Aspects of celestial amplitude and flat-space limit of AdS/CFT

A Thesis

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by

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort has been made to indicate this clearly, with due reference to literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor R. Loganayagam, at the International Center for Theoretical Sciences of the Tata Institute for Fundamental Research, Bengaluru.

Sarthak Duary

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Date: 18th July, 2024

In my capacity as the formal supervisor of record of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

R. Loganayagam.

Professor R. Loganayagam

Date: 18th JULY 2024.

Dedicated to

the Divine essence within myself

– may I sense its presence and follow its guidance!

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"Never say, "I cannot"; for you are infinite. Even time and space are as nothing compared with your nature. You can do anything and everything, you are almighty."

- Swami Vivekananda.

"The cosmos is within us. We are made of star-stuff. We are a way for the universe to know itself."

- Carl Sagan.

Abstract

In this thesis, we study two key aspects of flat-space holography: celestial holography and the flat-space limit of AdS/CFT.

In the first part of the thesis, we explore celestial amplitude corresponding to $2d$ bulk $\mathcal{S}$-matrix. We consider scalar particles with identical mass and show that the celestial amplitude becomes the Fourier transform of the $2d$ $\mathcal{S}$-matrix written in the rapidity variable. We translate the crossing and unitarity conditions into the conditions on the celestial amplitude. For the $2d$ Sinh-Gordon model, we calculate the celestial amplitude perturbatively in coupling constant and and identify two distinct types: retarded and advanced. These two types arise due to a pole at the origin of the complex rapidity plane. This particular pole is very important in perturbation theory. We will go into detail on how to deal with this pole within the perturbation theory framework. To do this, we will introduce two celestial amplitudes that correspond to two different $i\epsilon$ prescriptions. We check that the crossing and unitarity conditions are satisfied for the celestial amplitude. Imposing the crossing and unitarity conditions to the celestial amplitude, we want to find amplitudes to the higher order in perturbation theory from the lower order i.e., to provide a "*proof of principle*" to show we can apply the bootstrap idea to the celestial amplitude. We also study the gravitational dressing condition of the $\mathcal{S}$-matrix in terms of the celestial amplitude and see that for the dressed celestial amplitude, the poles on the right half-plane get erased for several ansatzes.

The infrared (IR) divergence in the $\mathcal{S}$-matrix arises from the assumption of asymptotic decoupling. This assumption treats the asymptotic Hamiltonian as free, suggesting that the asymptotic states are Fock space states, and the fields behave as free fields in the asymptotic region of flat spacetime. Relaxing this assumption allows for the introduction of the Faddeev-Kulish state, resulting in an IR-finite $\mathcal{S}$-matrix. The Faddeev-Kulish state includes soft photon modes to dress the scattering state within the Fock space, thereby addressing the long-range effects of the electromagnetic interaction. In the second part of the thesis, we construct the AdS correction to the Faddeev-Kulish dressed state. A salient

feature of AdS spacetime is that it serves as a built-in infrared regulator, and when we take the flat-space limit, IR divergences will show up unless the asymptotic dynamics of fields is examined appropriately. The guiding principle in studying AdS radius-corrected Faddeev-Kulish dressing is the equivalence established between Wilson line dressing and Faddeev-Kulish dressing. We construct modes for the massive scalar field dressed by the Wilson line using bulk operator reconstruction. We establish a mapping between AdS radius-corrected soft photon modes and CFT current operators. This mapping allows us to express the AdS radius-corrected Faddeev-Kulish dressed state by utilizing it in the Wilson line dressing.

List of publications

Publications relevant to the thesis.

1. S. Duary, *Celestial amplitude for 2d theory*, *JHEP* **12** (2022) 060, [2209.02776].

2. S. Duary, *AdS correction to the Faddeev-Kulish state: migrating from the flat peninsula*, *JHEP* **05** (2023) 079, [2212.09509].

Other publications.

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2. S. Duary, E. Hijano and M. Patra, *Towards an IR finite $\mathcal{S}$-matrix in the flat limit of AdS/CFT*, [2211.13711].

3. S. Duary, *Flat limit of massless scalar scattering in* AdS_2, [2305.20037].

4. S. Duary and S. Maji, *Spectral representation in Klein space: simplifying celestial leaf amplitudes*, [2406.02342].

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Chapter 1

Introduction

"*The Pole. Yes, but under very different circumstances from those expected ... Great God! This is an awful place and terrible enough for us to have labored to it without the reward of priority.*"

–Robert Falcon Scott, Scott of the Antarctic.

The holographic principle stands out as a highly effective tool for understanding quantum gravity. It says that all information about a dynamic system with gravity is stored in a lower-dimensional boundary of the system. This principle is initially proposed by 't Hooft [1] and later expanded upon in the context of string theory by Susskind [2]. A striking manifestation of the holographic principle is exemplified by the entropy of black holes. Counterintuitively, the entropy of a black hole, which encodes its microscopic configuration details, is proportional to the area of its event horizon rather than its volume [3, 4]. This connection between entropy and area highlights the holographic nature of gravity, where the information content is holographically encoded on the boundary surface.

The precise realization of the holographic principle was formulated by Maldacena [14] based on Type IIB string theory. The duality is between the Type IIB string theory in the spacetime $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ super Yang-Mills theory (SYM) in four dimensions. Subsequently, Gubser, Klebanov, and, Polyakov (GKP) [16], and Witten [15] introduced the holographic dictionary for the Anti-de Sitter/Conformal Field Theory correspondence (AdS/CFT), enabling precise computations of physical observables to test the correspondence. More concrete examples can be derived from string and M theory, for instance, $\mathrm{AdS}_7/\mathrm{CFT}_6$, and $\mathrm{AdS}_4/\mathrm{CFT}_3$ [7, 14]. These are top-down holographic models due to their evident ultraviolet (UV) origin in string and M theory. Despite not explicitly invoking a particular UV completion like string theory, AdS/CFT, also known as holography or gauge/gravity (or sometimes gauge/string) duality, is broadly applicable. It says that quantum gravity in AdS can be equally described by a CFT on the boundary.

Now, across various energy scales, from particle colliders to gravitational wave detectors, the physics we observe can be effectively described by flat spacetime. Though we understand the holographic principle in asymptotically AdS spacetime, flat-space holography is still largely unexplored. Celestial holography is an approach to flat-space holography. Celestial holography proposes that quantum gravity, or quantum field theory (QFT) in asymptotically flat spacetime can be represented by a dual theory that lives on the codimension 2 sphere at the boundary. The theory is referred to as the celestial conformal field theory (CCFT) [5, 6]. In asymptotically flat spacetimes, the fundamental observable is the $\mathcal{S}$-matrix, which describes how particles scatter and interact with each other. Remarkably, the $\mathcal{S}$-matrix displays an inherent holographic structure, as it is defined using on-shell momenta evaluated at the boundary of spacetime at infinity. This specification of the $\mathcal{S}$-matrix in terms of asymptotic data at the boundary hints at an intrinsic holographic encoding of the scattering information onto a lower-dimensional surface.

Celestial holography introduces a novel reformulation of quantum field theory scattering processes by recasting the $\mathcal{S}$-matrix into a celestial amplitude expressed in terms of boost eigenstates. This transformation enables the $\mathcal{S}$-matrix to be interpreted as a correlator involving operators defined at specific points on the celestial sphere at the boundary of spacetime. The key aspect of this approach involves considering conformal primary wavefunctions that transform covariantly under the Lorentz group. By expressing the $\mathcal{S}$-matrix elements in this conformal primary basis, instead of the usual plane wave basis, the $\mathcal{S}$-matrix itself transforms as a conformal correlator under the conformal group. Effectively, celestial holography trades the conventional plane wave basis, typically used to describe scattering processes, for a basis of conformal primary wavefunctions. This basis transformation recasts the $\mathcal{S}$-matrix into a celestial amplitude, allowing it to be interpreted as a correlator of operators living on the celestial sphere at the boundary of spacetime. This new way of looking at scattering amplitudes in celestial holography suggests a deep connection between the fundamental observables of quantum field theory in flat spacetime and the holographic encoding of this information onto the celestial sphere, in line with the principles of holography.

The use of conformal representations to quantum field theory amplitudes, as employed in the celestial holography approach, can be traced back to an early concept introduced by Dirac [8]. He noted that the most suitable framework for the conformal group is an embedding space, where the group is represented linearly. This embedding space formalism aligns well with the techniques commonly utilized in the study of the conformal bootstrap program in conformal field theories, see e.g., [13] and references therein.

Recent insights of celestial amplitude mainly come from a bottom-up approach, translating properties of scattering amplitude into properties of celestial amplitude. The analyses so far involves taking perturbative scattering amplitude as input and producing celestial amplitude. The analyses leave the non-perturbative defining properties of CCFT unknown, lacking an intrinsic definition. Recent works [9–12] explores 4d models with explicit CCFT duals. In the end, our goal is to develop a top-down stringy construction of CCFT. Alternatively, it can be seen as a CFT-inspired approach to analyzing amplitudes. With an intrinsic description of consistent CCFT, we aim to utilize CFT-bootstrap techniques for computing and constraining scattering.

Now, evaluating the celestial amplitude for a massive scalar in dimensions higher than $2d$ is technically challenging. In $2d$, we have the integrable $\mathcal{S}$-matrices, and it would be interesting to compute the celestial amplitudes. In the second chapter of the thesis chapter 2, we initiate the exploration of the celestial amplitude for the $2d$ bulk $\mathcal{S}$-matrix. As a first step towards bootstrapping celestial amplitude, we focus on $2d$ scattering. We show that the celestial amplitude is essentially the Fourier transform of the $2d$ $\mathcal{S}$-matrix in terms of rapidity variables. Translating the crossing and unitarity conditions, we establish their counterparts for the celestial amplitude. Perturbatively in the coupling constant for the $2d$ Sinh-Gordon model, we verify that the celestial amplitude satisfies the crossing and unitarity conditions. For Sinh-Gordon model, due to the presence of the pole in the origin of the complex rapidity-plane there are two types of celestial amplitude, the retarded and the advanced corresponding to $\pm i\epsilon$ prescriptions. We aim to apply the bootstrap idea, utilizing the crossing and unitarity conditions, to derive higher-order celestial amplitudes from lower-order ones. Also, we translate the gravitational dressing condition of the $\mathcal{S}$-matrix in terms of the celestial amplitude. Through various ansatzes, we observe the elimination of poles on the right half-plane for the dressed celestial amplitude.

Now, we will move to the other aspect of flat-space holography. A key insight here is that the flat-space limit of AdS provides a holographic representation of the flat-space $\mathcal{S}$-matrix through the correlator of a boundary CFT [45–50].

To understand the flat-space limit, we consider an observer situated deep within the central region of an AdS spacetime, whose observations are limited to physics below the AdS length scale. From this observer's perspective, the geometry the observer perceive will appear flat, even though the observer is residing within the full AdS spacetime. The key point here is that flat spacetime is essentially a part of the AdS spacetime. Consequently, the physics of flat spacetime must be encoded within the physics of the AdS spacetime.

Given the AdS/CFT correspondence, which establishes a duality between physics in AdS spacetime and a CFT living on its boundary, it follows that the physics of flat spacetime must also be encoded within these boundary CFTs, albeit in one higher dimension. This line of reasoning forms the general logic behind the flat-space limit of the AdS/CFT correspondence. However, the challenge lies in understanding precisely how to decode and extract the physics of flat spacetime from the boundary CFT description. The question is: knowing that flat-space physics is encoded in the CFT, how can we explicitly calculate and obtain the flat-space observables, such as the $\mathcal{S}$-matrix, from the boundary theory? Understanding flat-space holography means having a way to calculate flat-space $\mathcal{S}$-matrix in a lower-dimensional theory. Now, in a general theory of gravity, computing observables is difficult due to the presence of diffeomorphism invariance. However, in the context of AdS spacetime, we can circumvent this issue by considering boundary observables. This is particularly simple in the AdS/CFT correspondence, where we have a well-defined boundary on which we can place conformal operators and assemble correlation functions. These correlation functions in the boundary CFT are the key observables, and their main ingredients are the operators used to construct them, as well as the specific states of the CFT in which these operators are evaluated. In the AdS, these correlators correspond to insertions of operators at the boundary, which can be computed using bulk techniques like Witten diagrams in the AdS interior, yielding equivalent results. However, it is crucial to note that these boundary correlators are asymptotic constructs, meaning they are defined at the boundary of spacetime. The only precise observable we can define in the flat-space is the $\mathcal{S}$-matrix, which represents the overlap between scattering states. In this context, the main ingredients for constructing the $\mathcal{S}$-matrix are the scattering states. To understand the connection between the AdS/CFT description and flat-space physics, we need to establish a relationship between the conformal operators in the boundary CFT and the scattering states in the bulk. In other words, we need to understand how to construct the scattering states by smearing or combining the conformal operators in a specific way. Various approaches in the literature address the flat-space limit of AdS/CFT through different representations of CFT: position space [54, 55, 58, 59], mellin space [50, 63, 65], and momentum space [56, 61]. See e.g., [54–62] for recent developments in the flat-space limit of AdS/CFT.

This phenomenon of perceiving flat geometry deep within a curved spacetime bears an intriguing parallel to our everyday experience on Earth. Despite the Earth's surface being globally spherical, we perceive it as essentially flat within our local environment and range of activities. This apparent discrepancy arises because the scale of our daily experiences and the distance to our observable horizon are minuscule compared to the vast radius of the Earth itself. Just as an observer situated deep within the AdS geometry perceives flatness due to being confined within a small region compared to the overall curvature scale, our perception of flatness on the Earth's surface mirrors this phenomenon, albeit on a macroscopic scale. The curvature of the Earth's spherical surface is negligible over the relatively small distances we traverse, leading to our everyday experience of a flat world.

Now, we will give a brief outline of how we construct the $\mathcal{S}$-matrix utilizing techniques from AdS/CFT. The goal of flat-space scattering theory is to compute the $\mathcal{S}$-matrix, which requires calculating the overlap between two different scattering states of the full Hamiltonian. In interacting theories where the states of the full Hamiltonian are generally intractable, we use approximation for the scattering states. The approximation is the assumption of asymptotic decoupling, which says that the Hamiltonian at null infinity, the asymptotic boundary of flat spacetime is effectively free. Now, the key question is: how can we construct these Fock space scattering states, which are approximated by free particle states at asymptotic infinity, using techniques from the AdS/CFT? The key tool we use is the bulk operator reconstruction. We define the scattering state as an operator acting on the vacuum state. We consider a local bulk operator deep within the AdS spacetime. We can identify a specific scattering region deep inside the AdS geometry, where the spacetime appears effectively flat. Within the AdS/CFT framework, we have the bulk operator reconstruction, which allows us to represent any bulk operator as a smearing of operator defined on the boundary of AdS spacetime. Applying this bulk reconstruction technique to the local operator within the flat scattering region, we can express it in terms of boundary operators in the CFT. At this point, we invoke the asymptotic decoupling approximation, which says that the field we are reconstructing in the asymptotic region of flat-space is free. This allows us to decompose the field into plane waves. Now, we extract the modes like the creation, and annihilation operators through a Fourier transform, and then take a flat-space limit, and that will tell what the map between creation, and annihilation operators in flat-space, is with CFT operators in the boundary of AdS.

In the third chapter of the thesis chapter 3, we explore infrared flat-space physics from the flat-space limit of AdS/CFT. The infrared (IR) divergence in the $\mathcal{S}$-matrix is due to the assumption of asymptotic decoupling. This assumption treats the asymptotic Hamiltonian as free, implying that the asymptotic states are Fock space states and the fields behave as free fields in the asymptotic region of flat spacetime. By relaxing this assumption, Faddeev-Kulish state can be introduced, leading to an $\mathcal{S}$-matrix that is IR finite. Faddeev-Kulish state incorporates soft photon modes to dress the scattering state in the Fock space. The intuition is that there are always some soft photons present in the initial and final states due to the long-range nature of electromagnetic interactions. To obtain infrared-finite observables, one must account for these soft photons and modify the basis of asymptotic states to include them. The correct scattering states should be the asymptotic dressed states, which incorporate the soft radiation associated with the charged particles involved in the scattering processes. In the third chapter of the thesis chapter 3, we construct the AdS correction to the Faddeev-Kulish dressed state. A salient feature of AdS spacetime is that it serves as a built-in infrared regulator, and when we take the flat-space limit, infrared divergences will show up unless the asymptotic dynamics of fields is examined appropriately. The guiding principle in studying AdS radius-corrected Faddeev-Kulish dressing is the equivalence established between Wilson line dressing and Faddeev-Kulish dressing. We construct modes for the massive scalar field dressed by the Wilson line using vanilla bulk operator reconstruction. We establish a mapping between AdS radius-corrected soft photon modes and CFT current operators. This mapping allows us to express the AdS radius-corrected Faddeev-Kulish dressed state by inverting and utilizing it in the Wilson line dressing.

Organization of the thesis.

Now, we give the organizations of the remaining chapters of the thesis. The chapters 2 and 3 are based on published papers [116, 117] (publications in `Journal of High Energy Physics`). Each chapter begins with a broad introduction. To avoid repetition, those introductions are not included in this current chapter. Finally, we conclude the thesis by discussing conclusions, and open questions in the chapter 4.

1 Organization of the chapter 2.

The chapter 2 is organized as follows. First, in the section 2.2, we review the construction of massive celestial amplitude. In the section 2.3, we define the celestial 4-point amplitude for $2 \to 2$ scattering of massive scalar particles in $2d$. The massive celestial amplitude in $2d$ is given by the Fourier transform of the $\mathcal{S}$-matrix written in the rapidity variable. Subsequently, in the section 2.4, we evaluate the celestial amplitude in $2d$ Sinh-Gordon model by perturbatively expanding the $\mathcal{S}$-matrix. In the section 2.5, we translate the crossing and unitarity conditions in celestial space and check the crossing and unitarity conditions for the Sinh-Gordon model. Moving on to the section 2.6, we discuss about reconstructing the higher-order correction to the 4-point celestial amplitude from the lower-order amplitude using the crossing and unitarity conditions. In the section 2.7, we translate the gravitational dressing condition of the $2d$ QFT amplitude into celestial space, noting that this condition removes poles from the right half-plane, corresponding to celestial amplitudes for functions with multiple poles on the right half-plane. In the section 2.8, we summarize our main results.

2 Organization of the chapter 3.

The chapter 3 is organized as follows. In the section 3.2, we review various things to make the chapter 3 self-contained. In the section 3.2.1, we discuss the flat-space limit. We discuss how to extract the creation and annihilation operators for the free massive scalar field in terms of the CFT operators in this flat-space limit. We use the equivalence between the Faddeev-Kulish dressing and the Wilson line dressing as our main strategy to construct the AdS correction to the Faddeev-Kulish dressed state. Moving on to the section 3.2.2, we study the soft Wilson line dressed field in AdS and evaluate the CFT operator corresponding to this dressed field. The soft Wilson line dressed scalar field turns itself into the free field and thus can be reconstructed implementing the vanilla bulk operator reconstruction. In the section 3.3, we study the CFT representation as well as the mixed representation of the Faddeev-Kulish dressed state. In the section 3.4, we calculate the AdS corrected Faddeev-Kulish dressed state. We express the CFT current operators in terms of AdS radius-corrected photon creation/annihilation operators in the section 3.4.1, and 3.4.2. We express the dressed creation operator in terms of the AdS radius-corrected creation/annihilation modes of the photon. The dressed creation operator acting on the vacuum gives the AdS radius-corrected Faddeev-Kulish dressed state. Finally, we save the section 3.5 for summarizing our conclusions.

Chapter 2

Celestial amplitude for $2d$ theory

"*If you look up at the sky on a clear cloudless night, you appear to see a hemispherical dome above you, punctuated by myriads of stars.*"

–Roger Penrose, The road to reality.

This chapter is based on the paper

- S. Duary, *Celestial amplitude for 2d theory*, *JHEP* **12** (2022) 060, [2209.02776].

2.1 Introduction

While evaluating Quantum field theory (QFT) scattering amplitudes, we generally express the asymptotic states in terms of asymptotic plane wave solutions to the free wave equation which is energy-momentum eigenstates. This conventional plane wave basis makes spacetime translation invariance manifest due to energy-momentum conservation but obscures conformal invariance.

The Lorentz group in $\mathbb{R}^{1,d-1}$ is the same as the Euclidean conformal group in $(d-2)$-dimensions. This indicates that the d-dimensional $\mathcal{S}$-matrix is related to $(d-2)$-dimensional conformal field theory correlation functions.

As opposed to the plane wave basis which is energy-momentum eigenstate, in [5, 6], a new basis called the conformal primary basis which is the boost eigenstate is constructed for both massive and massless particles. In this basis, free fields transform as conformal primaries under the Lorentz group and the $\mathcal{S}$-matrix elements transform manifestly as conformal correlation functions on the celestial sphere i.e., the sphere at null infinity. The scattering amplitude computed in this basis is known as celestial amplitude, and we can express the bulk $\mathcal{S}$-matrix in terms of a boundary correlation function of 'celestial CFT' that lives in the sky as in fig.2.1.

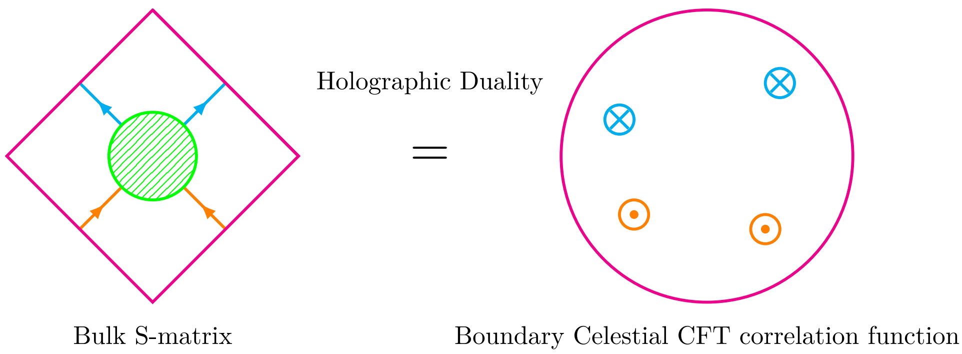

Figure 2.1: Bulk $\mathcal{S}$-matrix mapped to Boundary Celestial CFT correlation function

The motivation for seeking this conformal basis is to understand the holographic nature of quantum gravity in asymptotically flat spacetimes. Realization of the holographic duality [14–16] from the bottom up by finding the symmetries that both sides of the holographic dual pair obey [24, 25] gives rise to the enhancement of the Lorentz symmetry to full Virasoro [26] and the existence of a stress tensor in a $2d$ CFT obeying Ward identity constructed from subleading soft-graviton mode in the bulk [27]. These observations along with the equivalence between soft theorems and asymptotic symmetries i.e., soft theorems recasted as conservation laws associated with large gauge symmetries lead to the proposal that there exists a holographic duality between the theory of gravity in four-dimensional asymptotically flat spacetimes and some sort of exotic CFT living on the two-dimensional celestial sphere at null infinity.

The flat-space holography is initiated by the work of de Boer and Solodukhin [29]. From dS and AdS slicing of Minkowski space, they postulate that the flat-space in four dimensions has some description in terms of a theory living on the boundary of these dS and AdS slices which is identified with the celestial sphere.

In this celestial holography paradigm, we study the celestial amplitude for the $2d$ bulk scattering. We construct the map to evaluate celestial amplitude from bulk $\mathcal{S}$-matrix which is implemented by the change of basis from energy-momentum eigenstates to boost eigenstates. To summarize, for massive scalar particles the celestial amplitude is the fourier transform of the $2d$ $\mathcal{S}$-matrix written in the rapidity variable. We calculate the perturbative celestial amplitude for the $2d$ Sinh-Gordon model. One subtle point here is that in $2d$, the perturbative expansion of the Sinh-Gordon $\mathcal{S}$-matrix contains pole at rapidity $\theta = 0$. Therefore while evaluating the perturbative celestial amplitude, we should implement $i\epsilon$ prescription and as a consequence of it there are two types of celestial amplitude which we indicate by the retarded and advanced celestial amplitude. In the celestial space, we map the crossing and unitarity conditions to calculate the celestial amplitude to the higher order in perturbation theory from the lower one by imposing these constraints which we refer as "celestial bootstrap". We check the crossing and unitarity conditions for the Sinh-Gordon model.

In AdS, solving conformal crossing equation, the one-loop correction to the four point amplitude is evaluated for scalar ϕ^4 theory in [30]. To apply the CFT-bootstrap technique in flat space to find the amplitude, we map the crossing and unitarity conditions in the celestial space for $2d$ scattering. For massive scalar case, in higher than $2d$, technically it is a bit challenging to evaluate the celestial amplitude even at tree level say for massive ϕ^4 theory in $4d$, for that as a stepping stone towards bootstrapping celestial amplitude we restrict to $2d$ scattering. We also find that imposing the crossing and unitarity conditions to the celestial amplitude is not enough to determine the higher-order perturbative celestial amplitude from the lower order. There is an extra term which cannot be fixed by the crossing and unitarity conditions. We study the gravitational dressing condition of the $\mathcal{S}$-matrix in terms of the celestial amplitude. We see that the poles on the right half-plane of the dressed celestial amplitude get erased for the functions having multiple poles on the right half-plane.

Organization of the chapter.

The chapter is organized as follows. In the section 2.2, we review the construction of massive celestial amplitude. In the section 2.3, we define the celestial 4-point amplitude for $2 \to 2$ scattering of massive scalar particles in $2d$. The massive celestial amplitude in $2d$ is given by the Fourier transform of the $\mathcal{S}$-matrix written in the rapidity variable. In the section 2.4, we evaluate the celestial amplitude in $2d$ Sinh-Gordon model by perturbatively expanding the $\mathcal{S}$-matrix. In the section 2.5, we translate the crossing and unitarity conditions in celestial space and check the crossing and unitarity conditions for the Sinh-Gordon model. In the section 2.6, we discuss about reconstructing the higher-order correction to the 4-point celestial amplitude from the lower-order amplitude using the crossing and unitarity conditions. In the section 2.7, we translate the gravitational dressing condition of the $2d$ QFT amplitude in celestial space. We see that the gravitational dressing condition translated to celestial space acts as a pole eraser from the right half-plane corresponding to the celestial amplitude for the functions having multiple poles on the right half-plane. In the section 2.8, we summarize our main results and discuss future directions.

2.2 Massive Celestial amplitude

In this section, we review the construction of massive celestial amplitude described in [6]. We define a massive scalar conformal primary wavefunction $\phi^{\pm}_{\Delta}(X^{\mu}; \vec{w})$ as a solution to the massive Klein-Gordon equation of mass m in $\mathbb{R}^{1,d-1}$ that under a Lorentz group $SO(1, d-1)$ transformation transforms covariantly as a scalar conformal primary operator in $(d-2)$-dimensions. The massive scalar conformal primary wavefunction $\phi^{\pm}_{\Delta}(X^{\mu};\vec{w})$ in $\mathbb{R}^{1,d-1}$ admits the Fourier expansion on the plane waves

$$\phi^{\pm}_{\Delta}(X^{\mu};\vec{w}) = \int_{H_{d-1}} [d\hat{p}] \;\; G_{\Delta}(\hat{p};\vec{w}) \, \exp[\pm \; im \; \hat{p}.X], \tag{2.1}$$

where the on-shell momenta per mass, a unit timelike vector $\hat{p}(y,\vec{z})$ satisfying $\hat{p}^2 = -1$ can be parametrized using the $(d-1)$-dimensional hyperbolic space H_{d-1} coordinates y $(y > 0)$ and $\vec{z} \in \mathbb{R}^{d-2}$ as

$$\hat{p}(y,\vec{z}) = \left(\frac{1+y^2+|\vec{z}|^2}{2y}, \frac{\vec{z}}{y}, \frac{1-y^2-|\vec{z}|^2}{2y} \right). \tag{2.2}$$

The H_{d-1} metric is

$$ds^2_{H_{d-1}} = \frac{dy^2 + d\vec{z}.d\vec{z}}{y^2}. \tag{2.3}$$

Here, $[d\hat{p}]$ is the $SO(1,d-1)$ invariant measure on H_{d-1}

$$\begin{aligned} \int_{H_{d-1}} [d\hat{p}] &\equiv \int \frac{d^{d-1}\hat{p}^i}{\hat{p}^0} \\ &= \int_0^{\infty} \frac{dy}{y^{d-1}} \int d^{d-2}\vec{z}, \quad i = 1,\dots,d-1, \quad \hat{p}^0 = \sqrt{\hat{p}^i\hat{p}^i + 1}. \end{aligned} \tag{2.4}$$

Demanding the conformal covariance of $\phi^{\pm}_{\Delta}(X^{\mu};\vec{w})$ determines the Fourier coefficient, as the scalar bulk-to-boundary propagator in H_{d-1}. The scalar bulk-to-boundary propagator, $G_{\Delta}(\hat{p};\vec{w})$ in H_{d-1} is given by [15]

$$G_{\Delta}(\hat{p};\vec{w}) = \left(\frac{y}{y^2 + |\vec{z}-\vec{w}|^2} \right)^{\Delta}, \tag{2.5}$$

where $\vec{w} \in \mathbb{R}^{d-2}$ lies on the boundary of H_{d-1}. The scalar bulk-to-boundary propagator written in terms of $\hat{p}^{\mu}(y,\vec{z})$ and null momentum $q^{\mu}(\vec{w}) = (1+|\vec{w}|^2, 2\vec{w}, 1-|\vec{w}|^2)$ in $\mathbb{R}^{1,d-1}$ is given by [13]

$$G_{\Delta}(\hat{p};q) = \frac{1}{(-\hat{p}.q)^{\Delta}}. \tag{2.6}$$

Using the mapping from the plane wave to the conformal primary wavefunction given by eq.(2.1), the $\mathcal{S}$-matrix elements in the conformal prinary basis is given in terms of an integral transform

$$\tilde{\mathcal{A}}(\Delta_i, \vec{w}_i) = \prod_{k=1}^{n} \int_{H_{d-1}} [d\hat{p}_k] G_{\Delta_k}(\hat{p}_k; \vec{w}_k) \mathcal{A}(\pm m_i \hat{p}_i^{\mu}), \tag{2.7}$$

where $\pm m_i \hat{p}_i^{\mu}$ parametrization depends on whether the particle is incoming or outgoing. In the r.h.s., $\mathcal{A}(\pm m_i \hat{p}_i^{\mu})$ is the momentum space amplitude along with the momentum conserving delta function. We define $\tilde{\mathcal{A}}(\Delta_i, \vec{w}_i)$ by the massive celestial amplitude. Under the conformal symmetry action the massive celestial amplitude transforms covariantly as a $(d-2)$-dimensional CFT n-point function of scalar primaries with dimension Δ_i

$$\tilde{\mathcal{A}}(\Delta_i, \vec{w}_i'(\vec{w}_i)) = \prod_{k=1}^{n} \left| \frac{\partial \vec{w}_k'}{\partial \vec{w}_k} \right|^{\frac{-\Delta_k}{d-2}} \tilde{\mathcal{A}}(\Delta_i, \vec{w}_i). \tag{2.8}$$

Here, eq.(2.1) represents the massive scalar conformal primary wavefunction $\phi^{\pm}_{\Delta}(X^{\mu}; \vec{w})$ in $\mathbb{R}^{1,d-1}$ as a Fourier expansion in plane waves. The celestial amplitude involves an integral over the bulk coordinate X, and utilizing the Fourier expansion of the conformal primary wavefunction allows us to evaluate these bulk integrals. This results in a momentum-conserving delta function, which is incorporated into the definition in eq.(2.7) as $\mathcal{A}(\pm m_i \hat{p}_i^{\mu})$, representing the momentum space amplitude alongside the momentum-conserving delta function.

2.3 Massive Celestial amplitude for $2 \to 2$ scattering: The Celestial point

In this section, we define the celestial 4-point amplitude for $2 \to 2$ scattering of massive scalar particles in $2d$. In $2d$, the on-shell momenta of particles of mass m can be written in the rapidity parametrization θ

$$\begin{aligned} p^\mu &= (p^0 = E,\ p^1 = p) \\ &= m(\cosh\theta, \sinh\theta). \end{aligned} \tag{2.9}$$

We consider the elastic scattering process of identical real scalar particles of mass m with rapidities θ_1 and θ_2. Energy and momentum conservation for identical particles give

$$\theta_1 = \theta_4 \quad , \quad \theta_2 = \theta_3. \tag{2.10}$$

The $2 \to 2$ $\mathcal{S}$-matrix is given by

$$\begin{aligned} \mathcal{S}_{2\to 2} = \langle \theta_4, \theta_3 \,|\theta_1, \theta_2\rangle &= 4p_1^0 p_2^0 \times (2\pi)^2 \delta(p_1^1 - p_4^1)\delta(p_2^1 - p_3^1) + i(2\pi)^2 \delta^{(2)}(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu)\ \mathcal{T} \\ &= 4(2\pi)^2 \delta(\theta_1 - \theta_4)\delta(\theta_2 - \theta_3)\left(1 + \frac{i\ \mathcal{T}(\theta_1 - \theta_2)\,\mathrm{csch}(\theta_1 - \theta_2)}{4m^2}\right). \end{aligned} \tag{2.11}$$

We define $\mathcal{S}$-matrix which is dependent on the difference of rapidities as

$$S(\theta_1 - \theta_2) \equiv 1 + \frac{i\mathcal{T}(\theta_1 - \theta_2)\,\mathrm{csch}(\theta_1 - \theta_2)}{4m^2}.$$

The Mandelstem variables are

$$s = (p_1 + p_2)^2 = 4m^2 \cosh^2\left(\frac{\theta}{2}\right), \quad t = 4m^2 - s, \quad u = 0.$$

s is the center of mass energy squared, t-channel gives $\theta \to i\pi - \theta$. In $2d$, the celestial 4-point amplitude of the massive conformal primary wavefunction is

$$\tilde{\mathcal{A}} = \left(\prod_{i=1}^{4} \int \frac{d\hat{p}_i^1}{\hat{p}_i^0}\right) \times \prod_{i=1}^{4} G_{\Delta_i}(\hat{p}_i)\ \mathcal{S}_{2\to 2}, \tag{2.12}$$

where, $\hat{p}_i^\mu \equiv \hat{p}^\mu(\theta_i) = \frac{p_i^\mu}{m} = (\cosh\theta_i, \sinh\theta_i)$ and $G_{\Delta_i}(\hat{p}_i)$ is the bulk-to-boundary propagator in H_1. The scalar bulk-to-boundary propagator in H_1 is

$$\begin{aligned} G_\Delta(\hat{p}; q) &= \frac{1}{(-\hat{p}.q)^\Delta} = \frac{1}{(\cosh\theta - \sinh\theta)^\Delta} \\ &= e^{\Delta\theta}. \end{aligned} \tag{2.13}$$

Here, $q = (1, 1)$ since, conformal boundary of H_1 are specified by 2 dimensional points on the projective null cone

$$-(q^0)^2 + (q^1)^2 = 0 \quad , \quad q \sim \lambda q.$$

The massive scalar conformal primary wavefunctions in d dimensions are delta-function normalizable when Δ belongs to the principal continuous series of the irreducible unitary $SO(1, d-1)$ representations,

$$\Delta \in \frac{d-2}{2} + i\mathbb{R}.$$

Therefore, for the $2d$ scattering, Δ is pure imaginary. Now, we redefine $i\Delta$ as the conformal dimension of the conformal primary wavefunction in $2d$. The H_1 scalar bulk-to-boundary propagator $G_\Delta(\hat{p}) = e^{i\Delta\theta}$. The normalization condition for the H_1 scalar bulk-to-boundary propagator is

$$\int_{-\infty}^{\infty} d\theta e^{i\Delta_i\theta} e^{i\Delta_j\theta} = 2\pi\ \delta(\Delta_i + \Delta_j). \tag{2.14}$$

Now, the integral measure is

$$\prod_{i=1}^{4} \int \frac{d\hat{p}_i^1}{\hat{p}_i^0} = \prod_{i=1}^{4} \int_{-\infty}^{\infty} d\theta_i.$$

We can express celestial 4-point amplitude in eq.(2.12) as

$$\begin{aligned}\tilde{\mathcal{A}} &= 4(2\pi)^2 \prod_{i=1}^{4} \int_{-\infty}^{\infty} d\theta_i \;\; e^{i\Delta_i\theta_i} S(\theta_1-\theta_2)\delta(\theta_1-\theta_4)\delta(\theta_2-\theta_3) \\ &= 4(2\pi)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\theta_4 \; d\theta_3 \; S(\theta_4-\theta_3)\; e^{i\Delta_1\theta_4} e^{i\Delta_2\theta_3} e^{i\Delta_3\theta_3} e^{i\Delta_4\theta_4} \\ &= 4(2\pi)^2 \frac{1}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\theta_+ \; d\theta_- \; S(\theta_-) e^{\frac{i}{2}(\Delta_1+\Delta_4+\Delta_2+\Delta_3)\theta_+} e^{\frac{i}{2}(\Delta_1+\Delta_4-\Delta_2-\Delta_3)\theta_-} \\ &= 16\pi^3 \; \delta(\Delta_+) \; \mathcal{A}(\Delta_-).\end{aligned} \tag{2.15}$$

where, we define light-cone coordinates $\theta_{\pm} = \theta_4 \pm \theta_3$, $\Delta_{\pm} = \frac{1}{2}(\Delta_1 + \Delta_4 \pm \Delta_2 \pm \Delta_3)$ and

$$\mathcal{A}(\Delta_-) \equiv \int_{-\infty}^{\infty} d\theta e^{i\Delta_-\theta_-} S(\theta_-).$$

The delta function $\delta(\Delta_+)$ from the integration over the collective repidity coordinates $\sum_i \theta_i$. This delta function $\delta(\Delta_+)$ reflects the boost symmetry in the bulk. Now, we strip off the overall delta function $\delta(\Delta_+)$ and name the conjugate pair of θ_- as $\omega \equiv \Delta_- = \frac{1}{2}(\Delta_1 + \Delta_4 - \Delta_2 - \Delta_3)$, the $2d$ celestial amplitude is the fourier transform of the S-matrix with respect to rapidity.

$$\mathcal{A}(\omega) \equiv \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(\theta). \tag{2.16}$$

Physically, the rapidity θ shifts under the boost as $\theta \to \theta + c$. Therefore, in order to diagonalize the boost which is achieved by the conformal basis, we need to perform the Fourier transform of the $\mathcal{S}$-matrix $S(\theta)$ with respect to rapidity. Here, there is no celestial coordinate and the dual theory is zero-dimensional which we refer as the "celestial point". The CFT correlation function "dual" to the bulk $\mathcal{S}$-matrix would be a zero-dimensional CFT correlation function or said differently, an operator algebra without coordinates. We will now briefly explain the definition and significance of the rapidity variable in the context of scattering. The on-shell two-momenta of particles with mass m can be expressed in the rapidity parametrization θ, for example

$$p^{\mu} = m(\cosh\theta, \sinh\theta). \tag{2.17}$$

An important point to note is that boosts act very simply in terms of the rapidity θ. Under a boost, the rapidity θ shifts as $\theta \to \theta + c$. Rapidity is the variable that is conjugate to boost. The eigenstates of boosts are essentially Fourier transforms with respect to rapidity. Thus, physically to diagonalize the boost using the conformal primary basis, and construct celestial amplitude, we must Fourier transform the $\mathcal{S}$-matrix $S(\theta)$ with respect to rapidity.

2.4 Celestial amplitude in 2d Sinh-Gordon model

In this section, we evaluate the celestial amplitude in $2d$ Sinh-Gordon model by perturbatively expanding the $\mathcal{S}$-matrix of the Sinh-Gordon model. The Lagrangian of the Sinh-Gordon model [18, 28] is

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{b^2}(\cosh(b\phi) - 1). \tag{2.18}$$

The $\mathcal{S}$-matrix for Sinh-Gordon model with the parameter α related to the coupling b is

$$S(\theta) = \frac{\sinh\theta - i\sin\alpha}{\sinh\theta + i\sin\alpha}, \quad \alpha = \frac{\pi b^2}{8\pi + b^2}. \tag{2.19}$$

For the exact S-matrix for the Sine-Gordon model see [17]. We can verify the $\mathcal{S}$-matrix by expanding the Lagrangian perturbatively in coupling b

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{m^2b^2}{4!}\phi^4 + \frac{m^2b^4}{6!}\phi^6 + \cdots, \tag{2.20}$$

and calculate the perturbative expansion of the $\mathcal{S}$-matrix to first few orders in b^2. The perturbative expansion of the $\mathcal{S}$-matrix with respect to b^2 is given by

$$\begin{aligned}
S^{(0)}(\theta) &= 1\\
S^{(1)}(\theta) &= -\frac{1}{4} i b^2 \mathrm{csch}\theta\\
S^{(2)}(\theta) &= -\frac{b^4 \mathrm{csch}\theta(\pi \mathrm{csch}\theta - i)}{32\pi}\\
S^{(3)}(\theta) &= \frac{i b^6 \mathrm{csch}\theta \left(6\pi^2 \mathrm{csch}^2\theta - 12 i\pi \mathrm{csch}\theta + \pi^2 - 6\right)}{1536\pi^2}\\
S^{(4)}(\theta) &= \frac{b^8 \mathrm{csch}\theta \left(6\pi^3 \mathrm{csch}^3\theta - 18 i\pi^2 \mathrm{csch}^2\theta + 2\pi\left(\pi^2 - 9\right)\mathrm{csch}\theta - 3i\left(\pi^2 - 2\right)\right)}{12288\pi^3}\\
S^{(5)}(\theta) &= -i b^{10}\,\mathrm{csch}\,\theta\\
&\times \frac{\left(120\pi^4 \mathrm{csch}^4\theta - 480 i\pi^3 \mathrm{csch}^3\theta + 60\pi^2(\pi^2 - 12)\mathrm{csch}^2\theta - 160 i\pi(\pi^2 - 3)\mathrm{csch}\theta + \pi^4 - 120\pi^2 + 120\right)}{1966080\pi^4}.
\end{aligned} \tag{2.21}$$

Since, the perturbative expansion of the $\mathcal{S}$-matrix $S(\theta)$ contains poles at $\theta = 0$, we define the celestial amplitude using $i\epsilon$ prescription as

$$\mathcal{A}^{\pm}(\omega) \equiv \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(\theta \pm i\epsilon)$$

where

$$\mathcal{A}^{\pm}(\omega) = 2\pi\left[\delta(\omega) + b^2 f_1^{\pm}(\omega) + b^4 f_2^{\pm}(\omega) + \cdots\right], \tag{2.22}$$

$$f_n^{\pm}(\omega) = \frac{1}{2\pi(b^2)^n}\int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S^{(n)}(\theta \pm i\epsilon). \tag{2.23}$$

Here, we call $\mathcal{A}^{+}(\omega)$ as the retarded celestial amplitude and $\mathcal{A}^{-}(\omega)$ as the advanced celestial amplitude. Here, eq.(2.22) can be understood as the perturbative expansion of the retarded and advanced celestial amplitude and we call $f_n^{+}(\omega)$ and $f_n^{-}(\omega)$ as the perturbative retarded and advanced celestial amplitude. Now, we evaluate the Fourier transform by $+i\epsilon$ prescription. We shift the pole at $\theta = 0$ to $\theta = -i\epsilon$ and enclose the contour in the upper half-plane. Evaluating the Fourier transform using $+i\epsilon$ prescription, we get the perturbative retarded celestial amplitude

$$
\begin{aligned}
f_1^+(\omega) &= -\frac{1}{4(1+e^{\pi\omega})} \\
f_2^+(\omega) &= \frac{(\pi\omega + e^{\pi\omega}(\pi\omega+1)-1)(\coth(\pi\omega)-1)}{64\pi} \\
f_3^+(\omega) &= -\frac{e^{\frac{\pi\omega}{2}}(\coth(\pi\omega)-1)\left[(\pi^2(6\omega^2+5)+6)\sinh\left(\frac{\pi\omega}{2}\right)+12\pi\omega\cosh\left(\frac{\pi\omega}{2}\right)\right]}{1536\pi^2} \\
f_4^+(\omega) &= \frac{e^{\frac{\pi\omega}{2}}(\coth(\pi\omega)-1)}{12288\pi^3} \\
&\times\left[3\left(\pi^2\left(6\omega^2+5\right)+2\right)\sinh\left(\frac{\pi\omega}{2}\right)+\pi\omega\left(\pi^2\left(\omega^2+2\right)+18\right)\cosh\left(\frac{\pi\omega}{2}\right)\right] \\
f_5^+(\omega) &= -\frac{e^{\frac{\pi\omega}{2}}(\coth(\pi\omega)-1)}{1966080\pi^4}\times \\
&\left[\left(120\pi^2\left(6\omega^2+5\right)+\pi^4\left(5\omega^2\left(\omega^2-2\right)-14\right)+120\right)\sinh\left(\frac{\pi\omega}{2}\right)\right. \\
&\left.+80\pi\omega\left(\pi^2\left(\omega^2+2\right)+6\right)\cosh\left(\frac{\pi\omega}{2}\right)\right].
\end{aligned}
\tag{2.24}
$$

Next, we evaluate the integral by $-i\epsilon$ prescription. We shift the pole at $\theta = 0$ to $\theta = i\epsilon$ and enclose the contour in the lower half-plane. Evaluating the Fourier transform using $-i\epsilon$ prescription, we get the perturbative advanced celestial amplitude

$$
\begin{aligned}
f_1^-(\omega) &= \frac{1}{4(e^{-\pi\omega}+1)} \\
f_2^-(\omega) &= \frac{e^{\pi\omega}\left(\pi\omega + e^{\pi\omega}(\pi\omega-1)+1\right)(\coth(\pi\omega)-1)}{64\pi} \\
f_3^-(\omega) &= \frac{e^{\pi\omega}(\coth(\pi\omega)-1)\left[e^{\pi\omega}\left(\pi^2\left(3\omega^2+2\right)-12\pi\omega+6\right)-3\pi\omega(\pi\omega+4)-2\left(3+\pi^2\right)\right]}{3072\pi^2}.
\end{aligned}
\tag{2.25}
$$

We move the formulas for $f_4^-(\omega)$, $f_5^-(\omega)$ to the next page due to space constraints on the current page caused by the lengthy equation.

$$\begin{aligned}
f_4^-(\omega) &= \frac{e^{\pi\omega}(\coth(\pi\omega)-1)}{24576\pi^3} \\
&\times \left[\pi(\pi\omega+3)\left(\pi\left(\omega^2+2\right)+6\omega\right)+e^{\pi\omega}\left(\pi(\pi\omega-3)\left(\pi\left(\omega^2+2\right)-6\omega\right)-6\right)+6\right] \\
f_5^-(\omega) &= \frac{e^{\frac{3\pi\omega}{2}}(\coth(\pi\omega)-1)}{1966080\pi^4} \\
&\times \Bigg[\left(120\pi^2\left(3\omega^2+2\right)+\pi^4\left(5\left(\omega^2+4\right)\omega^2+16\right)+120\right)\sinh\left(\frac{\pi\omega}{2}\right) \\
&- 80\pi\omega\left(\pi^2\left(\omega^2+2\right)+6\right)\cosh\left(\frac{\pi\omega}{2}\right)\Bigg].
\end{aligned} \tag{2.26}$$

In appendix 2.A, we give the details of the computation of the perturbative retarded and advanced celestial amplitude.

2.5 Crossing and unitarity conditions in celestial space

In this section, our aim is to translate the crossing and unitarity conditions into the conditions on the celestial amplitude. In terms of rapidity, θ we express the crossing and the unitarity conditions as [18–20, 66]

$$\begin{aligned}
&S(\theta) = S(i\pi-\theta) \\
&|S(\theta)|^2 = 1,
\end{aligned} \tag{2.27}$$

where $\theta = \theta_1 - \theta_2$. The crossing condition physically implies the symmetry of the $\mathcal{S}$-matrix under the exchange of the s and t channels. Unitarity condition physically implies the probablity of getting 2-particle final state given initial state should be less than or equal to one, i.e., $|S(\theta)|^2 \leq 1$. Assuming integrability condition, we have $|S(\theta)|^2 = 1$. The crossing condition in celestial space is obtained by taking the Fourier transform of both sides of the crossing condition in the rapidity variable. Since, the perturbative expansion of the $\mathcal{S}$-matrix $S(\theta)$ contains poles at $\theta = 0$, therefore, perturbatively if we expand up to a given order we should take the Fourier transform of $S(\theta \pm i\epsilon)$. The crossing condition in celestial space is obtained by taking the Fourier transform of both sides

$$\begin{aligned}
\int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(\theta + i\epsilon) &= \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(i\pi - \theta + i\epsilon) \\
&= -\int_{i\pi+\infty}^{i\pi-\infty} d\theta' e^{i\omega(i\pi-\theta')} S(\theta' + i\epsilon) \quad (i\pi - \theta \equiv \theta') \\
&= \int_{i\pi-\infty}^{i\pi+\infty} d\theta' e^{i\omega(i\pi-\theta')} S(\theta' + i\epsilon) \\
&= e^{-\omega\pi} \int_{-\infty}^{+\infty} d\theta' e^{-i\omega\theta'} S(\theta' + i\epsilon) \\
&= e^{-\omega\pi} \mathcal{A}^{+}(-\omega) \\
\implies \mathcal{A}^{+}(\omega) = e^{-\omega\pi} \mathcal{A}^{+}(-\omega).
\end{aligned} \tag{2.28}$$

where, in the last step we use

$$\int_{i\pi-\infty}^{i\pi+\infty} d\theta' e^{i\omega(i\pi-\theta')} S(\theta') = \int_{-\infty}^{+\infty} d\theta' e^{i\omega(i\pi-\theta')} S(\theta'). \tag{2.29}$$

which is valid when we have no poles in the physical strip $0 < \mathrm{Im}(\theta) < \pi$. Similarly, the crossing condition while taking the Fourier transform of

$$\int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(\theta - i\epsilon)$$

becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(\theta - i\epsilon) &= \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(-i\pi - \theta - i\epsilon) \\
\implies \mathcal{A}^{-}(\omega) &= e^{\omega\pi} \mathcal{A}^{-}(-\omega).
\end{aligned} \tag{2.30}$$

The crossing condition relates the retarded (advanced) celestial amplitude of positive ω to the retarded (advanced) celestial amplitude of negative ω and vice-versa. Perturbatively expanding $\mathcal{A}^{\pm}(\omega)$ as

$$\mathcal{A}^{\pm}(\omega) = 2\pi\Big[\delta(\omega) + b^2 f_1^{\pm}(\omega) + b^4 f_2^{\pm}(\omega) + b^6 f_2^{\pm}(\omega) + \cdots\Big],$$

$f_n^{\pm}(\omega)$ satisfy the crossing condition

$$f_n^{\pm}(\omega) = e^{\mp\omega\pi} f_n^{\pm}(-\omega). \tag{2.31}$$

The unitarity condition gives

$$\begin{aligned} &S(\theta + i\epsilon)S(\theta + i\epsilon)^* = 1 \\ \implies &S(\theta + i\epsilon)S(-\theta - i\epsilon) = 1. \end{aligned} \tag{2.32}$$

Here, the unitarity condition $|S(\theta)|^2 = 1$ becomes $S(\theta + i\epsilon)S(-\theta - i\epsilon)$. We combine the original unitarity condition with the real analyticity of the S-matrix

$$S(\theta + i\epsilon)^* = S(-\theta - i\epsilon).$$

The unitarity condition in celestial space is obtained by taking the Fourier transform of both sides of

$$S(\theta + i\epsilon)S(-\theta - i\epsilon) = 1.$$

Now, the multiplication of functions $S(\theta+i\epsilon)S(-\theta-i\epsilon)$ gets converted into the convolution under the Fourier transform as follows

$$\begin{aligned} \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(\theta + i\epsilon)S(-\theta - i\epsilon) &= \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(\theta + i\epsilon)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{i\omega'\theta} \mathcal{A}^-(\omega')\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \mathcal{A}^-(\omega') \int_{-\infty}^{\infty} S(\theta + i\epsilon) e^{i(\omega+\omega')\theta} d\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \mathcal{A}^-(\omega')\mathcal{A}^+(\omega + \omega'). \end{aligned} \tag{2.33}$$

Here,

$$\mathcal{A}^{\pm}(\omega) \equiv \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(\theta \pm i\epsilon).$$

Here, $\mathcal{A}^+(\omega)$ is the retarded celestial amplitude which is the Fourier transform using $+i\epsilon$ prescription enclosing the contour in the upper half-plane in counterclockwise way and $\mathcal{A}^-(\omega)$ is the advanced celestial amplitude which is the Fourier transform using $-i\epsilon$ prescription enclosing the contour in the lower half-plane in clockwise way. Therefore, the unitarity condition in celestial space becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \mathcal{A}^{+}(\omega+\omega')\mathcal{A}^{-}(\omega') = 2\pi\delta(\omega). \tag{2.34}$$

Perturbatively expanding $\mathcal{A}^{\pm}(\omega)$ as

$$\mathcal{A}^{\pm}(\omega) = 2\pi\Big[\delta(\omega) + b^2 f_1^{\pm}(\omega) + b^4 f_2^{\pm}(\omega) + b^6 f_2^{\pm}(\omega) + \cdots\Big],$$

$$f_n^{\pm}(\omega) = \frac{1}{2\pi(b^2)^n}\int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S^{(n)}(\theta \pm i\epsilon), \tag{2.35}$$

and then put it in eq.(2.34) we have the unitarity condition order by order in perturbation theory

$$f_1^{+}(\omega) + f_1^{-}(-\omega) = 0,$$

$$f_n^{+}(\omega) + f_n^{-}(-\omega) + \sum_{j=1}^{n-1}\int_{-\infty}^{\infty} d\omega' f_{n-j}^{+}(\omega+\omega') f_j^{-}(\omega') = 0 \quad (n>1). \tag{2.36}$$

One important thing to note is that while translating the crossing and unitarity conditions in celestial space the retarded and the advanced celestial amplitudes naturally appears. Now, since

$$S^{(n)}(\theta + i\epsilon) + S^{(n)}(-\theta - i\epsilon) = 2 \ \mathrm{Re}\ S^{(n)}(\theta + i\epsilon)$$

$f_n^{+}(\omega) + f_n^{-}(-\omega)$ is related ro the fourier transform of the real part of $S^{(n)}(\theta + i\epsilon)$

$$f_n^{+}(\omega) + f_n^{-}(-\omega) = 2\ \frac{1}{2\pi(b^2)^n}\int_{-\infty}^{\infty} d\theta e^{i\omega\theta}\mathrm{Re}S^{(n)}(\theta + i\epsilon). \tag{2.37}$$

Therefore, we can say that the unitarity condition in perturbation theory translated into celestial amplitude relates the Fourier transform of the real part of the perturbative $\mathcal{S}$-matrix at a given order to the convolution of the retarded and advanced celestial amplitude of lower-orders. We illustrate the unitarity condition by the diagram 2.2.

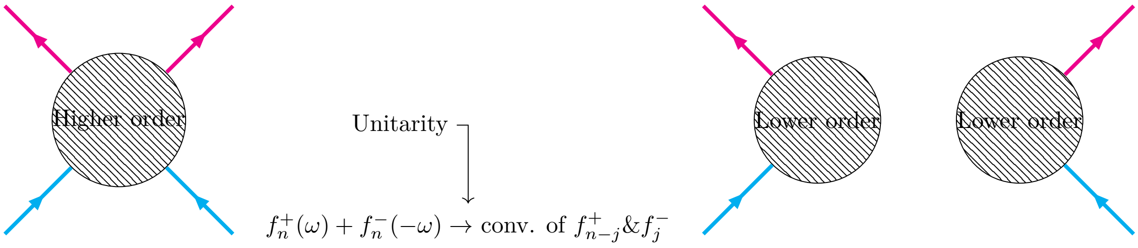

Figure 2.2: Pictorial representation of the unitarity condition in celestial space

2.5.1 Checking the crossing & unitarity conditions in celestial space for the 2d Sinh-Gordon model

We check the crossing and unitarity conditions for the $2d$ Sinh-model using celestial amplitudes obtained in section 2.4. $f_n^{\pm}(\omega)$ satisfy the crossing condition

$$f_n^{\pm}(\omega) = e^{\mp\omega\pi} f_n^{\pm}(-\omega). \tag{2.38}$$

The unitarity conditions are satisfied

$$f_1^+(\omega) + f_1^-(-\omega) = 0,$$

$$f_2^+(\omega) + f_2^-(-\omega) + \int_{-\infty}^{\infty} d\omega' f_1^+(\omega+\omega') f_1^-(\omega') = 0, \tag{2.39}$$

where the integral is given by

$$\int_{-\infty}^{\infty} d\omega' f_1^+(\omega+\omega') f_1^-(\omega') = \frac{\omega}{16(1-e^{\pi\omega})}. \tag{2.40}$$

In deriving the crossing and unitarity conditions in celestial space we should be extremely careful about the $i\epsilon$ prescription. The convolution of the perturbative retarded and retarded celestial amplitude and the perturbative advanced and advanced celestial amplitude diverges

$$\int_{-\infty}^{\infty} d\omega' f_1^{\pm}(\omega+\omega') f_1^{\pm}(\omega') \to \infty.$$

The proper $i\epsilon$ prescription involving the convolution of the retarded and advanced celestial amplitude cures this divergence.

2.6 Bootstrapping Celestial amplitude

In this section, we see that using the crossing and unitarity conditions, how much we can get for the higher order celestial amplitude from the lower order celestial amplitude. Crossing condition translated in celestial space gives

$$f_n^{\pm}(\omega) = e^{\mp\pi\omega} f_n^{\pm}(-\omega). \tag{2.41}$$

Unitarity condition translated in celestial space gives

$$f_n^{+}(\omega) + f_n^{-}(-\omega) + \sum_{j=1}^{n-1} \int_{-\infty}^{\infty} d\omega' f_{n-j}^{+}(\omega+\omega') f_j^{-}(\omega') = 0. \tag{2.42}$$

Now, the relation between $f_n^+(\omega)$ and $f_n^-(\omega)$ is

$$f_n^{+}(\omega) - f_n^{-}(\omega) + \frac{1}{2\pi(b^2)^n}\, 2\pi i \ \mathrm{Res}\Big[e^{i\omega\theta} S^{(n)}(\theta)\Big]_{\theta=0} = 0. \tag{2.43}$$

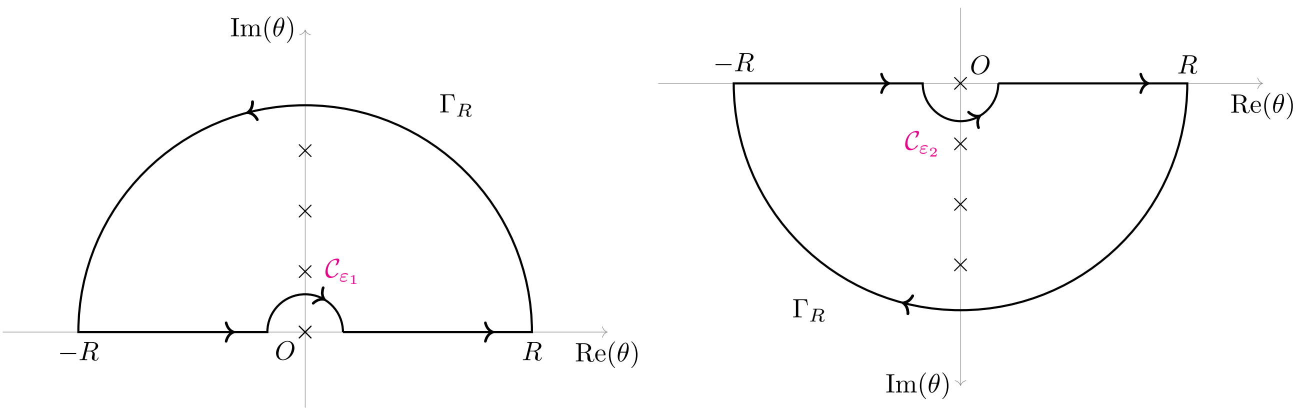

Figure 2.3: Contours for perturbative retarded and advanced celestial amplitudes

From the fig.2.3, we see that the difference between $f_n^+(\omega)$ and $f_n^-(\omega)$ is same as the negative of $\frac{1}{2\pi(b^2)^n}\, 2\pi i \ \mathrm{Res}\Big[e^{i\omega\theta} S^{(n)}(\theta)\Big]_{\theta=0}$. This is because when we enclose the contour in the upper half-plane, the small semicircle $\mathcal{C}_{\varepsilon_1}$ is in clockwise sense which adds to the

$\mathcal{C}_{\varepsilon_2}$ semicircle in clockwise sense after taking the difference of $f_n^+(\omega)$ and $f_n^-(\omega)$ and by convention while calculating the residue at $\theta = 0$ we enclose the semicircle in anticlockwise sense. As a consistency check we satisfy this equation by evaluating the residue of $e^{i\omega\theta}S^{(n)}(\theta)$ at $\theta = 0$ for $n = 1, \ldots, 5$.

$$\begin{aligned}
\frac{1}{2\pi b^2}\ 2\pi i\ \text{Res}\Big[e^{i\omega\theta}S^{(1)}(\theta)\Big]_{\theta=0} &= \frac{1}{4} \\
\frac{1}{2\pi (b^2)^2}\ 2\pi i\ \text{Res}\Big[e^{i\omega\theta}S^{(2)}(\theta)\Big]_{\theta=0} &= \frac{\pi\omega - 1}{32\pi} \\
\frac{1}{2\pi (b^2)^3}\ 2\pi i\ \text{Res}\Big[e^{i\omega\theta}S^{(3)}(\theta)\Big]_{\theta=0} &= \frac{3\pi^2\omega^2 - 12\pi\omega + 2\pi^2 + 6}{1536\pi^2} \\
\frac{1}{2\pi (b^2)^4}\ 2\pi i\ \text{Res}\Big[e^{i\omega\theta}S^{(4)}(\theta)\Big]_{\theta=0} &= \frac{\pi^3\omega^3 - 9\pi^2\omega^2 + 2\pi^3\omega + 18\pi\omega - 6\pi^2 - 6}{12288\pi^3}
\end{aligned} \tag{2.44}$$

$$\begin{aligned}
&\frac{1}{2\pi (b^2)^5}\ 2\pi i\ \text{Res}\Big[e^{i\omega\theta}S^{(5)}(\theta)\Big]_{\theta=0} \\
&= \frac{5\pi^4\omega^4 - 80\pi^3\omega^3 + 20\pi^4\omega^2 + 360\pi^2\omega^2 - 160\pi^3\omega - 480\pi\omega + 16\pi^4 + 240\pi^2 + 120}{1966080\pi^4}.
\end{aligned}$$

Using crossing and unitarity conditions in eq.(2.41) and eq.(2.42) along with eq.(2.43) we get

$$f_n^+(\omega) = \frac{1}{(1+e^{-\pi\omega})}\Bigg[\underbrace{-\frac{e^{-\pi\omega}}{2\pi(b^2)^n}\ 2\pi i\ \text{Res}\Big[e^{i\omega\theta}S^{(n)}(\theta)\Big]_{\theta=0}}_{\text{not fixed by crossing and unitarity conditions}} \underbrace{-\sum_{j=1}^{n-1}\int_{-\infty}^{\infty} d\omega' f_{n-j}^+(\omega+\omega') f_j^-(\omega')}_{\text{fixed by crossing and unitarity conditions}} \Bigg]. \tag{2.45}$$

The second term in eq.(2.45) is fixed by the crossing and unitarity conditions while the first term is not fixed by the crossing and unitarity conditions. Therefore, we can calculate f_2^+, f_3^+ and so on

$$\begin{aligned}
f_2^+(\omega) &= \frac{1}{(1+e^{-\pi\omega})}\Bigg[-\frac{e^{-\pi\omega}}{2\pi(b^2)^n}\ 2\pi i\ \text{Res}\Big[e^{i\omega\theta}S^{(2)}(\theta)\Big]_{\theta=0} - \int_{-\infty}^{\infty} d\omega' f_1^+(\omega+\omega') f_1^-(\omega')\Bigg] \\
f_3^+(\omega) &= \frac{1}{(1+e^{-\pi\omega})}\Bigg[-\frac{e^{-\pi\omega}}{2\pi(b^2)^n}\ 2\pi i\ \text{Res}\Big[e^{i\omega\theta}S^{(3)}(\theta)\Big]_{\theta=0} \\
&\qquad - \int_{-\infty}^{\infty} d\omega' \Big[f_1^+(\omega+\omega') f_2^-(\omega') + f_2^+(\omega+\omega') f_1^-(\omega')\Big]\Bigg].
\end{aligned} \tag{2.46}$$

We move the formulas for $f_4^-(\omega)$, $f_5^-(\omega)$ to the next page due to space constraints on the current page caused by the lengthy equation.

$$\begin{aligned}
f_4^+(\omega) = \frac{1}{(1+e^{-\pi\omega})}\Bigg[&- \frac{e^{-\pi\omega}}{2\pi(b^2)^n}\ 2\pi i\ \text{Res}\Big[e^{i\omega\theta}S^{(4)}(\theta)\Big]_{\theta=0} \\
&- \int_{-\infty}^{\infty} d\omega' \Big[f_1^+(\omega+\omega')f_3^-(\omega') + f_2^+(\omega+\omega')f_2^-(\omega') \\
&+ f_3^+(\omega+\omega')f_1^-(\omega')\Big]\Bigg] \\
f_5^+(\omega) = \frac{1}{(1+e^{-\pi\omega})}\Bigg[&- \frac{e^{-\pi\omega}}{2\pi(b^2)^n}\ 2\pi i\ \text{Res}\Big[e^{i\omega\theta}S^{(5)}(\theta)\Big]_{\theta=0} \\
&- \int_{-\infty}^{\infty} d\omega' \Big[f_1^+(\omega+\omega')f_4^-(\omega') + f_2^+(\omega+\omega')f_3^-(\omega') \\
&+ f_3^+(\omega+\omega')f_2^-(\omega') + f_4^+(\omega+\omega')f_1^-(\omega')\Big]\Bigg].
\end{aligned} \tag{2.47}$$

It is crucial to note that the $i\epsilon$ prescription introduces an additional term that cannot be fixed by crossing and unitarity conditions. Here, $f_n^+(\omega)$ represents the perturbative retarded celestial amplitude, evaluated using a Fourier transform with a $+i\epsilon$ prescription, enclosing the contour in the upper half-plane counterclockwise. Conversely, $f_n^-(\omega)$ is the advanced celestial amplitude obtained via a Fourier transform with a $-i\epsilon$ prescription, enclosing the contour in the lower half-plane clockwise. Now, when the contour is enclosed in the upper half-plane, the small semicircle is traversed clockwise, which, after taking the difference between $f_n^+(\omega)$ and $f_n^-(\omega)$, contributes equally to the clockwise semicircle, resulting in the residue at $\theta = 0$.

2.7 Gravitational dressing of the $2d$ QFT amplitude in celestial space

For integrable field theories, in presence of irrelevant deformation $T\bar{T}$, the $2d$ QFT $\mathcal{S}$-matrix is modified by a pure phase [21]

$$S_{ij}^{kl}(\theta) \to S_{ij}^{kl}(\theta) e^{i\delta_{ij}^{(t)}(\theta)}, \tag{2.48}$$

where, the diagonal phase shift $\delta_{ij}^{(t)}(\theta)$ is given by deformation parameter t and difference of rapidities $\theta = \theta_i - \theta_j$

$$\delta_{ij}^{(t)}(\theta) = t m_i m_j \sinh\theta. \tag{2.49}$$

The deformation parameter t is related to the string length $t = 2l_s^2$ in effective string theory. For same mass particles, we have

$$S(\theta) \to S(\theta) e^{2il_s^2 m^2 \sinh\theta}. \tag{2.50}$$

This factor is also discussed in the paper [66] as a solution to the crossing and unitarity conditions i.e., $S(\theta) = S(i\pi - \theta)$ and $S(\theta)S(-\theta) = 1$.

In terms of s-variable we have

$$S(s) \to S(s) e^{il_s^2\sqrt{s(s-4m^2)}}, \tag{2.51}$$

the dressing factor for massless particle reduces to $e^{il_s^2 s}$.

This dressing factor introduced in [22] in the context of gravitational scattering of relativistic point particles in trans-Planckian regime and large impact parameter is referred to as the gravitational dressing factor.

In this section, we study the gravitational dressing of the $2d$ QFT amplitude in the celestial space restricting to massless particles.

The gravitational dressing of the $2d$ QFT amplitude for massless particles is given by

$$S(s) \to S(s)S_{\text{grav}}(s), \tag{2.52}$$

where, the gravitational dressing factor is

$$S_{\text{grav}}(s) = e^{il_s^2 s}. \tag{2.53}$$

The $\mathcal{S}$-matrix is a function of mandelstem variable s which is center-of-mass energy squared. In the celestial space we define the celestial amplitude $\mathcal{A}(\omega)$ as

$$\begin{aligned} \mathcal{A}(\omega) &\equiv \int_0^\infty ds s^{\omega-1} S(s) \\ \implies S(s) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d\omega s^{-\omega}\mathcal{A}(\omega), \end{aligned} \tag{2.54}$$

where, we trade the mandelstem variable s for a rindler energy or conformal dimension ω diagonalizing boosts in the directions of the particles.
Now, Mellin transform of $e^{il_s^2 s}$ is

$$\int_0^\infty ds s^{\omega-1} e^{il_s^2 s} = (-il_s^2)^{-\omega}\Gamma(\omega). \tag{2.55}$$

The gravitational dressing gives

$$\int_0^\infty ds s^{\omega-1}\ S(s) \to \int_0^\infty ds s^{\omega-1}\ S(s)e^{il_s^2 s} \tag{2.56}$$

The gravitational dressing condition becomes

$$\mathcal{A}(\omega) \to \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d\omega'(-il_s^2)^{-\omega'}\Gamma(\omega')\mathcal{A}(\omega-\omega'). \tag{2.57}$$

Now, we take several ansatzes for $\mathcal{A}(\omega)$ that has a pole on the right half-plane of ω and see what we get after the convolution.

Ansatz for $\mathcal{A}(\omega)$

Ansatz 1

We take $\mathcal{A}(\omega)$ as

$$\mathcal{A}(\omega) = \csc \pi\omega. \tag{2.58}$$

The function has poles at $\omega = n\ ,\quad n \in \mathbb{Z}\ $. Performing the convolution we get

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} d\omega' (-il_s^2)^{-\omega'} \Gamma(\omega') \csc(\pi(\omega - \omega')) \\ &= \frac{e^{-il_s^2}\Gamma(\omega)\Gamma(1-\omega, -il_s^2)}{\pi}. \end{aligned} \tag{2.59}$$

where we use the Mellin-Barnes integral representation of the upper incomplete gamma function $\Gamma(a, z)$[**Eq.(3.4.11)** in [23], p. 113]

$$\Gamma(a, z) = -\frac{z^{a-1}e^{-z}}{\Gamma(1-a)}\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} ds\ \Gamma(s + 1 - a)\pi z^{-s} \csc \pi s. \tag{2.60}$$

we put $s = \omega' - \omega,\ a = 1 - \omega,\ z = -il_s^2$ in eq.(2.60) to prove eq.(2.59). Now, the upper incomplete gamma function $\Gamma(a, z)$ is an entire function of a when $z \neq 0$. Therefore, for $l_s^2 \neq 0$, $\Gamma(1 - \omega, -il_s^2)$ is an entire function of $1 - \omega$. The function $\Gamma(\omega)$ has poles at $\omega = n\ ,\quad n = \mathbb{Z}^- \cup \{0\}$. $\csc \pi\omega$ has poles at $\omega = n\ ,\quad n \in \mathbb{Z}$, after gravitational dressing the function $\frac{e^{-il_s^2}\Gamma(\omega)\Gamma(1-\omega,-il_s^2)}{\pi}$ has poles at $\omega = n\ ,\quad n = \mathbb{Z}^- \cup \{0\}$. We can see that after gravitational dressing the poles on the right half-plane of ω are absent.

Ansatz 2

We take $\mathcal{A}(\omega)$ as

$$\mathcal{A}(\omega) = \frac{1}{\omega - \mathfrak{C}}, \tag{2.61}$$

where, $\mathfrak{C}$ is located on the right half plane, i.e., $\text{Re}(\mathfrak{C}) > 0$. Performing the convolution we get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d\omega'(-il_s^2)^{-\omega'}\Gamma(\omega')\frac{1}{\omega-\omega'-\mathfrak{C}}.\\&=(-il_s^2)^{\mathfrak{C}-\omega}\gamma(\omega-\mathfrak{C},-il_s^2).\end{aligned} \tag{2.62}$$

where we use the Mellin-Barnes integral representation of the lower incomplete gamma function $\gamma(a,z)$ [**Eq.(3.4.10)** in [23], p. 113]

$$\gamma(a,z)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\ \frac{\Gamma(-s)}{s+a}z^{s+a}ds. \tag{2.63}$$

Now, the lower incomplete gamma function $\gamma(\omega-\mathfrak{C},-il_s^2)$ is meromorphic with simple poles at

$$\begin{aligned}&\omega-\mathfrak{C}=-n, \quad n=\mathbb{Z}^+\cup\{0\}\\&\implies \omega=\mathfrak{C}-n.\end{aligned} \tag{2.64}$$

For $n=0$, $\text{Re}(\mathfrak{C})>0$ therefore we have a pole at $\omega=\mathfrak{C}$ which lies on the right half-plane. Now, if we impose $\text{Re}(\mathfrak{C}-n)<0$, $n=\mathbb{Z}^+$ then we say that there are no other poles on the right half-plane but still there is a pole at $\omega=\mathfrak{C}$, $\text{Re}(\mathfrak{C})>0$.

Ansatz 3

We take $\mathcal{A}(\omega)$ as

$$\mathcal{A}(\omega)=\Gamma(-\omega). \tag{2.65}$$

The function has poles at $\omega=n$, $n=\mathbb{Z}^+\cup\{0\}$. Performing the convolution we get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d\omega'(-il_s^2)^{-\omega'}\Gamma(\omega')\Gamma(-(\omega-\omega'))\\&=2\left(-il_s^2\right)^{-\frac{\omega}{2}}K_\omega\left(2\sqrt{-il_s^2}\right).\end{aligned} \tag{2.66}$$

where we use the Mellin-Barnes integral representation of the modified Bessel function of the second kind $K_\nu(z)$[**Eq.(3.4.18)** in [23], p. 114]

$$K_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{4\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} ds\ \Gamma(s)\Gamma(s-\nu)\left(\frac{z}{2}\right)^{-2t}. \tag{2.67}$$

Now, the modified Bessel function of the second kind $K_\nu(z)$ has only one singular point at $\nu = \infty$ for fixed z. Here, for fixed l_s^2, $K_\omega(2\sqrt{-il_s^2})$ has pole when $\omega \to \infty$.

We can see that after gravitational dressing the poles on the right half-plane of ω are absent.

From the gravitational dressed celestial amplitude, we observe that if the function initially contains multiple poles, these poles are absent in the right half-plane after gravitational dressing. This observation is based on selecting an ansatz that includes poles. In the case of massless particles involved in a $2 \to 2$ process, the scattering amplitude can be described using the Mandelstam invariant s, from which the celestial amplitude can be constructed. For the sake of simplicity in calculations, we focused on the massless case for the analysis of gravitational dressing. In a follow up work [112], the authors explored gravitational dressing for massive particles and noted that dressing smooths out specific singular aspects of the celestial amplitude. It is important to note that the parametrization differs between massive and massless particles, preventing a direct massless limit to map between massive and massless celestial amplitudes.

Therefore, from this analysis we see that the poles on the right half-plane are absent after gravitational dressing if the function contains multiple poles. From the gravitational dressed celestial amplitude, it is observed that if the function initially includes multiple poles, these poles are no longer present in the right half-plane after gravitational dressing. We can think the gravitational dressing condition in $2d$ as a constraint which gives some constraint on the analytic structure of the celestial amplitude.

2.8 Conclusions

In this chapter, we study celestial amplitude for $2d$ bulk $\mathcal{S}$-matrix and show that for massive scalar particles the celestial amplitude is just the Fourier transform of the $\mathcal{S}$-matrix written in the rapidity variable. For the Sinh-Gordon $\mathcal{S}$-matrix we evaluate the perturbative celestial amplitude and see that there should be two types of celestial amplitude, the retarded and the advanced due to the presence of the pole in the origin of the complex rapidity-plane. Here, for the Sinh-Gordon model, the exact $\mathcal{S}$-matrix has no pole at rapidity, $\theta = 0$, since α is real. The rapidity, $\theta = 0$ pole is a perturbative manifestation, for this there are these two perturbative celestial amplitudes corresponding to two $i\epsilon$ prescriptions. The difference between these two perturbative amplitudes is related to the residue of the pole at $\theta = 0$.

In the celestial space, we translate the crossing and unitarity conditions and check these conditions for the Sinh-Gordon model. From the celestial CFT perspective, we ask about determining the higher order celestial amplitude from the lower order i.e., bootstrapping celestial amplitude by imposing the crossing and unitarity conditions. Finally, we analyze the gravitational dressing condition for the $2d$ QFT amplitude in celestial space restricted to massless particles. We see that this condition manifests itself as eraser of poles from the right half-plane in the celestial space. We will now discuss the massless limit of the study.

In any dimension, for massive particles, the conformal primary wavefunction can be expressed as a Fourier expansion in terms of plane waves, where the Fourier coefficients correspond to the scalar bulk-to-boundary propagator in hyperbolic space. Specifically for massive particles, this Fourier expansion takes the form of an integral over all possible on-shell momenta, each associated with a hyperbolic space. We can take the massless limit of the massive conformal primary wavefunction. The massless scalar conformal primary wavefunction is then obtained as the Mellin transform of the plane wave. Consequently, the transformation from momentum space amplitudes to celestial amplitudes for massless particles is characterized by this Mellin transform with respect to all external energies of the particles. Now in two dimensions, the on-shell momenta of massive particles can be expressed using rapidity parametrization. We have demonstrated that the celestial amplitude corresponds to the Fourier transform of the $2d$ $\mathcal{S}$-matrix formulated in terms of rapidity variables. For massless particles in a $2 \to 2$ process, we can use the Mandelstam invariant s to describe the scattering amplitude and construct the celestial amplitude from it. Notably, the parametrization differs between massive and massless particles in this context.

2.A Perturbative retarded and advanced celestial amplitude computation in $2d$ Sinh-Gordon model

In this appendix 2.A, we calculate the perturbative retarded and advanced celestial amplitude in $2d$ Sinh-Gordon model.

In the complex θ-plane, the perturbative $\mathcal{S}$-matrices $S^{(n)}(\theta)$ for $n = 1, \dots, 5$ contain poles at $\theta = n\pi i$, $n \in \mathbb{Z}$.

While calculating the perturbative retarded celestial amplitude $f_n^+(\omega)$, we choose the contour in the upper half-plane as in fig.2.4 and enclose the poles as $\theta = n\pi i$, $n \in \mathbb{Z}^+$.

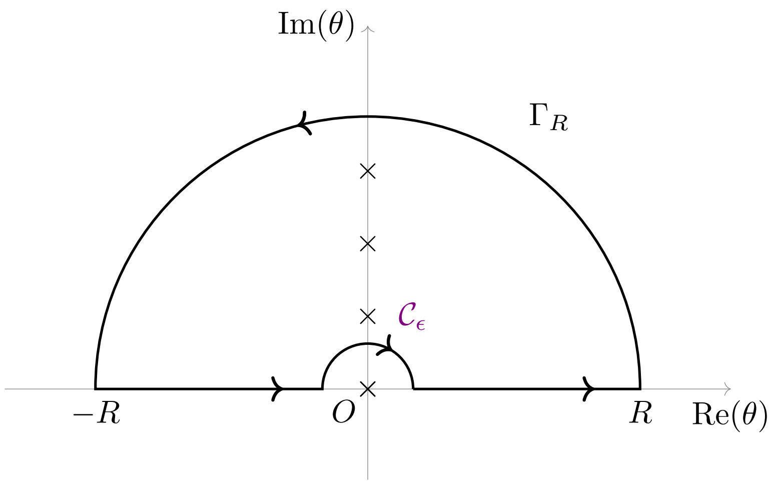

Figure 2.4: Contour for evaluating perturbative retarded celestial amplitude

While calculating the perturbative advanced celestial amplitude $f_n^-(\omega)$, we choose the contour in the lower half-plane as in fig.2.5 and enclose the poles as $\theta = n\pi i$, $n \in \mathbb{Z}^-$.

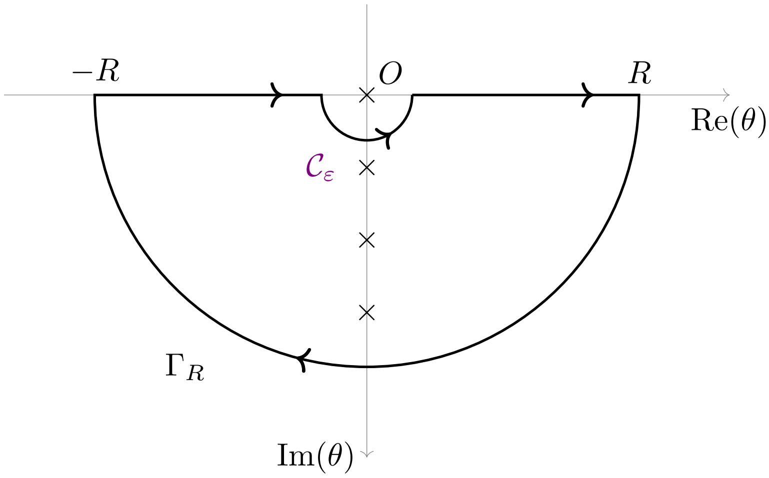

Figure 2.5: Contour for evaluating perturbative advanced celestial amplitude

The residues of the functions appeared in the perturbative expansion of the $\mathcal{S}$-matrix at the poles as $\theta = n\pi i$, $n \in \mathbb{Z}$ are given by

$$\begin{aligned}
\mathrm{Res}\Big[\operatorname{csch}\theta\, e^{i\omega\theta}\Big]_{\theta=n\pi i} &= (-1)^n e^{-n\pi\omega} \\
\mathrm{Res}\Big[\operatorname{csch}^2\theta\, e^{i\omega\theta}\Big]_{\theta=n\pi i} &= i\omega\; e^{-n\pi\omega} \\
\mathrm{Res}\Big[\operatorname{csch}^3\theta\, e^{i\omega\theta}\Big]_{\theta=n\pi i} &= -\frac{1}{2}(-1)^n(\omega^2+1)e^{-n\pi\omega} \\
\mathrm{Res}\Big[\operatorname{csch}^4\theta\, e^{i\omega\theta}\Big]_{\theta=n\pi i} &= -\frac{1}{6}i\omega(\omega^2+4)e^{-n\pi\omega} \\
\mathrm{Res}\Big[\operatorname{csch}^5\theta\, e^{i\omega\theta}\Big]_{\theta=n\pi i} &= \frac{1}{24}(-1)^n(\omega^2+1)(\omega^2+9)e^{-n\pi\omega}.
\end{aligned} \tag{2.68}$$

Chapter 3

AdS correction to the Faddeev-Kulish state

"*From a drop of water, a logician could infer the possibility of an Atlantic or a Niagara without having seen or heard of one or the other.*"

–Arthur Conan Doyle, Sherlock Holmes: A Study in Scarlet.

This chapter is based on the paper

- S. Duary, *AdS correction to the Faddeev-Kulish state: migrating from the flat peninsula*, *JHEP* **05** (2023) 079, [2212.09509].

3.1 Introduction

Scattering amplitudes in QED vanish in four spacetime dimensions in flat-space because of IR divergences. The soft photon interchange between the external legs due to long-range interactions causes IR divergences. In perturbation theory, to all loop orders, the scattering process suffers from IR divergences. The soft contribution of each of the diagrams exponentiates after resumming the series [34]. Therefore, the non-perturbative amplitude, $\mathcal{A}$ vanishes after taking the infrared regulator, $\Lambda_{\text{IR}}^{\text{reg}}$ to zero

$$\mathcal{A} \to 0 \quad , \quad \Lambda_{\text{IR}}^{\text{reg}} \to 0. \tag{3.1}$$

This means the probability of scattering is essentially nil. This is a quantum mechanical statement, but this is really just a reflection of a classical fact that the power or energy radiated does not vanish for soft photons, basically soft photons are produced in order to match the classical answer [35, 36]. The typical textbook solution to this IR divergence problem is to consider inclusive cross sections [36, 37] by taking a trace over the soft modes of the photons in the scattering states. This trace shifts the zero and yields a finite quantity. Since the trace is determined by the detector resolution, this is typically fine for phenomenological scenarios. Numerous soft photon modes evade detection and are thus regarded as unobservable. Nevertheless, in order to explore fine-grained issues regarding the unitarity of $\mathcal{S}$-matrix while taking soft modes of photon into consideration, $\mathcal{S}$-matrix must be defined appropriately [44].

An upshot of the resolution of the IR divergence alternative to employing inclusive cross sections is to use “dressed states” as physical scattering states. While computing the $\mathcal{S}$-matrix, the “in” and “out” scattering states we choose reside in a Fock space. This choice of approximation is a nice one since while looking at timelike infinity (for massive particles) or at null infinity (for massless particles), particles are so far apart from one other that they barely interact and therefore are free. Nevertheless, in this scenario, the states corresponding to the Fock space basis is the “sick basis”, leading the $\mathcal{S}$-matrix to become IR divergent. In order to construct an IR finite $\mathcal{S}$-matrix, the basis of scattering states has to be modified to incorporate the soft modes of the photons. The intuition is that because electro-magnetic interactions are long-range interactions, soft modes of photons in the “in” and “out” scattering states are always present. These dressed states by soft modes of the photons is referred as Faddeev-Kulish dressed state [38]. The Faddeev-Kulish state is such that it precisely cancels the IR divergences in the $\mathcal{S}$-matrix, resulting in an IR finite $\mathcal{S}$-matrix [39–43]. Recently, it was observed that the Faddeev-Kulish state arises as a consequence of asymptotic symmetries [90–94], which indicates the existence of selection sectors [95–98]. For discussion of Faddeev-Kulish dressing in celestial space which is achieved with dressing by edge modes, see e.g., [99–101]. For a more recent discussion of Faddeev-Kulish state, see e.g., [102]. In this chapter, our goal is to explore the AdS radius correction to the Faddeev-Kulish state. We construct the Faddeev-Kulish dressing in AdS/CFT from the standpoint of the Wilson line dressing. The equivalence of the Faddeev-Kulish dressing and the Wilson line dressing involving the soft modes of photons or gravitons has been studied in [82–85]. In flat spacetime, the Wilson line path is time-like geodesic for massive scattering states and for this geodesic, the Wilson line dressing describes the Faddeev-Kulish dressing [82–85]. We now turn to discussion on motivation and physical significance of the work.

Motivation.

In this chapter, we construct the AdS correction to the Faddeev-Kulish state which can be thought of as a perturbation of the flat spacetime result. If we consider some scattering process which involves photons, then the leading AdS correction to the process will also suffer from an IR divergence. The IR divergence coming from the leading AdS correction to the $\mathcal{S}$-matrix is cancelled by the AdS corrected Faddeev-Kulish state. Therefore, the AdS corrected Faddeev-Kulish dressed state serves as an useful tool to understand the leading AdS correction to the IR structure of the flat-space $\mathcal{S}$-matrix. The subtle point to note here is that the leading AdS correction to the $\mathcal{S}$-matrix will not be IR finite, the resummation of all AdS corrections will give an IR finite result for scattering process.

Physical significance.

The physical significance of constructing the AdS correction to the Faddeev-Kulish state is as follows. We assume that there is a place deep inside the AdS spacetime where a bulk observer is conducting scattering experiments. If the bulk observer could only have access to the physics below the AdS length scale, then all that the bulk observer would observe is flat spacetime. In the flat spacetime, the observer can calculate $\mathcal{S}$-matrix and its AdS radius correction suffers from an IR divergence. The IR divergence in the AdS corrected $\mathcal{S}$-matrix will be cancelled by the AdS corrected Faddeev-Kulish state.

The important point here is that the boundary observables in AdS are IR finite because the boundary observables have access to the entire AdS length scale. Scattering amplitudes involving photons in flat spacetime and AdS corrections to the scattering amplitudes are IR divergent. As a result, if we limit ourselves to bulk observables deep inside the AdS spacetime that can probe physics below the AdS length scale, we should be concerned about IR divergence.

Another essential point to note here is that the presence of an asymptotic null boundary allows us to define the $\mathcal{S}$-matrix constructed from the "in" and "out" scattering states. We define the scattering states in the flat spacetime region around the center of AdS spacetime. Therefore, the Faddeev-Kulish state we construct can be realized as the AdS radius correction which is a perturbative correction of the flat spacetime. If we want to understand the AdS correction to the soft theorem, the IR divergence in the AdS corrected $\mathcal{S}$-matrix will arise due to the soft photon exchange between the external legs. The AdS corrected Faddeev-Kulish state will cancel the IR divergence in the AdS corrected $\mathcal{S}$-matrix.

We now talk about observables connected to the scattering process in the AdS/CFT and how it relates to observables in flat spacetime. The boundary observables in the AdS, thanks to the AdS/CFT correspondence are CFT correlation functions. The CFT correlation functions can be computed using the Witten diagrams in the bulk of AdS spacetime. Due to the absence of boundaries, the flat spacetime observables is defined asymptotically. The particular observable for scattering amplitude in flat spacetime is the $\mathcal{S}$-matrix. Zooming in around the center of the AdS spacetime, AdS spacetime manifests itself into flat spacetime. In this flat-space limit, the $\mathcal{S}$-matrix can be obtained from the CFT correlation functions using (i) position space, (ii) mellin space, and (iii) momentum space representations of the CFT correlation functions, see e.g., [45–53] for earlier developments, see e.g., [54–61] for recent developments. Different CFT representations like position space [54, 55, 58, 59], mellin space [50, 63, 65], and momentum space [56, 61] elucidate different roadmap to reach to the flat space $\mathcal{S}$-matrix. In recent years, the flat-space limit of position space CFT correlation function using bulk reconstruction has been studied in [88, 89] to realize IR sector physics in flat spacetime from techniques in AdS/CFT. This is essentially a statement of deciphering the physics of flat spacetime that is already stored in AdS/CFT. In AdS spacetime, causality is attributed to analyticity and unitarity of CFT correlation functions [72–81]. These conditions in the flat space $\mathcal{S}$-matrix have spawned intriguing implementations in the "$\mathcal{S}$-matrix Bootstrap program" from tools in "CFT Bootstrap program" [63–70]. In a very recent paper [71], focusing bound from ANEC for the CFT correlation function and it's connection to the flat-space $\mathcal{S}$-matrix is studied.

Now, we turn to the discussion on Faddeev-Kulish dressing. In this chapter, we explore the AdS radius corrections to the Faddeev-Kulish dressed state which captures the outcome of cosmological constant on the flat spacetime state. We know the fact of life that the AdS radius acts as an IR regulator. In the flat-space limit, the scattering amplitudes will have IR divergences. If we consider the scattering states dressed by the soft modes of the photons (Faddeev-Kulish dressed state), then we can get rid of the IR divergences. With these dressed states, we need to understand how the $\mathcal{S}$-matrix becomes the IR finite one from AdS/CFT. To understand this, first we need to understand how IR divergences manifest themselves after taking the flat-space limit from the CFT correlation function using the Fock-space scattering states. The AdS radius corrections to this Faddeev-Kulish dressed state will provide new insight into an IR finite $\mathcal{S}$-matrix.

Now, we discuss the strategy to construct the AdS radius correction to the Faddeev-Kulish dressed state. We choose the Wilson line dressing as our guiding principle to arrive at the Faddeev-Kulish dressing in AdS/CFT. We are interested in the kinematic regime in which the scalar field is dressed by the soft modes of the photons. The dressed scalar field are free fields and the modes of the field can be reconstructed simply implementing the vanilla bulk operator reconstruction. Upon taking the flat-space limit, the creation/annihilation modes of the dressed field can be expressed in terms of the CFT operator corresponding to the undressed field which is dressed by the boundary-to-boundary Wilson line, which is the CFT representation. We can study the flat-space representation taking into account AdS radius corrections as well by reexpressing the CFT operator corresponding to the undressed field in terms of the undressed mode of the scalar field. In order to have a fully fledged flat-space representation, we must also express the Wilson line operator having CFT current operator in terms of photon creation/annihilation modes. In this chapter, we explore AdS radius correction to the Faddeev-Kulish dressed state. To accomplish so, we have to invert the map between the soft modes of the photon as a smearing of the CFT current operators which shows up in the Wilson line operator.

The creation mode of the soft Wilson line dressed massive scalar field is constructed implementing vanilla bulk operator reconstruction since the dressed field is simply free field. The result we highlight in eq.(3.2) is the expression for the dressed creation mode expressed in terms of the undressed creation mode with smearing over frequency and global time coordinate. The dressed mode dressed by the soft modes of photon acting on the vacuum state $|0\rangle$ is the Faddeev-Kulish state. The expression is the following

$$\begin{aligned}\sqrt{2\omega_{\vec{p}}}\;\tilde{a}^{\dagger}_{\omega_{\vec{p}}} = \tilde{\mathfrak{C}}\;L\int d\Delta_{\vec{p}}\;e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\;e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]}\\ \times\int d\tau\;e^{-iL\tau(\omega_{\vec{p}}-\Delta_{\vec{p}})}\;e^{iq\int_{\Gamma(\tau,\hat{p})}j_a dx^a}\;\sqrt{2\Delta_{\vec{p}}}\;\frac{a^{\dagger}_{\Delta_{\vec{p}}}}{\mathcal{C}}.\end{aligned} \tag{3.2}$$

Now, this expression of eq.(3.2) is written in terms of creation mode of the scalar field but the Wilson line which is expressed in terms of the CFT current operator. Therefore, this is in the "mixed representation" (mixed between the "CFT representation" and the "flat-space representation"). To write this dressed creation mode in the full fledged flat-space representation, we have to express the CFT current operators in the Wilson line in terms of creation/annihilation operators of photon. We express the Wilson line for a particular path where global time coordinate varies from 0 to τ keeping the angular direction fixed

$$e^{iq\int_{\Gamma(\tau,\hat{p})}j_a dx^a} = e^{iq\int_0^{\tau}\left(j^+_{\tau'}+j^-_{\tau'}\right)d\tau'}\quad, \tag{3.3}$$

where, $j^+_{\tau'}$ and $j^-_{\tau'}$ corresponds to the positive and negative frequency modes of the photon when mapped to the photon modes. We evaluate the CFT current operators in terms of AdS corrected photon creation/annihilation modes. After that, we find the inverse mapping. That means we express the CFT current operators in terms of the AdS corrected modes of the photon. Now, choosing a particular path we evaluate the Wilson line and for that we express the global time component of the CFT current operator in terms of the photon creation/annihilation modes. Finally, we express the dressed creation mode of the scalar field in terms of the AdS corrected creation/annihilation modes of the photon. The dressed creation operator acting on the vacuum state gives the AdS radius-corrected Faddeev-Kulish dressed state. The roadmap is summerized in the flowchart fig.3.1.

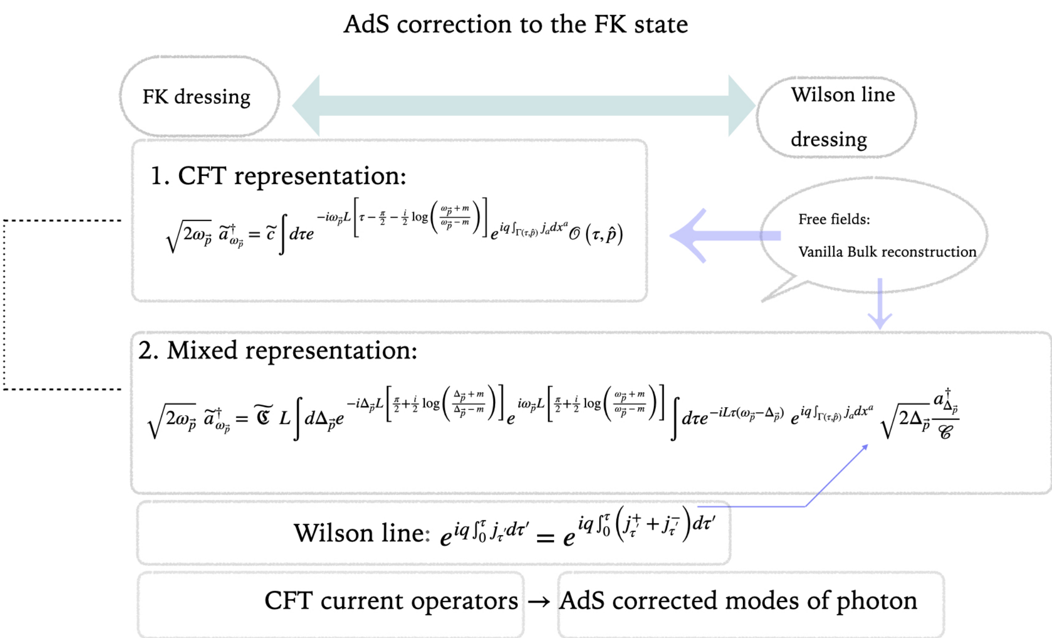

Figure 3.1: Flowchart: Roadmap of the AdS correction to the Faddeev-Kulish (FK) state

In [89, 114], the flat-space limit of bulk operator reconstruction techniques is utilized to derive the Weinberg's soft photon theorems and their AdS corrections through the application of Ward identities involving conserved CFT current. Using the flat-space limit of bulk operator reconstruction techniques, we construct asymptotic states for fields interacting with soft photon modes in AdS, utilizing Wilson line dressing.

Organization of the chapter.

The chapter is organized as follows. In the section 3.2 we review various things to make the chapter self-contained. In the section 3.2.1, we discuss the flat-space limit. We discuss how to extract the creation and annihilation operators for the free massive scalar field in terms of the CFT operators in this flat-space limit. We use the equivalence between the Faddeev-Kulish dressing and the Wilson line dressing as our main strategy to construct the AdS correction to the Faddeev-Kulish dressed state. In the section 3.2.2, we study the soft Wilson line dressed field in AdS and evaluate the CFT operator corresponding to this dressed field. The soft Wilson line dressed scalar field turns itself into the free field and thus can be reconstructed implementing the vanilla bulk operator reconstruction. In the section 3.3, we study the CFT representation as well as the mixed representation of the Faddeev-Kulish dressed state. In the section 3.4, we calculate the AdS corrected Faddeev-Kulish dressed state. We express the CFT current operators in terms of AdS radius-corrected photon creation/annihilation operators in the section 3.4.1. Next, we express the global time component of the CFT current operator in terms of the photon creation/annihilation modes in the section 3.4.2. Then, we express the dressed creation operator in terms of the AdS radius-corrected creation/annihilation modes of the photon. The dressed creation operator acting on the vacuum gives the AdS radius-corrected Faddeev-Kulish dressed state. Finally, we save the section 3.5 for summarizing our conclusions.

3.2 Preliminaries

In this section, we revisit some known things in order to make the chapter self-contained.

3.2.1 Flat Peninsula inside AdS Lake: The flat-space limit

In this section, we review the flat-space limit of AdS/CFT and how to extract the creation and annihilation operators for the massive scalar scalar field in terms of the CFT operator in this limit using the vanilla bulk operator reconstruction described in [89]. At the level of geometry, we take the large AdS radius limit such that the global AdS_4 metric becomes that of flat spacetime. The AdS_4 lorentzian metric in global coordinates is given by

$$ds^2 = \frac{L^2}{\cos^2 \rho}(-d\tau^2 + d\rho^2 + \sin^2 \rho \; d\Omega_2^2) \quad , \tag{3.4}$$

where its boundary CFT is located at $\rho = \frac{\pi}{2}$. The CFT is described by coordinates $\{\tau, \Omega_2\}$, with L representing the radius of AdS. The coordinate replacement

$$\tau = \frac{t}{L} \quad \text{and} \quad \tan \rho = \frac{r}{L} \quad ,$$

develops the AdS_4 metric into that of flat spacetime metric upon taking the limit $L \to \infty$

$$ds^2 \xrightarrow[L\to\infty]{} -dt^2 + dr^2 + r^2 d\Omega_2^2. \tag{3.5}$$

Essentially, the flat peninsula can be thought of as being inside the AdS lake, and this operation can be thought of as flat-space limit.

Massive scalar field modes in the flat-space limit of AdS/CFT

Having discussed the flat-space limit at the level of the geometry, we now turn to a discussion on fields. The creation and annihilation modes for a free massive scalar field in flat spacetime can be constructed in terms of the CFT operator in the flat-space limit of AdS/CFT. For normalizable modes of the bulk AdS scalar field, the bulk AdS scalar field $\phi(\rho, x)$ is related to the dual boundary CFT operator $\mathcal{O}(x)$ through the fall-off condition

$$\phi(\rho, x) \xrightarrow[\rho\to\frac{\pi}{2}]{} (\cos \rho)^{\Delta} \mathcal{O}(x). \tag{3.6}$$

We have

$$\begin{aligned} m^2 L^2 &= \Delta(\Delta - 3) \\ \implies \; \Delta &= \frac{3}{2} + mL + \mathcal{O}(L)^{-1}. \end{aligned} \tag{3.7}$$

To extract the creation and annihilation modes in flat spacetime in terms of CFT operators, first we reconstruct bulk operators as operators in the CFT using the vanilla bulk operator reconstruction prescription. We can extract the creation/annhilation modes of the scalar field from the position space field operator of the scalar field. The approach we follow is the following. First, we construct the free local bulk operators in the CFT using the free bulk operator reconstruction. Next, we extract the creation/annhilation modes using Fourier transform and then we take a large AdS radius limit which is the flat-space limit. The flat-space outgoing creation operator for the free massive scalar field ϕ is given by [89]

$$\sqrt{2\omega_{\vec{p}}}\, a^{\dagger}_{\omega_{\vec{p}}} = \mathcal{C}\int d\tau\, e^{-i\omega_{\vec{p}}L\left[\tau-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]}\mathcal{O}\left(\tau,\hat{p}\right) \quad , \tag{3.8}$$

where

$$\mathcal{C} = \frac{1}{2\pi}\left(\frac{mL}{\pi^3}\right)^{\frac{1}{4}}\left(\frac{2m}{i|\vec{p}|}\right)^{mL+\frac{1}{2}} L. \tag{3.9}$$

In the formula of eq.(3.8), the exponential part in the integrand is highly oscillatory as we take the flat-space limit, $L\to\infty$, therefore the insertion points of the operators are in windows of size $\mathcal{O}(1/L)$ at the complex points

$$\mathrm{Re}(\tau) = \frac{\pi}{2}\ , \ \text{and} \quad \mathrm{Im}(\tau) = \frac{1}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right). \tag{3.10}$$

Now, depending on the outgoing, and incoming particles, we have the insertion points $\tau = \pm\frac{\pi}{2} + i\tilde{\tau}$, where

$$\begin{aligned}\tilde{\tau} &= \frac{1}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right) \quad \text{(for outgoing modes)}\\ \tilde{\tau} &= -\frac{1}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right) \quad \text{(for incoming modes).}\end{aligned} \tag{3.11}$$

Here, in eq.(3.11) $\tilde{\tau}$ is a coordinate in the Euclidean half-sphere. To ensure the analyticity of scattering amplitudes, it is useful to shift the τ contour in the complex plane. This is done by shifting to $\pm\frac{\pi}{2} + i\tilde{\tau}$. Specifically, for positive $\omega_{\vec{p}}$ (outgoing modes), the shift is

$$\tilde{\tau} = \frac{1}{2}\log\left(\frac{\omega_{\vec{p}} + m}{\omega_{\vec{p}} - m}\right),$$

and for negative $\omega_{\vec{p}}$ (incoming modes), the shift is

$$\tilde{\tau} = -\frac{1}{2}\log\left(\frac{\omega_{\vec{p}} + m}{\omega_{\vec{p}} - m}\right).$$

In fig.3.2, we illustrate the correspondence between asymptotic regions in flat-space and the boundary of AdS.

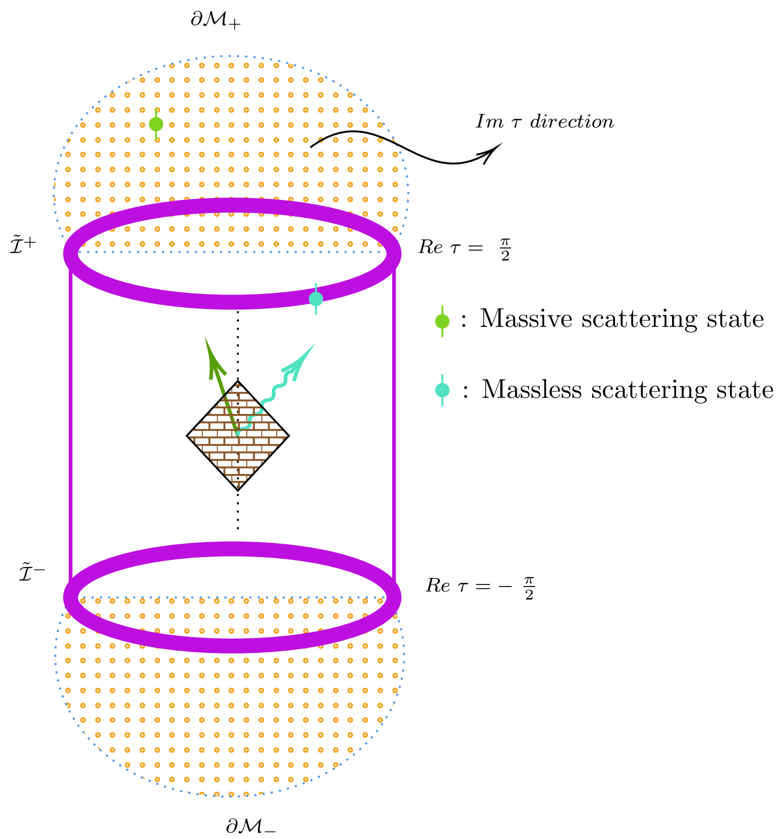

Figure 3.2: A schematic picture linking the boundary of AdS and the asymptotic regions of flat spacetime. The euclidean domes $\partial\mathcal{M}_\pm$, which play the roles of future/past timelike infinity $i^\pm$, are analytic continuations of the boundary CFT in the imaginary global time direction. Regions around, real part of global time at $\pm\frac{\pi}{2}$, Re $\tau = \pm\frac{\pi}{2}$ in an $\mathcal{O}(1/L)$ window, play the role of null infinity $\widetilde{\mathcal{I}}^\pm$.

In fig.3.2, the outgoing massive scattering state is depicted by the green arrow, piercing the boundary at a complex point in global time. Similarly, the outgoing massless scattering state is represented by the blue arrow, piercing the boundary when $\mathrm{Re}(\tau) = \frac{\pi}{2}$. Consequently, two regions in the CFT play pivotal roles: the regions surrounding $\pm\frac{\pi}{2}$, termed null infinity, and the euclidean domes, denoting future/past timelike infinity. The fig.3.2 on the boundary of AdS provides a holographic perspective of flat-space, particularly relevant in terms of particle propagation. Massless particles propagate along light-like geodesics, piercing the boundary of AdS at global time $\frac{\pi}{2}$ (outgoing particle). In contrast, massive particles follow time-like geodesics, therefore never pierce the boundary, or, to put it another way, they pierce the boundary at a complex point.

3.2.2 Soft Wilson line dressed scalar field in AdS and dual CFT operator

In this section, we review the set up of the Faddeev-Kulish dressed state in the flat-space limit of AdS/CFT from the paper [113].

First, we explain the soft Wilson line dressed scalar field in AdS and evaluate the CFT operator corresponding to this soft Wilson line dressed field. The bulk massive scalar field dressed by the bulk-to-boundary Wilson line $U_{\mathfrak{B}\partial}(y,x)$ is given by

$$\widetilde{\phi}(y) = U_{\mathfrak{B}\partial}(y,x)\phi(y) \quad , \tag{3.12}$$

where the bulk-to-boundary Wilson line is

$$\begin{aligned} U_{\mathfrak{B}\partial}(y,x) &= \mathcal{P}\left\{ e^{iq\int_\Gamma dx^M A_M} \right\} \\ &= \mathcal{P}\left\{ e^{iq\int_y^x dx^M A_M} \right\}. \end{aligned} \tag{3.13}$$

Here, Γ is the path in AdS that joins bulk point point y to boundary point x, $y \to x$. Now, considering scalar electrodynamics with action

$$S = \int d^4x\sqrt{-g}\Big(- D_M\phi^\dagger D^M\phi - \frac{1}{4}F_{MN}F^{MN} - m^2\phi^\dagger\phi\Big), \tag{3.14}$$

where $D_M = \partial_M - iqA_M$. Now, the Wilson line dressed field $\widetilde{\phi}$ satisfies [103]

$$\left(\Box - m^2\right)\widetilde{\phi} = -iq\widetilde{\phi}\nabla^M \int_\Gamma F_{MP}dy^P - 2iq\nabla^M\widetilde{\phi}\int_\Gamma F_{MP}\ dy^P + q^2\widetilde{\phi}\ g^{MN}\int_\Gamma F_{MP}dy^P\int_\Gamma F_{NQ}dy^Q. \tag{3.15}$$

We choose the Wilson line dressing in such a way that the field strength F_{MN} becomes $\mathcal{O}\left(\frac{1}{L}\right)$. We dress the scalar field with soft modes of the photon in the Wilson line and in AdS, the minimum frequency of photon is of $\mathcal{O}\left(\frac{1}{L}\right)$. This dressed scalar field we refer as soft Wilson line dressed scalar field. As a consequence of this dressing, the field $\widetilde{\phi}$ is free field.[1] Using this simplification, $\widetilde{\phi}$ can be obtained using vanilla bulk operator reconstruction. The boundary CFT operator dual to the Wilson line dressed field $\widetilde{\phi}$ we denote by $\widetilde{\mathcal{O}}$ and the operator $\widetilde{\mathcal{O}}$ is non-local since it involves boundary-to-boundary Wilson line. The Wilson line dressed scalar field $\widetilde{\phi}$ is given by

$$\widetilde{\phi}(y) = \mathcal{P}\left\{e^{iq\int_y^x A_M dx^M}\right\}\phi(y). \tag{3.16}$$

The fall-off conditions of the photon field and the scalar bulk fields are

$$\begin{aligned} A_a(\rho, x) &\xrightarrow[\rho\to\frac{\pi}{2}]{} j_a(x)\ \cos\rho \\ \phi(\rho, x) &\xrightarrow[\rho\to\frac{\pi}{2}]{} (\cos\rho)^\Delta \mathcal{O}(x) \\ \widetilde{\phi}(\rho, x) &\xrightarrow[\rho\to\frac{\pi}{2}]{} (\cos\rho)^\Delta \widetilde{\mathcal{O}}(x). \end{aligned} \tag{3.17}$$

Therefore, the CFT operator at the boundary corresponding to the soft Wilson line dressed field is given by

$$\widetilde{\mathcal{O}}(x') = U_{\partial\partial}(x', x)\mathcal{O}(x'). \tag{3.18}$$

The boundary-to-boundary Wilson line $U_{\partial\partial}$ is given by

$$U_{\partial\partial}(x', x) = \mathcal{P}\left\{e^{iq\int_{x'}^x dx^a j_a}\right\}. \tag{3.19}$$

[1] There is an alternative way to get the Faddeev-Kulish dressed state by considering a 'Soft-collinear effective theory(SCET)' lagrangian [87] to construct an asymptotic Hamiltonian. While constructing Faddeev-Kulish states we consider the soft part only, and put a hard cutoff on the photon energy. See ref. [86] for the construction of Faddeev-Kulish states within the framework of SCET.

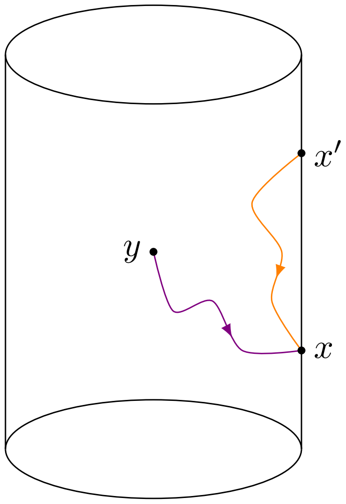

Figure 3.3: Wilson line dressing: Bulk-to-bulk and bulk-to-boundary Wilson lines are represented by violet and orange wiggly lines respectively.

3.3 CFT and Mixed representations of the Faddeev-Kulish dressed state

3.3.1 CFT representation

In this section, we express the Faddeev-Kulish dressed state in CFT representation. As discussed in the previous section, considering the field strength $F_{MN} = \mathcal{O}\left(\frac{1}{L}\right)$, the Wilson line dressed scalar field $\widetilde{\phi}$ is a free field and can be obtained using vanilla bulk operator reconstruction. The creation mode for the free massive scalar field $\widetilde{\phi}$ was constructed in [89] and it is given by

$$\sqrt{2\omega_{\vec{p}}}\,\widetilde{a}^{\dagger}_{\omega_{\vec{p}}} = \widetilde{c}\int d\tau e^{-i\omega_{\vec{p}}L\left[\tau-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]}\widetilde{\mathcal{O}}\left(\tau,\hat{p}\right). \tag{3.20}$$

where

$$\widetilde{c} = \frac{1}{2\pi}\left(\frac{mL}{\pi^3}\right)^{\frac{1}{4}}\left(\frac{2m}{i|\vec{p}|}\right)^{mL+\frac{1}{2}} L. \tag{3.21}$$

Now, the CFT operator of the dressed scalar field $\widetilde{\mathcal{O}}\left(\tau,\hat{p}\right)$ is related to the CFT operator $\mathcal{O}\left(\tau,\hat{p}\right)$ as

$$\widetilde{\mathcal{O}}\left(\tau,\hat{p}\right) = e^{iq\int_{\Gamma(\tau,\hat{p})} j_a dx^a}\mathcal{O}\left(\tau,\hat{p}\right). \tag{3.22}$$

Using eq.(3.22), we get the creation mode of the dressed massive scalar field in terms of $\mathcal{O}(\tau,\hat{p})$

$$\sqrt{2\omega_{\vec{p}}}\,\widetilde{a}^{\dagger}_{\omega_{\vec{p}}} = \widetilde{c}\int d\tau e^{-i\omega_{\vec{p}}L\left[\tau-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} e^{iq\int_{\Gamma(\tau,\hat{p})} j_a dx^a}\mathcal{O}(\tau,\hat{p})\,. \tag{3.23}$$

The Wilson line dressed creation mode $\widetilde{a}^{\dagger}_{\omega_{\vec{p}}}$ acting on the vacuum $|0\rangle$ is the CFT representation of the Faddeev-Kulish dressed state.

Now, in the expression of the CFT current operator in the Wilson line we implement $F_{MN}\to 0$ limit and the Wilson line does not depend on path as a consequence of this limit. Therefore, this enables us to choose a particular path for the Wilson line $\Gamma(\tau,\hat{p})$ which connects the two points in the boundary to evaluate $e^{iq\int_{\Gamma(\tau,\hat{p})} j_a dx^a}$. We choose the path to be global time coordinate varies from 0 to τ while keeping the coordinates of the angular direction fixed. Now, we evaluate the global time integral. Denoting the global time component of the CFT current operator as

$$j_\tau(\tau',\hat{p}) = \sum_m e^{i\omega_m\tau'}\hat{\chi}_m(\hat{p}) \quad , \tag{3.24}$$

where $\hat{\chi}_m$ involves spherical harmonics $Y_l^m(\hat{\Omega})$. Here, we denote the sum over modes by m. Integrating over global time yields

$$\int_0^\tau j_\tau(\tau',\hat{p})d\tau' = \sum_m \frac{1}{i\omega_m}(e^{i\omega_m\tau}-1)\hat{\chi}_m(\hat{p}). \tag{3.25}$$

We expand the Wilson line operator $e^{iq\int_0^\tau j_\tau(\tau',\ \hat{p})\ d\tau'}$ in series to $\mathcal{O}(q)$

$$e^{iq\int_0^\tau j_\tau(\tau',\ \hat{p})\ d\tau'} = 1 + q\sum_m \frac{1}{\omega_m}(e^{i\omega_m\tau}-1)\hat{\chi}_m(\hat{p}) + \mathcal{O}(q^2) \quad , \tag{3.26}$$

where, we use the expansion

$$e^{iq\int_0^\tau j_\tau(\tau',\ \hat{p})\ d\tau'} = 1 + iq\int_0^\tau j_\tau(\tau',\hat{p})d\tau' + \mathcal{O}(q^2). \tag{3.27}$$

Scalar/vector soft modes have frequency $\omega_m = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} + l + 2\kappa$, $\kappa \in \mathbb{Z}^+$.[2] Now, there are two different alternatives of the soft modes in the flat-space limit. One is that $\kappa \sim \mathcal{O}(1)$, and the frequency $\omega_m/L \to 0$. The mode is given by

$$\omega_m \sim \mathcal{O}(1) \quad , \quad \frac{\omega_m}{L} \to 0. \tag{3.28}$$

Another alternative is to denote $\omega_m = kL$ and take the flat-space limit, $L \to \infty$, and then to take soft limit with $k \to 0$. The mode is given by

$$\omega_m = Lk \quad , \quad \frac{\omega_m}{L} = k \to 0. \tag{3.29}$$

In fig.3.4, we draw a frequency scale to denote two different alternatives of the soft modes in the flat-space limit.

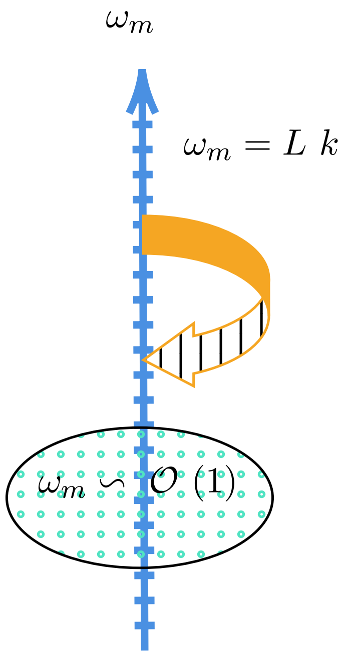

Figure 3.4: Frequency scale ω_m: $\omega_m \sim \mathcal{O}(1)$ and $\omega_m = Lk$.

[2]We refer the reader to the paper [89] (the section "Reconstruction of $U(1)$ gauge fields in global AdS_4") for details regarding the scalar/vector modes.

Now, $\omega_m \sim 2\kappa = Lk$ dominates the sum over modes, and therefore

$$\sum_{\kappa} \to \frac{L}{2} \int dk. \tag{3.30}$$

The Wilson line dressed creation mode of the massive scalar field at $\mathcal{O}(q)$ is given by

$$\sqrt{2\omega_{\vec{p}}}\, \tilde{a}^{\dagger}_{\omega_{\vec{p}}} = \frac{q}{2}\tilde{c} \int d\tau \int \frac{dk}{k} e^{-i\omega_{\vec{p}}L\left[\tau - \frac{\pi}{2} - \frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} \left(e^{iLk\tau} - 1\right) \hat{\chi}_k(\hat{p})\; \mathcal{O}\left(\tau, \hat{p}\right). \tag{3.31}$$

This expression for the dressed creation mode acting on the vacuum state $|0\rangle$ gives the "CFT representation" of the Faddeev-Kulish dressed state.

3.3.2 Mixed representation

We express the creation mode for the Wilson line dressed massive scalar field with the soft modes of the photon in terms of undressed mode. To accomplish so, we insert a delta function $\delta(\tau - \tau')$ and express the delta function in terms of integral over frequency, $\Delta_{\vec{p}}$ which is the corresponding frequency of the undressed mode of the massive scalar field

$$\int d\tau' \delta(\tau' - \tau) = \int d\tau' \int d\Delta_{\vec{p}}\, \frac{L}{2\pi} e^{-i\Delta_{\vec{p}}L(\tau'-\tau)}. \tag{3.32}$$

Therefore, the creation mode of the soft Wilson line dressed massive scalar field is expressed as

$$\begin{aligned}
\sqrt{2\omega_{\vec{p}}}\,\tilde{a}^{\dagger}_{\omega_{\vec{p}}} &= \tilde{c}\int d\tau e^{-i\omega_{\vec{p}}L\left[\tau-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} e^{iq\int_{\Gamma(\tau,\hat{p})} j_a dx^a}\,\mathcal{O}\left(\tau,\hat{p}\right) \\
&= \tilde{c}\int d\tau\int d\tau'\int d\Delta_{\vec{p}}\,\frac{L}{2\pi}e^{-i\Delta_{\vec{p}}L(\tau'-\tau)}\; e^{-i\omega_{\vec{p}}L\left[\tau-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} \\
&\qquad\times e^{iq\int_{\Gamma(\tau,\hat{p})} j_a dx^a}\,\mathcal{O}\left(\tau',\hat{p}\right) \\
&= \tilde{\mathfrak{C}}L\int d\Delta_{\vec{p}}\; e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\;\int d\tau e^{-i\omega_{\vec{p}}L\left[\tau-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} e^{i\Delta_{\vec{p}}L\tau} \\
&\qquad\times e^{iq\int_{\Gamma(\tau,\hat{p})} j_a dx^a}\int d\tau'\; e^{-i\Delta_{\vec{p}}L\left[\tau'-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\;\mathcal{O}\left(\tau',\hat{p}\right),
\end{aligned} \tag{3.33}$$

where, in the last step we use

$$e^{-i\Delta_{\vec{p}}L(\tau'-\tau)} = e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\; e^{i\Delta_{\vec{p}}L\tau}\; e^{-i\Delta_{\vec{p}}L\left[\tau'-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}. \tag{3.34}$$

where,

$$\begin{aligned}
\tilde{c} &= \frac{1}{2\pi}\left(\frac{mL}{\pi^3}\right)^{\frac{1}{4}}\left(\frac{2m}{i|\vec{p}|}\right)^{mL+\frac{1}{2}} L \\
\tilde{\mathfrak{C}} &= \frac{\tilde{c}}{2\pi}.
\end{aligned} \tag{3.35}$$

Now, using the expression for the creation mode $a^{\dagger}_{\Delta_{\vec{p}}}$ corresponing to frequency of the undressed mode, $\Delta_{\vec{p}}$ in terms of the boundary CFT operator $\mathcal{O}\left(\tau',\hat{p}\right)$

$$\sqrt{2\Delta_{\vec{p}}}\,a^{\dagger}_{\Delta_{\vec{p}}} = \mathcal{C}\int d\tau'\; e^{-i\Delta_{\vec{p}}L\left[\tau'-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\mathcal{O}\left(\tau',\hat{p}\right)\quad, \tag{3.36}$$

eq.(3.33) is expressed as

$$\begin{aligned}\sqrt{2\omega_{\vec{p}}}\ \widetilde{a}^{\dagger}_{\omega_{\vec{p}}} = \widetilde{\mathfrak{C}}\ L \int d\Delta_{\vec{p}}\ e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\ e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} \\ \times \int d\tau\ e^{-iL\tau(\omega_{\vec{p}}-\Delta_{\vec{p}})}\ e^{iq\int_{\Gamma(\tau,\hat{p})} j_a dx^a}\ \sqrt{2\Delta_{\vec{p}}}\ \frac{a^{\dagger}_{\Delta_{\vec{p}}}}{\mathcal{C}}.\end{aligned} \tag{3.37}$$

In eq.(3.37), we express the creation mode of the soft Wilson line dressed massive scalar field in terms of the undressed creation mode. We choose a particular path for the Wilson line $\Gamma(\tau,\hat{p})$ as in the previous section 3.3.1. We mode expand the τ component of the CFT current operator as

$$j_{\tau}(\tau',\hat{p}) = \sum_m e^{i\omega_m\tau'}\hat{\chi}_m(\hat{p}). \tag{3.38}$$

Now, we perform the τ integral and get

$$\begin{aligned}&\int d\tau\ e^{-iL\tau(\omega_{\vec{p}}-\Delta_{\vec{p}})}\ e^{iq\int_0^{\tau} j_{\tau}(\tau',\ \hat{p})\ d\tau'} \\ &= \int d\tau\ e^{-iL\tau(\omega_{\vec{p}}-\Delta_{\vec{p}})}\left[1+q\sum_m\frac{1}{\omega_m}(e^{i\omega_m\tau}-1)\hat{\chi}_m(\hat{p})+\mathcal{O}(q^2)\right] \\ &= \frac{2\pi}{L}\delta(\omega_{\vec{p}}-\Delta_{\vec{p}})+\frac{2\pi}{L}\sum_m\frac{q\ \hat{\chi}_m}{\omega_m}\left(\delta\left[(\omega_{\vec{p}}-\Delta_{\vec{p}})+\frac{\omega_m}{L}\right]-\delta(\omega_{\vec{p}}-\Delta_{\vec{p}})\right)+\mathcal{O}(q^2).\end{aligned} \tag{3.39}$$

Scalar/vector soft modes have frequency $\omega_m = \begin{Bmatrix}1\\2\end{Bmatrix} + l + 2\kappa,\ \kappa \in \mathbb{Z}^+$. In the flat-space limit, $\omega_m \sim 2\kappa = Lk$ dominates the sum over modes, and therefore

$$\sum_{\kappa} \rightarrow \frac{L}{2}\int dk. \tag{3.40}$$

We write eq.(3.37) by substituting $\omega_m = kL$ and replacing sum over modes by integration over k at $\mathcal{O}(q)$

$$\begin{aligned}\sqrt{2\omega_{\vec{p}}}\,\tilde{a}^{\dagger}_{\omega_{\vec{p}}} &= \tilde{\mathfrak{C}}\; L \int d\Delta_{\vec{p}}\; e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\; e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} \\ &\quad \frac{2\pi}{L}\sum_m \frac{q\,\hat{\chi}_m}{\omega_m}\left(\delta\left[(\omega_{\vec{p}}-\Delta_{\vec{p}})+\frac{\omega_m}{L}\right]-\delta(\omega_{\vec{p}}-\Delta_{\vec{p}})\right)\sqrt{2\Delta_{\vec{p}}}\;\frac{a^{\dagger}_{\Delta_{\vec{p}}}}{\mathcal{C}} \\ &= \tilde{\mathfrak{C}}\; L\frac{2\pi q}{L}\;\frac{L}{2}\int_0^{\infty} dk\frac{1}{L\;k}\int_{-\infty}^{\infty} d\Delta_{\vec{p}}\;\; e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\; e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} \\ &\qquad \times\left(\delta\left[(\omega_{\vec{p}}-\Delta_{\vec{p}})+k\right]-\delta(\omega_{\vec{p}}-\Delta_{\vec{p}})\right)\hat{\chi}_k\sqrt{2\Delta_{\vec{p}}}\;\frac{a^{\dagger}_{\Delta_{\vec{p}}}}{\mathcal{C}}.\end{aligned} \tag{3.41}$$

Now, we can perform the $\Delta_{\vec{p}}$ integral to simplify the expression and finally the dreesed creation mode is given by

$$\begin{aligned}\sqrt{2\omega_{\vec{p}}}\,\tilde{a}^{\dagger}_{\omega_{\vec{p}}} = \tilde{\mathfrak{C}}\;\pi\; q\;\int_0^{\infty} dk\frac{1}{k}\Bigg(& e^{-i(\omega_{\vec{p}}+k)L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+k+m}{\omega_{\vec{p}}+k-m}\right)\right]}\; e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} \\ & \hat{\chi}_k\sqrt{2(\omega_{\vec{p}}+k)}\frac{a^{\dagger}_{\omega_{\vec{p}}+k}}{\mathcal{C}}-\sqrt{2\omega_{\vec{p}}}\;\hat{\chi}_k\;\frac{a^{\dagger}_{\omega_{\vec{p}}}}{\mathcal{C}}\Bigg).\end{aligned} \tag{3.42}$$

The dressed creation mode acting on the vacuum $|0\rangle$ is the Faddeev-Kulish dressed state. In this way of writing the expression, the frequency of the creation mode of the scalar field will get shifted from $\omega_{\vec{p}}$ to $\omega_{\vec{p}} + k$. The external leg of the scalar field will get dressed due to the soft Wilson line and this has a nice pictorial representation in terms of the frequency of the scalar field mode, see fig.3.5. After further reexpressing the CFT operator in terms of the scalar field mode and perform the frequency integral $\Delta_{\vec{p}}$, we get this representation. We refer eq.(3.42) as "mixed representation" (mixed between the "CFT representation" and the "flat-space representation") since the Wilson line operator still contain the CFT current operator. For understanding the soft modes of photon in a pictorial way, it is better to express the expression in the mixed representation like in eq.(3.42). In terms of CFT operators, the soft modes will correspond to the CFT current operator dual to the gauge field.

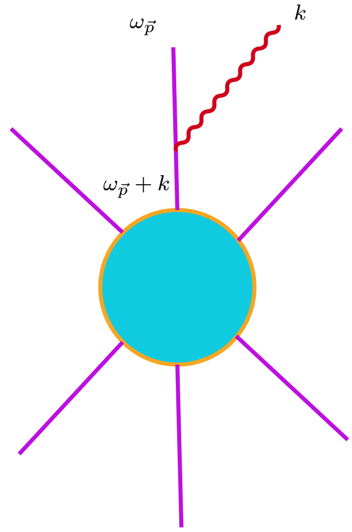

Figure 3.5: Dressing with soft modes of Wilson line: The frequency of the creation mode of the scalar field will get shifted from $\omega_{\vec{p}}$ to $\omega_{\vec{p}} + k$ as a consequence of the dressing.

3.4 AdS correction to the Faddeev-Kulish dressed state

In this section, we will explore AdS radius correction to the Faddeev-Kulish dressed state. From the previous subsection 3.3.2 (ref.eq.(3.37)), we note the expression for the dressed creation mode written in terms of the undressed mode

$$\begin{aligned}\sqrt{2\omega_{\vec{p}}}\,\widetilde{a}^{\dagger}_{\omega_{\vec{p}}} &= \widetilde{\mathfrak{C}}\;L\int d\Delta_{\vec{p}}\;e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\;e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]}\\ &\quad\times\int d\tau\;e^{-iL\tau(\omega_{\vec{p}}-\Delta_{\vec{p}})}\;e^{iq\int_{\Gamma(\tau,\hat{p})}j_a dx^a}\;\sqrt{2\Delta_{\vec{p}}}\;\frac{a^{\dagger}_{\Delta_{\vec{p}}}}{\mathcal{C}}.\end{aligned} \tag{3.43}$$

Now, in this expression the CFT current operator appears in the Wilson line which makes the formula to be in the mixed representation. We map the CFT current operators to creation/annhilation modes of the photon to express our AdS correction to the Faddeev-Kulish dressed state. Now, we have to write the CFT current operator that appears in the expression for the Wilson line

$$e^{iq\int_0^{\tau} j_{\tau'}d\tau'} = e^{iq\int_0^{\tau}\left(j^{+}_{\tau'}+j^{-}_{\tau'}\right)d\tau'}$$

in terms of the photon creation/annihilation operatos. In the expression of $j_{\tau'}$, we use $j^+_{\tau'}$ and $j^-_{\tau'}$ to denote the positive and negative frequency modes of the photon. Here, we choose a particular path for the Wilson line. The path is such that the global time coordinate varies from 0 to τ and the angular direction remains unchanged. To emphasize, we can do this simplification since we are interested in the soft Wilson line dressing to dress the field and in that soft limit the Wilson line is independent of the path because the field strength term can be ignored. Now, in next section 3.4.1, we explore the AdS corrected modes of the photon in terms of CFT current operators. Then, we invert this mapping in the next to next section 3.4.2, to express the CFT current operators to AdS corrected modes of the photon.

3.4.1 AdS corrected photon modes in terms of CFT current operators

Free photon fields in flat spacetime can be mode expanded as

$$A_\mu(x) = \int \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{q}}}}\sum_{\lambda=\pm}\left(\varepsilon^{(\lambda)}_\mu \hat{a}^{(\lambda)}_{\vec{q}}\, e^{iq\cdot x} + \varepsilon^{(\lambda)*}_\mu \hat{a}^{(\lambda)\dagger}_{\vec{q}}\, e^{-iq\cdot x}\right) \quad , \tag{3.44}$$

where, $\varepsilon^{(\lambda)}_\mu$ are polarization vectors. The creation/annihilation modes of the photon are given by

$$\begin{aligned} \sqrt{2\omega_{\vec{q}}}\,\mathbf{a}^{(\lambda)\dagger}_{\vec{q}} &= -\,i\int d^3\vec{x}\,\varepsilon^{(\lambda),\mu}e^{iq\cdot x}\overleftrightarrow{\partial_0}A_\mu(x) \\ \sqrt{2\omega_{\vec{q}}}\,\mathbf{a}^{(\lambda)}_{\vec{q}} &= i\int d^3\vec{x}\,(\varepsilon^{(\lambda),\mu})^* e^{-iq\cdot x}\overleftrightarrow{\partial_0}A_\mu(x) \quad . \end{aligned} \tag{3.45}$$

Now, we express the creation/annihilation modes of the photon in terms of the CFT current operators. We consider the magnetic boundary condition which implies the the gauge field is related to a the bundary CFT current as

$$A_\mu(\rho, x) \xrightarrow[\rho\to\frac{\pi}{2}]{} \cos\rho\, j_\mu(x) \quad . \tag{3.46}$$

Now, implementing bulk operator reconstruction, we get the free photon felds in AdS expressed in terms of the CFT current operator. The expression of the AdS correction to the annihilation operator of an outgoing photon with negative helicity is given by [114]

$$\begin{aligned}&\sqrt{2\omega_{\vec{q}}}\;\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\\&=\frac{1}{32\pi\omega_{\vec{q}}^2L^2}\frac{1+z_q\bar{z}_q}{\sqrt{2\omega_{\vec{q}}}}\int d\tau'\;e^{-i\omega_{\vec{q}}L\left(\frac{\pi}{2}-\tau'\right)}\int d^2z'\int d^2z_w\left[\frac{(1+z'\bar{z}')^2(1+z_w\bar{z}_w)^2}{(\bar{z}_w-\bar{z}')^2(z_q-z_w)^3}\right]\\&\quad\times\partial_{z'}j^-_{\bar{z}'}(\tau',z',\bar{z}')\;.\end{aligned} \tag{3.47}$$

The $1/L^2$ corrected mode is expressed in terms of a CFT current operator smeared over the boundary S^2 and we denote by $\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}$.

3.4.2 Inverse mapping: CFT current operators mapped to AdS corrected photon modes

In this section, we evaluate the inverse mapping. That means, we express the CFT current operators in terms of the AdS corrected creation/annihilation modes of the photon. From the previous section, we note that the annihilation operator of an outgoing photon of negative helicity is given by

$$\begin{aligned}&\sqrt{2\omega_{\vec{q}}}\;\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\\&=\frac{1}{32\pi\omega_{\vec{q}}^2L^2}\frac{1+z_q\bar{z}_q}{\sqrt{2\omega_{\vec{q}}}}\int d\tau'\;e^{-i\omega_{\vec{q}}L\left(\frac{\pi}{2}-\tau'\right)}\int d^2z'\int d^2z_w\left[\frac{(1+z'\bar{z}')^2(1+z_w\bar{z}_w)^2}{(\bar{z}_w-\bar{z}')^2(z_q-z_w)^3}\right]\\&\quad\times\partial_{z'}j^-_{\bar{z}'}(\tau',z',\bar{z}')\;.\end{aligned} \tag{3.48}$$

Acting with a $\partial_{\bar{z}_q}$ on both sides of the eq.(3.48) we have

$$\begin{aligned}\partial_{\bar{z}_q}\left(\frac{64\pi\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\right)&=\int d\tau'\;e^{-i\omega_{\vec{q}}L\left(\frac{\pi}{2}-\tau'\right)}\int d^2z'\;\int d^2z_w\;\partial_{\bar{z}_q}\left[\frac{(1+z'\bar{z}')^2(1+z_w\bar{z}_w)^2}{(\bar{z}_w-\bar{z}')^2(z_q-z_w)^3}\right]\\&\quad\times\partial_{z'}j^-_{\bar{z}'}\,(\tau',z',\bar{z}')\,.\end{aligned} \tag{3.49}$$

Now, using the identity

$$
\begin{aligned}
\partial_{\bar{z}_q} \frac{1}{(z_q - z_w)^3} &= (2\pi)\frac{(-1)^2}{2!}\partial_{z_q}^2 \delta^{(2)}(z_q, z_w) \\
&= (2\pi)\frac{(-1)^2}{2!}\partial_{z_w}^2 \delta^{(2)}(z_q, z_w) \quad ,
\end{aligned}
\tag{3.50}
$$

we have

$$
\begin{aligned}
\partial_{\bar{z}_q}\left(\frac{64\pi\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\right) &= \int d\tau' \; e^{-i\omega_{\vec{q}}L\left(\frac{\pi}{2}-\tau'\right)}\int d^2z' \; \int d^2z_w \; \pi \; \partial_{z_w}^2\delta^{(2)}(z_q, z_w) \\
&\times \left[\frac{(1+z'\bar{z}')^2(1+z_w\bar{z}_w)^2}{(\bar{z}_w - \bar{z}')^2}\right]\partial_{z'} j_{\bar{z}'}^{-}(\tau', z', \bar{z}') \, .
\end{aligned}
\tag{3.51}
$$

After performing the d^2z_w integral using the delta function $\delta^{(2)}(z_q, z_w)$ to simplify the expression a bit. The steps of the detailed calculation we save in the appendix 3.A, section 3.A.1. After simplification the expression we get is

$$
\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\right) = \int d\tau' \; e^{-i\omega_{\vec{q}}L\left(\frac{\pi}{2}-\tau'\right)}\int d^2z' \; (1+z'\bar{z}')^2\frac{2\bar{z}_q^2}{(\bar{z}_q-\bar{z}')^2} \;\; \partial_{z'} j_{\bar{z}'}^{-}(\tau', z', \bar{z}') \, .
\tag{3.52}
$$

Now, further acting ∂_{z_q} on the simplified expression eq.(3.52) we finally get

$$
\partial_{z_q}\left[\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\right)\right] = -\int d\tau' \; e^{-i\omega_{\vec{q}}L\left(\frac{\pi}{2}-\tau'\right)}\left(8\pi z_q(1+z_q\bar{z}_q)^2\right)\partial_{z_q} j_{\bar{z}_q}^{-}(\tau', z_q, \bar{z}_q) \, .
\tag{3.53}
$$

The details is saved in the appendix 3.A, section 3.A.1. The inverse mapping of the CFT current operator for the negative frequency in terms of the AdS corrected photon annihilation mode is given by

$$
\begin{aligned}
\partial_{z_q} j_{\bar{z}_q}^{-}(\tau', z_q, \bar{z}_q) = -\frac{L}{(2\pi)^2}\int d\omega_{\vec{q}} \; e^{-i\omega_{\vec{q}}L\left(\tau'-\frac{\pi}{2}\right)} \;\; \frac{1}{4z_q(1+z_q\bar{z}_q)^2} \\
\times \, \partial_{z_q}\left[\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\right)\right].
\end{aligned}
\tag{3.54}
$$

Now, we multiply both sides by $\frac{1}{\bar{z}_q - \bar{z}'}$ and integrate with respect to $d^2 z_q$ and obtain

$$\begin{aligned} j^-_{\bar{z}'}(\tau', z', \bar{z}') = \frac{L}{(2\pi)^3} \int d\omega_{\vec{q}}\, e^{-i\omega_{\vec{q}} L\left(\tau' - \frac{\pi}{2}\right)} \int d^2 z_q \frac{1}{\bar{z}_q - \bar{z}'} \frac{1}{4 z_q (1 + z_q \bar{z}_q)^2} \\ \times \partial_{z_q} \left[\partial_{\bar{z}_q} \left(\frac{64 \omega_{\vec{q}}^{7/2} L^2}{1 + z_q \bar{z}_q} \mathbf{a}^{\text{AdS}(-)}_{\vec{q}} \right) \right]. \end{aligned} \tag{3.55}$$

Now, we can express eq.(3.54) in terms of the coordinates z' and $\bar{z}'$

$$\begin{aligned} \partial_{z'} j^-_{\bar{z}'}(\tau', z', \bar{z}') = -\frac{L}{(2\pi)^2} \int d\omega_{\vec{q}}\, e^{-i\omega_{\vec{q}} L\left(\tau' - \frac{\pi}{2}\right)} \frac{1}{4 z' (1 + z' \bar{z}')^2} \\ \times \partial_{z_q} \left[\partial_{\bar{z}_q} \left(\frac{64 \omega_{\vec{q}}^{7/2} L^2}{1 + z_q \bar{z}_q} \mathbf{a}^{\text{AdS}(-)}_{\vec{q}} \right) \right] \Bigg|_{(z_q, \bar{z}_q) = (z', \bar{z}')}. \end{aligned} \tag{3.56}$$

For the creation mode of the photon we have CFT current operators $j^+_{\bar{z}'}$ and $j^+_{z'}$. Here, the + in the superscript of j's denotes the positive frequency mode of the photon. We have the following expressions for the CFT current operators

$$\begin{aligned} &\partial_{z'} j^+_{\bar{z}'}(\tau', z', \bar{z}') \\ &= -\frac{L}{(2\pi)^2} \int d\omega_{\vec{q}}\, e^{i\omega_{\vec{q}} L\left(\tau' - \frac{\pi}{2}\right)} \frac{1}{4 z' (1 + z' \bar{z}')^2} \partial_{z_q} \left[\partial_{\bar{z}_q} \left(\frac{64 \omega_{\vec{q}}^{7/2} L^2}{1 + z_q \bar{z}_q} \mathbf{a}^{\dagger \text{AdS}(-)}_{\vec{q}} \right) \right] \Bigg|_{(z_q, \bar{z}_q) = (z', \bar{z}')} \\ &j^+_{\bar{z}'}(\tau', z', \bar{z}') \\ &= \frac{L}{(2\pi)^3} \int d\omega_{\vec{q}}\, e^{i\omega_{\vec{q}} L\left(\tau' - \frac{\pi}{2}\right)} \int d^2 z_q \frac{1}{\bar{z}_q - \bar{z}'} \frac{1}{4 z_q (1 + z_q \bar{z}_q)^2} \partial_{z_q} \left[\partial_{\bar{z}_q} \left(\frac{64 \omega_{\vec{q}}^{7/2} L^2}{1 + z_q \bar{z}_q} \mathbf{a}^{\dagger \text{AdS}(-)}_{\vec{q}} \right) \right] \\ &\partial_{\bar{z}'} j^+_{z'}(\tau', z', \bar{z}') \\ &= -\frac{L}{(2\pi)^2} \int d\omega_{\vec{q}}\, e^{i\omega_{\vec{q}} L\left(\tau' - \frac{\pi}{2}\right)} \frac{1}{4 \bar{z}' (1 + z' \bar{z}')^2} \partial_{\bar{z}_q} \left[\partial_{z_q} \left(\frac{64 \omega_{\vec{q}}^{7/2} L^2}{1 + z_q \bar{z}_q} \mathbf{a}^{\dagger \text{AdS}(+)}_{\vec{q}} \right) \right] \Bigg|_{(z_q, \bar{z}_q) = (z', \bar{z}')} \\ &j^+_{z'}(\tau', z', \bar{z}') \\ &= \frac{L}{(2\pi)^3} \int d\omega_{\vec{q}}\, e^{i\omega_{\vec{q}} L\left(\tau' - \frac{\pi}{2}\right)} \int d^2 z_q \frac{1}{z_q - z'} \frac{1}{4 \bar{z}_q (1 + z_q \bar{z}_q)^2} \partial_{\bar{z}_q} \left[\partial_{z_q} \left(\frac{64 \omega_{\vec{q}}^{7/2} L^2}{1 + z_q \bar{z}_q} \mathbf{a}^{\dagger \text{AdS}(+)}_{\vec{q}} \right) \right]. \end{aligned} \tag{3.57}$$

3.4.3 Global time component of the CFT current operator mapped to photon modes

In this section, we evaluate the global time component of the CFT current operator in terms of the photon creation/annihilation modes. The path for Wilson line we choose in such a way that the global time coordinate varies from 0 to τ and the angular direction remains unchanged. The CFT current operator in the Wilson line operator

$$e^{iq\int_0^\tau j_{\tau'}d\tau'} = e^{iq\int_0^\tau (j^+_{\tau'}+j^-_{\tau'})d\tau'} \quad , \tag{3.58}$$

can be expressed in terms of the photon creation/annihilation modes. We use the current conservation equation in the boundary CFT $\partial_a j^a = 0$ to relate the global time component of the CFT current operator j_τ in terms of z and $\bar{z}$ components, j_z and $j_{\bar{z}}$. Using the current conservation equation we get,

$$\begin{aligned} &\partial_\tau j^\tau + \partial_z j^z + \partial_{\bar{z}} j^{\bar{z}} = 0 \\ \implies &-\partial_\tau j_\tau + \partial_z(g^{z\bar{z}} j_{\bar{z}}) + \partial_{\bar{z}}(g^{\bar{z}z} j_z) = 0 \\ \implies &\partial_\tau j_\tau = \frac{1}{2}(1+z\bar{z})^2(\partial_z j_{\bar{z}} + \partial_{\bar{z}} j_z) + \bar{z}(1+z\bar{z})j_{\bar{z}} + z(1+z\bar{z})j_z. \end{aligned} \tag{3.59}$$

Here, the S^2 part of the boundary metric is $ds^2 = \frac{4}{(1+z\bar{z})^2}dzd\bar{z}$ which gives $g_{z\bar{z}} = g_{\bar{z}z} = \frac{2}{(1+z\bar{z})^2}$ & $g^{z\bar{z}} = g^{\bar{z}z} = \frac{1}{2}(1+z\bar{z})^2$. Using the current conservation equation, global time component of the CFT current operators corresponding to positive and negative frequency modes j^+_τ and j^-_τ are given by

$$\begin{aligned} j^+_\tau &= \int_0^\tau d\tau' \left[\frac{1}{2}(1+z'\bar{z}')^2(\partial_{z'} j^+_{\bar{z}'} + \partial_{\bar{z}'} j^+_{z'}) + \bar{z}'(1+z'\bar{z}')j^+_{\bar{z}'} + z'(1+z'\bar{z}')j^+_{z'}\right] \\ j^-_\tau &= \int_0^\tau d\tau' \left[\frac{1}{2}(1+z'\bar{z}')^2(\partial_{z'} j^-_{\bar{z}'} + \partial_{\bar{z}'} j^-_{z'}) + \bar{z}'(1+z'\bar{z}')j^-_{\bar{z}'} + z'(1+z'\bar{z}')j^-_{z'}\right]. \end{aligned} \tag{3.60}$$

3.4.4 Faddeev-Kulish dressed state

Now, we express the dressed creation mode for the massive scalar field in terms of the photon creation/annihilation modes and creation mode of the undressed scalar operator. We substitute the expression for CFT current operators, $j^\pm_\tau$ as

$$e^{iq\int_{\Gamma(\tau,\hat{p})} j_a dx^a} = e^{iq\int_0^\tau (j^+_{\tau'}+j^-_{\tau'})d\tau'}$$

in the expression of the dressed creation mode of the massive scalar field

$$\begin{aligned}\sqrt{2\omega_{\vec{p}}}\,\widetilde{a}^{\dagger}_{\omega_{\vec{p}}} = \widetilde{\mathfrak{C}}\; L \int d\Delta_{\vec{p}}\; e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\; e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} \\ \times \int d\tau\; e^{-iL\tau(\omega_{\vec{p}}-\Delta_{\vec{p}})}\; e^{iq\int_{\Gamma(\tau,\hat{p})} j_a dx^a}\; \sqrt{2\Delta_{\vec{p}}}\; \frac{a^{\dagger}_{\Delta_{\vec{p}}}}{\mathcal{C}} \quad ,\end{aligned} \tag{3.61}$$

and express the AdS radius-corrected dressed operator at $\mathcal{O}(q)$. The AdS corrected dressed creation operator at $\mathcal{O}(q)$ is given by

$$\begin{aligned}\sqrt{2\omega_{\vec{p}}}\,\widetilde{a}^{\dagger}_{\omega_{\vec{p}}} &= \widetilde{\mathfrak{C}}\; L \int d\Delta_{\vec{p}}\; e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\; e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} \\ &\quad \times \int d\tau\; e^{-iL\tau(\omega_{\vec{p}}-\Delta_{\vec{p}})}\; iq\left(\int_0^{\tau} j^{+}_{\tau}(\tau',\hat{p})d\tau' + \int_0^{\tau} j^{-}_{\tau}(\tau',\hat{p})d\tau'\right)\sqrt{2\Delta_{\vec{p}}}\; \frac{a^{\dagger}_{\Delta_{\vec{p}}}}{\mathcal{C}} \\ &= \widetilde{\mathfrak{C}}\; L \int d\Delta_{\vec{p}}\; e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\; e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} \int d\tau\; e^{-iL\tau(\omega_{\vec{p}}-\Delta_{\vec{p}})} \\ &\times iq\Bigg[\Bigg\{-\frac{1}{2}(1+z'\bar{z}')^2 \\ &\times \left[\frac{1}{(2\pi)^2}\int d\omega_{\vec{q}}\; \mathcal{I}(\tau,\omega_q)\frac{1}{4z'(1+z'\bar{z}')^2}\partial_{z_q}\left[\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}^{\dagger \mathrm{AdS}(-)}_{\vec{q}}\right)\right]\Bigg|_{(z_q,\bar{z}_q)=(z',\bar{z}')} \right. \\ &\quad \left. + \frac{1}{(2\pi)^2}\int d\omega_{\vec{q}}\; \mathcal{I}(\tau,\omega_q)\frac{1}{4\bar{z}'(1+z'\bar{z}')^2}\partial_{\bar{z}_q}\left[\partial_{z_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}^{\dagger \mathrm{AdS}(+)}_{\vec{q}}\right)\right]\Bigg|_{(z_q,\bar{z}_q)=(z',\bar{z}')}\right]\Bigg\} \\ &+ \bar{z}'(1+z'\bar{z}') \\ &\left[\frac{1}{(2\pi)^3}\int d\omega_{\vec{q}}\; \mathcal{I}(\tau,\omega_q)\int d^2 z_q \frac{1}{\bar{z}_q-\bar{z}'}\frac{1}{4z_q(1+z_q\bar{z}_q)^2}\partial_{z_q}\left[\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}^{\dagger \mathrm{AdS}(-)}_{\vec{q}}\right)\right]\right] \\ &+ z'(1+z'\bar{z}') \\ &\left[\frac{1}{(2\pi)^3}\int d\omega_{\vec{q}}\; \mathcal{I}(\tau,\omega_q)\int d^2 z_q \frac{1}{z_q-z'}\frac{1}{4\bar{z}_q(1+z_q\bar{z}_q)^2}\partial_{\bar{z}_q}\left[\partial_{z_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}^{\dagger \mathrm{AdS}(+)}_{\vec{q}}\right)\right]\right]\Bigg] \\ &\quad \times \sqrt{2\Delta_{\vec{p}}}\; \frac{a^{\dagger}_{\Delta_{\vec{p}}}}{\mathcal{C}} \quad ,\end{aligned} \tag{3.62}$$

where, $\mathcal{I}(\tau,\omega_q)$ is given by

$$\mathcal{I}(\tau,\omega_q) = \frac{1}{i\omega_{\vec{q}}}e^{-\frac{1}{2}i\pi L\omega_{\vec{q}}}\Big(\frac{1}{iL\omega_{\vec{q}}}(e^{iL\tau\omega_{\vec{q}}}-1)-\tau\Big). \tag{3.63}$$

The dressed creation operator $\widetilde{a}^{\dagger}_{\omega_{\vec{p}}}$ acting on the vacuum $|0\rangle$ gives the AdS corrected Faddeev-Kulish dressed state.

In appendix 3.A, in eq.(3.64) we have written the expression by dropping the terms for the negative frequency mode involving annihilation modes of photon $\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(\pm)}$. We can further perform the global time, τ integral to simplify the expression. After performing the global time, τ integral the integral evaluates to delta function, using which we can further perform frequency, $\Delta_{\vec{p}}$ integral. We have shifted the final expression to appendix 3.A. After performing the frequency, $\Delta_{\vec{p}}$ integral we have the AdS corrected dressed creation operator in eq.(3.66) of appendix 3.A.

3.5 Conclusions

In this chapter, we construct the AdS radius correction to the Faddeev-Kulish dressed state. We follow the philosophy of Wilson line dressing in the context of AdS/CFT to arrive at the Faddeev-Kulish dressed state. We study the CFT representation and the mixed representation of the Faddeev-Kulish dressed state. We use bulk operator reconstruction prescription to construct the soft photon modes in terms of the CFT current operators. Then, after expressing the $1/L^2$ correction to the soft photon modes, we implement AdS radius correction to the Wilson line dressing. We invert the mapping of the AdS radius-corrected soft photon modes in terms of CFT current operators, that means we evaluate the CFT current operators in terms of the AdS radius-corrected modes of the photon. In the Wilson line dressing, we use this inverse mapping between the CFT current operators and soft photon modes to construct the AdS radius-corrected creation mode of the Wilson line dressed scalar field. The dressed mode acting on the vacuum is the desired Faddeev-Kulish dressed state, which takes into account the AdS radius correction.

Now, we discuss some aspects of the AdS correction to the Faddeev-Kulish dressed state. In AdS spacetime, we do not have scattering states because everything is confined by the AdS potential. However, we can consider various states that resemble flat scattering wavepackets in a sufficiently small region in both space and time. In an appropriate limit, the evolution of such states will be determined by flat spacetime physics to leading order.

In any real scattering process, the scattering state is a wavepacket with some finite spatial extent (a superposition of plane waves), and we can certainly construct such states on the flat spacetime region around the center of the AdS spacetime, with the effect of the small AdS potential treated perturbatively.

It does not make sense to talk about "the $\mathcal{S}$-matrix" in full AdS spacetime, because the wave packet arrives at the asymptotic boundary and bounces back in finite time. But it is unambiguous to talk about a correlation function, and with well-chosen kinematics the correlation function might be determined by the flat spacetime $\mathcal{S}$-matrix in the flat space limit. And, since the correlation function is defined unambiguously, we can consider corrections to this result; subleading terms for the correlation function in the flat spacetime.

In this chapter, we study a particular choice of states which is very natural from the perspective of CFT, since the states are created by local operator insertions. The outcome is a CFT correlation function that is determined by a flat-space scattering amplitude to leading order in the flat limit. The CFT operator dual to dressed field is written in terms of the boundary-to-boundary Wilson line and the CFT operator dual to the undressed field at the boundary.

Once we have made this choice, the states and correlation function have an unambiguous definition beyond the flat-space limit. So in that context it makes sense to ask about the AdS corrections to the correlation function in the flat space limit. We have expressed the $1/L^2$ corrected soft photon modes in terms of the CFT current operators, the soft photon modes will receive corrections via photon kernels while accounting for the small AdS potential. In section 3.4.2, we express the CFT current operators in terms of the AdS corrected modes of the photon. The primary intent of the result is to determine the inverse mapping which is a novel result of the chapter. To create the AdS radius-corrected Faddeev-Kulish state, we employ this inverse mapping between the CFT current operators and soft photon modes.

The issues of IR divergences and AdS as an IR cutoff are as follows. If we want to talk about bulk observables in a small enough region where flat spacetime physics applies, then of course there is no effect from being in AdS; the physics of the IR divergence is no different from being in flat spacetime. On the other hand, if we want to talk about CFT quantities like a correlation function, the IR divergence will change the precise nature of the flat spacetime limit, but the correlation function will still be finite. Since, we are around the flat spacetime region, we can as well think of the physical observable as the AdS correction to the $\mathcal{S}$-matrix. We continue to examine the scattering process around the flat spacetime region even after implementing the correction. The scattering process will not be moved beyond the region of flat spacetime by the soft modes in which we are concerned. The IR divergence in the AdS corrected $\mathcal{S}$-matrix will result from the soft photon exchange between the external legs, which is required in order to understand the AdS correction to the soft theorem. The IR divergence in the AdS corrected $\mathcal{S}$-matrix will be cancelled by the AdS corrected Faddeev-Kulish state.

3.A AdS corrected Faddeev-Kulish dressed state: some simplifications

In this appendix 3.A, we express the AdS corrected Faddeev-Kulish dressed state.

The AdS corrected dressed creation operator at $\mathcal{O}(q)$ is given by

$$
\begin{aligned}
\sqrt{2\omega_{\vec{p}}}\ \widetilde{a}^{\dagger}_{\omega_{\vec{p}}} &= \widetilde{\mathfrak{C}}\ L \int d\Delta_{\vec{p}}\ e^{-i\Delta_{\vec{p}} L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\ e^{i\omega_{\vec{p}} L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} \int d\tau\ e^{-iL\tau(\omega_{\vec{p}}-\Delta_{\vec{p}})} \\
&\times iq\Bigg[\Bigg\{-\frac{1}{2}(1+z'\bar{z}')^2 \\
&\times\Bigg[\frac{1}{(2\pi)^2}\int d\omega_{\vec{q}}\ \mathcal{I}(\tau,\omega_q)\frac{1}{4z'(1+z'\bar{z}')^2}\partial_{z_q}\left[\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}^{\dagger\mathrm{AdS}(-)}_{\vec{q}}\right)\right]\Bigg|_{(z_q,\bar{z}_q)=(z',\bar{z}')} \\
&+\frac{1}{(2\pi)^2}\int d\omega_{\vec{q}}\ \mathcal{I}(\tau,\omega_q)\frac{1}{4\bar{z}'(1+z'\bar{z}')^2}\partial_{\bar{z}_q}\left[\partial_{z_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}^{\dagger\mathrm{AdS}(+)}_{\vec{q}}\right)\right]\Bigg|_{(z_q,\bar{z}_q)=(z',\bar{z}')}\Bigg]\Bigg\} \\
&+\bar{z}'(1+z'\bar{z}') \\
&\left[\frac{1}{(2\pi)^3}\int d\omega_{\vec{q}}\ \mathcal{I}(\tau,\omega_q)\int d^2z_q\frac{1}{\bar{z}_q-\bar{z}'}\frac{1}{4z_q(1+z_q\bar{z}_q)^2}\partial_{z_q}\left[\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}^{\dagger\mathrm{AdS}(-)}_{\vec{q}}\right)\right]\right] \\
&+z'(1+z'\bar{z}') \\
&\left[\frac{1}{(2\pi)^3}\int d\omega_{\vec{q}}\ \mathcal{I}(\tau,\omega_q)\int d^2z_q\frac{1}{z_q-z'}\frac{1}{4\bar{z}_q(1+z_q\bar{z}_q)^2}\partial_{\bar{z}_q}\left[\partial_{z_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}^{\dagger\mathrm{AdS}(+)}_{\vec{q}}\right)\right]\right]\Bigg] \\
&\qquad\times\sqrt{2\Delta_{\vec{p}}}\frac{a^{\dagger}_{\Delta_{\vec{p}}}}{\mathcal{C}}\quad,
\end{aligned}
\tag{3.64}
$$

where, $\mathcal{I}(\tau,\omega_q)$ is given by

$$
\mathcal{I}(\tau,\omega_q)=\frac{1}{i\omega_{\vec{q}}}e^{-\frac{1}{2}i\pi L\omega_{\vec{q}}}\left(\frac{1}{iL\omega_{\vec{q}}}(e^{iL\tau\omega_{\vec{q}}}-1)-\tau\right). \tag{3.65}
$$

After performing the frequency, $\Delta_{\vec{p}}$ integral we have the AdS corrected dressed creation operator

$$
\sqrt{2\omega_{\vec{p}}}\ \widetilde{a}^{\dagger}_{\omega_{\vec{p}}}=\frac{2\pi\widetilde{\mathfrak{C}}}{\mathcal{C}}\ \left[-iq\left\{\frac{1}{2}(1+z'\bar{z}')^2\mathfrak{t}_1-\bar{z}'(1+z'\bar{z}')\mathfrak{t}_2-z'(1+z'\bar{z}')\mathfrak{t}_3\right\}\right]|0\rangle\ . \tag{3.66}
$$

Here,

$$
\begin{aligned}
\mathfrak{t}_1 &= \int d\omega_{\vec{q}}\ \mathfrak{t}_\omega\times\frac{1}{(2\pi)^2}\frac{1}{i\omega_{\vec{q}}}e^{-\frac{1}{2}i\pi L\omega_{\vec{q}}}\ \frac{1}{4z'(1+z'\bar{z}')^2}\partial_{z_q}\left[\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}^{\dagger\mathrm{AdS}(-)}_{\vec{q}}\right)\right]\Bigg|_{(z_q,\bar{z}_q)=(z',\bar{z}')} \\
&-\int d\omega_{\vec{q}}\ \mathfrak{t}_\omega\times\frac{1}{(2\pi)^2}\frac{1}{i\omega_{\vec{q}}}e^{-\frac{1}{2}i\pi L\omega_{\vec{q}}}\frac{1}{4\bar{z}'(1+z'\bar{z}')^2}\partial_{\bar{z}_q}\left[\partial_{z_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}^{\dagger\mathrm{AdS}(+)}_{\vec{q}}\right)\right]\Bigg|_{(z_q,\bar{z}_q)=(z',\bar{z}')},
\end{aligned}
\tag{3.67}
$$

$$\mathfrak{t}_\omega = \frac{1}{iL\omega_{\vec{q}}}\Big(e^{\frac{L}{2}\left[-i\pi\omega_{\vec{q}}-\omega_{\vec{p}}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)+(\omega_{\vec{p}}+\omega_{\vec{q}})\log\left(\frac{\omega_{\vec{p}}+\omega_{\vec{q}}+m}{\omega_{\vec{p}}+\omega_{\vec{q}}-m}\right)\right]}\sqrt{2(\omega_{\vec{p}}+\omega_{\vec{q}})}a^\dagger_{\omega_{\vec{p}}+\omega_{\vec{q}}} - \sqrt{2\omega_{\vec{p}}}a^\dagger_{\omega_{\vec{p}}}\Big)$$
$$+ ie^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]}\frac{\partial}{\partial\Delta_{\vec{p}}}\Big(\sqrt{2\Delta_{\vec{p}}}a^\dagger_{\Delta_{\vec{p}}}e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\Big)\Big|_{\Delta_{\vec{p}}=\omega_{\vec{p}}}, \tag{3.68}$$

$$\mathfrak{t}_2 = \int d\omega_{\vec{q}}\ \mathfrak{t}_\omega \times \mathfrak{t}_u, \tag{3.69}$$

$$\mathfrak{t}_3 = \int d\omega_{\vec{q}}\ \mathfrak{t}_\omega \times \mathfrak{t}_v, \tag{3.70}$$

$$\mathfrak{t}_u = \frac{1}{(2\pi)^3}\frac{1}{i\omega_{\vec{q}}}e^{-\frac{1}{2}i\pi L\omega_{\vec{q}}}\int d^2z_q\frac{1}{\bar{z}_q-\bar{z}'}\frac{1}{4z_q(1+z_q\bar{z}_q)^2}\partial_{z_q}\left[\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\dagger\mathrm{AdS}(+)}\right)\right], \tag{3.71}$$

$$\mathfrak{t}_v = \frac{1}{(2\pi)^3}\frac{1}{i\omega_{\vec{q}}}e^{-\frac{1}{2}i\pi L\omega_{\vec{q}}}\int d^2z_q\frac{1}{z_q-z'}\frac{1}{4\bar{z}_q(1+z_q\bar{z}_q)^2}\partial_{\bar{z}_q}\left[\partial_{z_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\dagger\mathrm{AdS}(+)}\right)\right]. \tag{3.72}$$

3.A.1 More detailed steps of the derivation of the inverse mapping

In this section 3.A.1, we provide the detailed steps to derive the inverse mapping between the CFT current operators and the photon modes.

The annihilation operator of an outgoing photon of negative helicity is expressed in terms of an integrated expression of the boundary CFT current operator

$$\sqrt{2\omega_{\vec{q}}}\ \mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}$$
$$= \frac{1}{32\pi\omega_{\vec{q}}^2L^2}\frac{1+z_q\bar{z}_q}{\sqrt{2}\omega_{\vec{q}}}\int d\tau'\ e^{-i\omega_{\vec{q}}L\left(\frac{\pi}{2}-\tau'\right)}\int d^2z'\int d^2z_w\left[\frac{(1+z'\bar{z}')^2(1+z_w\bar{z}_w)^2}{(\bar{z}_w-\bar{z}')^2(z_q-z_w)^3}\right] \tag{3.73}$$
$$\times\ \partial_{z'}j^-_{\bar{z}'}(\tau',z',\bar{z}')\ .$$

Now, we act with a $\partial_{\bar{z}_q}$ on both sides of the eq.(3.73) and get

$$\partial_{\bar{z}_q}\left(\frac{64\pi\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\right) = \int d\tau'\ e^{-i\omega_{\vec{q}}L\left(\frac{\pi}{2}-\tau'\right)}\int d^2z'\ \int d^2z_w\ \partial_{\bar{z}_q}\left[\frac{(1+z'\bar{z}')^2(1+z_w\bar{z}_w)^2}{(\bar{z}_w-\bar{z}')^2(z_q-z_w)^3}\right]$$
$$\times\ \partial_{z'}j^-_{\bar{z}'}(\tau',z',\bar{z}')\,. \tag{3.74}$$

Now, using the identity

$$\begin{aligned}\partial_{\bar{z}_q}\frac{1}{(z_q-z_w)^3} &= (2\pi)\frac{(-1)^2}{2!}\partial^2_{z_q}\delta^{(2)}(z_q,z_w) \\ &= (2\pi)\frac{(-1)^2}{2!}\partial^2_{z_w}\delta^{(2)}(z_q,z_w) \quad ,\end{aligned} \tag{3.75}$$

we simplify the expression and get

$$\begin{aligned}\partial_{\bar{z}_q}\left(\frac{64\pi\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\right) = \int d\tau'\ e^{-i\omega_{\vec{q}}L\left(\frac{\pi}{2}-\tau'\right)}\int d^2z'\ \int d^2z_w\ \pi\ \partial^2_{z_w}\delta^{(2)}(z_q,z_w) \\ \times\left[\frac{(1+z'\bar{z}')^2(1+z_w\bar{z}_w)^2}{(\bar{z}_w-\bar{z}')^2}\right]\partial_{z'}j_{\bar{z}'}^{-}(\tau',z',\bar{z}')\,.\end{aligned} \tag{3.76}$$

Now, we perform the d^2z_w integral using the delta function $\delta^{(2)}(z_q,z_w)$ to simplify the expression a bit

$$\begin{aligned}&\int d^2z_w\ \partial^2_{z_w}\delta^{(2)}(z_q,z_w)\left[\frac{(1+z_w\bar{z}_w)^2}{(\bar{z}_w-\bar{z}')^2}\right] \\ &= \frac{2\bar{z}_q^2}{(\bar{z}_q-\bar{z}')^2} \quad ,\end{aligned} \tag{3.77}$$

where, in the intermediate steps of eq.(3.77) we use the product rule for double differentiation and expand each term by the following way

$$\begin{aligned}\partial^2_{z_w}\frac{(1+z_w\bar{z}_w)^2}{(\bar{z}_w-\bar{z}')^2} &= \frac{1}{(\bar{z}_w-\bar{z}')^2}\partial^2_{z_w}\left[(1+z_w\bar{z}_w)^2\right]+(1+z_w\bar{z}_w)^2\ \partial^2_{z_w}\frac{1}{(\bar{z}_w-\bar{z}')^2} \\ &\quad +2\partial_{z_w}[(1+z_w\bar{z}_w)^2]\ \partial_{z_w}\frac{1}{(\bar{z}_w-\bar{z}')^2} \\ &= \frac{2\bar{z}_w^2}{(\bar{z}_w-\bar{z}')^2}-2\pi(1+z_w\bar{z}_w)^2\ \partial_{z_w}\partial_{\bar{z}_w}\delta^{(2)}(z_w,z') \\ &\quad -4\pi\bar{z}_w(1+z_w\bar{z}_w)\ \partial_{\bar{z}_w}\delta^{(2)}(z_w,z').\end{aligned} \tag{3.78}$$

After simplification the expression becomes

$$\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\right) = \int d\tau'\ e^{-i\omega_{\vec{q}}L\left(\frac{\pi}{2}-\tau'\right)}\int d^2z'\ (1+z'\bar{z}')^2\frac{2\bar{z}_q^2}{(\bar{z}_q-\bar{z}')^2}\ \ \partial_{z'}j_{\bar{z}'}^{-}(\tau',z',\bar{z}')\,. \tag{3.79}$$

Now, further acting ∂_{z_q} on the simplified expression eq.(3.79) and using the identity

$$\partial_{z_q} \frac{1}{(\bar{z}_q - \bar{z}')^2} = 2\pi(-1)\partial_{\bar{z}_q}\delta^{(2)}(z_q, z') \quad , \tag{3.80}$$

we get

$$\partial_{z_q}\left[\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\right)\right] = -\int d\tau'\ e^{-i\omega_{\vec{q}}L\left(\frac{\pi}{2}-\tau'\right)}\left(8\pi z_q(1+z_q\bar{z}_q)^2\right)\partial_{z_q}j^{-}_{\bar{z}_q}(\tau', z_q, \bar{z}_q)\,. \tag{3.81}$$

Now, we get the inverse mapping of the CFT current operator for the negative frequency in terms of the AdS corrected photon annihilation mode

$$\partial_{z_q}j^{-}_{\bar{z}_q}(\tau', z_q, \bar{z}_q) = -\frac{L}{(2\pi)^2}\int d\omega_{\vec{q}}\, e^{-i\omega_{\vec{q}}L\left(\tau'-\frac{\pi}{2}\right)}\ \frac{1}{4z_q(1+z_q\bar{z}_q)^2}\partial_{z_q}\left[\partial_{\bar{z}_q}\left(\frac{64\omega_{\vec{q}}^{7/2}L^2}{1+z_q\bar{z}_q}\mathbf{a}_{\vec{q}}^{\mathrm{AdS}(-)}\right)\right]. \tag{3.82}$$

The useful identities used to derive the inverse mapping, we list in appendix, section 3.A.2.

3.A.2 Useful identities used to derive the inverse mapping

Here, we list the useful identities used to derive the inverse mapping

$$\begin{aligned}
\partial_z^n \delta(z) &= \frac{(-1)^n n!}{z^n}\delta(z) \\
\partial_z \delta(z,w) &= -\partial_w \delta(z,w) \\
\int \partial_z^n \delta(z,w) f(z) dz &= (-1)^n \partial_z^n f(z)\Big|_{z=w} \\
\partial_z \delta(z,w) f(w,\bar{w}) &= \partial_z \Big[\delta(z,w) f(w,\bar{w})\Big] \\
&= \partial_z \Big[\delta(z,w) f(z,\bar{z})\Big] \\
&= \partial_z \delta(z,w) f(z,\bar{z}) + \delta(z,w)\partial_z f(z,\bar{z}) \\
\partial_z^2 \delta(z,w) f(w,\bar{w}) &= \partial_z^2 \Big[\delta(z,w) f(w,\bar{w})\Big] \\
&= \partial_z^2 \Big[\delta(z,w) f(z,\bar{z})\Big] \\
&= \partial_z^2 \delta(z,w) f(z,\bar{z}) + \delta(z,w)\partial_z^2 f(z,\bar{z}) + 2\partial_z \delta(z,w)\partial_z f(z,\bar{z}) \\
\partial_{\bar{z}} \frac{1}{(z-w)^{n+1}} &= (2\pi)\frac{(-1)^n}{n!}\partial_z^n \delta(z,w) \quad ,
\end{aligned} \tag{3.83}$$

where, the delta function $\delta(z,w)$ in the last equation is a complex plane delta function.

Chapter 4

Conclusions & Open Questions

"*The end is where we start from.*"

–T. S. Eliot.

In this thesis, we explore two key aspects of flat-space holography. First, we study the celestial amplitude, which takes a bottom-up approach. We consider conformal primary wavefunctions that transform as conformal primaries under the Lorentz group. Consequently, the resulting $\mathcal{S}$-matrix transforms covariantly as conformal correlators, trading the plane wave basis with a basis of conformal primary wavefunction. We study celestial holography ideas in $2d$. This setting serves as an excellent testing ground, as we have exact $\mathcal{S}$-matrices to play with in $2d$ and try to learn lessons from. Second, we study the flat-space limit of the AdS/CFT. We consider a scenario where we have a AdS geometry, and within this geometry, there is an observer who can only examine physical phenomena occurring at length scales smaller than the characteristic AdS length scale. From the perspective of this observer, confined to probing only these smaller length scales, the geometry they perceive and experience would appear to be flat, rather than the true underlying AdS geometry. In other words, if an observer is limited to studying physics within a certain distance scale in a AdS geometry, their observations and measurements would be indistinguishable from those made in a flat-space, as the curvature effects of the AdS geometry would be negligible at those small distance scales. The key idea is that flat spacetime is essentially a part of AdS spacetime. Consequently, the physics of flat spacetime must be encoded within the framework of AdS spacetime. Since the physics in AdS spacetime is known to be dual to CFTs via the AdS/CFT correspondence, it follows that CFTs must also encode the physics of flat spacetime. This reasoning forms the general logic behind the concept of flat-space limit of AdS/CFT, which establishes a connection between flat spacetime physics and CFTs through the intermediate step of flat-space limit of AdS spacetime.

In chapter 2, we study the celestial amplitude associated with the $2d$ bulk $\mathcal{S}$-matrix. We show that, in the case of massive scalar particles, the celestial amplitude is essentially the Fourier transform of the $\mathcal{S}$-matrix expressed in terms of rapidity. Considering the Sinh-Gordon $\mathcal{S}$-matrix, we compute the perturbative celestial amplitude, identifying the existence of two distinct types: the retarded and the advanced, attributed to a pole at the origin of the complex rapidity-plane. This pole holds significant importance within perturbation theory, and we elaborate extensively on how to address it within the framework of perturbation theory by introducing two celestial amplitudes that correspond to two different $i\epsilon$ prescriptions. By translating the crossing and unitarity conditions, we establish their equivalents for the celestial amplitude. Through perturbative analysis of the coupling constant in the $2d$ Sinh-Gordon model, we verify that the celestial amplitude satisfies the crossing and unitarity conditions. We employ the bootstrap idea to derive higher-order celestial amplitudes based on lower-order ones. Finally, we translate the gravitational dressing condition of the $\mathcal{S}$-matrix in terms of the celestial amplitude. We note, through various ansatzes, the removal of poles from the right half-plane for the dressed celestial amplitude.

In chapter 3, we construct the AdS radius correction to the Faddeev-Kulish dressed state. The infrared (IR) divergence in the $\mathcal{S}$-matrix arises due to the assumption of asymptotic decoupling. This assumption considers the asymptotic Hamiltonian as free, implying that the asymptotic states are described by Fock space states, and the fields behave like free fields in the asymptotic region of flat spacetime. By relaxing this assumption, the Faddeev-Kulish state can be introduced, leading to an IR-finite $\mathcal{S}$-matrix. The Faddeev-Kulish state incorporates soft photon modes that dress the scattering state within the Fock space, thereby accounting for the long-range effects of the electromagnetic interaction. Using the bulk operator reconstruction, we establish modes for the massive scalar field dressed by the Wilson line and examine both the CFT representation and the mixed representation of the Faddeev-Kulish dressed state. Using the bulk operator reconstruction, we create soft photon modes in terms of CFT current operators. Then, after incorporating the AdS correction into the soft photon modes, we apply AdS radius correction to the Wilson line dressing. We invert the mapping of the AdS radius-corrected soft photon modes in terms of CFT current operators, essentially evaluating the CFT current operators in relation to the AdS radius-corrected photon modes. In the Wilson line dressing, we utilize this inverse mapping between the CFT current operators and soft photon modes to construct the AdS radius-corrected creation mode of the Wilson line dressed scalar field. The resulting dressed mode, acting on the vacuum, represents the desired Faddeev-Kulish dressed state, incorporating the AdS radius correction.

We end by outlining several open questions.

- **Connecting the flat-space limit of AdS amplitude with the celestial amplitude.** It would be nice to connect the celestial amplitude to the flat-space limit of AdS amplitude. For this, we can utilize the dictionary that relates the positions of operators at the AdS boundary to the momenta of particles in the flat-space limit of AdS [54]. The celestial amplitude, expressed in the conformal primary basis, serves as an integral transform of the flat-space amplitude in momentum space [6], and by translating this celestial amplitude, we can interpret it as an integral transform of the positions of operators at the AdS boundary. This opens up possibilities to understand the interpretation of the integral transform in terms of a CFT residing on the boundary. In this line of thought, we can explore the conformal block expansion for the celestial amplitude. In AdS, this involves convoluting the conformal block expansion of the boundary four-point function with an integral transform, allowing us to observe how the block is altered. By exclusively working in flat-space and utilizing the partial wave expansion of the four-point amplitude with Gegenbauer polynomials as a basis [31], we can perform the integral transform. The main question is whether, after the integral transform, the transformed object demonstrates a well-defined structure in terms of conformal block decomposition, thereby establishing a connection with the conformal block expansion in the celestial CFT of [33].

An alternative approach can also be considered. By utilizing the mapping between the creation/annihilation operators and the CFT operators in the boundary of AdS of [89], we have the means to construct the $\mathcal{S}$-matrix based on CFT correlator. Next, we can convolute it with the bulk-to-boundary propagators to obtain the massive celestial amplitude. The combination of these two mappings allows for the computation of the celestial amplitude from AdS amplitude. It would be nice to connect with the recent works [109–111].

- **Asymptotic charges and states from AdS/CFT.** The asymptotic symmetries lead to the Faddeev-Kulish dressed state since the amplitudes that preserve the asymtotic charge are IR finite. The presence of the selection sectors can be inferred from the presence of asymptotic symmetries, as in [96], the explicit construction of Faddeev-Kulish state is shown in [98]. Now, the study of generalizations of asymptotic symmetries known as Λ-BMS$_4$ in AdS is done using leaky boundary condition instead of standard dirichlet boundary condition in works of [104]. It would be fascinating to have the states from AdS/CFT and incorporate AdS corrections into them. This approach would offer an alternative pathway of the work in chapter 3.

- **Faddeev-Kulish state involving soft graviton modes from AdS/CFT.** As an extension of the work in chapter 3, it would be interesting to construct the Faddeev-Kulish state involving soft graviton modes in the flat-space limit of AdS/CFT and examine its AdS corrections. This involves creating flat-space graviton scattering states using CFT_3 stress tensor operators. By applying Wilson line dressing, it would be nice to construct the Faddeev-Kulish state and by establishing the connection between flat-space graviton scattering states and CFT stress tensor operators allows for the derivation of the soft graviton theorem and the AdS correction to the soft graviton theorem from the CFT_3 stress tensor Ward identities.

- **Integrable bootstrap from the flat-space limit of AdS/CFT.** It would be nice to study whether integrability persists in some manner when exactly solvable models are placed in AdS_2. One can wonder if the subleading corrections to the flat-space $\mathcal{S}$-matrix retain their integrable properties. Furthermore, it would be nice to calculate non-perturbative boundary correlators in AdS_2 for integrable $\mathcal{S}$-matrices in flat-space. In [115], we analyze the flat-space limit of scalar scattering in AdS_2 using bulk-operator reconstruction in AdS/CFT. By mapping CFT operators on the AdS boundary to scattering states in flat-space, we compute the $\mathcal{S}$-matrix. We establish the tree-level factorization of the $n \to n$ $\mathcal{S}$-matrix for integrable models in the flat-space limit. Exploring the relationship between celestial CFT_0 (CCFT_0) amplitudes, as in [116], and the flat-space limit of CFT_1 correlators is an interesting avenue.

The ultimate aim is to understand quantum gravity in flat-space, either through the flat-space limit of the AdS/CFT framework or via celestial holography. There is still a long way to go ahead in this regard, and I really hope that my thesis will make a meaningful contribution. I would like to conclude my thesis by the quote

"The larger the island of knowledge, the longer the shoreline of wonder."

–Ralph W. Sockman.

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