].

[174] E. T. Newman and R. Penrose. "Note on the Bondi-Metzner-Sachs group". In: J. Math. Phys. 7 (1966), pp. 863–870. DOI: 10.1063/1.1931221.

[175] J. N. Goldberg et al. "Spin s spherical harmonics and edth". In: J. Math. Phys. 8 (1967), p. 2155. DOI: 10.1063/1.1705135.

[176] S. W. Hawking and Gary T. Horowitz. "The Gravitational Hamiltonian, action, entropy and surface terms". In: Class. Quant. Grav. 13 (1996), pp. 1487–1498. DOI: 10.1088/0264-9381/13/6/017. arXiv: gr-qc/9501014.

[177] Vivek Iyer and Robert M. Wald. "Some properties of Noether charge and a proposal for dynamical black hole entropy". In: Phys. Rev. D 50 (1994), pp. 846–864. DOI: 10.1103/PhysRevD.50.846. arXiv: gr-qc/9403028.

[178] Thomas Faulkner et al. "Gravitation from Entanglement in Holographic CFTs". In: JHEP 1403 (2014), p. 051. DOI: 10.1007/JHEP03(2014)051. arXiv: 1312.7856 [hep-th].

[179] Thomas Faulkner et al. "Nonlinear Gravity from Entanglement in Conformal Field Theories". In: JHEP 08 (2017), p. 057. DOI: 10.1007/JHEP08(2017)057. arXiv: 1705.03026 [hep-th].

[180] Stefan Hollands and Robert M. Wald. "Stability of Black Holes and Black Branes". In: Commun. Math. Phys. 321 (2013), pp. 629–680. DOI: 10.1007/s00220-012-1638-1. arXiv: 1201.0463 [gr-qc].

[181] Daniel Kapec, Vyacheslav Lysov, and Andrew Strominger. "Asymptotic Symmetries of Massless QED in Even Dimensions". In: Adv. Theor. Math. Phys. 21 (2017), pp. 1747–1767. DOI: 10.4310/ATMP.2017.v21.n7.a6. arXiv: 1412.2763 [hep-th].