On the Formal Moduli of $\mathbb{E}_n$-Monoidal Categories

A Thesis

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Bhanu Kiran Sandepudi

International Centre for Theoretical Sciences
Tata Institute of Fundamental Research

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Pranav Pandit at the International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bengaluru.

Bhanu Kiran Sandepudi

In my capacity as the formal supervisor of record of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

PPandit

Pranav Pandit

Date: March 6, 2025

Abstract

Topological quantum field theories encode an abundance of algebraic data: codimension p operators have an $\mathbb{E}_p$-algebra structure. Motivated by this, we study the deformation theory of $\mathbb{E}_n$-structures.

For a fixed ground field k, we use the framework of m-proximate formal moduli problems developed by Jacob Lurie to study the formal moduli of $\mathbb{E}_n$-algebras in: a) k-module spectra and b) k-linear ∞-categories, up to Morita equivalences of $\mathbb{E}_n$-algebras, as well as equivalences of $\mathbb{E}_n$-algebras in the appropriate categories. Two $\mathbb{E}_n$-algebras in a symmetric monoidal ∞-category are Morita equivalent if the higher category of n-fold right-modules over them are equivalent. Our main results are as follows:

1. The functor classifying the deformations of a homologically bounded below $\mathbb{E}_n$-algebra over k (upto Morita equivalences) is a formal moduli problem given by its $\mathbb{E}_n$-Hochschild cohomology

2. The functor classifying deformations of an $\mathbb{E}_n$-monoidal k-linear ∞-category (upto Morita equivalences) which is Morita equivalent to the $\mathbb{E}_n$-monoidal k-linear ∞-category of right modules over $\mathbb{E}_{n+1}$-algebra B is a $(n+2)$-proximate formal moduli problem. The corresponding formal completion is given by the $\mathbb{E}_{n+1}$-Hochschild cohomology of B. Convention, $\mathbb{E}_0$-monoidal category = category; Morita equivalence of $\mathbb{E}_0$-monoidal categories = equivalence of categories.

   Assuming B is homologically bounded below and considering deformations over the formal power series ring $k[[t]]$: the space of deformations is homotopy equivalent

to the space of $k[\beta]$-linear structures on RMod_B^n. Here β is a variable in degree $(-n-1)$.

Being a formal moduli problem implies that the formal neighbourhood is described by 'solutions to Maurer-Cartan equation'.

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Chapter 1

Introduction

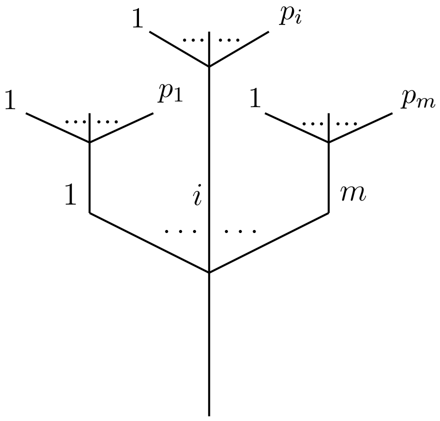
1.1 Preface

Consider an open subspace of the n-dimensional Euclidean space which is diffeomorphic to the standard unit open ball, called an n-disk. For every integer $m \geq 0$, there is an associated space of embeddings of m-many n-disks in another n-disk, denoted $E(n;m)$ The connectivity of $E(n;m)$ increases with n: $E(1;m)$ is not connected, $E(2;m)$ is connected, $E(3;m)$ is simply connected and so on. In general, the connectivity of $E(n;m)$ is related to the connectivity of the $(n-1)$-dimensional sphere.

The collection of all these spaces $\{E(n;m_1)\,|m_1 \geq 0\}$ has a natural 'composition law':

$$E(n;m) \times E(n;p_1) \times \cdots \times E(n;p_m) \longrightarrow E(n;p_1 + \cdots + p_m)$$

In plain words, pick a point in the space $E(n;m)$, i.e. an embedding of m-many n-disks in another n-disk. Choose an arbitrary ordering on these m-many n-disks. For every $1 \leq i \leq p$, pick a point in $E(n,p_i)$, i.e. an embedding of p_i-many n-disks in the disk labelled by i. This process yields an embedding of $p_1 + \cdots + p_m$-many n-disks in another n-disk, i.e. a point in the space $E(n;p_1 + \cdots + p_m)$. This composition can be schematically represented as a concatenation of trees



This collection of spaces along with this composition law is called the $\mathbb{E}_n$*-operad*. The $\mathbb{E}_n$-operad encodes an algebraic structure, the structure of an $\mathbb{E}_n$*-algebra*. The archetypal example of an $\mathbb{E}_n$-algebra is the n-fold loop space of a topological space. Indeed, it is well known that the concatenation of based loops in a topological space is homotopy associative, endowing the based 1-fold loop space with the structure of a homotopy associative algebra in spaces. The structure of an $\mathbb{E}_1$-algebra over a field k is equivalent to that of an A_∞-algebra or a differential graded (dg) associative algebra. The structure of an $\mathbb{E}_\infty$-algebra over a field of characteristic zero is equivalent to the structure of a commutative dg algebra.

The intermediate cases $1 < n < \infty$ interpolate between the *non-commutative* ($n = 1$) and *fully commutative* ($n = \infty$) cases. As n increases, so does the connectivity of the spaces $E(n; m)$. This observation translates to the increasing commutativity of the multiplication with increasing n. The case $n = 1$ reflects the fact that the 0-sphere is not connected. While the case of $n = \infty$ reflects the contractibility of the infinite dimensional sphere. The increasing commutativity of multiplication with increasing n reflects the increasing connectivity of the n-sphere with increasing n.

Topological quantum field theories are closely related to $\mathbb{E}_n$-structures. The algebra of local observables in an n-dimensional TQFT has an $\mathbb{E}_n$-structure; extended operators also have a similar structure. The local operators have a multiplication which is given by the OPE data of the theory: computed by ‘integrating the appropriate fields’ over certain cycles in the space of configurations of points in discs. This endows local observables with an $\mathbb{E}_n$-structure. More generally, one may include defects/extended operators to obtain a richer structure. For example, the OPE data corresponding to line operators is an $\mathbb{E}_{n-1}$-structure. Crudely, this boils down to the fact that a line has one less direction to move in compared to a point. In fact, the algebraic data of all the extended operators in a TQFT can be organised to form a higher category. This observation serves as a motivation for this work. Broadly, we would like to study extended operators in TQFTs via *deformation quantization* of the algebraic structure of the corresponding *classical theory*. This leads us to the study the formal deformations of $\mathbb{E}_n$-structures.

The problem at hand is to classify the deformations of B, a given algebraic object. We call this the *deformation problem* of B. We follow an approach to deformation theory via dg Lie algebras, as promoted by Drinfeld, Feigin, Kontsevich and others. We note that over a field of characteristic zero, there is a dg Lie algebra $\mathfrak{g}$, associated to every deformation problem.[1] Given a dg Lie algebra, there is the Maurer-Cartan equation

$$d\mu + \frac{1}{2}[\mu, \mu] = 0$$

Each deformation of B corresponds to a Maurer-Cartan element, i.e. a solution to the above equation, in the associated dg Lie algebra $\mathfrak{g}_B$. We say that $\mathfrak{g}_B$ *controls* the deformations of B. This observation maps the problem of classifying deformations of B to the problem of classifying Maurer-Cartan elements in $\mathfrak{g}_B$.

In what follows, we work over an arbitrary field k. Consequently, the dg Lie approach to deformation theory doesn't exactly apply here. In [Lur11b], Jacob Lurie developed a framework of deformation theory which not only generalizes this approach to arbitrary fields, but also is flexible enough to allow deformations over non-commutative bases. We review this framework in §2.2.2.

In the context of $\mathbb{E}_n$-structures over a field of characteristic zero, the correspondence

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[1] This observation has been independently sharpened to a theorem by Lurie and Pridham. See [Lur11b, theorem 2.0.2]

between deformations and Maurer-Cartan elements is not 1-1. An arbitrary Maurer-Cartan element may induce a *curvature* term, i.e. not every Maurer-Cartan element can be associated to an *honest* deformation. We illustrate this in the case $n = 1$, using the language of A_∞-algebra:

An A_∞-algebra consists of a graded k-vector space V and a sequence of k-linear maps $\{m_i : V^{\otimes i} \longrightarrow V\}_{i\geq 1}$ with $\deg(m_i) = 2 - i$, such that the following holds for all $N \geq 1$

$$\sum_{\substack{r,t\geq 0\\ N=r+s+t}} (-1)^{r+st} m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$$

where $u = r + 1 + t$. The above quadratic equation encodes the homotopy coherent associative multiplication on V. In particular, when $N = 1$ the above sequence of equations gives a differential on V: $m_1^2 = 0$. We recover associative algebras by assuming V is concentrated in degree zero and $m_i = 0 \quad \forall\, i \neq 2$.

In this language, a deformation of the A_∞-structure is a deformation of the maps $\{m_i\}_i$ that satisfies the above equation.[2] For instance, a first order deformation of this data is given by a sequence of maps $\{m_i + \epsilon m_i'\}_{i\geq 1}$ such that $\epsilon^2 = 0$ and

$$\sum_{\substack{r,t\geq 0\\ N=r+s+t}} (-1)^{r+st} (m_u + \epsilon m_u')(1^{\otimes r} \otimes (m_s + \epsilon m_s') \otimes 1^{\otimes t}) = 0$$

Given the $\{m_i\}_{i\geq 1}$ already satisfy the above equation, we are left with the condition that

$$\sum_{\substack{r,t\geq 0\\ N=r+s+t}} (-1)^{r+st} m_u(1^{\otimes r} \otimes m_s' \otimes 1^{\otimes t}) + \sum_{\substack{r,t\geq 0\\ N=r+s+t}} (-1)^{r+st} m_u'(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0 \quad (1.1)$$

The dg Lie algebra associated to this deformation problem is the Hochschild cochain complex of the A_∞-algebra, shifted by 1 so that the Gerstenhaber bracket becomes a Lie bracket. Moreover, equation 1.1 precisely says that $\{m_i'\}_{i\geq 1}$ must be a solution to the

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[2] We refer the reader to [PS94, KS02, KL09] for further details on the deformation theory of A_∞-algebras.

Maurer-Cartan equation (upto terms of order ϵ^2) in the Hochschild cochain complex. In [KL09, example 3.14] it was demonstrated that an arbitrary Maurer-Cartan element in the Hochschild complex of a A_∞ -algebra may deform the A_∞-algebra to a curved A_∞-algebra, i.e. introduce a map $m_0 : k \longrightarrow V$ in the deformed A_∞-algebra. This is a problem because in this case $m_1^2 = m_0$. A non-zero m_0 forces us out of the world of homological algebra.

Therefore, the first question should be about characterizing the situations where Maurer-Cartan elements do not induce a curvature in deformations.

> When can we *gauge away* the curvature?

Addressing this question over the ring of formal power series $k[[t]]$ forms the core motivation behind this work.

In order to address the question about deformations over $k[[t]]$, we begin by first analyzing the broader problem of characterizing the infinitesimal moduli spaces of $\mathbb{E}_n$-algebras and $\mathbb{E}_n$-monoidal ∞-categories. Following Grothendieck, we adopt the functor of points approach to study these infinitesimal moduli spaces. The framework of *functor of Artin rings*, developed by Schlessinger, is an approach to deformation theory using the functor of points approach [Sch68]. A functor of Artin rings is a functor

$$X : \mathrm{CAlg}_k^{\mathrm{art}} \longrightarrow \mathrm{Set}$$

from the category of commutative local artinian k-algebras to the category of sets, such that $X(k)$ is a singleton set. Lurie's framework of *formal moduli problems* is an adaptation of Schlessinger's framework to the setting of derived, non-commutative algebraic geometry. A functor of artinian (small) $\mathbb{E}_n$-rings is a functor

$$X : \mathrm{Alg}_k^{n,\mathrm{small}} \longrightarrow \mathcal{S}$$

from the ∞-category of $\mathbb{E}_n$-algebra analogs of commutative local artinian k-algebras to the ∞-category of spaces, such that $X(k)$ is a contractible space. See [Lur11b] for details (or §2.2.2 for a short review).

Given an $\mathbb{E}_1$-algebra B there are three associated natural deformation problems: a) deform B as an $\mathbb{E}_1$-algebra upto equivalences, b) deform B as an $\mathbb{E}_1$-algebra upto Morita equivalences (definition 3.0.1), and c) deform B as a module over itself, i.e. as an object of LMod_B. It turns out that these three deformation problems are closely related. When made precise, this relation can be expressed as a fiber sequence of functors of small $\mathbb{E}_2$-rings (proposition 3.4.3).

$$\mathrm{ObjDef}_B^{\mathbb{E}_2} \longrightarrow \mathrm{SimDef}_{(\mathrm{RMod}_B, B)} \longrightarrow \mathrm{CatDef}_{\mathrm{RMod}_B}$$

In the above sequence, the middle term corresponds to the deformation problem a), the right-most term corresponds to the deformation problem b), the left-most term corresponds to the deformation problem c).

Moreover, there is a fiber sequence of non-unital $\mathbb{E}_2$-algebras associated with the above fiber sequence of functors.

$$B[-1] \longrightarrow T_B[-1] \longrightarrow \mathrm{HH}^*_{\mathbb{E}_1}(B)$$

where T_B is the tangent complex of B (definition 3.5.14), while $\mathrm{HH}^*_{\mathbb{E}_1}(B)$ is the Hochschild cohomology of B (definition 3.5.12). Think of these non-unital $\mathbb{E}_2$-algebras as the non-commutative versions of the dg Lie algebras which control the above three deformation problems.

In fact, the analogs of the three deformation problems exist for an $\mathbb{E}_n$-algebra B, for $n \geq 1$: a) deform B as an $\mathbb{E}_n$-algebra upto equivalences, b) deform B as an $\mathbb{E}_n$-algebra upto Morita equivalences, and c) deform the presentable $(\infty, n-1)$-category of $(n-1)$-fold iterated modules over B viewed as an object of the presentable (∞, n)-category of n-fold iterated modules over B.

The preceding fiber sequence of non-unital $\mathbb{E}_2$-algebras becomes a fiber sequence of the non-unital $\mathbb{E}_{n+1}$-algebras

$$B[-1] \longrightarrow T_B[-n] \longrightarrow \mathrm{HH}^*_{\mathbb{E}_n}(B)$$

here T_B is the tangent complex of B and $\mathrm{HH}^*_{\mathbb{E}_n}(B)$ is the $\mathbb{E}_n$-Hochschild cohomology of B.

This discussion is closely related to the work of John Francis on the deformations of $\mathbb{E}_n$-algebras [Fra13]. For B an $\mathbb{E}_n$-algebra in a symmetric monoidal k-linear ∞-category, theorem 1.1 of loc. cit established the existence of the preceding fiber sequence of non-unital $\mathbb{E}_{n+1}$-algebras. We recover the preceding fiber sequence of non-unital $\mathbb{E}_{n+1}$-algebras through corollary 3.4.5. See § 3.5.2 for a detailed discussion connecting our approach to Francis'.

The fiber sequence of functors of small $\mathbb{E}_{n+1}$-rings associated to an $\mathbb{E}_n$-algebra B (proposition 3.4.3) and the fiber sequence of associated non-unital $\mathbb{E}_{n+1}$-algebras of corollary 3.4.5 are central objects of this work. The main results of §3.5 depend vitally on the interrelation between the three deformation problems associated with an $\mathbb{E}_n$-algebra B.

1.2 Motivation: Deformation quantization

A major inspiration for this work arises from the objective of studying TQFTs as a deformation quantization of classical theories. Broadly, deformation quantization is a process which *takes a commutative algebraic structure and gives a non-commutative algebraic structure.*

The field of deformation quantization began several decades ago through the work of Bayen, Flato, Frøndsal, Lichnerowicz, and Sternheimer [BFF+78]. Over time, this area has seen numerous significant contributions. Our intention here is not to provide a detailed historical overview but rather to recall the aspects directly pertinent to this thesis.

In his seminal work, Maxim Kontsevich proved that any finite dimensional Poisson manifold admits a deformation quantization [Kon03], i.e the commutative algebra of functions on a Poisson manifold can be canonically deformed into an associative algebra. Given a smooth manifold M, a *star product* $\star$ on M is the structure of a $\mathbb{R}[[\hbar]]$-linear associative multiplication on the algebra of smooth functions $C^\infty(M)$. For every $f, g \in C^\infty(M)$

$$f \star g = fg + \sum_{i \geq 1} \hbar^i P_i(f, g)$$

where P_i are bilinear maps which behave like differential operators in each component operators. For $f(\hbar), g(\hbar) \in C^\infty(M)[[\hbar]]$, the star product is defined by $\hbar$-linearity. [Kon03, theorem 1.1] says that the set of equivalence classes of star products on a smooth manifold is in bijection with the equivalence classes of Poisson structures depending on the formal parameter $\hbar$. Kontsevich obtained the result about deformation quantization of Poisson manifold as a corollary of a much more general result about the quasiisomorphism of certain dg Lie algebras [Kon03, §4.6.2 Main Theorem], the Kontsevich formality theorem.

In the article [Toë14], Bertrand Toën advocated a unified approach to deformation quantization arising from derived algebraic geometry (developed in detail by [CPT$^+$17]). In this approach, deformation quantization of algebraic spaces is viewed from the point of view of the deformations of the associated category of sheaves (equipped with an appropriate monoidal structure).

Let X be a *nice* derived algebraic stack (eg. quotient of a commutative dg algebra by a linear algebraic group). X admits a tangent complex T_X, which is defined to be the $\mathcal{O}_X$-dual to the cotangent complex. One defines the complex of n-shifted polyvector fields on X

$$\mathcal{P}ol(X, n) := \bigoplus_{i \geq 0} \Gamma(X, \mathrm{Sym}^i_{\mathcal{O}_X}(T_X[-1-n]))$$

The Lie bracket on vector fields endows this complex the structure of a graded Poisson dg-algebra where the Lie bracket has cohomological degree $(-1-n)$ and weight (-1). The space of n-shifted Poisson structures on X is defined to be the space of maps of graded dg-Lie algebras [CPT+17, definition 3.1.1]

$$\mathrm{Poiss}(X,n) := \mathrm{Map}_{\mathrm{dgLie}_k^{\mathrm{gr}}}(k(2)[-1], \mathcal{P}ol(X,n+1)[n+1])$$

where $k(2)[-1]$ is the graded dg-Lie algebra which is k in cohomological degree 1, has zero bracket and pure weight 2. Roughly, graded dg-modules over k are k-dg-modules equipped with an action of $k[\epsilon]$ with $\deg(\epsilon) = -1$, $\epsilon^2 = 0$ and a direct sum decomposition with respect to this action, called the *weight decomposition.*

One has the higher formality conjecture [Toë14, conjecture 5.3]: For $n \geq 0$ an integer and a *nice* derived algebraic stack X, the dg-Lie algebra $\widehat{HH}^{\mathbb{E}_{n+1}}(X)[n+1]$ is quasiisomorphic to $\mathcal{P}ol(X,n)[n+1]$. Here $\widehat{HH}^{\mathbb{E}_{n+1}}(X) \simeq \mathrm{End}_{\widehat{\mathcal{L}^n(X)}}(j_*\mathcal{O}_X)$ is the formal version of the Hochschild complex of X, and $\widehat{\mathcal{L}^n(X)}$ is the formal completion along $j : X \longrightarrow \mathcal{L}^n(X)$, the natural map from X to the corresponding n-fold loop stack of X, $\mathcal{L}^n(A) := \mathrm{Map}(S^n, X)$. Here we view the topological n-sphere as a constant derived stack. The conjecture can be interpreted as the statement that the $\mathbb{E}_n$-Hochschild complex of a *nice* derived algebraic stack is quasiisomorphic to the complex of n-shifted polyvector fields. This conjecture reduces to Kontsevich's formality theorem in the case $n = 0$ [Kon03, §4.6.2]. The higher formality conjecture is known to be true when X is the quotient of a cdga by a linear algebraic group [Toë13, corollary 5.4].

Our result on formal deformations (theorem 3.5.10) is an extension of the result of Anthony Blanc, Ludmil Katzarkov and Pranav Pandit [BKP18, theorem 4.27]. It is in the spirit of a result that would be required for the deformation quantization of algebraic stacks. In ongoing work, we attempt to extend our result regarding formal deformations, making it applicable to the derived category of a quasi-compact and quasi-separated scheme. More precisely, we attempt to generalize the result to any derived category which admits a single compact generator.

Following the argument of [Toë14, §5.3]: let X be a derived Artin stack for which the formality conjecture holds. Given a n-shifted Poisson structure $k(2)[-n-2] \xrightarrow{p}$

$\mathcal{P}ol(X, n+1)$ on X, the formality conjecture gives a map $k(2)[-n-2] \xrightarrow{q} \widehat{\mathrm{HH}}^{*}_{\mathbb{E}_{n+1}}(X) \longrightarrow \mathrm{HH}^{*}_{\mathbb{E}_{n+1}}(X)$. Now, if a version of theorem 3.5.10 holds for the derived category of X, it will follow that q provides a formal deformation of the derived category of X, viewed as an $\mathbb{E}_n$-monoidal k-linear ∞-category. This deformation is the formal deformation quantization of the pair (X, p).

Deformation quantization of an algebraic stack X equipped with a n-shifted Poisson structure p was extensively discussed in [CPT$^+$17]. The authors work in characteristic zero and use the technical tools of formal algebraic geometry to define a deformation quantization of the pair (X, p). The essential idea is to view X as a family of formal derived stacks, which translated questions about the stack X into questions about certain dg Lie algebras that encode the formal geometry of points of X. Given such a stack X, there is an associated stack X_{DR}, called the de Rham stack of X. As a functor on the ∞-category of cdgas, $X_{\mathrm{DR}}(A) \coloneqq X(A_{\mathrm{black}})$, where $A_{\mathrm{black}} = \pi_0(A)_{\mathrm{black}}$. There is a natural functor $X \longrightarrow X_{\mathrm{DR}}$ that allows us to view X as a family of formal derived stacks over X_{DR} [CPT$^+$17, corollary 2.1.9]. The authors define a certain sheaf of graded mixed cdga $\mathcal{B}_X$ over X_{DR} (see [CPT$^+$17, definition 2.4.11]) such that the ∞-category of perfect complexes of $\mathcal{O}_X$-modules is equivalent to the ∞-category of perfect modules over $\mathcal{B}_X$, $\mathcal{B}_X - \mathrm{Mod}^{\mathrm{perf}}_{\epsilon-\mathrm{dg}^{gr}}$ [CPT$^+$17, Main result B]. The quantization of (X, p) is constructed in terms of a deformation of the ∞-category $\mathcal{B}_X - \mathrm{Mod}^{\mathrm{perf}}_{\epsilon-\mathrm{dg}^{gr}}$, induced the n-shifted Poisson structure p [CPT$^+$17, definition 3.5.5]. One would expect that this construction of the deformation quantization of the pair (X, p) and the one explained in the preceding paragraph must coincide when both are valid.

1.3 Relation to other works

Throughout this work, we are over a ground field k. We study the deformations of $\mathbb{E}_n$-algebras in: a) the ∞-category of k-modules, b) the ∞-category of k-linear ∞-categories, up to Morita equivalences (definition 3.0.1), as well as equivalences of $\mathbb{E}_n$-algebras.

We situate our work in the context of deformations of monoidal categories.

1.3.1 Deformations of monoidal categories

Over the years, there has been a continuous interest in studying the formal moduli spaces of algebraic objects: associative algebras [Ger64], abelian categories [Ane06, LVdB05], prestacks of linear categories [VL18], A_∞-algebras [PS94, KS02, KL09] dg-categories [KL09] (equivalently, k-linear ∞-categories [Lur11b]). Following previous works [Dav97, CY98, Yet98, Yet03], there has been a recent resurgence of interest to study deformations of more intricate structures in the form of monoidal categories [PS22, ITC23, FGS24]. (Braided) Monoidal categories arise in the study of knot invariants and quantum groups both of which are of interest to mathematicians and physicists alike. We summarise some of these results and sketch a comparison with our work.

In [PS22], Piergiorgio Panero and Boris Shoikhet construct a generalization of the Davidov-Yetter complex of a k-linear monoidal (dg-) category $\mathcal{C}$ as a totalization of a 2-cocellular dg vector space. In fact, they show that this complex is an $\mathbb{E}_2$-algebra [PS22, theorem 4.6] and conjecture a lift of this $\mathbb{E}_2$-algebra to an $\mathbb{E}_3$-algebra. They define an abelian category of 2-bimodules over a k-linear bicategory [PS22, §3.6], and show that the complex they constructed may be realised as the internal hom-object in this category [PS22, proposition 3.14]. They argue that the third cohomology of this complex may be identified with the equivalence class of infinitesimal deformations of $\mathcal{C}$ [PS22, theorem 5.3]. We note that in their work they only consider deformations of: a) the composition b) the monoidal product on morphisms c) the associator of the monoidal structure and d) the left and right units of the monoidal structure. In particular, the "set theoretic data" of objects and the monoidal structure on objects is held fixed under deformations [PS22, §5.1]. They classify deformations up to equivalence of monoidal categories, which is a strictly stronger notion compared to the notion of a Morita equivalence of monoidal categories (definition 3.0.1): every monoidal equivalence induces a Morita equivalence, while the converse need not be true.

A large part of this thesis is devoted to studying the deformations of monoidal ∞-categories upto Morita equivalences. Another difference between loc. cit. and this is that here the "set theoretic" data is not held fixed under deformations. This means that some objects may "disappear" under deformations. Let $\mathcal{C}$ be the ∞-category of right modules over an $\mathbb{E}_2$-algebra B. It is reasonable to expect a map from the (space of) deformations of $\mathcal{C}$ as defined by Panero-Shoikhet to the (space of) Morita deformations of $\mathcal{C}$ as defined in this work (construction 3.2.1). Conjecturally, there is a natural map of $\mathbb{E}_3$-algebras from the complex defined by Panero-Shoikhet to the $\mathbb{E}_2$-Hochschild cohomology of B, $\mathrm{HH}^*_{\mathbb{E}_2}(B)$ (definition 3.5.12). In fact, this map should factor via the natural map of $\mathbb{E}_3$-algebras $T_B \longrightarrow \mathrm{HH}^*_{\mathbb{E}_2}(B)$ (see proposition 3.5.16), where T_B is the tangent complex of B (definition 3.5.14). Consequently, it seems reasonable to expect a natural functor from the category of 2-bimodules of [PS22] to the underlying homotopy category of the ∞-category of bimodules over $\mathcal{C}$ (as defined in [Lur17, §4.3]). In this thesis we do not study the deformations of monoidal ∞-category upto monoidal equivalences.

In [ITC23], Angel Toledo studies the deformations of a monoidal triangulated categories. The focus is on derived categories arising in algebraic geometry which admit an essentially unique dg-enhancement. The strategy employed by Toledo involves constructing a lift of the monoidal structure on a triangulated category to a (homotopy-truncated) monoidal structure on the dg-enhancement, called a pseudo dg-tensor structure on the dg-category [ITC23, §2.2]. Any pseudo dg-tensor structure on a dg-category induces an associative monoidal structure in the homotopy 2-category of dg-categories [ITC23, theorem 3.16]. As the name suggests, a pseudo dg-tensor structure is not compatible with the homotopy theory of dg-categories, i.e it is not the right notion of a monoidal structure on a dg-category. Nonetheless, the focus of Toledo's work is tensor triangulated categories, hence this notion suffices. Combined with Toën's Morita theorem, Toledo views this structure in terms of bimodules, proceeds to construct a double complex from this data, such that the 4th cohomology of the corresponding total complex gives first order deformations of the pseudo dg-tensor structure [ITC23, theorem 3.15]. To be precise, only the associativity of the pseudo dg-tensor structure is deformed. In contrast to [PS22], Toledo's work takes into consideration the notion of homotopy equivalence of dg-categories.

A difference between this thesis and Toledo's work is that we focus on classifying deformations up to Morita equivalences, which is a weaker notion of equivalence compared to weak equivalence of monoidal dg-categories. It is not immediately clear how to compare deformations of $\mathcal{C}$ as defined by Toledo and Morita deformations of $\mathcal{C}$ (as defined in construction 3.2.1).

In [FGS24], Matthieu Faigt, Azat Gainutdinov and Christoph Schweigert use the methods of homological algebra to develop methods to explicitly study the Davydov-Yetter cohomology of a tensor category which is equivalent to the category of modules over a finite dimensional algebra over a field k. They describe the Davydov-Yetter cohomology in terms of the cohomology of an internal hom-object, i.e Ext-groups, of the unit of the Drinfeld center of the tensor category [FGS24, theorem 1]. This allows for an explicit characterization of the Davydov-Yetter cohomology, even describing an associative monoid structure arising from the Yoneda product. They develop a method to explicitly compute the cocycles which give rise to the deformation of the tensor structure [FGS24, §5.4]. The Davydov-Yetter deformation theory characterizes only those deformations where just the associativity of the monoidal structure is deformed. In particular, the objects, composition and the monoidal structure on the objects is held fixed under deformations. If $\mathcal{C}$ is the ∞-category of right modules over a finite dimensional (classical) commutative k-algebra B, then there should be a map from the Davydov-Yetter complex of $\mathcal{C}$ to T_B (computed by viewing B as an $\mathbb{E}_2$-algebra over k). Possibly, this map could allow a partial description of the cohomology of T_B in terms of the explicit calculations of [FGS24].

Here we provide (also see [PS24]) different perspective on the problem of deformations of monoidal categories by applying the framework of formal moduli problems of Jacob Lurie. An advantage of our approach is the availability of powerful theoretical machinery. We study deformations of monoidal categories more comprehensively as compared to the other approaches. For instance, we do not fix the set of objects of the category under deformations; we work with the full ∞-categorical data, not passing to truncations at any point; also address the question of deformations up to Morita equivalences. A drawback is that the results are abstract; do not precisely connect to the more explicit computations performed in the above three works. Although our project begun independently, we note that a recent work by Fei Yu Chen also uses Lurie's framework to study deformations of objects in higher categories [Che23]. The key departure of our work from Chen's work is our treatment of deformations over the formal power series ring $k[[t]]$, which is crucial for application to deformation quantization. In addition, the consideration given to 'compact generators' in this thesis is something missing in [Che23]. A compact generator allows one to gain a better control over the space of deformations, as demonstrated in §3.5 . We have included the full treatment without reference to Chen to keep the thesis self contained. Part of the reason is also because some of Chen's work is not entirely clear to me. For instance, in the proof of [Che23, proposition 2.2.5] Chen shows that there is a $(n-3)$-truncated map of spaces

$$\Omega\mathrm{ObjDef}_M(A) \longrightarrow \Omega\mathrm{ObjDef}_M(B) \times_{\Omega\mathrm{ObjDef}_M(B')} \Omega\mathrm{ObjDef}_M(A')$$

from which the Chen concludes that removing the loop spaces, gives a $(n-2)$-truncated map of spaces

$$\mathrm{ObjDef}_M(A) \longrightarrow \mathrm{ObjDef}_M(B) \times_{\mathrm{ObjDef}_M(B')} \mathrm{ObjDef}_M(A')$$

The conclusion cannot follow from such a general argument because it is agnostic to those connected components of $\mathrm{ObjDef}_M(A)$ which do not contain the base point. It is not hard to cook up examples to illustrate this problem. Consider a map $S^2 \sqcup S^2 \xrightarrow{f} S^2 \sqcup \{*\}$ where the first copy of S^2 maps to S^2 via the identity, while the second copy of S^2 maps to $\{*\}$. Choose a base point in the the first copy of S^2. Then $\Omega(S^2 \sqcup S^2) \xrightarrow{\Omega f} \Omega(S^2 \sqcup \{*\})$

is a homotopy equivalence. Clearly, the map f is not (-1)-truncated, i.e. f does not induce a homotopy equivalence on the image. The proof of theorem 3.2.3 takes this issue into account and provides a slightly different proof.

1.4 Overview

In this section, we set the notation to be used in the rest of this document, summarise the content, and highlight the main results of this work.

1.4.1 Notation

- We work over a field, denoted k. This is fixed once and for all.
- Fix a nested system of three universes $U_0 \in U_1 \in U_2$. The elements of U_0 are called *small*, the elements of U_1 are called *large* and those of U_3 are called *very large*. Let κ_0 be the smallest cardinal which is large but not small.
- By an ∞-category we always mean a quasicategory (weak Kan complex), as in [Lur09, definition 1.1.2.4].
- $\mathcal{S}$ is the ∞-category of small spaces (Kan Complexes), as in [Lur09, definition 1.2.16.1]. $\widehat{\mathcal{S}}$ is the ∞-category of large spaces.
- $\widehat{\mathcal{C}\mathrm{at}}_\infty$ is the ∞-category of large ∞-categories admitting all small colimits and colimit preserving functors. This is a symmetric monoidal ∞-category where the monoidal structure preserves colimits in each variable.
- $\mathrm{Pr}^L \subset \widehat{\mathcal{C}\mathrm{at}}_\infty$ is the ∞-category of presentable ∞-categories and colimit preserving functors, as in [Lur09, definition 5.5.3.1]. This is a symmetric monoidal ∞-category where the monoidal structure, denoted $\otimes$, preserves colimits in each variable.

- Mod_k is the stable ∞-category of k-modules in spectra. By [Lur17, corollary 4.5.1.6], this is canonically equivalent to the ∞-categories of left k-module spectra and right k-module spectra [Lur17, definition 7.1.1.2]. This is a symmetric monoidal ∞-category where the tensor product is an exact functor.

- For $n \geq 1$ an integer, Alg_k^n is the ∞-category of $\mathbb{E}_n$-algebra objects in Mod_k. While $\mathrm{Alg}_k^{n,\mathrm{aug}} \simeq (\mathrm{Alg}_k^n)_{/k}$ is the ∞-category of augmented $\mathbb{E}_n$-algebras over k.

- $\mathrm{Pr}_k^L := \mathrm{Mod}_{\mathrm{Mod}_k}(\mathrm{Pr}^L)$ is the ∞-category of k-linear ∞-categories. This has a symmetric monoidal structure induced from Pr^L.

- For an integer $n \geq 1$, $n\mathrm{Pr}_k^{\mathrm{L}} = \mathrm{Mod}_{\mathrm{pr},(n-1)\mathrm{Pr}_k^{\mathrm{L}}}(\widehat{\mathcal{C}\mathrm{at}_\infty})$ is the presentable $(\infty, n+1)$-category of presentable k-linear (∞, n)-categories. We adopt the convention that $0\mathrm{Pr}_k^{\mathrm{L}} = \mathrm{Mod}_k$. See definition 2.3.8.

- For an $\mathbb{E}_n$-algebra $A \in \mathrm{Alg}_k^n$, $\mathrm{LMod}_{\mathrm{pr},A}^n := \mathrm{LMod}_{\mathrm{LMod}_A^{n-1}}((n-1)\mathrm{Pr}_k^{\mathrm{L}})$ is the presentable k-linear (∞, n)-category of n-fold iterated left A-modules. We adopt the convention that $\mathrm{LMod}_A^0 = A$. In situations where it is unambiguous, we use A instead of $\mathrm{LMod}_{\mathrm{pr},A}^n$ to avoid notational clutter. $\mathrm{RMod}_{\mathrm{pr},A}^n$ is defined analogously.

1.4.2 Outline and Main Results

We present an outline of the thesis and highlight the main results to aid the reader.

Chapter 2 is a review of the theoretical framework used in this work. This part presents the language in which the results of this thesis are stated. No novelty is claimed here, and appropriate citations have been given.

The first section recalls the Lurie's framework of derived deformation theory [Lur11b]. We recall the notion of a formal moduli problem, which is a vast generalization of Schelssinger's notion of a functor of Artin rings [Sch68]. A formal moduli problem is a functor which satisfies the derived analogs of Schlessinger's criterion. Intuitively, a formal moduli problem is the functor of points of an infinitesimal moduli space, and there is an algebraic description of this space (for example, as solutions to the Maurer-Cartan equation in an appropriate dg Lie algebra).

The deformation problems we study in this work are not formal moduli problems. Although this discrepancy can be quantified using the notion of an m-proximate formal moduli problem, where $m \geq 0$ is an integer. A 0-proximate formal moduli problem is a formal moduli problem. We recall the notion of an m-proximate formal moduli problem, originally developed in [Lur11b].

Lurie's framework is flexible enough to allow deformations over non-commutative bases, i.e. $\mathbb{E}_n$-algebras. The deformation context of $\mathbb{E}_n$-algebras is essential to us as all results in chapter 3 are formulated in this context. We recall the relevant aspects of this deformation context.

The second section recalls the theory of presentable (∞, n)-categories of Germán Stefanich [Ste20]. To talk about Morita equivalences of $\mathbb{E}_n$-algebras (definition 3.0.1) we use the notion of the presentable higher category of iterated modules over them.

Chapter 3 is the heart of this thesis. This work is in collaboration with Pranav Pandit (currently in preparation [PS24]). The chapter's introduction provides an in-depth explanation of all findings. Here we highlight the key results.

We are concerned with the deformation theory of $\mathbb{E}_n$-algebras and $\mathbb{E}_n$-monoidal ∞-categories, over a field k. Let $n \geq 1$ be an integer and B be an $\mathbb{E}_n$-algebra over k. There are two natural deformation problems associated with B.

a) Deformations of B up to equivalences of $\mathbb{E}_n$-algebras.

b) Deformations of B up to Morita equivalences of $\mathbb{E}_n$-algebras. This notion of equivalence is described in definition 3.0.1).

In this chapter, we investigate both of these deformation problems via their functor of points. The functor corresponding to problem a) is denoted AlgDef_B, and it is studied in §3.1; the functor corresponding to b) is denoted $\mathrm{CatDef}_{\mathrm{RMod}^n_{\mathrm{pr},B}}$, and it is addressed in §3.5. In general, neither of these two functors satisfies the derived analogs of Schlessinger's criterion, i.e. neither of these two is a formal moduli problem. We characterize this failure using the language of proximate formal moduli problems. Additionally, under a boundedness condition on B, we prove each of these two functors is a formal moduli problem:

Theorem A (proposition 3.1.3) Let $n \geq 1$ and $m \leq 0$ be integers, B be an $\mathbb{E}_n$-algebra over k such that the underlying spectrum of k is m-connective. Then AlgDef_B as in construction 3.1.1 is a formal moduli problem.

Theorem B (theorem 3.5.10) Let $n \geq 1$ and $m \leq 0$ be integers, B be an $\mathbb{E}_n$-algebra over k such that the underlying spectrum of k is m-connective. Then the space $\mathrm{CatDef}_{\mathrm{RMod}^n_{\mathrm{pr},B}}(k[[t]])$ is homotopy equivalent to the space as $\mathrm{CatDef}^{\wedge}_{\mathrm{RMod}^n_{\mathrm{pr},B}}(k[[t]])$.

Warning 1.4.1. We use the convention that an $\mathbb{E}_0$-monoidal category is a category, while a Morita equivalence of $\mathbb{E}_0$-monoidal categories is an equivalence of categories.

There is a third deformation problem which is naturally associated to an $\mathbb{E}_n$-algebra B.

e) Deformation $\mathrm{RMod}^{n-1}_{\mathrm{pr},B}$ viewed as an object of $\mathrm{RMod}^n_{\mathrm{pr},B}$. In case $n = 1$, this is the same as deforming B viewed as a module over itself.

The functor corresponding to this deformation problem is denoted $\mathrm{ObjDef}_{\mathrm{RMod}^{n-1}_{\mathrm{pr},B}}$. Unsurprisingly, this functor is not a formal moduli problem. The three deformation problems *a*), *b*) and *e*) are not entirely unrelated.

Theorem C (proposition 3.4.3) There is a fiber sequence of functors

$$\mathrm{ObjDef}_{\mathrm{RMod}^{n-1}_{\mathrm{pr},B}} \longrightarrow \mathrm{SimDef}_{(\mathrm{RMod}^n_{\mathrm{pr},B},\mathrm{RMod}^{n-1}_{\mathrm{pr},B})} \longrightarrow \mathrm{CatDef}_{\mathrm{RMod}^n_{\mathrm{pr},B}}$$

here $\mathrm{SimDef}_{(\mathrm{RMod}^n_{\mathrm{pr},B},\mathrm{RMod}^{n-1}_{\mathrm{pr},B})}$ is the deformation problem of the pair $(\mathrm{RMod}^n_{\mathrm{pr},B}, \mathrm{RMod}^{n-1}_{\mathrm{pr},B})$, where $\mathrm{RMod}^{n-1}_{\mathrm{pr},B}$ is viewed as an object of $\mathrm{RMod}^n_{\mathrm{pr},B}$. Note that when $n = 1$, this becomes the deformation problem of the pair (RMod_B, B), i.e. a category with a chosen object. Morally, one can identify $\mathrm{SimDef}_{(\mathrm{RMod}^n_{\mathrm{pr},B},\mathrm{RMod}^{n-1}_{\mathrm{pr},B})}$ with AlgDef_B. This identification can be made precise, but we do not attempt that in this work. This fiber sequence of functors is important to this thesis. There is an associated sequence of formal moduli problems (corollary 3.4.5; also see proposition 3.5.16) which makes a recurring appearance in the proofs of various results in chapter 3.

Chapter 2

Preliminaries

In this chapter we review the necessary background from the literature. No novelty is claimed here.

2.1 ∞-Operads

Operads are a mathematical gadget which allows one to conveniently package algebraic structures. The algebraic structure of a Lie algebra, an associative algebra, a commutative algebra, a left module over an associative algebra are some examples which admit a description in terms of operads. This thesis extensively uses the framework of ∞-operads as developed in [Lur17, §2]. In this section, we review some aspects of this framework.

Notation 2.1.1. Let $\mathcal{F}in_*$ be the skeleton of the category of finite pointed sets and their morphisms. The objects of $\mathcal{F}in_*$ are denoted $\langle n\rangle$ where $n \geq 0$. For $n > 0$ the object $\langle n\rangle$ is to be understood as $\{*, 1, 2, ..., n\}$, while $\langle 0\rangle$ is $\{*\}$. Let $f : \langle m\rangle \longrightarrow \langle n\rangle$ be a morphism in $\mathcal{F}in_*$, then

a. f is *inert* is for every $i \in \{1, ..., n\}$, the inverse image $f^{-1}(i)$ is a singleton set.

b. f is *active* if $f^{-1}(*)$ is a singleton set.

For every $n > 0$ and $1 \leq i \leq n$ there is an inert morphism, denoted $\rho^i : \langle n\rangle \longrightarrow \langle 1\rangle$, defined as

$$\rho^i(j) = \begin{cases} 1 & \text{if } i = j \\ * & \text{otherwise} \end{cases}$$

By $\mathrm{N}(\mathcal{F}in_*)$ be the ∞-category obtained by taking the nerve of $\mathcal{F}in_*$.

Definition 2.1.2. [Lur17, definition 2.1.1.10] An *∞-operad* is a functor $p : \mathcal{O}^{\otimes} \longrightarrow \mathrm{N}(\mathcal{F}in_*)$ between ∞-categories which satisfies the following conditions:

1. For every inert morphism $f : \langle m\rangle \longrightarrow \langle n\rangle$ in $\mathrm{N}(\mathcal{F}in_*)$ and every object $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$, there exists a p-cocartesian arrow $\bar{f} : C \longrightarrow C'$ in $\mathcal{O}^{\otimes}$ lifting f.

2. Let $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$ and $C' \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ be objects, let $f : \langle m\rangle \longrightarrow \langle n\rangle$ be a morphism in $\mathcal{F}in_*$, and let $\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C')$ be the union of those connected components of $\mathrm{Map}_{\mathcal{O}^{\otimes}}(C, C')$ which lie over $f \in \mathrm{Hom}_{\mathcal{F}in_*}(\langle m\rangle, \langle n\rangle)$. Choose a p-cocartesian morphism $C' \longrightarrow C'_i$ lying over the inert morphisms $\rho^i : \langle n\rangle\langle 1\rangle$ for $1 \leq i \leq n$. Then the induced map

$$\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C') \longrightarrow \prod_{1 \leq i \leq n} \mathrm{Map}^{\rho^i \circ f}_{\mathcal{O}^{\otimes}}(C, C')$$

is a homotopy equivalence.

3. For every finite collection of objects $C_1, ..., C_n \in \mathcal{O}^{\otimes}_{\langle 1\rangle}$, there exists an object $C \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ and a collection of p-cocartesian morphisms $C \longrightarrow C_i$ covering $\rho^i : \langle n\rangle \longrightarrow \langle 1\rangle$.

Example 2.1.3. [Lur17, definition 2.0.0.7] Let $\mathcal{C}^{\otimes} \xrightarrow{p} \mathrm{N}(\mathcal{F}in_*)$ be an ∞-operad and a cocartesian fibration. Let $\mathcal{C} \simeq \mathcal{C}^{\otimes}_{\langle 1\rangle}$ be the fiber of p over $\langle 1\rangle$. Then the cocartesian fibration p endows $\mathcal{C}$ with the structure of a symmetric monoidal ∞-category.

Definition 2.1.4. [Lur17, definition 2.1.2.3] Let $p : \mathcal{O}^{\otimes} \longrightarrow \mathrm{N}(\mathcal{F}in_*)$ be an ∞-operad.

A morphism f in $\mathcal{O}^{\otimes}$ is *inert* if $p(f)$ is inert and f is p-cocartesian. While f is *active* if $p(f)$ is active.

Definition 2.1.5. [Lur17, definition 2.1.2.7] Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ be ∞-operads. A *morphism of ∞-operads* is a map of simplicial sets $f : \mathcal{O}^{\otimes} \longrightarrow \mathcal{O}'^{\otimes}$ such that

1. The diagram

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \arrow[rr, "f"] \arrow[dr] & & \mathcal{O}'^{\otimes} \arrow[dl] \\
& \mathrm{N}(\mathcal{F}in_*) &
\end{tikzcd}$$

   commutes.

2. The map f preserves inert morphisms.

The ∞-category of ∞-operad morphisms from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$ is denoted $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$.

Example 2.1.6. In the above definition, let $\mathcal{O}'$ be a symmetric monoidal ∞-category and let $\mathcal{O} = \mathrm{N}(\mathcal{F}in_*)$. Then $\mathrm{Alg}_{\mathrm{N}(\mathcal{F}in_*)}(\mathcal{O}')$ is the ∞-category of commutative algebra objects in $\mathcal{O}'$.

Definition 2.1.7. [Lur17, definition 2.1.3.7] Let $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ and $q : \mathcal{D}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be maps of ∞-operads which are also cocartesian fibrations. Then $\mathcal{C}$ and $\mathcal{D}$ are *$\mathcal{O}$-monoidal ∞-categories.* A map of ∞-operads $f \in \mathrm{Alg}_{\mathcal{C}}(\mathcal{D})$ is a *$\mathcal{O}$-monoidal functor* if it carries p-cocartesian arrows to q-cocartesian arrows.

2.1.1 The left module ∞-operad

We follow [Lur17, §4.2] to review some aspects of left-module objects in ∞-categories. An analogous treatment can be done for right-module objects.

Definition 2.1.8. [Lur17, definition 2.1.1.1] A *coloured operad* $\mathcal{O}$ is defined as follows:

1. A collection of *objects or colours* of $\mathcal{O}$, $\{X, Y, Z, ..\}$.

2. Given a finite set I, an I-indexed collection of colours $\{X_i \in \mathcal{O}\}_{i\in I}$ and a colour $Y \in \mathcal{O}$, a *set of morphisms from* $\{X_i\}_{i\in I}$ *to* Y denoted $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$.

3. Given a map of finite sets $I \longrightarrow J$ let the fibres be $\{I_j\}_{j\in J}$. Consider a collection of colours $\{X_i\}_{i\in I}$, $\{Y_j\}_{j\in J}$ and $Z \in \mathcal{O}$. There is a composition
$$\prod_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I_j}, Y_j) \times \mathrm{Mul}_{\mathcal{O}}(\{Y_j\}_{j\in J}, Z) \longrightarrow \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Z)$$

4. A collection of morphisms $\{id_X \in \mathrm{Mul}_{\mathcal{O}}(\{X\}, X)\}_{X\in\mathcal{O}}$ which are left and right units of the composition on $\mathcal{O}$.

5. The composition is associative.

The left module operad is an example of a coloured operad.

Example 2.1.9. [Lur17, definition 4.2.1.1] The left module coloured operad **LM** is defined as follows:

1. There are two colours: $\mathfrak{a}$ and $\mathfrak{m}$.

2. Let $\{X_i\}_{i\in I}$ be a finite collection of colours and Y be another colour. If $Y = \mathfrak{a}$ and $X_i = \mathfrak{a}$ for every $i \in I$, then $\mathrm{Mul}_{\mathbf{LM}}(\{X_i\}_{i\in I}, Y)$ is the set of linear orderings of I.

   If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$ is the collection of linear orderings $\{i_1 \leq ... \leq i_n\}$ such that $X_{i_n} = \mathfrak{m}$ and $X_{i_j} = \mathfrak{a}$ for every $j < n$.

   All other morphisms sets are empty.

3. Composition is given by composition of linear orderings. Intuitively, one can imagine concatenation of trees with vertices labelled by $\mathfrak{a}$ and $\mathfrak{m}$.

Another example of a coloured operad is the associative operad.

Example 2.1.10. [Lur17, definition 4.1.1.1] The associative coloured operad **Assoc** is the full sub-operad of **LM** spanned by the colour $\mathfrak{a}$.

Definition 2.1.11. [Lur17, notation 4.2.1.6] Let $\mathbf{LM}^{\otimes}$ be the category obtained from the colour operad $\mathbf{LM}$. This category is defined as follows:

1. The objects are finite sequences of colours $X_1, .., X_n \in \mathbf{LM}$.
2. A morphism from $\{X_1, .., X_m\}$ to $\{Y_1, .., Y_n\}$ is the data of a morphism of pointed sets $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$ along with a collection

$$\{\phi_j \in \mathrm{Mul}_{\mathbf{LM}}(\{X_i\}_{i \in \alpha^{-1}(j)}, Y_j)\}_{j \in J}$$

3. Composition in $\mathbf{LM}^{\otimes}$ is given by compostion of $\mathbf{LM}$.

Definition 2.1.12. [Lur17, definition 4.2.1.7] There is an evident forgetful functor $\mathbf{LM}^{\otimes} \longrightarrow \mathcal{F}in_*$ which induces a functor of ∞-categories $N(\mathbf{LM}^{\otimes}) \longrightarrow N(\mathcal{F}in_*)$. The ∞-operad $\mathcal{LM}^{\otimes}$ is given by taking the nerve of the category $\mathbf{LM}^{\otimes}$.

Definition 2.1.13. The full sub ∞-operad of $\mathcal{LM}^{\otimes}$ spanned by the colour $\mathfrak{a}$ is an ∞-operad $\mathrm{Assoc}^{\otimes}$.

Remark 2.1.14. [Lur17, remark 4.2.1.10] The natural inclusion of coloured operads $\mathbf{Assoc} \hookrightarrow \mathbf{LM}$ induces a fully faithful map of ∞-categories $\mathrm{Assoc}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$ which is a map of ∞-operads.

Definition 2.1.15. [Lur17, definition 4.2.1.19] Let $\mathcal{C}^{\otimes} \xrightarrow{q} \mathcal{LM}^{\otimes}$ be a map of ∞-operads which is a cocartesian fibration. Then q exhibits $\mathcal{C}_{\mathfrak{m}} = q^{-1}(\mathfrak{m})$ as *left-tensored over* $\mathcal{C}_{\mathfrak{a}} = q^{-1}(\mathfrak{a})$.

Definition 2.1.16. [Lur17, definition 4.1.1.10] A monodial ∞-category is a map of ∞-operads $\mathcal{C}^{\otimes} \longrightarrow \mathrm{Assoc}^{\otimes}$ which is also a cocartesian fibration.

Example 2.1.17. Let $\mathcal{C}^{\otimes} \longrightarrow \mathrm{N}(\mathcal{F}in_*)$ be a symmetric monoidal ∞-category. Then Assoc-monoids in $\mathcal{C}$ are maps of ∞-operads $\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{C})$. The objects of $\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{C})$ are associative algebra objects of $\mathcal{C}$.

Example 2.1.18. Let $\mathcal{C}^{\otimes} \longrightarrow \mathrm{N}(\mathcal{F}in_*)$ be a symmetric monoidal ∞-category. Then $\mathcal{LM}$-monoids in $\mathcal{C}$ are maps of ∞-operads $\mathrm{Alg}_{\mathcal{LM}}(\mathcal{C})$. The objects of $\mathrm{Alg}_{\mathcal{LM}}(\mathcal{C})$ may be viewed as pairs (A, M) where A is an associative algebra object of $\mathcal{C}$ and M is an object of $\mathcal{C}$ equipped with a left action of A.

2.1.2 Little l-disk ∞-operad

We follow [Lur17, §5] to review some aspects of $\mathbb{E}_n$-algebras.

Definition 2.1.19. [Lur17, definition 5.1.0.2] Let $l > 1$ be an integer, $\square^l = (-1, 1)^l$ be the open cube of dimension l and ${}^t\mathbb{E}_l^{\otimes}$ be a topological category defined as follows:

1. The objects are $\langle n \rangle \in \mathcal{F}in_*$.
2. A morphisms $\langle m \rangle \longrightarrow \langle n \rangle$ in ${}^t\mathbb{E}_l^{\otimes}$ is the following data
    a. A morphism $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$ in $\mathcal{F}in_*$.
    b. For each $j \in \{1, ..., n\}$ an embedding $\square^l \times \alpha^{-1}(j) \xrightarrow{\phi} \square^l$. Moreover, for every $x \in \alpha^{-1}(j)$, $\phi(-, x) : \square^k \longrightarrow \square^k$ is given by
    $$(y_1, ..., y_l) \mapsto (a_1 y_1 + b_1, ..., a_l y_l + b_l)$$
    for some real constants $a_i > 0$ and b_i. ϕ is called a *rectilinear embedding.*
3. For every pair of objects $\langle m \rangle, \langle n \rangle \in {}^t\mathbb{E}_l^{\otimes}$, the space $\mathrm{Hom}_{{}^t\mathbb{E}^{\otimes}_l}(\langle m \rangle, \langle n \rangle)$ has the presentation
$$\mathrm{Hom}_{{}^t\mathbb{E}_l^{\otimes}}(\langle m \rangle, \langle n \rangle) = \coprod_{f:\langle m \rangle \longrightarrow \langle n \rangle} \prod_{1 \leq j \leq n} \mathrm{Rect}(\square^l \times f^{-1}(j), \square^l)$$
4. The composition of morphisms is defined in the obvious way, in terms of concatenation of trees.

Let $\mathbb{E}_l^{\otimes}$ be the topological nerve of ${}^t\mathbb{E}_l{}^{\otimes}$.

Proposition 2.1.20. *[Lur17, proposition 5.1.0.3] There is a forgetful functor ${}^t\mathbb{E}_l{}^{\otimes} \longrightarrow \mathcal{F}in_*$, which induces a functor $\mathbb{E}_l^{\otimes} \longrightarrow \mathrm{N}(\mathcal{F}in_*)$. This induced functor exhibits $\mathbb{E}_l^{\otimes}$ as an ∞-operad.*

Definition 2.1.21. $\mathbb{E}_l^{\otimes}$ is called the *∞-operad of little l-cubes*

Construction 2.1.22. [Lur17, construction 5.1.2.1] Let $l, l' > 0$ be integers. There is a topological functor $\rho : {}^t\mathbb{E}_l{}^{\otimes} \times {}^t\mathbb{E}_{l'}{}^{\otimes} \longrightarrow {}^t\mathbb{E}_{l+l'}{}^{\otimes}$ defined as follows:

1. The diagram

$$\begin{tikzcd}
{}^t\mathbb{E}_l{}^{\otimes} \times {}^t\mathbb{E}_{l'}{}^{\otimes} \arrow[r, "\rho"] \arrow[d] & {}^t\mathbb{E}_{l+l'}{}^{\otimes} \arrow[d] \\
\mathrm{N}(\mathcal{F}in_*) \times \mathrm{N}(\mathcal{F}in_*) \arrow[r, "\#"] & \mathrm{N}(\mathcal{F}in_*)
\end{tikzcd}$$

   commutes. Here, $\# : \mathrm{N}(\mathcal{F}in_*) \times \mathrm{N}(\mathcal{F}in_*) \longrightarrow \mathrm{N}(\mathcal{F}in_*)$ is as defined in [Lur17, notation 2.2.5.1]:

   a. $\langle m \rangle \# \langle n \rangle = \langle mn \rangle$.

   b. Given $f : \langle m \rangle \longrightarrow \langle m' \rangle$ and $g : \langle n \rangle \longrightarrow \langle n' \rangle$ two morphisms of finite pointed sets,

$$f\#g(an + b - n) := \begin{cases} * & \text{if } f(a) = * \text{ or } g(b) = * \\ f(a)n' + g(b) - n' & \text{otherwise} \end{cases}$$

2. Given morphisms $\bar{\alpha} : \langle m \rangle \longrightarrow \langle n \rangle$ in ${}^t\mathbb{E}_l{}^{\otimes}$ and $\bar{\beta} : \langle m' \rangle \longrightarrow \langle n' \rangle$ in ${}^t\mathbb{E}_{l'}{}^{\otimes}$, denote $\bar{\alpha} = (\alpha, \{f_j : \square^l \times \alpha^{-1}(j) \longrightarrow \square^l\}_{j \in \{1,..,n\}})$ and $\bar{\beta} = (\beta, \{f_{j'} : \square^{l'} \times \alpha^{-1}(j') \longrightarrow \square^{l'}\}_{j' \in \{1,..,n'\}})$. Define $\rho(\bar{\alpha}, \bar{\beta}) : \langle mm' \rangle \longrightarrow \langle nn' \rangle$

$$(\alpha\#\beta, \{f_j \times f_{j'} : \square^{l+l'} \times \alpha^{-1}(j) \times \beta^{-1}(j'), \square^{l+l'}\})_{j \in \{1,..,n\}, j' \in \{1,..,n'\}}$$

Taking the topological nerve induces a functor $\mathbb{E}_l^{\otimes} \times \mathbb{E}_{l'}^{\otimes} \xrightarrow{\rho} \mathbb{E}_{l+l'}^{\otimes}$.

The following is a version of the Dunn additivity theorem proven by Lurie. It plays an important role throughout the next chapter.

Theorem 2.1.23. *[Lur17, theorem 5.1.2.2] Let $l, l' > 0$ be integers. The functor $\mathbb{E}_l^{\otimes} \times \mathbb{E}_{l'}^{\otimes} \xrightarrow{\varrho} \mathbb{E}_{l+l'}^{\otimes}$ of construction 2.1.22 exhibits the ∞-operad $\mathbb{E}_{l+l'}^{\otimes}$ as a tensor product of the ∞-operads $\mathbb{E}_l^{\otimes}$ and $\mathbb{E}_{l'}^{\otimes}$.*

In other words, $\varrho(\bar{\alpha}, \bar{\beta})$ is inert if both $\bar{\alpha}$ and $\bar{\beta}$ are inert. Let $\mathcal{C}$ be a symmetric monoidal ∞-category such that the tensor product preserves colimits in each variable. For every integer $l > 0$, $\mathrm{Alg}^l(\mathcal{C}) := \mathrm{Alg}_{\mathbb{E}_l}(\mathcal{C})$ is the ∞-category of $\mathbb{E}_n$-algebra objects in $\mathcal{C}$. The above theorem (Dunn-Lurie additivity) implies that

$$\mathrm{Alg}^{l+l'}(\mathcal{C}) \simeq \mathrm{Alg}^l(\mathrm{Alg}^{l'}(\mathcal{C}))$$

Lemma 2.1.24. *[Lur17, example 5.1.0.7] There is an equivalence of ∞-operads $\mathbb{E}_1^{\otimes} \simeq$ Assoc$^{\otimes}$.*

2.2 Formal Moduli Problems

In Schlessinger's axiomatic approach to deformation theory [Sch68], the deformations of an algebro-geometric object are encoded in a functor from local Artin rings to sets, called a functor of Artin rings. A functor of Artin rings that behaves well with certain limits allows one to study its value on 'larger' Artin rings in terms of 'smaller' ones, for example, small extensions/square-zero extensions. A functor of Artin rings that does this may admit a hull [Sch68, definition 2.7], i.e. a surjective morphism from a 'minimal' pro-representable functor of Artin rings. It would be even better if such a functor is (pro)-representable, allowing an algebraic description of the formal neighbourhood of the object being deformed. Schlessinger's criterion [Sch68, theorem 2.11] provides sufficient conditions on a functor of Artin rings to guarantee the existence of a hull, and pro-representability. An example of a pro-representable functor of Artin rings is the formal completion of scheme at a closed point. If the scheme is a moduli scheme, then the formal completion encodes the data of deformations of the object represented by the closed point.

Analogously, one would like a 'derived functor of Artin rings' to behave well with homotopy limits and satisfy a derived variant of Schlessinger's criterion. When trying to formulate such a criterion, one is guided by the natural requirement that the formal completion of a derived algebraic stack at any closed point should satisfy that criterion. In this section, we review the axiomatic framework of derived deformation theory as developed by Lurie [Lur11b]. In this framework a derived functor of Artin rings that satisfies certain conditions is called a *formal moduli problem*. If a functor is a formal moduli problem it admits an algebraic description. For instance, in terms of a dg Lie algebra.

There are numerous derived functors of Artin rings that arise from algebro-geometric objects and which fail to be a formal moduli problem. The functors we study also show this behaviour. We review the notion of an *approximate formal moduli problem* as introduced in [Lur11b, §5.1]. This notion allows for a quantification of the failure of a deformation functor to being a formal moduli problem, giving a convenient language to formulate our results.

2.2.1 Stabilization and Spectrum Objects

We will see that the tangent space to a formal moduli problem is more naturally described as a tangent spectrum, from which the tangent space can be recovered. In view of this observation, we recall the definition of a spectrum object.

Definition 2.2.1. Let $\mathfrak{Y}$ be an ∞-category which admits finite limits. A *spectrum object* of $\mathfrak{Y}$ is a reduced, excisive functor $X : \mathcal{S}_*^{fin} \longrightarrow \mathfrak{Y}$. By definition, X is reduced if it preserves final object and X is excisive if it takes pushout squares $\mathcal{S}_*^{fin}$ to pullback squares in $\mathfrak{Y}$.

Notation 2.2.2. [Lur17, notation 1.4.2.20] Let $\mathrm{Sp}(\mathfrak{Y}) = \mathrm{Exc}_*(\mathcal{S}_*^{fin}, \mathfrak{Y})$ be the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{fin}\mathfrak{Y})$ spanned by the spectrum objects of $\mathfrak{Y}$. For each integer n, there is a functor

$$\Omega^{\infty-n} : \mathrm{Sp}(\mathfrak{Y}) \longrightarrow \mathfrak{Y}$$

which is given by $E \mapsto \Omega^\infty(E[n])$, where $E[n]$ is the shift of E be n. In case $n \geq 0$, the functor $\Omega^{\infty-n}$ is given by evaluation on the n-sphere.

Example 2.2.3. Let $\mathcal{S}$ be the ∞-category of spaces. The ∞-category of spectrum objects $\mathrm{Sp}(\mathcal{S})$ is equivalent to the ∞-category of spectra.

Remark 2.2.4. Let R be an $\mathbb{E}_\infty$-ring spectrum, $\mathfrak{Y} = (\mathrm{CAlg}_R)_{/R} = \mathrm{CAlg}_R^{\mathrm{aug}}$ be the ∞-category of augmented $\mathbb{E}_\infty$-algebras over R. By [Lur17, corollary 7.3.4.14], the ∞-category $\mathrm{Sp}(\mathfrak{Y})$ is equivalent to the ∞-category of R-module spectra Mod_R.

Let $E \in \mathrm{Mod}_R$. Following [Lur17, remark 7.3.4.16], for any $n \geq 0$ we can identify $\Omega^{\infty-n}E$ with the square-zero extension $R \oplus E[n]$.

Remark 2.2.5. Let $n \geq 1$ be an integer, R be an $\mathbb{E}_{n+1}$-ring spectrum, LMod_R be the ∞-category of left R-modules in $\mathrm{Sp}(\mathcal{S})$. As a consequence of [Lur17, corollary 4.8.5.20 & proposition 7.1.1.4] LMod_R is an $\mathbb{E}_n$-monoidal stable ∞-category.

Let Alg_R^n be the ∞-category of $\mathbb{E}_n$-algebras in LMod_R and $\mathrm{Alg}_R^{n,\mathrm{aug}} = (\mathrm{Alg}_R^{n,\mathrm{aug}})_{/R}$ be the ∞-category of augmented $\mathbb{E}_n$-algebras over R. It follows from [Lur17, theorem 7.3.4.13 & proposition 3.4.2.1] that

$$\mathrm{Sp}(\mathrm{Alg}_R^{n,\mathrm{aug}}) \simeq \mathrm{LMod}_R$$

is the ∞-category of spectrum objects of Alg_R^n.

Remark 2.2.6. A spectrum object in spaces can be intuitively thought of as an ∞-fold loop space of a space. This perspective is made precise by [Lur17, proposition 1.4.2.24].

This proposition says that the ∞-category $\mathrm{Sp}(\mathfrak{Y})$ is equivalent to the homotopy limit of the tower of ∞-categories

$$\ldots \longrightarrow \mathfrak{Y}_* \xrightarrow{\Omega} \mathfrak{Y}_* \xrightarrow{\Omega} \mathfrak{Y}_*$$

2.2.2 Formal Moduli Problems

We present an introduction to the framework of formal moduli problems.

Definition 2.2.7. A *deformation context* is a pair $(\mathfrak{Y}, \{E_\alpha\}_{\alpha \in T})$, where $\mathfrak{Y}$ is a presentable ∞-category and $\{E_\alpha\}_{\alpha \in T}$ is a set of objects of the stabilisation $\mathrm{Sp}(\mathfrak{Y})$.

A deformation context decides the test objects which we are allowed to probe the formal moduli of the objects of our interest. For instance, consider Schlessinger's framework of functor of Artin rings. In this setting one could consider a functor which maps local Artin k-algebras to sets. The test objects in this case are local Artin k-algebras. Lurie's framework is more flexible, allowing one to choose test objects from arbitrary presentable ∞-categories. In other words, the choice of a deformation context is the choice of the objects which act as the bases over which we deform.

Example 2.2.8. [Lur11b, example 1.1.4] Let R be an $\mathbb{E}_\infty$-ring spectrum. In view of remark 2.2.4, the pair $(\mathrm{CAlg}_R^{\mathrm{aug}}, R)$ is a deformation context.

View R as an $\mathbb{E}_{n+1}$-ring spectrum via the forgetful functor $\mathrm{CAlg} \longrightarrow \mathrm{Alg}^{n+1}$. Following remark 2.2.5, the pair (Alg_R^n, R) is also a deformation context for every integer $n \geq 1$.

For the remainder of this section, we focus on the deformation context (Alg_k^n, k), where k is a field. This is the deformation context relevant for chapter 3. The reader is referred to [Lur11b, §1] for a more general treatment.

Formal $\mathbb{E}_n$-Moduli Problems

Let $n \geq 1$ be an integer, k be a field.

Notation 2.2.9. Let $\mathrm{Alg}_k^{n,\mathrm{aug}} \simeq (\mathrm{Alg}_k^n)_{/k}$ be the ∞-category of *augmented* $\mathbb{E}_n$*-algebras over* k.

We fix an integer $n \geq 1$ and the deformation context $(\mathfrak{Y}, \{E\}) = (\mathrm{Alg}_k^{n,\mathrm{aug}}, \{E\})$ for the rest of this section. Here, $E \in \mathrm{Sp}(\mathrm{Alg}_k^{n,\mathrm{aug}})$ corresponds to $k \in \mathrm{Mod}_k$ (remark 2.2.5).

Now we specify the test objects in the deformation context of $\mathbb{E}_n$-algebras. These are analogs of local Artin k-algebras.

Definition 2.2.10. [Lur11b, definitions 1.1.5 & 1.1.8]

1. A morphism $\phi : A \to A'$ in $\mathrm{Alg}_k^{n,\mathrm{aug}}$ is called *elementary* if there is an integer $n > 0$ and a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\phi"] & k \arrow[d, "\phi_0"] \\ A' \arrow[r] & \Omega^{\infty-n}E \end{tikzcd}$$

   Here, ϕ_0 is the base point of $\Omega^{\infty-n}E = k \oplus k[n]$.

2. A morphism $\phi : A \to A'$ in $\mathrm{Alg}_k^{n,\mathrm{aug}}$ is called *small* if it can be written as a composition of finitely many elementary morphisms.

3. An object A of $\mathrm{Alg}_k^{n,\mathrm{aug}}$ is called *small* if the canonical map $A \to k$ is a small morphism. The full ∞-subcategory of small objects of $\mathrm{Alg}_k^{n,\mathrm{aug}}$ is denoted by $\mathrm{Alg}_k^{n,\mathrm{small}}$.

Proposition 2.2.11. *[Lur11b, proposition 4.5.1] An* $\mathbb{E}_n$*-algebra* A *over* k *is small if and only if*

1. *A is connective, i.e.* $\pi_i A \simeq 0$ *for all* $i < 0$.

2. $\pi_* A$ *is a finite dimensional* k*-vector space.*

3. *Let* $\mathfrak{n}$ *be the radical of* $\pi_0 A$*. Then unit map* $k \longrightarrow (\pi_0 A)/\mathfrak{n}$ *is an isomorphism of* k*-vector spaces.*

The analog of a functor of Artin rings in Lurie's framework is the following:

Definition 2.2.12. A *pre-formal moduli problem* is a functor $X : \mathrm{Alg}_k^{n,\mathrm{small}} \longrightarrow \mathcal{S}$ such that $X(k)$ is contractible.

The following is the analog of Schlessinger's criterion in Lurie's framework:

Definition 2.2.13. [Lur11b, definition 1.1.14] Let σ:

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & B_1 \arrow[d, "\phi"] \\ A_2 \arrow[r] & B_2 \end{tikzcd}$$

be any pullback in $\mathrm{Alg}_k^{n,\mathrm{small}}$. A pre-formal moduli problem X is called a *formal moduli problem* if $X(\sigma)$ is a Cartesian diagram in $\mathcal{S}$ whenever ϕ is small.

Notation 2.2.14. Let $\mathrm{Moduli}_k^n \subset \mathrm{Fun}(\mathrm{Alg}_k^{n,\mathrm{small}}, \mathcal{S})$ be the full subcategory spanned by formal moduli problems.

Formal moduli problems have an important invariant, called the tangent complex. Let X be an algebraic variety over $\mathbb{C}$, $\eta \in X(\mathbb{C})$ be a closed point of X. The Zariski tangent space of X at η is defined to be the fiber $T_{X,\eta} = \mathrm{fib}_\eta(X(\mathbb{C}[\epsilon]/\epsilon^2) \longrightarrow X(\mathbb{C}))$. Equivalently, this is the Zariski tangent space of the formal completion of X at the closed point η. The tangent complex of a formal moduli problem is a derived analog of the Zariski tangent space of an algebraic variety.

Remark 2.2.15. Let $X : \mathrm{Alg}_k^{n,\mathrm{small}} \longrightarrow \mathcal{S}$ be a formal moduli problem. It follows from [Lur11b, proposition 1.2.3] that $\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{E} \mathrm{Alg}_k^{n,\mathrm{aug}}$ factors via the full subcategory of small objects $\mathrm{Alg}_k^{n,\mathrm{small}}$. As a consequence of [Lur11b, proposition 1.2.4] we conclude that the composition

$$\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{E} \mathrm{Alg}_k^{n,\mathrm{small}} \xrightarrow{X} \mathcal{S}$$

is a strongly excisive, i.e. may be viewed as a spectrum.

Definition 2.2.16. [Lur11b, definition 1.2.5] Let $X : \mathrm{Alg}_k^{n,\mathrm{small}} \longrightarrow \mathcal{S}$ be a formal moduli problem. Let $X(E_\alpha)$ denote the composition $\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{E} \mathrm{Alg}_k^{n,\mathrm{small}} \xrightarrow{X} \mathcal{S}$. We view $X(E)$ as an object in the ∞-category of spectra, called the *tangent complex to* X, denoted TX.

The following result establishes that the tangent complex of a formal moduli problem is a complete invariant.

Proposition 2.2.17. *[Lur11b, proposition 1.2.10] Let $f : X \longrightarrow Y$ be a map of formal moduli problems. Then f is an equivalence if and only if it induces an equivalence of spectra $X(E) \longrightarrow Y(E)$.*

We recall the notion of $\mathbb{E}_n$ Koszul duality in terms of a universal property. This duality leads to the generalization of the relation between deformation problems and dg Lie algebras.

Let $n \geq 1$ be an integer, k be a field, A be an $\mathbb{E}_n$-algebra over k. Let $\mathrm{Aug}(A) = \mathrm{Map}_{\mathrm{Alg}_k^n}(A, k)$ be the space of augmentations of A. Let B be an $\mathbb{E}_n$-algebra over k and choose augmentations $\epsilon_A : A \to k$ and $\epsilon_B : B \to k$. Let $\mathrm{Pair}(A, B)$ be the fiber of the map $\mathrm{Aug}(A \otimes_k B) \longrightarrow \mathrm{Aug}(A) \times \mathrm{Aug}(B)$ over (ϵ_A, ϵ_B).

Definition 2.2.18. Define $\mathfrak{D}^n(A)$ as the $\mathbb{E}_n$ Koszul dual of A, specified by the universal property given below

$$\mathrm{Map}_{\mathrm{Alg}_k^{n,\mathrm{aug}}}(B, \mathfrak{D}^n(A)) \simeq \mathrm{Pair}(A, B)$$

Equivalently, the $\mathbb{E}_n$-Koszul dual represents the functor $B \mapsto \mathrm{Pair}(A, B)$.

By [Lur11b, proposition 4.4.1] the object $\mathfrak{D}^n(A)$ along with a universal pairing $\nu : A \otimes_k \mathfrak{D}^n(A) \longrightarrow k$ exists. $\mathfrak{D}^n : (\mathrm{Alg}_k^{n,\mathrm{aug}})^{op} \longrightarrow \mathrm{Alg}_k^{n,\mathrm{aug}}$ is called the $\mathbb{E}_n$ Koszul duality functor.

The following result is one of the most important results in the context of derived deformation theory. This is the non-commutative, positive characteristic analog of the statement that “dg Lie algebras control deformation problems over a field of characteristic zero” (see [Lur11b, theorem 2.0.2]).

Theorem 2.2.19. *[Lur11b, theorem 4.0.8] Let $n \geq 0$ be an integer and k be a field. Then there is an equivalence of ∞-categories $\Psi : Alg_k^{n,\ aug} \longrightarrow Moduli_k^n$, given by given by $A \mapsto \mathrm{Map}_{\mathrm{Alg}_k^{n,\mathrm{aug}}}(\mathfrak{D}^n(-), A)$. In addition, we have the following (homotopy) commutative diagram*

$$\begin{tikzcd}
Alg_k^{n,aug} \arrow[r, "\Psi"] \arrow[d, "\mathfrak{m}"] & Moduli_k^n \arrow[d, "T[-n]"] \\
Mod_k \arrow[r] & Sp
\end{tikzcd}$$

where $T : Moduli_k^n \to Sp$ is the tangent complex functor and $\mathfrak{m} : Alg_k^{n,aug} \to Mod_k$ is the augmentation ideal functor given by $(A \mapsto \mathrm{fib}(A \to k))$ with the fiber taken in Mod_k.

Definition 2.2.20. [Lur11b, definition 4.1.1] Let m and $n \geq 1$ be integers, k be a field, and A be an $\mathbb{E}_n$-algebra over k. Note that A is equipped with a unit, which is a map of k-modules $k \xrightarrow{e} A$. Then A is said to be *m-coconnective* if the homotopy groups $\pi_i \mathrm{cofib}(e)$ vanish for $i > -m$.

Note that when $m > 0$, A is m-coconnective if and only if the unit map $k \to A$ induces an isomorphism $k \longrightarrow \pi_0 A$ and $\pi_i A$ vanish for all $i > 0$ and $-m < i < 0$.

The following two results about $\mathbb{E}_n$ Koszul duality will be essential in §3.5 and §??.

Lemma 2.2.21. *[Lur11b, lemma 1.5.10] For every small object $A \in \mathrm{Alg}_k^{n,\mathrm{small}}$, $\mathfrak{D}^n(A)$ is a compact object of $\mathrm{Alg}_k^{n,\mathrm{aug}}$.*

Lemma 2.2.22. *[Lur11b, lemma 4.5.9] Let $n \geq 1$ be an integer, k be a field and A be an augmented $\mathbb{E}_n$-algebra over k. If A is connective, then the Koszul dual $\mathfrak{D}^n(A)$ is n-coconnective.*

2.2.3 Approximations to Formal Moduli Problems

Definition 2.2.23. Consider a (homotopy) commutative diagram of spaces:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

This diagram is said to be *n-Cartesian* if the homotopy fibres of the the natural map $X \to Y \times_{Y'} X'$ are $(n-2)$-truncated. We say that a space is (-2)-truncated space if and only if it is contractible. It follows that a 0-Cartesian square is a Cartesian square in the usual sense.

Definition 2.2.24. Let σ:

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & B_1 \arrow[d, "\phi"] \\ A_2 \arrow[r] & B_2 \end{tikzcd}$$

be any pullback in $\mathrm{Alg}_k^{n,\mathrm{small}}$. Let $n \geq 0$ be an integer. A pre-formal moduli problem X is called a *n-proximate formal moduli problem* if $X(\sigma)$ is a $(n-2)$-Cartesian diagram in $\mathcal{S}$ whenever ϕ is small.

Notation 2.2.25. As Moduli_k^n is an accessible localization of $\mathrm{Fun}(\mathrm{Alg}_k^{n,\mathrm{small}}, \mathcal{S})$, there exists a left adjoint $L : \mathrm{Fun}(\mathrm{Alg}_k^{n,\mathrm{small}}, \mathcal{S}) \longrightarrow \mathrm{Moduli}_k^n$ to the natural inclusion $i : \mathrm{Moduli}_k^n \longrightarrow \mathrm{Fun}(\mathrm{Alg}_k^{n,\mathrm{small}}, \mathcal{S})$. We note that the natural inclusion factors through the full subcategory of pre-formal moduli problems $\mathrm{Fun}_*(\mathrm{Alg}_k^{n,\mathrm{small}}, \mathcal{S})$. This leads to an adjunction (denoted using the same symbols)

$$\begin{tikzcd} \mathrm{Moduli}_k^n \arrow[r, "i"] & \mathrm{Fun}_*(\mathrm{Alg}_k^{n,\mathrm{small}}, \mathcal{S}) \arrow[l, bend right, "L"'] \end{tikzcd}$$

For every integer $m \geq 0$, Let $\mathrm{Prox}(m) \subset \mathrm{Fun}_*(\mathrm{Alg}_k^{n,\mathrm{small}}, \mathcal{S})$ be the full subcategory of m-proximate formal moduli problems. We will denote the composition

$$\mathrm{Prox}(m) \subset \mathrm{Fun}_*(\mathrm{Alg}_k^{n,\mathrm{small}}, \mathcal{S}) \xrightarrow{L} \mathrm{Moduli}_k^n$$

by $(-)^\wedge$.

Theorem 2.2.26. *[Lur11b, theorem 5.1.9] Let $X : \mathrm{Alg}_k^{n,\mathrm{small}} \to \mathcal{S}$ be a pre-formal moduli problem. Then the following are equivalent:*

1. *The functor X is a n-proximate formal moduli problem.*

2. *There exists an $(n-2)$-truncated map $\eta : X \to Y$, where Y is an n-proximate formal moduli problem.*

3. *Let L denote the left adjoint to the natural inclusion $Moduli_n^n \hookrightarrow Fun(\mathrm{Alg}_k^{n,\mathrm{small}}, \mathcal{S})$. Then the unit map $X \to L(X)$ is $(n-2)$-truncated.*

2.3 Presentable (∞, n)-categories

Notation 2.3.1. Fix a nested system of three universes $U_0 \in U_1 \in U_2$. The elements of U_0 are called *small*, the elements of U_1 are called *large* and those of U_3 are called *very large*. Let κ_0 be the smallest cardinal that is large.

Notation 2.3.2. Let $\widehat{\mathcal{C}\mathrm{at}}_\infty$ be the ∞-category of large cocomplete ∞-categories and colimit preserving functors.

The notion of a presentable (∞, n)-category relies on the observation that the ∞-category of presentable ∞-categories Pr^{L} is an essentially large ∞-category. Pr^L is both a subcategory and an object of $\widehat{\mathcal{C}\mathrm{at}}_\infty$. The observation that $\widehat{\mathcal{C}\mathrm{at}}_\infty$ is κ_0-compactly generated by presentable ∞-categories (see proposition 2.3.4) allows a construction of presentable higher categories as iterated modules over Pr^L without a need for an infinite nested sequences of universes. A detailed development of these ideas may be found in [Ste20]. Here we present a terse review relevant for our purpose.

Definition 2.3.3. [Ste20, definition 5.1.2] Let $\mathcal{C}$ be a very large, locally large ∞-category admitting all large colimits. An object $E \in \mathcal{C}$ is κ_0-*compact* if the corresponding corepresentable functor $\mathcal{C} \xrightarrow{h^E} \widehat{\mathcal{S}}$ preserves κ_0-filtered colimits. We say $\mathcal{C}$ is κ_0-*compactly generated* if it is generated under large colimits by the κ_0-compact objects, and the space of κ_0-compact objects is large.

Proposition 2.3.4. *[Ste20, proposition 5.1.4] The ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ is κ_0-compactly generated. An object of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ is κ_0-compact if and only if it is a presentable ∞-category.*

Proposition 2.3.5. *[Ste20, proposition 5.1.10] Let $\mathcal{E}$ be an algebra in $\widehat{\mathcal{C}\mathrm{at}}_\infty$, and A be an algebra in $\mathcal{E}$. Then the ∞-category RMod_A of right A-modules equipped with its natural left $\mathcal{E}$-module structure, is a presentable $\mathcal{E}$-module. In other words, RMod_A is κ_0-compact as an $\mathcal{E}$-module.*

Notation 2.3.6. Let $\mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ be the ∞-category of $\mathbb{E}_\infty$-algebra objects in $\widehat{\mathcal{C}\mathrm{at}}_\infty$. Following [Ste20, remark 5.1.11], we have a lax symmetric monoidal functor

$$\mathrm{Mod}_{\mathrm{pr},-} : \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$$

which sends an object $\mathcal{E}$ to $\mathrm{Mod}_{\mathrm{pr},\mathcal{E}}$ the ∞-category of κ_0-compact $\mathcal{E}$-module objects in $\widehat{\mathcal{C}\mathrm{at}}_\infty$.

We have a functor

$$\mathrm{Mod}_- : \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \to \mathrm{CAlg}(\mathrm{CAT}_\infty)$$

where CAT_∞ is the ∞-category of very large ∞-categories. The functor sends an object $\mathcal{E}$ to the ∞-category $\mathrm{Mod}_\mathcal{E}$ of $\mathcal{E}$-module in $\widehat{\mathcal{C}\mathrm{at}}_\infty$. By [Ste20, proposition 5.1.7] we know that the collection of free $\mathcal{E}$-modules $\mathcal{E} \otimes \mathcal{C}$ where $\mathcal{C}$ is a presentable ∞-category is a collection of κ_0-compact generators of $\mathrm{Mod}_\mathcal{E}$. By convention, we let the functor

$$\mathrm{Mod}^n : \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \to \mathrm{CAlg}(\mathrm{CAT}_\infty)$$

be the composition $-\mathrm{Mod} \circ -\mathrm{Mod}^{n-1}_{\mathrm{pr},-}$.

Definition 2.3.7. [Ste20, definition 5.2.2] We define the ∞-*category of presentable* (∞, n)-*categories* $n\mathrm{Pr}^L$ to be the ∞-category of presentable modules over $(n-1)\mathrm{Pr}^L$

$$n\mathrm{Pr}^L = \mathrm{Mod}_{\mathrm{pr},(n-1)\mathrm{Pr}^L}(\widehat{\mathcal{C}\mathrm{at}}_\infty) = \mathrm{Mod}^n_{\mathrm{pr},\mathcal{S}}$$

A presentable (∞, n)-category is an ∞-category that is a κ_0-compact left module over $(n-1)\mathrm{Pr}^L$. We adopt the convention that $0\mathrm{Pr}^L$ is the ∞-category of spaces $\mathcal{S}$.

Definition 2.3.8. [Ste20, variant of definition 5.2.5] Let k be a field. We define the *∞-category of k-linear (∞, n)-categories* to be the ∞-category of presentable modules over $(n-1)\mathrm{Pr}^L_k$.

$$n\mathrm{Pr}^{\mathrm{L}}_k = \mathrm{Mod}_{\mathrm{pr},(n-1)\mathrm{Pr}^{\mathrm{L}}_k}(\widehat{\mathcal{C}\mathrm{at}}_\infty) = \mathrm{Mod}^{n+1}_{\mathrm{pr},k}$$

A k-linear (∞, n)-category of an ∞-category that is a κ_0-compact left module over $(n-1)\mathrm{Pr}^L_k$. We adopt the convention that $0\mathrm{Pr}^{\mathrm{L}}_k$ is the ∞-category of k-module spectra Mod_k.

There are several examples of presentable k-linear (∞, n)-categories:

Example 2.3.9. Let $\mathcal{C}$ be an algebra object in $n\mathrm{Pr}^L_k$. It follows from proposition 2.3.5 that $\mathrm{RMod}_{\mathcal{C}}(n\mathrm{Pr}^L_k) \in (n+1)\mathrm{Pr}^L_k$ is a k-linear $(\infty, n+1)$-category.
If A is an $\mathbb{E}_n$-algebra over k, then $\mathrm{RMod}^n_A := \mathrm{RMod}_{\mathrm{RMod}^{n-1}_A}((n-1)\mathrm{Pr}^L_k)$ is the presentable k-linear (∞, n)-category of n-fold iterated right A-modules.

When $n = 2$, it follows from [Lur17, corollary 4.8.5.20] that RMod_A is an $\mathbb{E}_1$-monoidal k-linear ∞-category. It follows from proposition 2.3.5 that $\mathrm{RMod}^2_A = \mathrm{RMod}_{\mathrm{RMod}_A}(\mathrm{Pr}^L_k)$ is a presentable k-linear $(\infty, 2)$-category. When $n > 2$, iterating this argument shows that RMod^n_A is a presentable $(n-1)\mathrm{Pr}^L_k$-module.

Given a presentable ∞-category $\mathcal{M}$ tensored over a presentable symmetric monodial ∞-category $\mathcal{A}$ one may construct an ∞-category $\mathfrak{M}$ enriched in $\mathcal{A}$. This construction is in fact functorial (see [Ste20, §3.2] for a discussion, or alternatively [GH15, §7]). In case $\mathcal{A} = (n-1)\mathrm{Pr}^L$, we may view $\mathfrak{M}$ as an (∞, n)-category. We recall the definition of the $(\infty, n+1)$-category $n\mathfrak{Pr}^L$ of presentable (∞, n)-categories.

Remark 2.3.10. Let $n \geq 1$ be an integer, $\widehat{n\mathcal{C}\mathrm{at}_\infty}$ be the ∞-category of ∞-categories enriched in $(\infty, n-1)$-categories. By [Ste20, §5.3] we have a lax symmetric monoidal functor

$$\psi_n : \mathrm{Mod}^n_{\mathcal{S}} \to n\widehat{\mathcal{C}\mathrm{at}}_\infty$$

which turns ∞-categories tensored over $\mathrm{Mod}^{n-1}_{\mathrm{pr},\mathcal{S}}$ in to the associated ∞-categories enriched in $(\infty, n-1)$-categories. If $\mathcal{C} \in \mathrm{Mod}^n_{\mathcal{S}}$, then the underlying ∞-category of $\psi_n(\mathcal{C})$ is equivalent to $\mathcal{C}$.

Let $\mathcal{C}$ be a presentable (∞, n)-category. Given that there is the enrichment functor ψ_n, we can define the n-fold endomorphism object for an object $E \in \mathcal{C}$. We define this object iteratively.

Definition 2.3.11. Let $\mathcal{C}$ be a presentable (∞, n)-category and $E \in \mathcal{C}$ be an object. Then there is an associated (presentable[1]) $(\infty, n-1)$-category of endomorphisms of E, denoted $\underline{\mathrm{Map}}_{\mathcal{C}}(E, E)$. The n-fold endomorphism object of E, denoted $\mathfrak{Z}(E)$, is the $(n-1)$-fold endomorphism object of the identity $id_E \in \underline{\mathrm{Map}}_{\mathcal{C}}(E, E)$. It follows that $\mathfrak{Z}(E)$ has the structure of an $\mathbb{E}_n$-algebra.

Let $n\mathfrak{Cat}^L$ be the $(\infty, n+1)$-categorical enrichment of $\mathrm{Mod}^n_{\mathcal{S}}$. More precisely, we view $\mathrm{Mod}^n_{\mathcal{S}}$ as a module over itself, therefore $\mathrm{Mod}^n_{\mathcal{S}}$ may be enriched in itself. The functor $\psi_n : \mathrm{Mod}^n_{\mathcal{S}} \to n\widehat{\mathcal{C}\mathrm{at}}_\infty$ allows us to enrich $\mathrm{Mod}^n_{\mathcal{S}}$ in $n\widehat{\mathcal{C}\mathrm{at}}_\infty$. Note that the underlying ∞-category of $n\mathfrak{Cat}^L$ is equivalent to $\mathrm{Mod}^n_{\mathcal{S}}$.

Definition 2.3.12. The full subcategory of $n\mathfrak{Cat}^L$ spanned by presentable (∞, n)-categories is denoted $n\mathfrak{Pr}^L$. The underlying ∞-category of $n\mathfrak{Pr}^L$ is equivalent to $n\mathrm{Pr}^L$.

---

[1] In general, the endomorphism $(\infty, n-1)$-category is not κ_0-compact, i.e. it is not presentable. Although it can be realized as a κ_0-filtered colimit of κ_0-compact objects of $(n-1)\mathrm{Pr}^{\mathrm{L}}$.

Chapter 3

Deformations of $\mathbb{E}_n$-monoidal categories

This chapter serves as the core of this thesis. In the following sections, we will undertake an in-depth study of the deformation theory of $\mathbb{E}_n$-monoidal k-linear ∞-categories, considering both Morita equivalences and $\mathbb{E}_n$-monoidal equivalences.

Definition 3.0.1. Two $\mathbb{E}_n$-algebra objects B, B' in a symmetric monoidal ∞-category are *Morita equivalent* if $\mathrm{RMod}^n_{\mathrm{pr},B}$ and $\mathrm{RMod}^n_{\mathrm{pr},B'}$ are equivalent as n-fold iterated module ∞-categories.

Due to [Lur17, corollary 4.8.5.20] we know that for every $B \in \mathrm{Alg}^n_k$, an $\mathbb{E}_n$-algebra over k, the corresponding k-linear ∞-category of right B-modules, RMod_B, has an $\mathbb{E}_{n-1}$-monoidal structure. Stefanich, in [Ste20, §5], introduced the concept of a presentable (∞, n)-category. Based on [Ste20, §5], we define the presentable k-linear $(\infty, 2)$-category of right modules over RMod_B as $\mathrm{RMod}^2_{\mathrm{pr},B} \coloneqq \mathrm{RMod}_{\mathrm{RMod}_B}(\mathrm{Pr}^{\mathrm{L}}_k)$. Proceeding along these lines, we define the presentable k-linear (∞, n)-category of n-fold modules over B to be $\mathrm{RMod}^n_{\mathrm{pr},B} \coloneqq \mathrm{RMod}_{\mathrm{RMod}^{n-1}_{\mathrm{pr},B}}((n-1)\mathrm{Pr}^{\mathrm{L}}_k)$ (see definition 2.3.8 and example 2.3.9).

In section 3.1, we study the deformations of $\mathbb{E}_n$-algebras over k, up to equivalences of $\mathbb{E}_n$-algebras. We construct a functor AlgDef_B, called the deformation functor for an $\mathbb{E}_n$-

algebra B (construction 3.1.1), and establish that this functor is a 1-proximate formal moduli problem (lemma 3.1.2). We also prove that under a boundedness hypothesis on the underlying k-module of B, the functor AlgDef_B is a formal moduli problem (proposition 3.1.3). This result plays a crucial role in the proofs of our main results in the subsequent sections.

In section 3.2, we study the deformations of presentable k-linear (∞, n)-categories. This section extends the work of [Lur11b, §5.3], originally focusing on deformations of k-linear ∞-categories, to include presentable k-linear (∞, n)-categories where $n > 1$. We construct a functor $\mathrm{CatDef}_{\mathcal{C}}$, called the deformation functor of a presentable k-linear (∞, n)-category $\mathcal{C}$, for every integer $n \geq 0$ (construction 3.2.1). Per our established convention, $\mathcal{C}$ is a k-module in the case where $n = 0$. We show that $\mathrm{CatDef}_{\mathcal{C}}$ is a $(n+1)$-proximate formal moduli problem (theorem 3.2.3). We define $\mathfrak{Z}(\mathcal{C})$, a k-linear center of $\mathcal{C}$ (definition 3.2.5), and construct a natural transformation from $\mathrm{CatDef}_{\mathcal{C}}$ to the formal moduli problem given by $\mathfrak{Z}(\mathcal{C})$ (construction 3.2.7). We show that the formal completion of $\mathrm{CatDef}_{\mathcal{C}}$, denoted $\mathrm{CatDef}_{\mathcal{C}}^{\wedge}$, is equivalent to the formal moduli problem given by $\mathfrak{Z}(\mathcal{C})$ (corollary 3.2.10). As a consistency check, one can see that the results of this section reduce to those of [Lur11b, §5.3] upon setting $n = 1$.

In section 3.3, we study the deformations of objects in presentable k-linear (∞, n)-categories. This part is a common generalization of [Lur11b, §5.2 & §5.3], which deal with the deformations of objects in k-linear ∞-categories and the deformations of k-linear ∞-categories respectively. We construct ObjDef_E, the deformation functor for an object E in a presentable k-linear (∞, n)-category, for $n \geq 1$ (construction 3.3.1). We prove that ObjDef_E is a n-proximate formal moduli problem (theorem 3.3.3). It is noteworthy that the functor $\mathrm{CatDef}_{\mathcal{C}}$ can be seen as a specific instance of the functor ObjDef_E: if $\mathcal{C}$ is a presentable k-linear (∞, n)-category for $n \geq 0$, then consider $\mathcal{C}$ as an object in the presentable k-linear $(\infty, n+1)$-category $n\mathrm{Pr}_k^{\mathrm{L}}$. We adopt the convention that $0\mathrm{Pr}_k^{\mathrm{L}} = \mathrm{Mod}_k$.

In section 3.4, we study the deformations of pairs $(\mathcal{C}, E)$ such that E is an object of a presentable k-linear (∞, n)-category $\mathcal{C}$, for every integer $n \geq 1$. This part is a generalization of [BKP18, §4.1], which deals with the deformations of pairs $(\mathcal{C}, E)$ such that $\mathcal{C}$ is a k-linear ∞-category. We construct a functor $\mathrm{SimDef}_{(\mathcal{C},E)}$, the deformation functor of the pair $(\mathcal{C}, E)$ (construction 3.4.1). We demonstrate that there exists a fiber sequence of deformation functors that includes ObjDef_E, $\mathrm{SimDef}_{(\mathcal{C},E)}$, and $\mathrm{CatDef}_{\mathcal{C}}$ (as stated in proposition 3.4.3). This leads to the conclusion that $\mathrm{SimDef}_{(\mathcal{C},E)}$ is a $(n+1)$-proximate formal moduli problem (corollary 3.4.4). An immediate corollary that follows is that there is an associated fiber sequence of formal moduli problems associated with the pair $(\mathcal{C}, E)$ (corollary 3.4.5). Next, we undertake a closer analysis of the formal completion of $\mathrm{SimDef}_{(\mathcal{C},E)}$, which is a formal moduli problem denoted $\mathrm{SimDef}^{\wedge}_{(\mathcal{C},E)}$. We define $\mathfrak{Z}(\mathcal{C}, E)$, a k-linear center of the pair $(\mathcal{C}, E)$ (definition 3.4.6), and construct a natural transformation from $\mathrm{SimDef}_{(\mathcal{C},E)}$ to the formal moduli problem given by $\mathfrak{Z}(\mathcal{C}, E)$ (construction 3.4.10). We finally show that $\mathrm{SimDef}^{\wedge}_{(\mathcal{C},E)}$ is given by $\mathfrak{Z}(\mathcal{C}, E)$ (proposition 3.4.11). The last part of this section focuses on obtaining an explicit description of the functor $\mathrm{ObjDef}^{\wedge}_{E}$, using the fiber sequence of formal moduli problems obtained earlier (proposition 3.4.14). In pursuit of this proposition, we discuss the restriction of a formal $\mathbb{E}_n$-moduli problem to a formal $\mathbb{E}_{n+1}$-moduli problem via the natural forgetful functor $\mathrm{Alg}_k^{n+1} \longrightarrow \mathrm{Alg}_k^n$. To conclude this section, we make some conjectural observations, connecting to [Fra13, conjecture 4.50].

In section 3.5, we focus on studying the deformations of an $\mathbb{E}_n$-monoidal k-linear ∞-category $\mathcal{C}$, up to Morita equivalences. As we discussed above, this amounts to studying the deformations of the presentable k-linear $(\infty, n+1)$-category $\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}$. We assume that $\mathcal{C}$ arises as the ∞-category of modules over an $\mathbb{E}_{n+1}$-algebras B over k. We impose an additional boundedness hypothesis on the underlying k-module of B (definitions 3.5.2 & 3.5.3). We focus on deformations over the formal power series ring, $k[[t]]$, (theorem 3.5.10) We show that the space of deformations $\mathrm{CatDef}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(A)$ is equivalent to the space $\mathrm{CatDef}^{\wedge}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(A)$, where $A = k[[t]]$. To conclude this section, we show through proposition 3.5.16 that the fiber sequence of non-unital $\mathbb{E}_{n+1}$-algebras of [Fra13, theorem 1.1] can be seen as a specific case of the fiber sequence of formal moduli problems in corollary 3.4.5: in the notation of corollary 3.4.5, if we choose $\mathcal{C} = \mathrm{RMod}^{n+1}_{\mathrm{pr},B}$ and $E = \mathrm{RMod}^{n}_{\mathrm{pr},B}$, we recover the fiber sequence appearing in [Fra13]. Specifically, through lemma 3.5.13 we show that the $\mathbb{E}_{n+1}$-Hochschild cohomology of B (definition 3.5.12) is canonically equivalent to the k-linear center $\mathfrak{Z}(\mathrm{RMod}^{n+1}_{\mathrm{pr},B})$. Moreover, in lemma 3.5.15, we also show that the tangent complex of B (definition 3.5.14) is canonically related to the k-linear center $\mathfrak{Z}(\mathrm{RMod}^{n+1}_{\mathrm{pr},B}, \mathrm{RMod}^{n}_{\mathrm{pr},B})$.

We need a couple of results to show that the various deformation functors that we construct are m-proximate. We record these now:

The following lemma does not rely on any specific model of an (∞, m)-category. All we use is that an (∞, m)-category has an $(\infty, m-1)$-category of morphisms between any pair of objects. Following the standard practice, an $(\infty, 0)$-category is an ∞-groupoid/Kan complex. In addition, we use that there is a truncation functor from the ∞-category of (∞, m)-categories to the ∞-category of $(\infty, 0)$-categories which sends an (∞, m)-category to the underlying Kan complex. For instance, the model of (∞, m)-categories developed in [GH15] meets these requirements, so does the model used by [Ste20].

Lemma 3.0.2. *Let $m \geq 1$, $F : \mathcal{C} \to \mathcal{D}$ be a functor between (∞, m)-categories. If F induces an equivalence of m-morphism spaces $\mathcal{C}(x_1, y_1)(x_2, y_2)...(x_m, y_m) \xrightarrow{\simeq} \mathcal{D}(Fx_1, Fy_1)(Fx_2, Fy_2)...(Fx_m, Fy_m)$ for all $x_1, y_1 \in \mathcal{C}$, for all $x_i, y_i \in \mathcal{C}(x_{i-1}, y_{i-1})$, then the induced map of spaces $(F)^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is $(m-2)$-truncated.*

Proof. We prove this by induction on m. When $m = 1$, then $F : \mathcal{C} \to \mathcal{D}$ is a fully faithful map of $(\infty, 1)$-categories. We note that a fully faithful map is conservative. This implies that F induces an equivalence of spaces $\mathcal{C}(x, y)^{eq} \xrightarrow{F_{xy}} \mathcal{D}(Fx, Fy)^{eq}$ for all $x, y \in \mathcal{C}$. Here $\mathcal{C}(x, y)^{eq} \subset \mathcal{C}(x, y)$ and $\mathcal{D}(Fx, Fy)^{eq} \subset \mathcal{D}(Fx, Fy)$ are the subspaces of invertible morphisms, i.e. morphisms which are equivalences. In particular, for every $x \in \mathcal{C}$, the map $F^{\simeq}_x : \mathcal{C}^{\simeq}(x, x) \to \mathcal{D}^{\simeq}(Fx, Fx)$ is an equivalence of spaces. This gives an isomorphism of homotopy groups $\pi_i(\mathcal{C}^{\simeq}, x) \simeq \pi_i(\mathcal{D}^{\simeq}, Fx)$ for every $x \in \mathcal{C}$, for every $i \geq 1$. We conclude that the map $F^{\simeq}$ is (-1)-truncated when $m = 1$.

Assume that the result is true for some m. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $(\infty, m+1)$-categories satisfying the hypothesis. For every pair of objects $x, y \in \mathcal{C}$, $\mathcal{C}(x, y) \xrightarrow{F_{xy}} \mathcal{D}(Fx, Fy)$ is a functor of (∞, m)-categories such that F_{xy} induces an equivalence of m-morphism spaces: for every object $x_0 \in \mathcal{C}$ and every sequence $\{x_{i+1} \in \mathcal{C}(x_i, x_i)\}_{i=0}^{m}$, the induced map of spaces

$$\mathcal{C}(x_0, x_0)(x_1, x_1)..(x_m, x_m) \xrightarrow{F^{\simeq}_{x_1,..,x_m}} \mathcal{D}(Fx_0, Fx_0)(Fx_2, Fx_2)..(Fx_m, Fx_m)$$

is an equivalence. By the induction step, the map of spaces $\mathcal{C}(x_0, x_0)^{\simeq} \xrightarrow{F^{\simeq}_{x_0}} \mathcal{D}(Fx_0, Fx_0)^{\simeq}$ is $(m-2)$-truncated for every $x_0 \in \mathcal{C}$. We conclude that $\mathcal{C}^{\simeq} \xrightarrow{F^{\simeq}} \mathcal{D}^{\simeq}$ is a $(m-1)$-truncated map of spaces. □

Lemma 3.0.3. *Let $m \geq -2$ be an integer, $U \xrightarrow{f} V$ be a m-truncated map of space over a space Y. Then for any map $* \xrightarrow{y} Y$, the base change preserves m-truncated maps, i.e. the induced map $U_y := U \times_Y * \xrightarrow{f_y} V_y := V \times_Y *$ is m-truncated.*

Proof. We proceed by induction on m. The result is evident for $m = -2$. Let the result be true for some $m \geq -2$ and $U \to V$ be a $(m+1)$-truncated map of spaces over Y. Consider the pullback of spaces

$$\begin{tikzcd}
U_y \arrow[r, "g"] \arrow[d] & U \arrow[d, "h"] \\
* \arrow[r, "y"] & Y
\end{tikzcd}$$

Then for any $u \in U_y$, the induced map of loop spaces $U(gu, gu) \xrightarrow{f^*} V(fgu, fgu)$ is a m-truncated map of spaces over $Y(y, y)$. By the induction hypothesis, $U(gu, gu)_{\bar{y}} \to V(fgu, fgu)_{\bar{y}}$ is m-truncated. Here $\bar{y}$ is the constant loop at $y \in Y$. By commutativity of limits, we have that $U(gu, gu)_{\bar{y}} \simeq U_y(u, u)$ and $V(fgu, fgu)_{\bar{y}} \simeq V_y(f_y u, f_y u)$ for all $u \in U_y$. We conclude that $U_y(u, u) \xrightarrow{f_y} V_y(f_y u, f_y u)$ is m-truncated for all $u \in U_y$. So $U_y \to V_y$ is $m + 1$-truncated. □

3.1 Deformations of $\mathbb{E}_n$-algebras

In this section, for an integer $n \geq 1$, we will study deformations of an $\mathbb{E}_n$-algebra over a field k, viewed as an object of the ∞-category of $\mathbb{E}_n$-algebras over k. A key result to be used in §3.5 is proposition 3.1.3. The proposition says that the formal moduli space of a homologically bounded below $\mathbb{E}_n$-algebra B over the field k is characterized by a dg-Lie algebra, more precisely by an $\mathbb{E}_{n+1}$-algebra.

Construction 3.1.1. By [Lur17, theorem 4.8.5.16] there is a symmetric monoidal functor

$$\mathrm{Alg}_k \to \mathrm{Pr}^{\mathrm{L}}_k$$

which sends $A \mapsto \mathrm{LMod}_A$. By passing to $\mathbb{E}_n$-algebra objects and using [Lur17, theorem 5.1.2.2] (Dunn-Lurie additivity) we obtain a symmetric monoidal functor

$$\mathrm{Alg}^{n+1}_k \longrightarrow \mathrm{Alg}^n(\mathrm{Pr}^L_k)$$

which sends an $\mathbb{E}_{n+1}$-algebra A to the $\mathbb{E}_n$-monoidal k-linear ∞-category of left A-modules $\mathrm{LMod}^{\otimes}_A$. We compose the functor $\mathrm{Alg}^{n+1}_k \longrightarrow \mathrm{Alg}^n(\mathrm{Pr}^L_k)$ with the functor $\mathrm{Alg}^n(\mathrm{Pr}^{\mathrm{L}}_k) \to \mathrm{Pr}^{\mathrm{L}}$ given by taking $\mathbb{E}_n$-algebra objects, $\mathcal{D} \mapsto \mathrm{Alg}^n(\mathcal{D})$. This composition of functors classifies a cocartesian fibration $\mathrm{Alg}^n \xrightarrow{q} \mathrm{Alg}^{n+1}_k$. We restrict to q-cocartesian arrows to obtain a left fibration, $\mathrm{Alg}^{n,\mathrm{coCar}} \to \mathrm{Alg}^{n+1}_k$. Given an $\mathbb{E}_n$-algebra B over k with $n \geq 1$, [Lur09, proposition 2.1.2.1] gives an induced left fibration

$$\mathrm{Deform}[B] \coloneqq (\mathrm{Alg}^{n,\mathrm{coCar}})_{/(k,B)} \to (\mathrm{Alg}^{n+1}_k)_{/k} \simeq \mathrm{Alg}^{n+1,\mathrm{aug}}_k$$

classifying the deformations of B where equivalences are equivalences of algebras. We call $\mathrm{Deform}[B]$ the ∞-*category of deformations of* B. The objects of Deform[B] are tuples (A, B_A, μ) where A is an augmented $\mathbb{E}_{n+1}$-algebra over k and B_A is an $\mathbb{E}_n$-algebra over A such that $k \otimes_A B_A \xrightarrow{\mu} B$ is an equivalence in Alg^n_k. The restriction of the preceding left fibration to small $\mathbb{E}_{n+1}$-algebras along the inclusion $\mathrm{Alg}_k^{n+1,\mathrm{small}} \to \mathrm{Alg}_k^{n+1,\mathrm{aug}}$ is classified by the functor

$$\mathrm{AlgDef}_B : \mathrm{Alg}_k^{n+1,\mathrm{small}} \to \widehat{\mathcal{S}}$$

which we call the *deformation functor of* B.

Lemma 3.1.2. *Let $n \geq 1$ be an integer, k be a field, B be an $\mathbb{E}_n$-algebra over k. Then the deformation functor of B, AlgDef_B (as defined in construction 3.1.1) is a 1-proximate formal moduli problem. In particular, AlgDef_B can be regarded as a functor taking values in the ∞-category $\mathcal{S}$ of small spaces.*

Proof. The case $n = 1$ is [BKP18, corollary 4.18]. Their proof can be directly adapted to the case $n > 1$. The key input to the proof is that given a pullback in Alg_k^{n+1}

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

the image of this diagram in Alg_k under the forgetful functor $\mathrm{Alg}_k^{n+1} \to \mathrm{Alg}_k$ is a pullback square. Consequently, it follows from [Lur11a, proposition 7.4] that the map

$$\mathrm{LMod}_A \xrightarrow{F} \mathrm{LMod}_{A_0} \times_{\mathrm{LMod}_{A_{01}}} \mathrm{LMod}_{A_1}$$

is a fully faithful functor of k-linear ∞-categories. In other words, it admits a right adjoint G such that the unit of the adjunction is an equivalence. It is a straightforward observation that F has a natural lift to an $\mathbb{E}_n$-monoidal functor. The functor $\mathrm{Alg}^n : \mathrm{Alg}^n(\mathrm{Pr}_k^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$ takes a cocartesian fibration over the ∞-operad $\mathbb{E}_n^{\otimes}$ and assigns to it the ∞-category of its sections which are morphisms of ∞-operads (see [Lur17, definition 2.1.2.7]). It follows that $\mathrm{Alg}^n(-)$ preserves limits. The forgetful functor from $\mathbb{E}_n$-algebras to the underlying k-module spectra is conservative. This implies that the functor $\mathrm{Alg}^n(F)$ is fully faithful: note that F is an $\mathbb{E}_n$-monoidal functor and G is a lax monoidal functor, i.e. G is a map of ∞-operads. In fact, G preserves the $\mathbb{E}_n$-monoidal unit. It follows that $\mathrm{Alg}^n(F)$ admits a right adjoint, $\mathrm{Alg}^n(G)$, such that the unit of the adjunction is an equivalence. We conclude that the following map is fully faithful.

$$\mathrm{Alg}^n(\mathrm{LMod}_A) \xrightarrow{\mathrm{Alg}^n(F)} \mathrm{Alg}^n(\mathrm{LMod}_{A_1}) \times_{\mathrm{Alg}^n(\mathrm{LMod}_{A_{01}})} \mathrm{Alg}^n(\mathrm{LMod}_{A_0})$$

It now follows from lemmas 3.0.2 & 3.0.3 that

$$\mathrm{AlgDef}_B(A) \longrightarrow \mathrm{AlgDef}_B(A_1) \times_{\mathrm{AlgDef}_B(A_{01})} \mathrm{AlgDef}_B(A_0)$$

is (-1)-truncated.

We now show that AlgDef_B is valued in essentially small spaces. For $m \geq 0$ an integer, we note that $\mathrm{AlgDef}_B(k \oplus k[m]) \to \Omega\mathrm{AlgDef}_B(k \oplus k[m+1])$ is a homotopy equivalence on the essential image. In addition, the space $\Omega\mathrm{AlgDef}_B(k \oplus k[m+1])$ is the fibre of the map $\mathrm{Map}_{\mathrm{Alg}_A^{n+1}}(A \otimes_k B, A \otimes_k B) \to \mathrm{Map}_{\mathrm{Alg}_k^{n+1}}(B, B)$, implying that it is essentially small. We conclude that $\mathrm{AlgDef}_B(k \oplus k[m])$ is essentially small for any $m \geq 0$. Now given any $A \in \mathrm{Alg}_k^{n+1,\mathrm{small}}$, we choose a finite sequence $A = A_0 \to A_1 \to \cdots \to A_n = k$ such that there exists integers $m_i \geq 0$ and pullbacks

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d] & k \arrow[d] \\
A_{i+1} \arrow[r] & k \oplus k[m_i]
\end{tikzcd}$$

We note that the map $\mathrm{AlgDef}_B(A_i) \to \mathrm{AlgDef}_B(k) \times_{\mathrm{AlgDef}_B(k \oplus k[m_i])} \mathrm{AlgDef}_B(A_{i+1})$ is a homotopy equivalence on the image. By descending induction we conclude that $\mathrm{AlgDef}_B(A)$ is essentially small for any $A \in \mathrm{Alg}_k^{n+1,\mathrm{small}}$. □

Proposition 3.1.3. *Let $n \geq 1$ and $m \leq 0$ be integers, k be a field, B be an $\mathbb{E}_n$-algebra over k such the the underlying spectrum is m-connective. Then the deformation functor* AlgDef_B *is a formal moduli problem.*

Proof. Note that as B is an $\mathbb{E}_n$-algebra over k, it is equipped with the unit map $k \to B$, forcing $\pi_0 B$ to be non-zero. The case $n = 1$ is [BKP18, proposition 4.19]. Their proof can be directly adapted to the case $n > 1$. The key input is that for any pullback of connective objects of Alg_k^{n+1}

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_1 \arrow[d] \\ A_0 \arrow[r] & A_{01} \end{tikzcd}$$

such that $\pi_0 A_0 \to \pi_0 A_{01}$ and $\pi_0 A_1 \to \pi_0 A_{01}$ are surjective, the map

$$\mathrm{LMod}_A \xrightarrow{F} \mathrm{LMod}_{A_1} \times_{\mathrm{LMod}_{A_{01}}} \mathrm{LMod}_{A_0}$$

restricts to a map

$$(\mathrm{LMod}_A)_{\geq m} \xrightarrow{F_{\geq m}} (\mathrm{LMod}_{A_1})_{\geq m} \times_{(\mathrm{LMod}_{A_{01}})_{\geq m}} (\mathrm{LMod}_{A_0})_{\geq m}$$

which is an equivalence (variant of [Lur11a, proposition 7.6]). For every $A' \in \mathrm{Alg}_k^{n+1}$, $(\mathrm{LMod}_{A'})_{\geq m} \subset \mathrm{LMod}_{A'}$ is the full subcategory consisting of m-connective A'-modules. We note that F has a natural lift to an $\mathbb{E}_n$-monoidal functor, which in turn induces the following colimit preserving, fully faithful map (the argument supporting this is in the proof of lemma 3.1.2)

$$\mathrm{Alg}^n(\mathrm{LMod}_A) \xrightarrow{\mathrm{Alg}^n(F)} \mathrm{Alg}^n(\mathrm{LMod}_{A_1}) \times_{\mathrm{Alg}^n(\mathrm{LMod}_{A_{01}})} \mathrm{Alg}^n(\mathrm{LMod}_{A_0})$$

It follows from above that $\mathrm{Alg}^n(F)$ restricts to a fully faithful map

$$(\mathrm{Alg}^n(\mathrm{LMod}_A))_{\geq m} \xrightarrow{\mathrm{Alg}^n(F)_{\geq m}} (\mathrm{Alg}^n(\mathrm{LMod}_{A_1}))_{\geq m} \times_{(\mathrm{Alg}^n(\mathrm{LMod}_{A_{01}}))_{\geq m}} (\mathrm{Alg}^n(\mathrm{LMod}_{A_0}))_{\geq m}$$

Observe that for every $A' \in \mathrm{Alg}_k^{n+1}$, the forgetful map $\mathrm{Alg}^n(\mathrm{LMod}_{A'}) \to \mathrm{LMod}_{A'}$ is conservative and the functor $\mathrm{Alg}^n(-)$ preserves pullbacks. We conclude that the counit of the adjunction, where $\mathrm{Alg}^n(F)$ is the left adjoint, is an equivalence when restricted to m-connective algebras. We conclude that $\mathrm{Alg}^n(F)_{\geq m}$ is an equivalence. In addition, the proof of [Lur11b, proposition 5.2.14] implies that every deformation of an m-connective object over a small $\mathbb{E}_{n+1}$-algebra is m-connective. It follows that the below diagram is a cartesian square of spaces

$$\begin{tikzcd}
\mathrm{AlgDef}_B(A) \arrow[r] \arrow[d] & \mathrm{AlgDef}_B(A_1) \arrow[d] \\
\mathrm{AlgDef}_B(A_0) \arrow[r] & \mathrm{AlgDef}_B(A_{01})
\end{tikzcd}$$

and therefore AlgDef_B is a formal $\mathbb{E}_{n+1}$-moduli problem. □

3.2 Deformations of higher presentable categories

In this section, we study the deformations of presentable k-linear (∞, n)-categories. For the sake of technical simplicity, we focus on those presentable k-linear (∞, n)-categories where the (∞, n)-category of k-linear endofunctors is κ_0-compact, i.e. it is an object of $n\mathrm{Pr}^{\mathrm{L}}_k$. Additionally, we assume that the $(n+1)$-fold endomorphism objects of these higher categories are small (definition 2.3.11). An example of such objects is the n-fold iterated module category over an $\mathbb{E}_n$-algebra. The results in this section, which are crucial for §3.5, are: a) theorem 3.2.3, which implies that the deformations of a presentable k-linear (∞, n)-category $\mathcal{C}$ fail to be a formal moduli problem, but this failure is be quantified; and b) corollary 3.2.10 which gives the formal moduli problem which 'most closely approximates' the deformations of $\mathcal{C}$.

Construction 3.2.1. Let $n \geq 0$ be an integer, k be a field, $n\mathrm{Pr}^L_k$ be the ∞-category of presentable k-linear (∞, n)-categories, $\mathcal{C}$ be an object of $n\mathrm{Pr}^L_k$. We adopt the convention that $0\mathrm{Pr}^L_k = \mathrm{Mod}_k$ and $\mathrm{Alg}^0(\mathrm{Pr}^{\mathrm{L}}_k) = \mathrm{Pr}^{\mathrm{L}}_k$. Following [Lur17, theorem 4.8.5.16] there is a symmetric monoidal functor

$$\mathrm{Alg}_k \longrightarrow \mathrm{Pr}^L_k$$

which sends an algebra A to the ∞-category LMod_A of left modules over A. Using Dunn-Lurie additivity [Lur17, theorem 5.1.2.2] and taking $\mathbb{E}_n$-algebra objects (repeatedly apply the functor $\mathrm{Alg}_k(-)$), we get an induced symmetric monoidal functor

$$\mathrm{Alg}_k^{n+1} \longrightarrow \mathrm{Alg}^n(\mathrm{Pr}_k^L)$$

which sends $A \mapsto \mathrm{LMod}_A$, the $\mathbb{E}_n$-monoidal k-linear ∞-category of left A-modules. Following [Ste20, proposition 5.1.13], there is a lax symmetric monoidal functor

$$\mathrm{Alg}^n(\mathrm{Pr}_k^L) \longrightarrow (n+1)\mathrm{Pr}_k^L$$

which sends an $\mathbb{E}_n$-monoidal k-linear ∞-category $\mathcal{A}$ to the presentable $(\infty, n+1)$-category $\mathrm{LMod}^n_{\mathrm{pr},\mathcal{A}}$ of presentable modules over $\mathrm{LMod}^{n-1}_{\mathrm{pr},\mathcal{A}}$ in $n\mathrm{Pr}_k^L$. We adopt the convention that the functor $\mathrm{Alg}^0(\mathrm{Pr}_k^{\mathrm{L}}) \longrightarrow \mathrm{Pr}_k^{\mathrm{L}}$ is the identity. The composition gives a functor

$$\mathrm{Alg}_k^{n+1} \longrightarrow (n+1)\mathrm{Pr}_k^{\mathrm{L}}$$

which classifies a cocartesian fibration, denoted $n\mathrm{Cat}_k \xrightarrow{q} \mathrm{Alg}_k^{n+1}$. The fiber over $A \in \mathrm{Alg}_k^{n+1}$ consists of $\mathcal{D}$, an object of $\mathrm{LMod}^{n+1}_{\mathrm{pr},A}$, i.e. a presentable A-linear (∞, n)-category[1]. Restrict to allow only q-cocartesian arrows to obtain a left fibration $(n\mathrm{Cat}_k)^{\mathrm{cocar}} \to \mathrm{Alg}_k^{n+1}$. Note that $(k, \mathcal{C})$ is in the fiber over k and define $\mathrm{Deform}[\mathcal{C}] := (n\mathrm{Cat}_k)^{\mathrm{cocar}}_{/(k,\mathcal{C})}$. It follows from [Lur09, proposition 2.1.2.1] and stability of left fibrations under pullbacks and composition that there is an induced left fibration

$$\mathrm{Deform}[\mathcal{C}] \longrightarrow (\mathrm{Alg}_k^{n+1})_{/k} \simeq \mathrm{Alg}_k^{n+1,\mathrm{aug}}$$

This induced left fibration is classified by the functor

$$\chi : \mathrm{Alg}_k^{n+1,\mathrm{aug}} \to \widehat{\mathcal{S}}$$

---

[1] see example 2.3.9: $\mathrm{LMod}^{n+1}_{\mathrm{pr},A} = \mathrm{LMod}_{\mathrm{LMod}^n_{\mathrm{pr},A}}(n\mathrm{Pr}_k^{\mathrm{L}})$.

Restricting along the natural inclusion $\mathrm{Alg}_k^{n+1\mathrm{small}} \to \mathrm{Alg}_k^{n+1,\mathrm{aug}}$ gives the functor

$$\mathrm{CatDef}_{\mathcal{C}} : \mathrm{Alg}_k^{n+1,\mathrm{small}} \longrightarrow \widehat{\mathcal{S}}$$

called the *deformation functor of* $\mathcal{C}$.

Lemma 3.2.2. *Let κ_0 be the smallest large cardinal, $n \geq 0$ be an integer, k be a field, $\mathcal{C}$ be an object of $\mathrm{Ind}_{\kappa_0}((n+1)\mathrm{Pr}_k^{\mathrm{L}})$. Consider σ:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

a pullback square in Alg_k^{n+1}. Then the induced morphism in $\mathrm{Ind}_{\kappa_0}((n+1)\mathrm{Pr}_A^{\mathrm{L}})$

$$\mathrm{LMod}^{n+1}_{\mathrm{pr},A}(\mathcal{C}) \xrightarrow{F} \mathrm{LMod}^{n+1}_{\mathrm{pr},A'}(\mathcal{C}) \times_{\mathrm{LMod}^{n+1}_{\mathrm{pr},B'}(\mathcal{C})} \mathrm{LMod}^{n+1}_{\mathrm{pr},B}(\mathcal{C})$$

is $(n+1)$-fully faithful. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ between k-linear $(\infty, n+1)$-categories is $(n+1)$-fully faithful iff for every pair of objects $C, C' \in \mathcal{C}$, the induced map $F_{C,C} : \mathcal{C}(C, C') \to \mathcal{D}(F(C), F(C'))$ of k-linear (∞, n)-categories is n-fully faithful. We adopt the convention that a map $F : \mathcal{C} \to \mathcal{D}$ between k-linear $(\infty, 0)$-categories (i.e. k-module spectra) is 0-fully faithful iff it is an equivalence.

Proof. This proof is identical to that of [Che23, proposition 2.2.4]. When $n = 0$ and $\mathcal{C} \in \mathrm{Pr}_k^{\mathrm{L}}$, the result was proven by Lurie ([Lur11a, proposition 7.4]). In case $n = 0$ and $\mathcal{C} \in \mathrm{Ind}_{\kappa_0}(\mathrm{Pr}_k^{\mathrm{L}})$, the result follows from Lurie's proposition: let $\mathcal{C} \simeq \mathrm{colim}_i \mathcal{C}_i$ be a κ_0-filtered colimit of k-linear ∞-categories and consider the morphism

$$\begin{aligned} \mathrm{LMod}_A(\mathrm{colim}_i \mathcal{C}_i) \xrightarrow{F} & \mathrm{LMod}_{A'}(\mathrm{colim}_i \mathcal{C}_i) \times_{\mathrm{LMod}_{B'}(\mathrm{colim}_i \mathcal{C}_i)} \mathrm{LMod}_B(\mathrm{colim}_i \mathcal{C}_i) \\ \simeq & \mathrm{colim}_i (\mathrm{LMod}_{A'}(\mathcal{C}_i) \times_{\mathrm{LMod}_{B'}(\mathcal{C}_i)} \mathrm{LMod}_B(\mathcal{C}_i)) \end{aligned}$$

The preceding functor is induced by the functors

$$\mathrm{LMod}_A(\mathcal{C}_i) \xrightarrow{F_i} \mathrm{LMod}_{A'}(\mathcal{C}_i) \times_{\mathrm{LMod}_{B'}(\mathcal{C}_i)} \mathrm{LMod}_B(\mathcal{C}_i)$$

which are fully faithful for every i. Note that colimits in Pr^{L} can be computed as limits in the ∞-category of ∞-categories (see [Lur09, §5.5.3]). It follows from the definition of a cofiltered limit diagram of ∞-categories that the functor F is fully faithful when $n = 0$.

We induct on n to prove the result. Assume that the lemma is true for some $0 \leq n$, consider the case of $n+1$. Given a map of $\mathbb{E}_{n+1}$-algebras $k \to R$, denote $\mathcal{C}_R \coloneqq \mathrm{LMod}^{n+1}_{\mathrm{pr},R}(\mathcal{C})$. Consider the following map in $\mathrm{Ind}_{\kappa_0}((n+1)\mathrm{Pr}^{\mathrm{L}}_A)$

$$\mathcal{C}_A \xrightarrow{F} \mathcal{C}_{A'} \times_{\mathcal{C}_{B'}} \mathcal{C}_B$$

Following the argument for the case of $n = 0$ above, we may assume that $\mathcal{C} \in (n+1)\mathrm{Pr}^{\mathrm{L}}_k$, instead of $\mathrm{Ind}_{\kappa_0}((n+1)\mathrm{Pr}^{\mathrm{L}}_A)$. Let $C \in \mathcal{C}_A$, and consider following map arising due to the unit of the adjunction given by F and its right adjoint ('restriction of scalars')

$$C \xrightarrow{u_C} C_{A'} \times_{C_{B'}} C_B$$

here $C_{A'} = A' \otimes_A C$, with C_B and $C_{B'}$ defined similarly. To prove the lemma, it is enough to show that

$$\mathcal{C}_A(C', C) \xrightarrow{u_{C*}} \mathcal{C}_A(C', C_{A'} \times_{C_{B'}} C_B) \simeq \mathcal{C}_{A'}(C'_{A'}, C_{A'}) \times_{\mathcal{C}_{B'}(C'_{B'}, C_{B'})} \mathcal{C}_B(C'_B, C_B)$$

is a n-fully faithful map in $\mathrm{Ind}_{\kappa_0}(n\mathrm{Pr}^{\mathrm{L}}_k)$ for every $C' \in \mathcal{C}_A$. Note that u_{C*} preserves κ_0-small colimits and admits a right adjoint ('restriction of scalars'). Let $D, D' \in \mathcal{C}_A(C', C)$, and consider the following map induced by the unit of the adjunction arising due to u_{C*}

$$D \to D_{A'} \times_{D_{B'}} D_B$$

It is enough to show that the induced map

$$\begin{aligned}\mathcal{C}_A(C',C)(D',D) \to &\mathcal{C}_A(C',C)(D',D_{A'} \times_{D_{B'}} D_B)\\ \simeq &\mathcal{C}_A(C',C)_{A'}(D'_{A'},D_{A'}) \times_{\mathcal{C}_A(C',C)_{B'}(D'_{B'},D_{B'})} \mathcal{C}_A(C',C)_B(D'_B,D_B)\end{aligned}$$

is a $(n-1)$-fully faithful map in $\mathrm{Ind}_{\kappa_0}((n-1)\mathrm{Pr}^{\mathrm{L}}_k)$ for every $D' \in \mathcal{C}_A(C',C)$. Note that $\mathcal{C}_A(C',C)_{A'} \coloneqq A' \otimes_A \mathcal{C}_A(C',C)$ with $\mathcal{C}_A(C',C)_{B'}$ and $\mathcal{C}_A(C',C)_B$ similarly defined.

The preceding map is $(n-1)$-fully faithful because of the induction hypothesis.

□

Theorem 3.2.3. *Let $n \geq 0$ be an integer, k be a field, $\mathcal{C}$ be an object of $n\mathrm{Pr}^L_k$, i.e. a presentable k-linear (∞, n)-category. Then the deformation functor $\mathrm{CatDef}_{\mathcal{C}} : \mathrm{Alg}^{n+1}_k \to \widehat{\mathcal{S}}$ (as in construction 3.2.1) is a $(n+1)$-proximate formal moduli problem (after a change of universe).*

Proof. Consider a pullback square in $\mathrm{Alg}^{n+1,\mathrm{small}}_k$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

By lemma 3.2.2, the induced map

$$\mathrm{LMod}^{n+1}_{\mathrm{pr},A} \xrightarrow{F} \mathrm{LMod}^{n+1}_{\mathrm{pr},A'} \times_{\mathrm{LMod}^{n+1}_{\mathrm{pr},B'}} \mathrm{LMod}^{n+1}_{\mathrm{pr},B}$$

is $(n+1)$-fully faithful.[2] By lemma 3.0.2 the induced map of spaces

$$(\mathrm{LMod}^{n+1}_{\mathrm{pr},A})^{\simeq} \xrightarrow{F^{\simeq}} (\mathrm{LMod}^{n+1}_{\mathrm{pr},A'} \times_{\mathrm{LMod}^{n+1}_{\mathrm{pr},B'}} \mathrm{LMod}^{n+1}_{\mathrm{pr},B})^{\simeq} \simeq (\mathrm{LMod}^{n+1}_{\mathrm{pr},A'})^{\simeq} \times_{(\mathrm{LMod}^{n+1}_{\mathrm{pr},B'})^{\simeq}} (\mathrm{LMod}^{n+1}_{\mathrm{pr},B})^{\simeq}$$

---

[2] see example 2.3.9: $\mathrm{LMod}^{n+1}_{\mathrm{pr},A} = \mathrm{LMod}_{\mathrm{LMod}^{n}_{\mathrm{pr},A}}(n\mathrm{Pr}^{\mathrm{L}}_k)$.

is $(n-1)$-truncated. Note that $F^{\simeq}$ fits into a commutative triangle

$$\begin{tikzcd}
(\mathrm{LMod}^{n+1}_{\mathrm{pr},A})^{\simeq} \arrow[rr, "F^{\simeq}"] \arrow[dr, "(k\otimes_A -)^{\simeq}"'] & & (\mathrm{LMod}^{n+1}_{\mathrm{pr},A'})^{\simeq} \times_{(\mathrm{LMod}^{n+1}_{\mathrm{pr},B'})^{\simeq}} (\mathrm{LMod}^{n+1}_{\mathrm{pr},B})^{\simeq} \arrow[dl, "(k\otimes -)^{\simeq}"] \\
& (\mathrm{LMod}^{n+1}_{\mathrm{pr},k})^{\simeq} &
\end{tikzcd}$$

here $(k\otimes -)^{\simeq} : (\mathrm{LMod}^{n+1}_{\mathrm{pr},A'})^{\simeq} \times_{(\mathrm{LMod}^{n+1}_{\mathrm{pr},B'})^{\simeq}} (\mathrm{LMod}^{n+1}_{\mathrm{pr},B})^{\simeq} \to (\mathrm{LMod}^{n+1}_{\mathrm{pr},B'})^{\simeq} \xrightarrow{(k\otimes_{B'} -)^{\simeq}} (\mathrm{LMod}^{n+1}_{\mathrm{pr},k})^{\simeq}$. Taking the fibers over the point $\mathcal{C} \in (\mathrm{LMod}^{n+1}_{\mathrm{pr},k})^{\simeq}$ induces the following map of spaces

$$\mathrm{CatDef}_{\mathcal{C}}(A) \longrightarrow \mathrm{CatDef}_{\mathcal{C}}(A') \times_{\mathrm{CatDef}_{\mathcal{C}}(B')} \mathrm{CatDef}_{\mathcal{C}}(B)$$

which, due to lemma 3.0.3, is $(n-1)$-truncated. We conclude that $\mathrm{CatDef}_{\mathcal{C}}$ is a $(n+1)$-proximate formal $\mathbb{E}_{n+1}$-moduli problem. □

Corollary 3.2.4. *Let $n \geq 0$ be an integer, k be a field, $\mathcal{C}$ be an object of $n\mathrm{Pr}^L_k$. There exists a formal moduli problem $\mathrm{CatDef}^{\wedge}_{\mathcal{C}} : \mathrm{Alg}^{n+1,\mathrm{small}}_k \longrightarrow \mathcal{S}$ and a $(n-1)$-truncated natural transformation $\theta : \mathrm{ObjDef}_{\mathcal{C}} \longrightarrow \mathrm{ObjDef}^{\wedge}_{\mathcal{C}}$ (possibly after a change of universe).*

Proof. This follows from [Lur11b, theorem 5.1.9]. □

We now give an explicit description of the natural transformation $\theta : \mathrm{ObjDef}_{\mathcal{C}} \longrightarrow \mathrm{ObjDef}^{\wedge}_{\mathcal{C}}$.

Let $n \geq 0$ be an integer, k be a field, $\mathcal{C}$ be an object of $n\mathrm{Pr}^L_k$. We have the functors

a. $\mathrm{Alg}^{n+1}_k \to \mathrm{Alg}(n\mathrm{Pr}^L_k)$, sending $A \mapsto \mathrm{LMod}^n_{\mathrm{pr},A}$ ([Ste20, remark 5.1.13] and Dunn-Lurie additivity [Lur17, theorem 5.1.2.2]).

b. $\mathrm{RMod}(n\mathrm{Pr}^L_k) \to \mathrm{Alg}(n\mathrm{Pr}^L_k)$ is a cocartesian fibration (combination of [Lur17, corollaries 4.2.3.2 & 4.2.3.3] and [Lur09, corollary 5.5.2.9]), where the objects of the domain are given by pairs $(\mathcal{A}, \mathcal{D})$ such that $\mathcal{A}$ is an object of $\mathrm{Alg}(n\mathrm{Pr}^L_k)$ and $\mathcal{D}$ is a right module over $\mathcal{A}$. The functor sends the pair $(\mathcal{A}, \mathcal{D})$ to $\mathcal{A}$ (right module variant of [Lur17, definition 4.2.1.13]).

c. $\mathrm{RMod}(n\mathrm{Pr}^L_k) \to n\mathrm{Pr}^L_k$ is the forgetful functor which sends a pair $(\mathcal{A}, \mathcal{D})$ to $\mathcal{D}$.

Define

$$\mathrm{RCat}_n(k, \mathcal{C}) := (\mathrm{Alg}^{n+1}_k \times_{\mathrm{Alg}(n\mathrm{Pr}^L_k)} \mathrm{RMod}(n\mathrm{Pr}^L_k)) \times_{n\mathrm{Pr}^L_k} \{\mathcal{C}\}$$

to be the ∞ category of right actions of $\mathbb{E}_{n+1}$-algebras on $\mathcal{C}$. An object of $\mathrm{RCat}_n(k, \mathcal{C})$ is a pair $(A, \mathcal{C}_A)$, where $A \in \mathrm{Alg}^{n+1}_k$ and $\mathcal{C}_A$ is a right module over $\mathrm{LMod}^n_{\mathrm{pr},A}$ such that the underlying k-linear (∞, n)-category of $\mathcal{C}_A$ is equivalent to $\mathcal{C}$. We arrive at the following definition:

Definition 3.2.5. A *k-linear center of $\mathcal{C}$* is a final object of $\mathrm{RCat}_n(k, \mathcal{C})$.

Lemma 3.2.6. *Let $n \geq 0$ be an integer, k be a field, $\mathcal{C}$ be an object of $n\mathrm{Pr}^L_k$. Let $\mathcal{E} := \mathrm{Fun}_k(\mathcal{C}, \mathcal{C})$ be the presentable k-linear (∞, n)-category of k-linear endofunctors of $\mathcal{C}$.*[3] *Let the $\mathbb{E}_{n+1}$-algebra of $(n+1)$-fold endomorphisms of $\mathcal{C}$ be small. Then there exists a k-linear center of $\mathcal{C}$.*

Proof. Let $\mathfrak{Z}(\mathcal{C})$ be the $\mathbb{E}_{n+1}$-algebra of n-fold endomorphisms of $id_{\mathcal{C}} \in \mathcal{E}$ (definition 2.3.11). The cases $n = 0$ and $n = 1$ have been proven in [Lur11b, §5] (see proposition 5.3.12 of loc.cit.).

We have a lax symmetric monoidal functor given by the composition

$$\mathrm{Alg}^n_k \longrightarrow \mathrm{Alg}((n-1)\mathrm{Pr}^{\mathrm{L}}_k) \longrightarrow (n\mathrm{Pr}^L_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}$$

given by $A \mapsto (\mathrm{LMod}^n_{\mathrm{pr},A}, \mathrm{LMod}^{n-1}_{\mathrm{pr},A})$. The left functor in the composition arises due to [Ste20, remark 5.1.13], while the right functor arises due to a variant of [Lur17, theorem 4.8.5.5] for large ∞-categories. Applying the functor $\mathrm{Alg}(-)$, we obtain a functor $\mathrm{Alg}^{n+1}_k \xrightarrow{F} \mathrm{Alg}((n\mathrm{Pr}^{\mathrm{L}}_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}) \simeq \mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)$.

We show that F admits a right adjoint *at* $\mathcal{E}$. By [Lur09, lemma 5.2.4.1], it is enough to show that the right fibration $\mathcal{A}_{\mathcal{E}} := \mathrm{Alg}^{n+1}_k \times_{\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)} (\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k))_{/\mathcal{E}} \xrightarrow{q} \mathrm{Alg}^{n+1}_k$ is

---

[3] In general, $\mathcal{E}$ need not be presentable, i.e. it need not be a κ_0-compact object of $n\mathrm{Pr}^{\mathrm{L}}_k$. It should be possible to relax this hypothesis, but we do not do it here.

representable. An object of $\mathcal{A}_{\mathcal{E}}$ is given by a colimit preserving functor $\mathrm{LMod}^n_{\mathrm{pr},A} \xrightarrow{\phi} \mathcal{E}$ of monoidal presentable k-linear (∞, n)-categories. Following [Lur17, proposition 4.7.3.14][4], $\mathrm{LMod}^n_{\mathrm{pr},A}$ is generated under small colimits by objects of the form $\mathrm{LMod}^{n-1}_{\mathrm{pr},A} \otimes_k \mathcal{D}$, where $\mathcal{D}$ is a presentable k-linear $(\infty, n-1)$-category . This tells us that the k-linear functor ϕ is determined by the component

$$\phi_{\mathrm{LMod}^{n-1}_{\mathrm{pr},A}} : \underline{\mathrm{Map}}_{\mathrm{LMod}^n_{\mathrm{pr},A}}(\mathrm{LMod}^{n-1}_{\mathrm{pr},A}, \mathrm{LMod}^{n-1}_{\mathrm{pr},A}) \longrightarrow \underline{\mathrm{Map}}_{\mathcal{E}}(id_{\mathcal{C}}, id_{\mathcal{C}})$$

which is a colimit preserving functor of $\mathbb{E}_2$-monoidal k-linear $(\infty, n-1)$-categories. But

$$\underline{\mathrm{Map}}_{\mathrm{LMod}^n_{\mathrm{pr},A}}(\mathrm{LMod}^{n-1}_{\mathrm{pr},A}\mathrm{LMod}^{n-1}_{\mathrm{pr},A}) \simeq \mathrm{LMod}^{n-1}_{\mathrm{pr},A}$$

is an equivalence of $\mathbb{E}_2$-monoidal presentable k-linear $(\infty, n-1)$-categories. There is a map of $\mathbb{E}_2$-monodial k-linear $(\infty, n-1)$-categories

$$\phi_{\mathrm{LMod}^{n-1}_{\mathrm{pr},A}} : \mathrm{LMod}^{n-1}_{\mathrm{pr},A} \longrightarrow \underline{\mathrm{Map}}_{\mathcal{E}}(id_{\mathcal{C}}, id_{\mathcal{C}})$$

Now we repeat the argument for the $\mathbb{E}_2$-monoidal k-linear functor $\phi_{\mathrm{LMod}^{n-1}_{\mathrm{pr},A}}$ and, after repeating this argument n times, see that ϕ is determined by a map of $\mathbb{E}_{n+1}$-algebras, $A \to \mathfrak{Z}(\mathcal{C})$. This argument shows that all objects and morphisms in $\mathcal{A}_{\mathcal{E}}$ can be viewed as morphisms of algebras and their homotopies.

Note that there is a canonical functor of monoidal k-linear (∞, n)-categories $\mathrm{LMod}^n_{\mathrm{pr},\mathfrak{Z}(\mathcal{C})} \to \mathcal{E}$, arising from the identity morphism $\mathfrak{Z}(\mathcal{C}) \xrightarrow{id} \mathfrak{Z}(\mathcal{C})$. This shows that $\mathrm{Alg}^{n+1}_k \times_{\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)} (\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k))_{/\mathcal{E}}$ has a final object and the fibration $\mathrm{Alg}^{n+1}_k \times_{\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)} (\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k))_{/\mathcal{E}} \xrightarrow{q} \mathrm{Alg}^{n+1}_k$ is representable:

$$\mathrm{Map}_{\mathrm{Alg}^{n+1}_k}(A, \mathfrak{Z}(\mathcal{C})) \simeq \mathrm{Map}_{\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)}(\mathrm{LMod}^n_{\mathrm{pr},A}, \mathcal{E})$$

---

[4] To apply [Lur17, proposition 4.7.3.14], we first complete $\mathrm{LMod}^n_{\mathrm{pr},A}$ and $(n-1)\mathrm{Pr}^{\mathrm{L}}_k$ under κ_0-filtered colimits, use the adjoint functor theorem for κ_0-presentable ∞-categories, and use [Lur17, lemma 4.7.3.12].

Note that the space $\mathrm{Map}_{\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)}(\mathrm{LMod}^n_{\mathrm{pr},A}, \mathcal{E})$ is the fiber of q over A. By the universal property of the endomorphism object (see [Lur17, §4.7.1]), the space $\mathrm{Map}_{\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)}(\mathrm{LMod}^n_{\mathrm{pr},A}, \mathcal{E})$ classifies the actions of $\mathrm{LMod}^n_{\mathrm{pr},A}$ on $\mathcal{C}$. We note that

$$\mathrm{Alg}^{n+1}_k \times_{\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)} (\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k))_{/\mathcal{E}} \simeq \mathrm{RCat}_n(k, \mathcal{C})$$

The canonical morphism of monoidal presentable k-linear (∞, n)-categories $\mathrm{LMod}^n_{\mathrm{pr},\mathfrak{Z}(\mathcal{C})} \to \mathcal{E}$ endows $\mathcal{C}$ with a right $\mathfrak{Z}(\mathcal{C})$-module structure, and gives a final object of $\mathrm{RCat}_n(k, \mathcal{C})$.

□

Construction 3.2.7. Let $n \geq 0$, k be a field, $\mathcal{C}$ be an object of $n\mathrm{Pr}^L_k$. By [Lur11b, construction 4.4.6] we have a functor $\lambda^{n+1} : \mathcal{M}^{n+1} \longrightarrow \mathrm{Alg}^{n+1,\mathrm{aug}}_k \times \mathrm{Alg}^{n+1,\mathrm{aug}}_k$ which is a right fibration. The objects of $\mathcal{M}^{n+1}$ may be viewed as triples (A, B, ϵ), where $A, B \in \mathrm{Alg}^{n+1,\mathrm{aug}}_k$ and $\epsilon : A \otimes_k B \to k$ is an augmentation such that $A \to A \otimes_k B \xrightarrow{\epsilon} k$ is equivalent to the augmentation of A and similar holds for B. Let $\mathrm{Deform}[\mathcal{C}] \longrightarrow \mathrm{Alg}^{n+1,\mathrm{aug}}_k$ be the left fibration of construction 3.2.1, whose objects may be viewed as pairs $(A, \mathcal{D}, \mu)$, where $\mathcal{D}$ is an object of $n\mathrm{Pr}^L_A$ and $\mu : k \otimes_A \mathcal{D} \longrightarrow \mathcal{C}$ is an equivalence in $n\mathrm{Pr}^L_k$.

Let $(A, B, \epsilon) \in \mathcal{M}^{n+1}$ and $(A, \mathcal{D}, \mu) \in \mathrm{Deform}[\mathcal{C}]$. Then $\mathcal{D} \otimes_k B$ has a left $A \otimes_k B$-linear, and right B-linear structures. Therefore, $k \otimes_{A \otimes_k B} (\mathcal{D} \otimes_k B)$ has a right B-linear structure. As this construction is functorial in A, B and $\mathcal{D}$, we get a map

$$\mathrm{Deform}[\mathcal{C}] \times_{\mathrm{Alg}^{n+1,\mathrm{aug}}_k} \mathcal{M}^{n+1} \longrightarrow \mathrm{Deform}[\mathcal{C}] \times (\mathrm{Alg}^{n+1,\mathrm{aug}}_k \times_{\mathrm{Alg}(n\mathrm{Pr}^L_k)} \mathrm{RMod}(n\mathrm{Pr}^L_k) \times_{n\mathrm{Pr}^L_k} \{\mathcal{C}\})$$

which is a left representable fibration, i.e. for every object $(A, \mathcal{D}, \mu) \in \mathrm{Deform}[\mathcal{C}]$, the ∞-category $\mathrm{Deform}[\mathcal{C}] \times_{\mathrm{Alg}^{n+1,\mathrm{aug}}_k} \mathcal{M}^{n+1} \times_{\mathrm{Deform}[\mathcal{C}]} \{(A, \mathcal{D}, \mu)\}$ has a final object (see [Lur11b, definition 3.1.2]): the final object is the $\mathbb{E}_{n+1}$ Koszul dual of A, denoted $\mathfrak{D}^{n+1}(A)$ equipped with the universal augmentation $A \otimes_k \mathfrak{D}^{n+1}(A) \xrightarrow{\epsilon_{\mathfrak{D}}} k$. Due to [Lur11b, construction 3.1.3], this left representable fibration leads to a duality functor

$$\mathfrak{D}^{n+1}_{\mathcal{C}} : \mathrm{Deform}[\mathcal{C}]^{op} \longrightarrow \mathrm{Alg}^{n+1,\mathrm{aug}}_{k} \times_{\mathrm{Alg}(n\mathrm{Pr}^{L}_{k})} \mathrm{RMod}(n\mathrm{Pr}^{L}_{k}) \times_{n\mathrm{Pr}^{L}_{k}} \{\mathcal{C}\}$$

There is a canonical equivalence

$$\mathrm{Alg}^{n+1}_{k} \times_{\mathrm{Alg}(n\mathrm{Pr}^{L}_{k})} \mathrm{RMod}(n\mathrm{Pr}^{L}_{k}) \times_{n\mathrm{Pr}^{L}_{k}} \{\mathcal{C}\} \simeq (\mathrm{Alg}^{n+1}_{k})_{/\mathfrak{Z}(\mathcal{C})}$$

As a consequence of [Lur09, proposition 1.2.13.8], the natural functor $\mathrm{Alg}^{n+1,\mathrm{aug}}_{k} \xrightarrow{f} \mathrm{Alg}^{n+1}_{k}$ which forgets the augmentation preserves colimits. Hence this functor admits a right adjoint, which we denote as $k \oplus - : \mathrm{Alg}^{n+1}_{k} \longrightarrow \mathrm{Alg}^{n+1,\mathrm{aug}}_{k}$. We obtain the following equivalence by pulling back the preceding equivalence along f

$$\mathrm{Alg}^{n+1,\mathrm{aug}}_{k} \times_{\mathrm{Alg}(n\mathrm{Pr}^{L}_{k})} \mathrm{RMod}(n\mathrm{Pr}^{L}_{k}) \times_{n\mathrm{Pr}^{L}_{k}} \{\mathcal{C}\} \simeq (\mathrm{Alg}^{n+1,\mathrm{aug}}_{k})_{/k\oplus\mathfrak{Z}(\mathcal{C})}$$

We get a homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{Deform}[\mathcal{C}]^{op} \arrow[r, "\mathfrak{D}^{n+1}_{\mathcal{C}}"] \arrow[d] & (\mathrm{Alg}^{n+1,\mathrm{aug}}_{k})_{/k\oplus\mathfrak{Z}(\mathcal{C})} \arrow[d] \\
(\mathrm{Alg}^{n+1,\mathrm{aug}}_{k})^{op} \arrow[r, "\mathfrak{D}^{n+1}"] & \mathrm{Alg}^{n+1,\mathrm{aug}}_{k}
\end{tikzcd}$$

where $\mathfrak{D}^{n+1}(-) : (\mathrm{Alg}^{n+1,\mathrm{aug}}_{k})^{op} \longrightarrow \mathrm{Alg}^{n+1,\mathrm{aug}}_{k}$ is the $\mathbb{E}_{n+1}$-Koszul duality functor (see [Lur11b, §4.4]). The right vertical functor is the natural projection, sending an object $A \longrightarrow k \oplus \mathfrak{Z}(\mathcal{C})$ to A. Let

$$X(-) = \mathrm{Map}_{\mathrm{Alg}^{n+1,\mathrm{aug}}_{k}}(\mathfrak{D}^{n+1}(-), k \oplus \mathfrak{Z}(\mathcal{C})) \simeq \mathrm{Map}_{\mathrm{Alg}^{n+1}_{k}}(\mathfrak{D}^{n+1}(-), \mathfrak{Z}(\mathcal{C}))$$

The above commutative square leads to a natural transformation $\alpha : \mathrm{CatDef}_{\mathcal{C}} \longrightarrow X$.

Remark 3.2.8. By the adjunction between $(n+1)$-proximate formal $\mathbb{E}_{n+1}$-moduli problems and formal $\mathbb{E}_{n+1}$-moduli problems the map α factors as

$$\mathrm{CatDef}_{\mathcal{C}} \xrightarrow{\theta} \mathrm{CatDef}^{\wedge}_{\mathcal{C}} \xrightarrow{\beta} X$$

where θ is $(n-1)$-truncated (see [Lur11b, remark 5.1.11]). By [Lur11b, proposition 1.2.10], if β induces an equivalence of tangent complexes, then β is an equivalence of formal moduli problems. Following [Lur11b, remark 5.1.10 & lemma 5.1.12] we note that

$$\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(k \oplus k[m]) \simeq \mathrm{colim}_i \Omega^i \mathrm{CatDef}_{\mathcal{C}}(k \oplus k[m+i]) \simeq \Omega^{n+1}\mathrm{CatDef}_{\mathcal{C}}(k \oplus k[m+n+1])$$

the second equality follows from the fact that $\mathrm{CatDef}_{\mathcal{C}}$ is a $(n+1)$-proximate formal moduli problem.

If $\mathrm{CatDef}_{\mathcal{C}}(k \oplus k[m]) \xrightarrow{k\oplus k[m]} X(k \oplus k[m])$ is $(n-1)$-truncated for every $m > 0$, then $\Omega^{n+1}\mathrm{CatDef}_{\mathcal{C}}(k \oplus k[m+n+1]) \simeq X(k \oplus k[m])$. Therefore, $\beta : \mathrm{CatDef}^{\wedge}_{\mathcal{C}} \to X$ induces an equivalence of tangent complexes, therefore β must be an equivalence.

Lemma 3.2.9. *The natural transformation* $\alpha : \mathrm{CatDef}_{\mathcal{C}} \longrightarrow X$ *of construction 3.2.7 is* $(n-1)$*-truncated.*

Proof. Following remark 3.2.8, it suffices to only evaluate α on $k \oplus k[m]$ for $m > 0$ and exhibit that the fibers are $(n-1)$-truncated. Consider the homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{CatDef}_{\mathcal{C}}(k \oplus k[m]) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Alg}_k^{n+1}}(\mathfrak{D}^{n+1}(k \oplus k[m]), \mathfrak{Z}(\mathcal{C})) \arrow[d] \\
\Omega^{n+1}\mathrm{CatDef}_{\mathcal{C}}(k \oplus k[m+n+1]) \arrow[r, "\omega"] & \Omega^{n+1}\mathrm{Map}_{\mathrm{Alg}_k^{n+1}}(\mathfrak{D}^{n+1}(k \oplus k[m+n+1]), \mathfrak{Z}(\mathcal{C}))
\end{tikzcd}$$

the left vertical map is $(n-1)$-truncated because $\mathrm{CatDef}_{\mathcal{C}}$ is $(n+1)$-proximate, and the right vertical map is an equivalence because X is a formal moduli problem. It suffices to show that ω is an equivalence.

Denote $A = k \oplus k[m+n+1]$. Note that $\Omega^{n+1}\mathrm{ObjDef}_{\mathcal{C}}(A)$ is the homotopy fiber of

$$\xi : \mathrm{End}_{\mathcal{E}_A}(A \otimes id, A \otimes id) \simeq \mathrm{End}_{\mathcal{E}}(id, A \otimes id) \longrightarrow \mathrm{End}_{\mathcal{E}}(id, id)$$

where $\mathcal{E}$ is the k-linear ∞-category of n-fold endomorphisms of $\mathcal{C}$. Note the fiber sequence of k-modules

$$k[m+n+2] \to A \to k$$

this induces a fiber sequence in $\mathcal{E}$

$$k[m+n+1] \otimes id \to A \otimes id \to id$$

We conclude that $\Omega^{n+1}\mathrm{ObjDef}_{\mathcal{C}}(A) \simeq \mathrm{Map}_{\mathcal{E}}(k[m+n+1] \otimes id, id) \simeq \mathrm{Map}_{\mathcal{E}}(id, id) \otimes k[m+n+1]$. Note that

$$\begin{aligned} \mathrm{Map}_{\mathcal{E}}(id, id) \otimes k[m+n+1] &\simeq \mathrm{Map}_k(k[-m-n-1], \mathfrak{Z}(\mathcal{C})) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}_k^{n+1}}(\mathrm{Free}_k^{n+1}(k[-m-n-1]), \mathfrak{Z}(\mathcal{C})) \end{aligned}$$

By [Lur11b, proposition 4.5.6], $\mathfrak{D}^{n+1}(k \oplus k[m]) \simeq \mathrm{Free}_k^{n+1}(k[-m-n-1])$. One can conclude from the universal property of pullbacks and the co-Yoneda embedding that the map ω is induced by a map $\nu : \mathrm{Free}_k^{n+1}(k[-m-n-1]) \to \mathfrak{D}^{n+1}(k \oplus k[m])$ which by uniqueness has to be an equivalence. □

We conclude that

Corollary 3.2.10.

$$\mathrm{CatDef}_{\mathcal{C}}^{\wedge}(-) \xrightarrow{\beta} \mathrm{Map}_{\mathrm{Alg}_k^{n+1}}(\mathfrak{D}^{n+1}(-), \mathfrak{Z}(\mathcal{C}))$$

is an equivalence of formal $\mathbb{E}_{n+1}$*-moduli problems.*

3.3 Deformations of objects in higher presentable categories

Now we construct the deformation functor for an object $E \in \mathcal{C}$ in a presentable k-linear (∞, n)-category (construction 3.3.1). We prove that this functor is a n-proximate formal moduli problem (theorem 3.3.3). Further analysis of this functor will be carried out in section 3.4.

Construction 3.3.1. Let $n \geq 1$ be an integer, k a field, $\mathcal{C}$ a presentable k-linear (∞, n)-category, and $E \in \mathcal{C}$ an object. Following [Lur17, corollary 4.2.3.2] there is a cocartesian fibration[5]

$$\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}((n-1)\mathrm{Pr}^{\mathrm{L}}_k)$$

The fiber over any $\mathcal{D} \in \mathrm{Alg}((n-1)\mathrm{Pr}^{\mathrm{L}}_k)$ is the presentable k-linear (∞, n)-category of left $\mathcal{D}$-module objects of $\mathcal{C}$. Following [Ste20, remark 5.1.13], there is a lax symmetric monoidal functor

$$\mathrm{Alg}^n_k \to \mathrm{Alg}((n-1)\mathrm{Pr}^{\mathrm{L}}_k)$$

sending $A \mapsto \mathrm{LMod}^{n-1}_{\mathrm{pr},A}$. There is an induced cocartesian fibration

$$n\mathrm{LObj}_k \xrightarrow{q} \mathrm{Alg}^n_k$$

where $n\mathrm{LObj}_k \coloneqq \mathrm{LMod}(\mathcal{C}) \times_{\mathrm{Alg}((n-1)\mathrm{Pr}^{\mathrm{L}}_k)} \mathrm{Alg}^n_k$. The fiber of q over $A \in \mathrm{Alg}^{n+1}_k$ is $\mathrm{LMod}_{\mathrm{LMod}^{n-1}_{\mathrm{pr},A}}(\mathcal{C})$. Restrict to only cocartesian arrows in $n\mathrm{LObj}_k$ to obtain a left fibration $n\mathrm{LObj}^{\mathrm{cocar}}_k \to \mathrm{Alg}^n_k$. Note that (k, E) is in the fiber over k. Form the slice category with respect to this object, inducing a left fibration

$$(n\mathrm{LObj}^{\mathrm{cocar}}_k)_{/(k,E)} \to (\mathrm{Alg}^n_k)_{/k} \simeq \mathrm{Alg}^{n,\mathrm{aug}}_k$$

---

[5] The corollary actually provides a cartesian fibration. One has to use [Lur17, corollary 4.2.3.3] to conclude that each cartesian arrow induces a functor which preserves small limits, pass to completion under κ_0-filtered colimits and appeal to the adjoint functor theorem in order to deduce the existence of left adjoints. This gives the above cocartesian fibration.

$\mathrm{Deform}[E] \coloneqq (n\mathrm{LObj}_k^{\mathrm{cocar}})_{/(k,E)}$ is the ∞*-category of deformations of* E. Restricting the preceding left fibration along the natural map $\mathrm{Alg}_k^{n,\mathrm{small}} \to \mathrm{Alg}_k^{n,\mathrm{aug}}$ gives rise to a left fibration which classified by a functor

$$\mathrm{ObjDef}_E : \mathrm{Alg}_k^{n,\mathrm{small}} \to \widehat{\mathcal{S}}$$

This is called the *deformation functor of* E. Given $A \in \mathrm{Alg}_k^{n,\mathrm{small}}$, the points of the space $\mathrm{ObjDef}_E(A)$ represent pairs (E_A, μ), where E_A is an object of $\mathrm{LMod}_{\mathrm{LMod}^{n-1}_{\mathrm{pr},A}}(\mathcal{C})$ and $k \otimes_A E_A \xrightarrow{\mu} E$ is an equivalence in $\mathcal{C}$.

Remark 3.3.2. Notice that construction 3.2.1 is actually a specific instance of construction 3.3.1: by selecting $\mathcal{C} = (n+1)\mathrm{Pr}_k^{\mathrm{L}}$ in the latter, we recover the former. For reasons related to notation and clarity, we present these as separate constructions.

Theorem 3.3.3. *Let* $n \geq 1$*,* k *be a field,* $\mathcal{C}$ *be a presentable* k*-linear* (∞, n)*-category and* $E \in \mathcal{C}$ *be an object. The functor* $\mathrm{ObjDef}_E : \mathrm{Alg}_k^{n+1,\mathrm{small}} \to \widehat{\mathcal{S}}$ *of construction 3.3.1 is a* n*-proximate formal moduli problem (possibly after changing to the universe* U_1*).*

Proof. Consider the following pullback in $\mathrm{Alg}_k^{n+1,\mathrm{small}}$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A' \arrow[d] \\
B \arrow[r] & B'
\end{tikzcd}$$

It follows from lemma 3.2.2 that

$$\mathrm{LMod}^{n+1}_{\mathrm{pr},A}(\mathcal{C}) \xrightarrow{F} \mathrm{LMod}^{n+1}_{\mathrm{pr},A'}(\mathcal{C}) \times_{\mathrm{LMod}^{n+1}_{\mathrm{pr},B'}(\mathcal{C})} \mathrm{LMod}^{n+1}_{\mathrm{pr},B}(\mathcal{C})$$

is n-fully faithful. It follows from lemmas 3.0.2 and 3.0.3 that

$$\mathrm{ObjDef}_E(A) \to \mathrm{ObjDef}_E(A') \times_{\mathrm{ObjDef}_E(B')} \mathrm{ObjDef}_E(B)$$

is a $(n-2)$-truncated map of spaces. We conclude that ObjDef_E is a n-proximate formal moduli problem. □

We record the following for completeness. The proof of the statement is given in the next section.

Remark 3.3.4. By proposition 3.4.14, we have a description of the formal moduli problem ObjDef_E:

$$\mathrm{ObjDef}_E(-) \simeq \mathrm{Map}_{\mathrm{Alg}_k^{n,\mathrm{aug}}}(\mathfrak{D}^n(-), \mathfrak{Z}(E))$$

where $\mathfrak{Z}(E)$ is the $\mathbb{E}_n$-algebra of n-fold endomorphisms of $E \in \mathcal{C}$ (definition 2.3.11). Recall that $\mathfrak{D}^n : (\mathrm{Alg}_k^{n,\mathrm{aug}})^{op} \longrightarrow \mathrm{Alg}_k^{n,\mathrm{aug}}$ is the $\mathbb{E}_n$ Koszul duality functor (see [Lur11b, §4.4]).

3.4 Simultaneous deformations

Let $n \geq 1$ be an integer, k be a field, $\mathcal{C}$ a presentable k-linear (∞, n)-category and $E \in \mathcal{C}$ be an object. In this section, we study the deformations of the pair $(\mathcal{C}, E)$. The results which are most relevant for §3.5 here are: a) proposition 3.4.3 which establishes a natural fiber sequence, linking the deformations of $\mathcal{C}$, deformations of E, and the deformations of the pair $(\mathcal{C}, E)$; b) corollary 3.4.5 which relates the formal moduli problems associated to these deformation functors.

Let $n \geq 1$ be an integer, k be a field, $n\mathrm{Pr}_k^L$ be the presentable $(\infty, n+1)$-category of presentable k-linear (∞, n)-categories. Let $(\mathcal{C}, E)$ be an object of $(n\mathrm{Pr}_k^L)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}$. We construct the deformation functor for the pair $(\mathcal{C}, E)$.

Construction 3.4.1. Following [Ste20, remark 5.1.13], there is a lax symmetric monoidal functor

$$\mathrm{Alg}_k^{n+1} \to \mathrm{Alg}(n\mathrm{Pr}_k^{\mathrm{L}})$$

sending $A \mapsto \mathrm{LMod}^{n-1}_{\mathrm{pr},A}$. Recall that the underlying ∞-categories of the objects in $(n-1)\mathrm{Pr}_k^{\mathrm{L}}$ are large, cocomplete ∞-categories and the functors are colimit preserving.

Following [Lur17, theorem 4.8.5.16], there is a symmetric monoidal functor

$$\mathrm{Alg}((n-1)\mathrm{Pr}^{\mathrm{L}}_k) \longrightarrow (n\mathrm{Pr}^L_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}$$

sending a monoidal presentable k-linear $(\infty, n{-}1)$-category $\mathcal{D}$ to $(\mathrm{LMod}_{\mathcal{D}}((n{-}1)\mathrm{Pr}^{\mathrm{L}}_k), \mathcal{D})$. The symmetric monoidal structure on $(n\mathrm{Pr}^L_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}$ arises because $\mathrm{Mod}^n_{\mathrm{pr},k}$ is the monoidal unit of $n\mathrm{Pr}^L_k$ (see [Lur17, remark 2.2.2.5]). The composition of the above functors gives a lax symmetric monoidal functor

$$\mathrm{Alg}^n_k \to (n\mathrm{Pr}^L_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}$$

sending $A \mapsto (\mathrm{LMod}^n_{\mathrm{pr},A}, \mathrm{LMod}^{n-1}_{\mathrm{pr},A})$. Following [Lur17, corollary 4.2.3.2] (see footnote 5), there is a cocartesian fibration

$$\mathrm{LMod}((n\mathrm{Pr}^L_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/)} \to \mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)$$

leading to the following cocartesian fibration

$$\mathrm{LMod}((n\mathrm{Pr}^L_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}) \times_{\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)} \mathrm{Alg}^{n+1}_k \xrightarrow{q} \mathrm{Alg}^{n+1}_k$$

Let $n\mathrm{LCat}^*_k \coloneqq \mathrm{LMod}((n\mathrm{Pr}^L_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}) \times_{\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)} \mathrm{Alg}^{n+1}_k$. The fiber over an $\mathbb{E}_{n+1}$-algebra A consists of pairs $(\mathcal{D}, E)$ where $\mathcal{D}$ is a presentable A-linear (∞, n)-category and E is an object of $\mathcal{D}$. Consider the left fibration $(n\mathrm{LCat}^*_k)^{\mathrm{cocar}} \longrightarrow \mathrm{Alg}^{n+1}_k$ obtained by allowing only q-cocartesian arrows in $(n\mathrm{LCat}^*_k)$. The ∞*-category of deformations of* $(\mathcal{C}, E)$ is defined to be $\mathrm{Deform}[\mathcal{C}, E] \coloneqq (n\mathrm{LCat}^*_k)^{\mathrm{cocar}}_{/(k,\mathcal{C},E)}$.

Following [Lur09, proposition 2.1.2.1] and the stability of left fibrations under pullbacks and compositions which induces a left fibration, we obtain an induced left fibration

$$\mathrm{Deform}[\mathcal{C}, E] \xrightarrow{q} (\mathrm{Alg}^{n+1}_k)_{/k} \simeq \mathrm{Alg}^{n+1,\mathrm{aug}}_k$$

The left fibration q is classified by a functor $\chi : \mathrm{Alg}_k^{n+1,\mathrm{aug}} \longrightarrow \widehat{\mathcal{S}}$. Restricting along the natural map $\mathrm{Alg}_k^{n+1,\mathrm{small}} \to \mathrm{Alg}_k^{n+1,\mathrm{aug}}$ gives

$$\mathrm{SimDef}_{(\mathcal{C},E)} : \mathrm{Alg}_k^{n+1,\mathrm{small}} \longrightarrow \widehat{\mathcal{S}}$$

called the *deformation functor of the pair* $(\mathcal{C}, E)$. The points of the space $\mathrm{SimDef}_{(\mathcal{C},E)}(A)$ represent tuples $(\mathcal{C}_A, E_A, \mu, \nu)$ where $\mathcal{C}_A$ is an A-linear (∞, n)-category, E represents a map $\mathrm{Mod}^n_{\mathrm{pr},A} \longrightarrow \mathcal{C}_A$ such that $\mu : k \otimes_A \mathcal{C}_A \xrightarrow{\simeq} \mathcal{C}$ is an equivalence of presentable k-linear (∞, n)-categories and $\nu : \mu(E_A) \xrightarrow{\simeq} E$ is an equivalence in $\mathcal{C}$.

Remark 3.4.2. There is a forgetful functor $(n\mathrm{Pr}^L_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/} \to n\mathrm{Pr}^L_k$ which sends a pair $(\mathcal{C}, E)$ to $\mathcal{C}$. This induces a map $n\mathrm{LCat}^*_k \longrightarrow n\mathrm{Cat}_k$ over Alg_k^{n+1}. This induces a natural transformation

$$\tau_{(\mathcal{C},E)} : \mathrm{SimDef}_{(\mathcal{C},E)} \longrightarrow \mathrm{CatDef}_{\mathcal{C}}$$

The homotopy fiber of the map $(n\mathrm{Pr}^L_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/} \to n\mathrm{Pr}^L_k$ at the object $\mathcal{C}$ is underlying Kan complex of the ∞-category $\mathrm{Fun}_k(\mathrm{Mod}^n_{\mathrm{pr},k}, \mathcal{C}) \simeq \mathcal{C}$. There is a natural *evaluation* map $\mathrm{Fun}_k(\mathrm{Mod}^n_{\mathrm{pr},k}, \mathcal{C})^{\simeq} \to (n\mathrm{Pr}^L_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}$ given by $(F : \mathrm{Mod}^n_{\mathrm{pr},k} \to \mathcal{C}) \mapsto (\mathcal{C}, F(k))$. This evaluation map induces a functor $\mathrm{Deform}[E] \to \mathrm{Deform}[\mathcal{C}, E]$, such that there is an induced a natural transformation

$$\psi_{(\mathcal{C},E)} : \mathrm{ObjDef}_E^{\mathbb{E}_{n+1}} \to \mathrm{SimDef}_{(\mathcal{C},E)}$$

where $\mathrm{ObjDef}_E^{\mathbb{E}_{n+1}}$ is obtained by restricting ObjDef_E of construction 3.3.1 along the natural forgetful functor $\mathrm{Alg}_k^{n+1,\mathrm{small}} \to \mathrm{Alg}_k^{n,\mathrm{small}}$ which views an $\mathbb{E}_{n+1}$-algebra as an $\mathbb{E}_n$-algebra.

The following result establishes that the deformations of $\mathcal{C}$, E and the pair $(\mathcal{C}, E)$ arrange into a fiber sequence.

Proposition 3.4.3. *The sequence of natural transformations of remark 3.4.2*

$$\mathrm{ObjDef}_E^{\mathbb{E}_{n+1}} \xrightarrow{\psi_{(\mathcal{C},E)}} \mathrm{SimDef}_{(\mathcal{C},E)} \xrightarrow{\tau_{(\mathcal{C},E)}} \mathrm{CatDef}_{\mathcal{C}}$$

is a fiber sequence in $\mathrm{Fun}_*(\mathrm{Alg}_k^{n+1,\mathrm{small}}, \widehat{\mathcal{S}})$.

Proof. Following remark 3.4.2, we have a fiber sequence of ∞-categories

$$\mathrm{Deform}[E] \to \mathrm{Deform}[\mathcal{C}, E] \to \mathrm{Deform}[\mathcal{C}]$$

which commutes with projections to $\mathrm{Alg}_k^{n+1,\mathrm{aug}}$. The fiber sequence in the statement of the proposition follows from the preceding fiber sequence. □

Corollary 3.4.4. *Let* $n \geq 1$*,* k *be a field,* $\mathcal{C}$ *be a* k*-linear* (∞, n)*-category and* $E \in \mathcal{C}$ *be an object. Then the deformation functor* $\mathrm{SimDef}_{(\mathcal{C},E)} : \mathrm{Alg}_k^{n+1,\mathrm{small}} \longrightarrow \widehat{\mathcal{S}}$ *as defined in construction 3.4.1 is a* $(n+1)$*-proximate formal moduli problem.*

Proof. Proposition 3.4.3 gives a fiber sequence of functors

$$\mathrm{ObjDef}_E^{\mathbb{E}_{n+1}} \xrightarrow{\psi_{(\mathcal{C},E)}} \mathrm{SimDef}_{(\mathcal{C},E)} \xrightarrow{\tau_{(\mathcal{C},E)}} \mathrm{CatDef}_{\mathcal{C}}$$

Due to theorem 3.2.3 $\mathrm{CatDef}_{\mathcal{C}}$ is a $(n+1)$-proximate formal moduli problem. It follows from theorem 3.3.3 that $\mathrm{ObjDef}_E^{\mathbb{E}_{n+1}}$ is a n-proximate formal moduli problem.

Every pullback square in $\mathrm{Alg}_k^{n+1,\mathrm{small}}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

induces a homotopy commutative diagram of spaces

$$\begin{tikzcd} \mathrm{ObjDef}_E^{\mathbb{E}_{n+1}}(A) \arrow[r] \arrow[d] & \mathrm{SimDef}_{(\mathcal{C},E)}(A) \arrow[r] \arrow[d] & \mathrm{CatDef}_{\mathcal{C}}(A) \arrow[d] \\ O(E) \arrow[r] & S(\mathcal{C}, E) \arrow[r] & C(\mathcal{C}) \end{tikzcd}$$

where

$$\begin{aligned} S(\mathcal{C}, E) :=& \mathrm{SimDef}_{(\mathcal{C},E)}(A') \times_{\mathrm{SimDef}_{(\mathcal{C},E)}(B')} \mathrm{SimDef}_{(\mathcal{C},E)}(B) \\ C(\mathcal{C}) :=& \mathrm{CatDef}_{\mathcal{C}}(A') \times_{\mathrm{CatDef}_{\mathcal{C}}(B')} \mathrm{CatDef}_{\mathcal{C}}(B) \\ O(E) :=& \mathrm{ObjDef}_E^{\mathbb{E}_{n+1}}(A) \times_{\mathrm{ObjDef}_E^{\mathbb{E}_{n+1}}(B')} \mathrm{ObjDef}_E^{\mathbb{E}_{n+1}}(B) \end{aligned}$$

Both the top and the bottom rows in the above diagram are fiber sequences. The left-most vertical arrow is $(n-2)$-truncated, while the right-most vertical arrow is $(n-1)$-truncated. It follows that the middle vertical arrow is $(n-1)$-truncated: form the fiber sequence consisting of the fibers of the vertical maps and use the long exact sequence of homotopy groups. We conclude that $\mathrm{SimDef}_{(\mathcal{C},E)}$ is a $(n+1)$-proximate formal moduli problem. □

Corollary 3.4.5. *The fiber sequence of proposition 3.4.3 gives rise to a homotopy commutative diagram in the ∞-category of $(n+1)$-proximate formal moduli problems* $\mathrm{Prox}(n+1)_{\mathbb{E}_{n+1}}$

$$\begin{tikzcd} \mathrm{ObjDef}_E^{\mathbb{E}_{n+1}} \arrow[r, "\psi_{(\mathcal{C},E)}"] \arrow[d] & \mathrm{SimDef}_{(\mathcal{C},E)} \arrow[r, "\tau_{(\mathcal{C},E)}"] \arrow[d] & \mathrm{CatDef}_{\mathcal{C}} \arrow[d] \\ \mathrm{ObjDef}_E^{\mathbb{E}_{n+1},\wedge} \arrow[r] & \mathrm{SimDef}^{\wedge}_{(\mathcal{C},E)} \arrow[r] & \mathrm{CatDef}^{\wedge}_{\mathcal{C}} \end{tikzcd}$$

where the bottom row is a fiber sequence of formal moduli problems.

Proof. The bottom row is a fiber sequence of formal moduli problems because the functor $L_{n+1} : \mathrm{Prox}(n+1)_{\mathbb{E}_{n+1}} \to \mathrm{Moduli}_k^{\mathbb{E}_{n+1}}$ preserves small limits (lemma **??**). The homotopy commutative diagram arises from the unit of the adjunction $L_{n+1} \dashv i$, where $i : \mathrm{Moduli}_k^{\mathbb{E}_{n+1}} \to \mathrm{Prox}(n+1)_{\mathbb{E}_{n+1}}$ is the natural inclusion. □

Now we further investigate the formal moduli problem $\mathrm{SimDef}^{\wedge}_{(\mathcal{C},E)}(-)$ of corollary 3.4.5.

Let $n \geq 1$ be an integer, k be a field, $\mathcal{C}$ be an presentable k-linear (∞, n)-category, and $E \in \mathcal{C}$ an object. We have the functors

a. $\mathrm{Alg}_k^{n+1} \to \mathrm{Alg}(n\mathrm{Pr}_k^L)$, sending $A \mapsto \mathrm{LMod}^n_{\mathrm{pr},A}$ ([Ste20, remark 5.1.13] and Dunn-Lurie additivity [Lur17, theorem 5.1.2.2]).

b. $\mathrm{RMod}((n\mathrm{Pr}_k^{\mathrm{L}})_{\mathrm{Mod}^n_{\mathrm{pr},k}/}) \to \mathrm{Alg}(n\mathrm{Pr}_k^{\mathrm{L}})$ is a cocartesian fibration (combination of [Lur17, corollaries 4.2.3.2 & 4.2.3.3] and [Lur09, corollary 5.5.2.9]), where the objects of the domain are given by pairs $(\mathcal{A}, \mathcal{D})$ such that $\mathcal{A}$ is an object of $\mathrm{Alg}(n\mathrm{Pr}_k^L)$ and $\mathcal{D} \in (n\mathrm{Pr}_k^{\mathrm{L}})_{\mathrm{Mod}^n_{\mathrm{pr},k}/}$ is a right module over $\mathcal{A}$. The functor sends the pair $(\mathcal{A}, \mathcal{D})$ to $\mathcal{A}$ (right module variant of [Lur17, definition 4.2.1.13]).

c. $\mathrm{RMod}((n\mathrm{Pr}_k^{\mathrm{L}})_{\mathrm{Mod}^n_{\mathrm{pr},k}/}) \to (n\mathrm{Pr}_k^{\mathrm{L}})_{\mathrm{Mod}^n_{\mathrm{pr},k}/}$ which sends a pair $(\mathcal{A}, \mathcal{D})$ to $\mathcal{D}$.

Define

$$\mathrm{RCat}_n^*(k, \mathcal{C}, E) \coloneqq \mathrm{Alg}_k^{n+1} \times_{\mathrm{Alg}(n\mathrm{Pr}_k^{\mathrm{L}})} \mathrm{RMod}((n\mathrm{Pr}_k^{\mathrm{L}})_{\mathrm{Mod}^n_{\mathrm{pr},k}/}) \times_{(n\mathrm{Pr}_k^{\mathrm{L}})_{\mathrm{Mod}^n_{\mathrm{pr},k}/}} \{(\mathcal{C}, E)\}$$

to be the ∞ category of right actions of $\mathbb{E}_{n+1}$-algebras on the pair $(\mathcal{C}, E)$. An object of $\mathrm{RCat}_n^*(k, \mathcal{C}, E)$ is a pair $(A, \mathcal{C}_A, E_A)$, where $A \in \mathrm{Alg}_k^{n+1}$ and $(\mathcal{C}_A, E_A)$ is a right A-module object of $(n\mathrm{Pr}_k^{\mathrm{L}})_{\mathrm{Mod}^n_{\mathrm{pr},k}/}$, such that the underlying k-linear (∞, n)-category of $\mathcal{C}_A$ is equivalent to $\mathcal{C}$, with E_A corresponding to E. We arrive at the following definition:

Definition 3.4.6. The *k-linear center of the pair* $(\mathcal{C}, E)$ is a final object of $\mathrm{RCat}_n^*(k, \mathcal{C}, E)$.

Remark 3.4.7. Let $\mathcal{E} \coloneqq \mathrm{Fun}_k(\mathcal{C}, \mathcal{C})$ be the presentable k-linear (∞, n)-category of k-linear endofunctors of $\mathcal{C}$,[6] and $\mathrm{Fun}_k(\mathrm{Mod}^n_{\mathrm{pr},k}, \mathcal{C}) \simeq \mathcal{C}$ be the presentable k-linear (∞, n)-category of k-linear functors. An object $E \in \mathcal{C}$ can also be viewed as a functor $\mathrm{Mod}^n_{\mathrm{pr},k} \xrightarrow{E} \mathcal{C}$. Form the following pullback square in the ∞-category of large ∞-categories.

$$\begin{tikzcd}
\mathcal{E}_E \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "E^*"] \\
\{*\} \arrow[r, "E"] & \mathrm{Fun}_k(\mathrm{Mod}^n_{\mathrm{pr},k}, \mathcal{C}) \simeq \mathcal{C}
\end{tikzcd}$$

---

[6] In general, $\mathcal{E}$ need not be presentable, i.e. $\mathcal{E}$ doesn't have to be a κ_0-compact object of $n\mathrm{Pr}_k^{\mathrm{L}}$. We make this assumption on $\mathcal{E}$ throughout. With some additional work, it is possible to relax this assumption.

To be precise, we use the functor ψ_n of remark 2.3.10 to form the homotopy fiber of $\mathfrak{E} := \psi_n(\mathcal{E}) \xrightarrow{\psi_n(E^*)} \mathfrak{C} := \psi_n(\mathcal{C})$ at the object $E \in \mathfrak{C}$. Denote this fiber by $\mathfrak{E}_E$.

$$\mathfrak{E}_E \longrightarrow \mathfrak{E} \xrightarrow{\psi_n(E^*)} \mathfrak{C}$$

It is straightforward to see that the underlying ∞-category of $\mathfrak{E}_E$, obtained by truncation, is equivalent to $\mathcal{E}_E$: the truncation functor is a right adjoint.

Informally, objects of $\mathcal{E}_E$ are pairs (F, γ) such that $F : \mathcal{C} \to \mathcal{C}$ and $F(E) \xrightarrow{\gamma} E$ is an equivalence in $\mathcal{C}$. Let $\underline{\mathrm{Map}}_{(n\mathrm{Pr}^{\mathrm{L}}_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}}((\mathcal{C}, E), (\mathcal{C}, E))$ be the ∞-category of endofunctors of the pair $(\mathcal{C}, E)$. It follows from definition of the slice construction that

$$\mathcal{E}_E \simeq \underline{\mathrm{Map}}_{(n\mathrm{Pr}^{\mathrm{L}}_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}}((\mathcal{C}, E), (\mathcal{C}, E))$$

is the endomorphism object of the pair $(\mathcal{C}, E)$. More precisely, $\mathfrak{E}_E$ is the endomorphism object of the pair $(\mathcal{C}, E)$, while $\mathcal{E}_E$ is only its underlying ∞-category. In particular, $\mathcal{E}_E$ has a monoidal structure.

Lemma 3.4.8. *Let $\mathfrak{Z}(E)$ be the $\mathbb{E}_n$-algebra of n-fold endomorphisms of $E \in \mathcal{C}$. Let $\mathfrak{Z}(\mathcal{C}, E)$ be the $\mathbb{E}_{n+1}$-algebra of $(n+1)$-fold endomorphisms of the pair $(\mathcal{C}, E)$ and $\mathfrak{Z}(\mathcal{C})$ be the k-linear center of $\mathcal{C}$. There is $\mathfrak{Z}(\mathcal{C}, E) \simeq \mathfrak{Z}(\mathcal{C}) \times_{\mathfrak{Z}(E)} \{*\}$ is an equivalence of $\mathbb{E}_n$-algebras.*

Proof. Consider the fiber sequence of remark 3.4.7

$$\mathfrak{E}_E \longrightarrow \mathfrak{E} \xrightarrow{\psi_n(E^*)} \mathfrak{C}$$

The above is a fiber sequence in the ∞-categories enriched in $(\infty, n-1)$-categories. This fiber sequence induces a diagram of monoidal ∞-categories enriched in $(\infty, n-2)$-

categories:

$$\underline{\mathrm{Map}}_{\mathfrak{E}_E}(id_{(\mathcal{C},E)}, id_{(\mathcal{C},E)}) \longrightarrow \underline{\mathrm{Map}}_{\mathfrak{E}}(id_{\mathcal{C}}, id_{\mathcal{C}}) \longrightarrow \underline{\mathrm{Map}}_{\mathfrak{E}}(E, E)$$

We claim that the above diagram is a fiber sequence: follows from the definition of a fiber sequence in the ∞-category of ∞-categories enriched in $(\infty, n-1)$-categories. In turn there is a diagram of $\mathbb{E}_2$-monoidal ∞-categories enriched in $(\infty, n-2)$-categories:

$$\underline{\mathrm{Map}}_{\mathfrak{E}_E}(id_{(\mathcal{C},E)}, id_{(\mathcal{C},E)})(id, id) \longrightarrow \underline{\mathrm{Map}}_{\mathfrak{E}}(id_{\mathcal{C}}, id_{\mathcal{C}})(id, id) \longrightarrow \underline{\mathrm{Map}}_{\mathfrak{E}}(E, E)(id_E, id_E)$$

By the same argument, the above is a fiber sequence as well. Repeating this a total of n times, we arrive at a fiber sequence of $\mathbb{E}_n$-algebras over k:

$$\mathfrak{Z}(\mathcal{C}, E) \longrightarrow \mathfrak{Z}(\mathcal{C}) \longrightarrow \mathfrak{Z}(E)$$

$\square$

Lemma 3.4.9. *Let $n \geq 1$, k be a field, $\mathcal{C}$ be a presentable k-linear (∞, n)-category and $E \in \mathcal{C}$ an object. Assume that the $\mathbb{E}_{n+1}$-algebra of $(n+1)$-fold endomorphisms of the pair $(\mathcal{C}, E)$ is small. Then there exists a k-linear center of the pair $(\mathcal{C}, E)$.*

Proof. We follow the notation of remark 3.4.7. This proof is analogous to that of lemma 3.2.6. Let $\mathcal{D}$ be a monoidal presentable k-linear (∞, n)-category, and $\mathfrak{E}_E$ be as in remark 3.4.7. By the universal property of an endomorphism object [Lur17, §4.7.1], any right $\mathcal{D}$-module structure on $(\mathcal{C}, E)$ is classified by a monoidal functor $\psi_n(\mathcal{D}) \xrightarrow{\vartheta} \mathfrak{E}_E$. Such a functor gives a diagram

$$\begin{tikzcd}
\psi_n(\mathcal{D}) \arrow[dr, "\vartheta"] \arrow[drr, bend left, "\varrho"] & & \\
& \mathfrak{E}_E \arrow[r] & \mathfrak{E}
\end{tikzcd}$$

The functor ϱ induces a colimit preserving morphism of monoidal presentable k-linear (∞, n)-categories: $\mathcal{D} \longrightarrow \mathcal{E}$. If $\mathcal{D} \simeq \mathrm{LMod}^n_{\mathrm{pr},A}$ for some $A \in \mathrm{Alg}^{n+1}_k$, then every colimit preserving monoidal k-linear functor $\mathcal{D} \longrightarrow \mathcal{E}$ is equivalent to a map of $\mathbb{E}_{n+1}$-algebras $A \xrightarrow{\varphi} \mathfrak{Z}(\mathcal{C})$ (proof of lemma 3.2.6). Let $\mathfrak{Z}(\mathcal{C}, E)$ be the $\mathbb{E}_{n+1}$-algebra of n-fold endomorphisms of the identity functor $id_{(\mathcal{C},E)} \in \mathfrak{E}_E$. It follows that given a factorization

$$\begin{tikzcd}
\psi_n(\mathrm{LMod}^n_{\mathrm{pr},A}) \arrow[dr] \arrow[drr, bend left] & & \\
& \mathfrak{E}_E \arrow[r] & \mathfrak{E}
\end{tikzcd}$$

There is an induced map of $\mathbb{E}_{n+1}$-algebras

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, bend left, "\varphi"] & & \\
& \mathfrak{Z}(\mathcal{C}, E) \arrow[r] & \mathfrak{Z}(\mathcal{C})
\end{tikzcd}$$

As a consequence, in order for $\mathcal{D}$ to act on the pair $(\mathcal{C}, E)$, the map φ must factor as $A \longrightarrow \mathfrak{Z}(\mathcal{C}, E) \longrightarrow \mathfrak{Z}(\mathcal{C})$. It follows that every right A-module structure on $(\mathcal{C}, E)$ is given by a map of $\mathbb{E}_{n+1}$-algebras $A \to \mathfrak{Z}(\mathcal{C}, E)$. It follows that the following two ∞-categories are equivalent

$$\mathrm{Alg}^{n+1}_k \times_{\mathrm{Alg}(n\mathrm{Pr}^{\mathrm{L}}_k)} \mathrm{RMod}((n\mathrm{Pr}^{\mathrm{L}}_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}) \times_{(n\mathrm{Pr}^{\mathrm{L}}_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}/}} \{(\mathcal{C}, E)\} \simeq (\mathrm{Alg}^{n+1}_k)_{/\mathfrak{Z}(\mathcal{C},E)}$$

Hence $\mathfrak{Z}(\mathcal{C}, E)$ is a k-linear center of the pair $(\mathcal{C}, E)$. □

Construction 3.4.10. Let $n \geq 1$ be an integer, k a field, $\mathcal{C}$ a presentable k-linear (∞, n)-category and $E \in \mathcal{C}$ an object. By [Lur11b, construction 4.4.6] we have a right fibration $\lambda^{n+1} : \mathcal{M}^{n+1} \to \mathrm{Alg}^{n+1,\mathrm{aug}}_k \times \mathrm{Alg}^{n+1,\mathrm{aug}}_k$. The objects of $\mathcal{M}^{n+1}$ are triples (A, B, μ) such that $A, B \in \mathrm{Alg}^{n+1,\mathrm{aug}}_k$ and $A \otimes_k B \xrightarrow{\mu} k$ is an augmentation. Let $\mathrm{Deform}[\mathcal{C}, E] \to \mathrm{Alg}^{n+1,\mathrm{aug}}_k$ be the left fibration from construction 3.4.1.

Let $(A, B, \mu) \in \mathcal{M}^{n+1}$ and $(A, \mathcal{C}_A, E_A, \mu, \nu) \in \mathrm{Deform}[\mathcal{C}, E]$. Note that $\mathcal{C}_A \otimes_k B$ can be endowed an action of $A \otimes_k B$ where the action of B is trivial. Then $k \otimes_{A \otimes_k B} (\mathcal{C} \otimes_k B)$ has a right B-linear structure, more precisely it is an object of $\mathrm{RMod}_{\mathrm{LMod}^n_{\mathrm{pr},B}}((n\mathrm{Pr}^{\mathrm{L}}_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}})$.

The image of $k\otimes_{A\otimes_k B}(\mathcal{C}\otimes_k B)$ under the map $\mathrm{RMod}_{\mathrm{LMod}^n_{\mathrm{pr},B}}((n\mathrm{Pr}^{\mathrm{L}}_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}}) \to (n\mathrm{Pr}^{\mathrm{L}}_k)_{\mathrm{Mod}^n_{\mathrm{pr},k}}$ is equivalent to $(\mathcal{C}, E)$. By repeating construction 3.2.7, replacing $\mathfrak{Z}(\mathcal{C})$ with $\mathfrak{Z}(\mathcal{C}, E)$ we obtain a homotopy commutative square

$$\begin{tikzcd}
\mathrm{Deform}[\mathcal{C}, E]^{op} \arrow[r, "\mathfrak{D}^{n+1}_{(\mathcal{C},E)}"] \arrow[d] & (\mathrm{Alg}^{n+1,\mathrm{aug}}_k)_{/k\oplus\mathfrak{Z}(\mathcal{C},E)} \arrow[d] \\
(\mathrm{Alg}^{n+1,\mathrm{aug}}_k)^{op} \arrow[r, "\mathfrak{D}^{n+1}"] & \mathrm{Alg}^{n+1,\mathrm{aug}}_k
\end{tikzcd}$$

This commutative diagram induces a natural transformation

$$\mathrm{SimDef}_{(\mathcal{C},E)} \xrightarrow{\varphi} \mathrm{Map}_{\mathrm{Alg}^{n+1,\mathrm{aug}}_k}(\mathfrak{D}^{n+1}(-), k \oplus \mathfrak{Z}(\mathcal{C}, E))$$

where $\mathrm{Map}_{\mathrm{Alg}^{n+1,\mathrm{aug}}_k}(\mathfrak{D}^{n+1}(-), k \oplus \mathfrak{Z}(\mathcal{C}, E))$ is a formal moduli problem. Recall that $\mathfrak{D}^{n+1} : (\mathrm{Alg}^{n+1,\mathrm{aug}}_k)^{op} \longrightarrow \mathrm{Alg}^{n+1,\mathrm{aug}}_k$ is the $\mathbb{E}_{n+1}$ Koszul duality functor (see [Lur11b, §4.4]).

The following proposition gives an explicit description of the formal moduli problem $\mathrm{SimDef}^{\wedge}_{(\mathcal{C},E)}$, associated with the pair $(\mathcal{C}, E)$, in terms of the k-linear center $\mathfrak{Z}(C, E)$.

Proposition 3.4.11. *Let $n \geq 1$ be an integer, k a field, $\mathcal{C}$ a presentable k-linear (∞, n)-category and $E \in \mathcal{C}$ and object. Let the k-linear center of the pair $(\mathcal{C}, E)$ be small. Then*

$$\mathrm{SimDef}^{\wedge}_{(\mathcal{C},E)}(-) \simeq \mathrm{Map}_{\mathrm{Alg}^{n+1,\mathrm{aug}}_k}(\mathfrak{D}^{n+1}(-), \mathfrak{Z}(\mathcal{C}, E))$$

Proof. Following remark 3.2.8, it is enough to prove that the natural transformation of construction 3.4.10

$$\mathrm{SimDef}_{(\mathcal{C},E)} \xrightarrow{\varphi} \mathrm{SimDef}^{\wedge}_{(\mathcal{C},E)}$$

is such that $\mathrm{SimDef}_{(\mathcal{C},E)}(k\oplus k[m]) \xrightarrow{\alpha(k\oplus k[m])} \mathrm{SimDef}^{\wedge}_{(\mathcal{C},E)}(k\oplus k[m])$ is a $(n-1)$-truncated map of spaces for every $m > 0$.

The proof is a parallel to that of lemma 3.2.9. Let $\mathcal{E}^n_E$ be the $\mathbb{E}_n$-monoidal ∞-category of n-fold endomorphisms of $id_{(\mathcal{C},E)} \in \mathfrak{C}_E$ (notation as in remark 3.4.7) and $\mathfrak{Z}(\mathcal{C}, E)$ be the

$\mathbb{E}_{n+1}$-algebra of n-fold endomorphisms of $id \in \mathcal{E}_E$, i.e. $\mathfrak{Z}(\mathcal{C}, E) = \underline{\mathrm{Map}}_{\mathcal{E}^n_E}(id, id)$, where $id \in \mathcal{E}^n_E$ is the $\mathbb{E}_n$-monoidal unit.

For every integer $m \geq 0$, the space $\Omega^{n+1}\mathrm{SimDef}_{(\mathcal{C},E)}(k \oplus k[m])$ can be identified with the left-most space of the fiber sequence

$$\mathrm{Map}_{\mathcal{E}^n_E}(id, id[m+n+1]) \longrightarrow \mathrm{Map}_{\mathcal{E}^n_E}(id, (k \oplus k[m+n+1]) \otimes id) \longrightarrow \mathrm{Map}_{\mathcal{E}^n_E}(id, id)$$

We have the following equivalences

$$\begin{aligned} \mathrm{Map}_{\mathcal{E}^n_E}(id, id[m+n+1]) &\simeq \mathrm{Map}_k(k[-m-n-1], \mathfrak{Z}(\mathcal{C}, E)) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}^{n+1}_k}(\mathrm{Free}^{n+1}_k(k[-m-n-1]), \mathfrak{Z}(\mathcal{C}, E)) \end{aligned}$$

Due to [Lur11b, proposition 4.5.6], there is an equivalence

$$\mathrm{Map}_{\mathrm{Alg}^{n+1}_k}(\mathrm{Free}^{n+1}_k(k[-m-n-1]), \mathfrak{Z}(\mathcal{C}, E)) \simeq \mathrm{Map}_{\mathrm{Alg}^{n+1}_k}(\mathfrak{D}^{n+1}(k \oplus k[m]), \mathfrak{Z}(\mathcal{C}, E))$$

We have the following equivalence of spaces induced by φ

$$\Omega^{n+1}\mathrm{SimDef}_{(\mathcal{C},E)}(k \oplus k[m]) \simeq \mathrm{Map}_{\mathrm{Alg}^{n+1}_k}(\mathfrak{D}^{n+1}(k \oplus k[m]), \mathfrak{Z}(\mathcal{C}, E))$$

In addition, there is the following natural equivalence of spaces

$$\mathrm{Map}_{\mathrm{Alg}^{n+1}_k}(\mathfrak{D}^{n+1}(k \oplus k[m]), \mathfrak{Z}(\mathcal{C}, E)) \simeq \Omega^{n+1}\mathrm{Map}_{\mathrm{Alg}^{n+1}_k}(\mathfrak{D}^{n+1}(k \oplus k[m+n+1]), \mathfrak{Z}(\mathcal{C}, E))$$

arising from the fact that $\mathrm{Map}_{\mathrm{Alg}^{n+1}_k}(\mathfrak{D}^{n+1}(-), \mathfrak{Z}(\mathcal{C}, E))$ is a formal moduli problem.

It now follows from the same argument as in the proof of lemma 3.2.9 that

$$\mathrm{SimDef}_{(\mathcal{C},E)}(k \oplus k[m]) \xrightarrow{\alpha(k \oplus k[m])} \mathrm{SimDef}^{\wedge}_{(\mathcal{C},E)}(k \oplus k[m])$$

is $(n-1)$-truncated. This proves the proposition. □

The following discussion aims to analyze the formal moduli problem $\mathrm{ObjDef}_E^{\wedge}$.

Remark 3.4.12. Let $(\mathfrak{Y}_1, \{E_\alpha\}_{\alpha \in T})$ and $(\mathfrak{Y}_2, \{F_\alpha\}_{\alpha \in T})$ are two deformation contexts admitting deformation theories $\mathfrak{D}_1 : \mathfrak{Y}_1^{op} \longrightarrow \Xi_1$ and $\mathfrak{D}_2 : \mathfrak{Y}_2^{op} \longrightarrow \Xi_2$ respectively. Let $U : \mathfrak{Y}_2 \longrightarrow \mathfrak{Y}_1$ be a functor which preserves pullback squares and final objects. Assume that for every $\alpha \in T$, and every integer $n > 0$, there is an equivalence $U(\Omega^{\infty-n} E_\alpha) \simeq \Omega^{\infty-n} F_\alpha$.

Then there is an induced functor $U^{\mathrm{small}} : \mathfrak{Y}_1^{\mathrm{small}} \to \mathfrak{Y}_2^{\mathfrak{small}}$ which preserves final objects and pullback squares. Therefore

$$\Xi_2 \simeq \mathrm{Moduli}^{\mathfrak{Y}_2} \xrightarrow{(U^{\mathrm{small}})^*} \mathrm{Moduli}^{\mathfrak{Y}_2} \simeq \Xi_1$$

is the functor given by pullback by U^{small}.

There is a map of ∞-operads $\mathbb{E}_n^{\otimes} \xrightarrow{j} \mathbb{E}_{n+1}^{\otimes}$ [Lur17, §5.1.1]. Heuristically, this map is the identity on objects and given on morphism space by sending a configuration of n-dimensional open disks to the configuration of $(n+1)$-dimensional open disks obtained by taking a with the interval $(-1, 1)$.

This map of ∞-operads gives rise to the forgetful functor $\mathrm{Alg}_k^{n+1} \xrightarrow{U} \mathrm{Alg}_k^n$. Note that U admits a left adjoint, in particular U preserves small limits. In addition, for every integer $m \geq 0$, $U(k \oplus k[m]) \simeq k \oplus k[m]$ and $U(k) \simeq k$. It follows from remark 3.4.12 and theorem 2.2.19 that U induces a functor

$$\mathrm{Alg}_k^{n,\mathrm{aug}} \simeq \mathrm{Moduli}_k^{\mathbb{E}_n} \xrightarrow{U^*} \mathrm{Moduli}_k^{\mathbb{E}_{n+1}} \simeq \mathrm{Alg}_k^{n+1,\mathrm{aug}}$$

Lemma 3.4.13. *Let $n \geq 1$ be an integer, k a field, and $X : \mathrm{Alg}_k^{n,\mathrm{small}} \to \mathcal{S}$ be a formal $\mathbb{E}_n$-moduli problem. Consider the restriction $X^{\mathbb{E}_{n+1}} : \mathrm{Alg}_k^{n+1,\mathrm{small}} \to \mathcal{S}$ via the above map*

$\mathrm{Moduli}_k^{\mathbb{E}_n} \xrightarrow{U^*} \mathrm{Moduli}_k^{\mathbb{E}_{n+1}}$. *By theorem 2.2.19,*

$$X(-) \simeq \mathrm{Map}_{\mathrm{Alg}_k^{n,\mathrm{aug}}}(\mathfrak{D}^n(-), \mathfrak{Z})$$
$$X^{\mathbb{E}_{n+1}}(-) \simeq \mathrm{Map}_{\mathrm{Alg}_k^{n+1,\mathrm{aug}}}(\mathfrak{D}^{n+1}(-), \mathfrak{Z}')$$

for some $\mathfrak{Z} \in \mathrm{Alg}_k^{n,\mathrm{aug}}$ *and* $\mathfrak{Z}' \in \mathrm{Alg}_k^{n+1,\mathrm{aug}}$. *Let* $\{\mathfrak{m}_i : \mathrm{Alg}_k^{i,\mathrm{aug}} \longrightarrow \mathrm{Mod}_k\}_{i=n,n+1}$ *be the functors which map to the augmentation ideal.*

Then there is an equivalence $\mathfrak{m}_{n+1}(\mathfrak{Z}') \simeq \mathfrak{m}_n(\mathfrak{Z})[-1]$ *of non-unital* $\mathbb{E}_{n+1}$*-algebras.*

Proof. By definition, the tangent spectrum of X is unchanged under the map U^*. By computing the tangent spectrum of $X^{\mathbb{E}_{n+1}}$ and applying theorem 2.2.19, we find that $\mathfrak{m}_{n+1}(\mathfrak{Z}') \simeq \mathfrak{m}_n(\mathfrak{Z})[-1]$ is an equivalence of spectra. We have the following diagram

$$\begin{tikzcd}
\mathrm{Moduli}^{\mathbb{E}_{n+1}} & \mathrm{Alg}_k^{n+1,\mathrm{aug}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \mathrm{Alg}_k^{n+1,\mathrm{nu}} \\
\mathrm{Moduli}^{\mathbb{E}_n} \arrow[u, "U^*"] & \mathrm{Alg}_k^{n,\mathrm{aug}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[u, dashed] & \mathrm{Alg}_k^{n,\mathrm{nu}} \arrow[u, dashed]
\end{tikzcd}$$

The top and bottom rightward horizontal arrows are equivalences given by [Lur17, proposition 5.4.4.10]. We conclude that the right vertical arrow takes a non-unital $\mathbb{E}_n$-algebra $\mathfrak{a}$ and maps it to the non-unital $\mathbb{E}_{n+1}$-algebra whose underlying k-module is $\mathfrak{a}[-1]$. Hence $\mathfrak{m}_{n+1}(\mathfrak{Z}') \simeq \mathfrak{m}_n(\mathfrak{Z})[-1]$ is an equivalence of non-unital $\mathbb{E}_{n+1}$-algebras. □

A direct proof of the following proposition can be provided; however, we utilize the fiber sequence of corollary 3.4.5 to illustrate its use. Subsequent sections will further demonstrate the significant utility of this fiber sequence.

Proposition 3.4.14. *Let* $n \geq 1$ *be an integer,* k *be a field,* $\mathcal{C}$ *be a presentable* k*-linear* (∞, n)*-category and* $E \in \mathcal{C}$ *be an object. Then*

$$\mathrm{ObjDef}_E^{\wedge}(-) \simeq \underline{\mathrm{Map}}_{\mathrm{Alg}_k^{n,\mathrm{aug}}}(\mathfrak{D}^{n+1}(-), k \oplus \mathfrak{Z}(E))$$

where $\mathfrak{Z}(E)$ *is the* $\mathbb{E}_n$*-algebra of* n*-fold endomorphisms of* E*, viewed as a non-unital* $\mathbb{E}_n$*-algebra over* k*.*

Proof. Let

$$\mathrm{ObjDef}^{\wedge}_{E}(-) \simeq \underline{\mathrm{Map}}_{\mathrm{Alg}^{n,\mathrm{aug}}_{k}}(\mathfrak{D}^{n+1}(-), k \oplus \mathfrak{a}_E)$$

Following corollary 3.4.5, there is a fiber sequence of formal moduli problems

$$\mathrm{ObjDef}^{\mathbb{E}_{n+1},\wedge}_{E} \longrightarrow \mathrm{SimDef}^{\wedge}_{(\mathcal{C},E)} \longrightarrow \mathrm{ObjDef}^{\wedge}_{\mathcal{C}}$$

Consequentially, there is a fiber sequence of non-unital $\mathbb{E}_{n+1}$-algebras (also a fiber sequence of k-modules):

$$\mathfrak{a}_E \longrightarrow \mathfrak{Z}(\mathcal{C}, E) \longrightarrow \mathfrak{Z}(\mathcal{C})$$

The above sequence leads to the following fiber sequence of k-modules

$$\mathfrak{Z}(\mathcal{C}, E) \longrightarrow \mathfrak{Z}(\mathcal{C}) \longrightarrow \mathfrak{a}_E[1]$$

By lemma 3.4.8, $\mathfrak{Z}(\mathcal{C}, E) \simeq \mathfrak{Z}(\mathcal{C}) \times_{\mathfrak{Z}(E)} \{*\}$ is an equivalence of $\mathbb{E}_n$-algebras. Therefore, $\mathfrak{a}_E[1] \simeq \mathfrak{Z}(E)$ an equivalence of k-modules. So, the preceding fiber sequence of k-modules lifts to the fiber sequence of $\mathbb{E}_n$-algebras of lemma 3.4.8.

In turn, $\mathfrak{a}_E \simeq \mathfrak{Z}(E)[-1]$ is an equivalence of k-modules. We conclude that

$$\mathrm{ObjDef}^{\wedge,\mathbb{E}_{n+1}}_{E}(-) \simeq \underline{\mathrm{Map}}_{\mathrm{Alg}^{n,\mathrm{aug}}_{k}}(\mathfrak{D}^{n+1}(-), k \oplus \mathfrak{Z}(E)[-1])$$

It follows from lemma 3.4.13 that the non-unital $\mathbb{E}_{n+1}$-algebra structure on $\mathfrak{Z}(E)[-1]$ is induced by the $\mathbb{E}_n$-algebra structure on $\mathfrak{Z}(E)$ via the map $U^* : \mathrm{Moduli}^{\mathbb{E}_n}_k \longrightarrow \mathrm{Moduli}^{\mathbb{E}_{n+1}}_k$. This finishes the proof of the proposition. □

The following discussion connects to [Fra13, conjecture 4.50], giving insights into the restrictions of formal $\mathbb{E}_n$-moduli problems to formal $\mathbb{E}_m$-moduli problems for $m > n$.

The ∞-operad $\mathbb{E}^{\otimes}_n$ is Koszul self dual for all $n > 0$: consider the setting of dg-operads over k. Let E_n be a dg-operad model for the ∞-operad $\mathbb{E}^{\otimes}_n$, $\mathfrak{D}(\mathrm{E}_n)$ denote the Koszul dual dg-operad of E_n. A direct implication of [Fre11, theorem B] is that there is a morphism $\mathrm{E}_n[-n] \simeq \mathfrak{D}(\mathrm{E}_n)$ which is a weak equivalence of dg-operads over k.

Definition 3.4.15. Let p be an integer and M be a dg-module over k. An $\mathrm{E}_m[p]$-algebra structure on M is an E_m-algebra structure on $M[p]$.

There should exist an equivalence of ∞-categories $\mathrm{Alg}_k^{n,\mathrm{nu}} \simeq \mathrm{Alg}^{\mathrm{nu}}_{\mathfrak{D}(\mathrm{E}_n)}(\mathrm{Mod}_k)$.

The map $\mathbb{E}_n^{\otimes} \xrightarrow{j} \mathbb{E}_{n+1}^{\otimes}$ of ∞-operads will have an analog in the dg-operad model, $\mathrm{E}_n \xrightarrow{j} \mathrm{E}_{n+1}$. Under Koszul duality, the map j gives rise to a map of dg-operads $\mathrm{E}_{n+1}[-n-1] \xrightarrow{\mathfrak{D}(j)} \mathrm{E}_n[-n]$. Under the equivalence $\mathrm{Moduli}_k^{\mathbb{E}_n} \simeq \mathrm{Alg}_k^{n,\mathrm{nu}}$, the map $\mathbb{E}_{n+1}[-n-1] \xrightarrow{\mathfrak{D}(j)} \mathbb{E}_n[-n]$ should induce a functor $\mathrm{Moduli}_k^{\mathbb{E}_n} \xrightarrow{\mathfrak{D}(j)^*} \mathrm{Moduli}_k^{\mathbb{E}_{n+1}}$.

Conjecture 3.4.16. *[Fra13, conjecture 4.50] There exists a natural equivalence $U^* \simeq (\mathfrak{D}(j))^*$.*

Supporting argument: It follows from lemma 3.4.13 and the definition of a $\mathfrak{D}(\mathrm{E}_m)$-algebra that for every $\mathfrak{a} \in \mathrm{Alg}_k^{n,\mathrm{nu}}$, the under lying k-modules of the non-unital $\mathbb{E}_{n+1}$-algebras $U^*(\mathfrak{a})$ and $(\mathfrak{D}(j))^*(\mathfrak{a})$ are equivalent to $\mathfrak{a}[-1]$. Moreover, the functor $U : \mathrm{Alg}_k^{n+1,\mathrm{aug}} \to \mathrm{Alg}_k^{n,\mathrm{aug}}$ is induced by the morphism of operads $\mathrm{E}_n \xrightarrow{j} \mathrm{E}_{n+1}$.

3.5 Module categories over $\mathbb{E}_n$-algebras

Let $\mathcal{C}$ be the $\mathbb{E}_n$-monoidal k-linear ∞-category of right modules over an $\mathbb{E}_{n+1}$-algebra B. We study the deformations of $\mathcal{C}$ upto Morita equivalences. By definition, $\mathcal{C}$ admits a compact generator, which allows a better control over its deformations. We assume that B is m-connective for some integer $m \leq 0$. The main result here is theorem 3.5.10, proving which was the original objective of this thesis. This result implies that 'over $k[[t]]$, every Maurer-Cartan element in the k-linear center of $\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}$ induces a deformation of $\mathcal{C}$ (upto a Morita equivalence)'. In §3.5.2 we recover the fiber sequence of non-unital $\mathbb{E}_{n+1}$-algebras of [Fra13, theorem 1.1] using the language of formal moduli problems (proposition 3.5.16).

Warning 3.5.1. By an $\mathbb{E}_0$-monoidal k-linear ∞-category we mean a k-linear ∞-category. We emphasise that this convention is *non-standard*.

Definition 3.5.2. [Lur11b, definition 5.3.20] A presentable stable ∞-category $\mathcal{C}$ is said to be *tamely compactly generated* (tcg) if it is compactly generated and if for any choice of compact objects E, E' of $\mathcal{C}$ we have $\pi_{-m}\underline{\mathrm{Map}}_{\mathcal{C}}(E, E') = \mathrm{Ext}^m_{\mathcal{C}}(E, E') \simeq 0$ for sufficiently large m.

We say an $\mathbb{E}_n$-monoidal k-linear ∞-category $\mathcal{C}$ is tamely compactly generated if the underlying k-linear ∞-category of $\mathcal{C}$ is tamely compactly generated.

Definition 3.5.3. [BKP18, definition 3.2] Let $\mathcal{C}$ be an tamely compactly generated k-linear ∞-category. $\mathcal{C}$ *admits a single compact generator* $E \in \mathcal{C}$ if E is a compact object and for every $C \in \mathcal{C}$, $\underline{\mathrm{Map}}_{\mathcal{C}}(E, C) \simeq 0$ iff $C \simeq 0$.

Remark 3.5.4. Let $n \geq 1$ be an integer, $\mathcal{C}$ be a tamely compactly generated $\mathbb{E}_n$-monoidal k-linear ∞-category Assume that $\mathcal{C}$ admits a single compact generator E (definition 3.5.3). We assume that E is an idempotent, i.e. there exists an equivalence $E \otimes E \xrightarrow{\sim} E$.

This assumption endows $B := \underline{\mathrm{Map}}_{\mathcal{C}}(E, E)$ with the structure of an $\mathbb{E}_{n+1}$-algebra. This structure on B is non-canonical, depending on the choice of the equivalence $E \otimes E \xrightarrow{\sim} E$. This choice is precisely a choice of an equivalence $E \otimes E \xrightarrow{\sim} E$ in the homotopy category of $\mathcal{C}$.

In the scenario of interest, namely the ∞-category of modules over $\mathbb{E}_{n+1}$-algebras, there is a canonical choice for such an equivalence because E is the unit of the $\mathbb{E}_n$-monoidal structure in this case. We are *not* aware of any other interesting and natural examples of $\mathbb{E}_n$-monoidal ∞-categories which satisfy the hypothesis that the compact generator is idempotent.

Lemma 3.5.5. *Let $n \geq 0$ be an integer, k be a field, and $\mathcal{C}$ be a tamely compactly generated $\mathbb{E}_n$-monoidal k-linear ∞-category which admits a single compact generator E.*

Assume that $E \otimes E \simeq E$ *when* $n \geq 1$*, and denote the endomorphism object of* E *by* $B \coloneqq \underline{\mathrm{End}}_{\mathcal{C}}(E)$*. Then* $\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}$ *and* $\mathrm{RMod}^{n+1}_{\mathrm{pr},B}$ *are equivalent as presentable* k*-linear* $(\infty, n+1)$*-categories.*[7]

Proof. By [Lur17, theorem 7.1.2.1] we have an equivalence of k-linear ∞-categories $\mathrm{RMod}_B \xrightarrow{U} \mathcal{C}$, such that $B \mapsto E$. This proves the lemma for $n = 0$.

When $n = 1$, U allows us to endow RMod_B with an $\mathbb{E}_n$-monoidal structure coming from $\mathcal{C}$. This $\mathbb{E}_n$-monoidal structure has no reason to be equivalent to the canonical $\mathbb{E}_n$-monoidal structure on RMod_B. Using these two $\mathbb{E}_n$-monoidal structures, we view RMod_B as an object of ${}_{\mathrm{RMod}_B}\mathrm{Bimod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_k)$, i.e. a $\mathrm{RMod}_B - \mathcal{C}$-bimodule. We denote this object by ${}_B\mathcal{M}_{\mathcal{C}}$. In addition, we may view RMod_B as an object of ${}_{\mathcal{C}}\mathrm{Bimod}_{\mathrm{RMod}_B}(\mathrm{Pr}^{\mathrm{L}}_k)$, i.e. a $\mathcal{C} - \mathrm{RMod}_B$-bimodule. We denote this object by ${}_{\mathcal{C}}\mathcal{M}_B$. Note that, following [Lur17, theorems 4.3.2.7 & 4.8.4.1] an equivalence of ∞-categories

$$ {}_{\mathcal{C}}\mathcal{M}_B \in \underline{\mathrm{Map}}_{2\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_k), \mathrm{RMod}_{\mathrm{RMod}_B}(\mathrm{Pr}^{\mathrm{L}}_k)) \simeq {}_{\mathcal{C}}\mathrm{Bimod}_{\mathrm{RMod}_B}(\mathrm{Pr}^{\mathrm{L}}_k) $$

$$ {}_B\mathcal{M}_{\mathcal{C}} \in \underline{\mathrm{Map}}_{2\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}_{\mathrm{RMod}_B}(\mathrm{Pr}^{\mathrm{L}}_k), \mathrm{RMod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_k)) \simeq {}_{\mathrm{RMod}_B}\mathrm{Bimod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_k) $$

Under such equivalences of functor ∞-categories with bimodule ∞-categories, the diagonal bimodule ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$ corresponds to $id_{\mathrm{RMod}_{\mathcal{C}}}$ and the diagonal bimodule ${}_B\mathcal{M}_B$ corresponds to $id_{\mathrm{RMod}_{\mathrm{RMod}_B}}$. Note that ${}_{\mathcal{C}}\mathcal{M}_B \otimes_B {}_B\mathcal{M}_{\mathcal{C}} \simeq {}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$ and ${}_B\mathcal{M}_{\mathcal{C}} \otimes_{\mathcal{C}} {}_{\mathcal{C}}\mathcal{M}_B \simeq {}_B\mathcal{M}_B$, implying that $\mathrm{RMod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_k) \simeq \mathrm{RMod}^2_{\mathrm{pr},B}$ is an equivalence of presentable k-linear $(\infty, 2)$-categories. Recall that $\mathrm{RMod}_{\mathrm{pr},\mathcal{C}} \simeq \mathrm{RMod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_k)$ (follows from a variant of [Ste20, corollary 5.1.15]).

We induct on n and use a similar argument when $n > 1$ to finish the proof. Assume that $\mathrm{RMod}^{n-1}_{\mathrm{pr},\mathcal{C}} \xrightarrow{U_n} \mathrm{RMod}^n_{\mathrm{pr},B}$ is an equivalence of presentable k-linear $(\infty, n-1)$-categories. As $\mathcal{C}$ is an $\mathbb{E}_n$-monoidal k-linear ∞-category, the presentable k-linear (∞, n)-category $\mathrm{RMod}^{n-1}_{\mathrm{pr},\mathcal{C}}$ has an $\mathbb{E}_1$-monoidal structure[8], and so does $\mathrm{RMod}^n_{\mathrm{pr},B}$. These two

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[7] The proof relies an the hypothesis that E is a compact generator and B has the structure of an $\mathbb{E}_{n+1}$-algebra, not that $E \otimes E \simeq E$.

[8] This follows from [Lur17, corollary 4.8.5.20]. We have made extensive use of this corollary without reference throughout this chapter.

$\mathbb{E}_1$-monoidal structures are not necessarily equivalent. Using the equivalence U_n, we endow $\mathrm{RMod}^n_{\mathrm{pr},B}$ with the structure of a $\mathrm{RMod}^{n-1}_{\mathrm{pr},\mathcal{C}} - \mathrm{RMod}^n_{\mathrm{pr},B}$-bimodule, denoted ${}_{\mathcal{C}}\mathcal{M}_B$ and the structure of a $\mathrm{RMod}^n_{\mathrm{pr},B} - \mathrm{RMod}^{n-1}_{\mathrm{pr},\mathcal{C}}$-bimodule, denoted ${}_B\mathcal{M}_{\mathcal{C}}$. Note that

$$\begin{aligned}{}_{\mathcal{C}}\mathcal{M}_B \in \underline{\mathrm{Map}}_{n\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}_{\mathrm{RMod}^{n-1}_{\mathrm{pr},\mathcal{C}}}(n\mathrm{Pr}^{\mathrm{L}}_k), \mathrm{RMod}_{\mathrm{RMod}^n_{\mathrm{pr},B}}(n\mathrm{Pr}^{\mathrm{L}}_k))\\ \simeq {}_{\mathrm{RMod}^{n-1}_{\mathrm{pr},\mathcal{C}}}\mathrm{Bimod}_{\mathrm{RMod}_{\mathrm{RMod}^n_{\mathrm{pr},B}}}(n\mathrm{Pr}^{\mathrm{L}}_k)\end{aligned}$$

$${}_B\mathcal{M}_{\mathcal{C}} \in \underline{\mathrm{Map}}_{2\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}_{\mathrm{RMod}_B}(\mathrm{Pr}^{\mathrm{L}}_k), \mathrm{RMod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_k)) \simeq {}_{\mathrm{RMod}_B}\mathrm{Bimod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_k)$$

It follows that the above two bimodules establish an equivalence $\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}} \simeq \mathrm{RMod}^{n+1}_{\mathrm{pr},B}$.

□

Remark 3.5.6. Let $n \geq 0$ and k be a field. We have the following diagram of functors,

$$\mathrm{SimDef}_{(\mathrm{RMod}^{n+1}_{\mathrm{pr},B}, \mathrm{RMod}^n_{\mathrm{pr},B})} \xrightarrow{\mathcal{E}} \mathrm{AlgDef}_B \xrightarrow{\mathcal{M}} \mathrm{CatDef}_{\mathrm{RMod}^{n+1}_{\mathrm{pr},B}}$$

where the functor $\mathcal{E}$ takes the $(n+1)$-fold endomorphisms of the marked object, which is an $\mathbb{E}_{n+1}$-algebra. The functor $\mathcal{E}$ is induced by a right adjoint (variant of [Lur17, theorem 4.8.5.11] for κ_0-presentable ∞-categories) to the composition of the below functor

$$\begin{aligned}\mathrm{Alg}^{n+1}_k &\longrightarrow (n+1)\mathrm{Pr}^{\mathrm{L}}_k\\ A &\mapsto \mathrm{RMod}^{\mathrm{n+1}}_{\mathrm{pr,A}}\end{aligned}$$

with the functor $(n+1)\mathrm{Pr}^{\mathrm{L}}_k \xrightarrow{\mathrm{Ind}_{\kappa_0}(-)} ((n+1)\mathrm{Pr}^{\mathrm{L}}_k)^{\wedge}$. The functor Ind_{κ_0}, which sends $\mathcal{D} \mapsto \mathrm{Ind}_{\kappa_0}(\mathcal{D})$, is an equivalence because of [Lur09, theorem 5.5.7.10].

On the other hand, the functor $\mathcal{M}$ takes $(n+1)$-fold presentable modules over each algebra deformation of B, induced by $\mathrm{Alg}^{n+1}_k \longrightarrow (n+1)\mathrm{Pr}^{\mathrm{L}}_k$ which sends $A \mapsto \mathrm{RMod}^{n+1}_{\mathrm{pr},A}$.

3.5.1 Formal deformations

We characterize formal deformations of $\mathbb{E}_n$-monodidal k-linear ∞-categories which are of the form RMod_B, where $B \in \mathrm{Alg}^{n+1}_k$. We assume the B is m-connective for integer

$m \leq 0$ (theorem 3.5.10).

Lemma 3.5.7. *[BKP18, lemma 4.23] Let k be a field and*

$$\cdots \to B_2 \to B_1 \to B_0$$

be an inverse system of discrete local artinian commutative k-algebras having same residue field k. Let $B = \lim_i B_i$ be the limit. Let $n \geq 2$ and consider for each i the algebra B_i as an small (artinian) $\mathbb{E}_n$-algebra and B as an augmented $\mathbb{E}_n$-algebra via the forgetful functor $\mathrm{CAlg}_k^{\mathrm{aug}} \to \mathrm{Alg}_k^{n,\mathrm{aug}}$. Then the natural map $\mathrm{colim}_i \mathrm{Spf}(B_i) \to \mathrm{Spf}(B)$ is an equivalence of formal $\mathbb{E}_n$-moduli problems. Equivalently, the natural map $\mathrm{colim}_i \mathfrak{D}^n(B_i) \to \mathfrak{D}^n(B)$ is an equivalence of augmented $\mathbb{E}_n$-algebras over k.

Remark 3.5.8. It follows from lemma 3.5.7 that a point in $\mathrm{CatDef}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[[t]]) \simeq \lim_i \mathrm{CatDef}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[t]/t^i)$ is given by a family $\{\mathcal{M}_i\}$ with $\mathcal{M}_i$ a deformation of $\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}$ over $k[t]/t^i$ for each $i \geq 1$ and $k[t]/t^i \otimes_{k[t]/t^{i+1}} \mathcal{M}_{i+1} \simeq \mathcal{M}_i$ a an equivalence of $k[t]/t^i$-linear $(\infty, n+1)$-categories. Further, we have an equivalence $\mathrm{CatDef}^{\wedge}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[[t]]) \simeq \mathrm{Map}_{\mathrm{Alg}_k^{n+2}}(\mathfrak{D}^{n+2}(k[[t]]), \mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}))$.

Lemma 3.5.9. *Let $n \geq 0$ be an integer, k be a field, $\mathcal{C}$ be a tamely compactly generated $\mathbb{E}_n$-monoidal k-linear ∞-category admitting a single compact generator E. Assume that $E \otimes E \simeq E$ when $n \geq 1$. Then*

$$\mathrm{CatDef}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[[t]]) \xrightarrow{\theta(k[[t]])} \mathrm{CatDef}^{\wedge}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[[t]])$$

induces a bijection on connected components.

Proof. Let $B := \underline{\mathrm{End}}_{\mathcal{C}}(E)$ be the $\mathbb{E}_{n+1}$-algebra of endomorphisms of E. By lemma 3.5.5, we have an equivalence $\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}} \simeq \mathrm{RMod}^{n+1}_{\mathrm{pr},B}$, such that $\mathrm{RMod}^{n-1}_{\mathrm{pr},\mathcal{C}} \mapsto \mathrm{RMod}^n_{\mathrm{pr},B}$. In turn, we have an equivalence of $\mathbb{E}_{n+2}$-algebras $\mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}) \xrightarrow{\sim} \mathfrak{Z}(\mathrm{RMod}^{n+1}_{\mathrm{pr},B})$. Denote $\mathfrak{Z}_n(B) := \mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}})$.

We compute that $\mathfrak{D}^{n+2}(k[[t]]) = k[\beta]$ with $\deg(\beta) = -n-2$ is the free $\mathbb{E}_\infty$-algebra on one generator viewed as an $\mathbb{E}_{n+2}$-algebra via the forgetful functor $\mathrm{CAlg}_k \longrightarrow \mathrm{Alg}_k^{n+2}$. Let $\eta \in \mathrm{CatDef}^{\wedge}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[[t]])$ correspond to a $\mathrm{LMod}^{n+1}_{k[\beta]}$-linear structure on $\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}$. Note that η may be viewed as a $\mathcal{C}$-linear natural transformation $\eta : id_{\mathcal{C}} \longrightarrow id_{\mathcal{C}}$. For instance, in case $n=1$:

$$\mathrm{Fun}_k(\mathrm{RMod}_{\mathrm{pr},\mathcal{C}}, \mathrm{RMod}_{\mathrm{pr},\mathcal{C}}) \simeq {}_{\mathcal{C}}\mathrm{Bimod}_{\mathcal{C}}$$

and $k[\beta] \overset{\eta}{\to} \mathfrak{Z}_n(B) \simeq \underline{\mathrm{Map}}_{{}_{\mathcal{C}}\mathrm{Bimod}_{\mathcal{C}}}(\mathcal{C},\mathcal{C})(id_{\mathcal{C}}, id_{\mathcal{C}})$.

In addition, $\mathrm{RMod}^{n-1}_{\mathrm{pr},\mathcal{C}} \in \mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}$ is an object in a presentable k-linear $(\infty, n+1)$-category and $\mathfrak{Z}(\mathrm{RMod}^{n-1}_{\mathrm{pr},\mathcal{C}}) \overset{\simeq}{\to} \mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},\mathcal{B}}) \simeq B$ is an equivalence of $\mathbb{E}_{n+1}$-algebras.

This data induces a composition of maps $k[\beta] \overset{\eta}{\to} \mathfrak{Z}_n(B) \to B$ of $\mathbb{E}_{n+1}$-algebras, which we denote $k[\beta] \xrightarrow{\eta(E)} B$. Note that the map $\mathfrak{Z}_n(\mathcal{C}) \to B$ is from lemma 3.4.8. Consider the image of the generator β under this map, which induces a map $E \xrightarrow{\phi(E)} E[n+2]$. Consider the cocone of this map in $\mathcal{C}$, i.e. the fiber sequence:

$$E^\eta \longrightarrow E \xrightarrow{\phi(E)} E[n+2]$$

We note that E^η is a compact generator of $\mathcal{C}$: let $F \in \mathcal{C}$ be such that

$$\underline{\mathrm{Map}}_{\mathcal{C}}(E^\eta, F) \simeq 0 \implies \underline{\mathrm{Map}}_{\mathcal{C}}(E[n+2], F) \overset{\simeq}{\to} \underline{\mathrm{Map}}_{\mathcal{C}}(E, F)$$

$$\underline{\mathrm{Map}}_{\mathcal{C}}(E, F)[-n-2] \overset{\simeq}{\to} \underline{\mathrm{Map}}_{\mathcal{C}}(E, F) \implies \underline{\mathrm{Map}}_{\mathcal{C}}(E, F) \simeq 0$$

the preceding implication is due to the tamely compactly generated hypothesis on $\mathcal{C}$. Since E is a compact generator, $F \simeq 0$. The converse is always true.

Compact objects of $\mathcal{C}$ are closed under finite colimits and finite limits.

Moreover by naturality and $\mathcal{C}$-linearity of η, we have

$$\begin{tikzcd}
E^\eta \otimes E \arrow[r] \arrow[d] & E \otimes E \simeq E \arrow[r, "\phi(E\otimes E)"] \arrow[d, "\simeq"] & E[n+2] \otimes E \simeq E[n+2] \arrow[d, "\simeq"] \\
E^\eta \otimes E \arrow[r] & E \otimes E \simeq E \arrow[r, "\phi(E)\otimes id_E"] & E[n+2] \otimes E \simeq E[n+2]
\end{tikzcd}$$

a fiber sequence, hence $E^\eta \otimes E \simeq E^\eta$. Moreover

$$E^\eta \otimes E^\eta \longrightarrow E \otimes E^\eta \simeq E^\eta \xrightarrow{\phi(E^\eta)} E[n+2] \otimes E^\eta \simeq E^\eta[n+2]$$

$\phi(E^\eta)$ is homotopic to zero because the action of $k[\beta]$ on E^η is trivial by construction. We see that $E^\eta \otimes E^\eta \simeq E^\eta \oplus E^\eta[n+2]$. The $\mathbb{E}_n$-monoidal structure on $\mathcal{C}$ induces a map of k-modules

$$\begin{tikzcd}
\underline{\mathrm{Map}}_{\mathcal{C}}(E^\eta, E^\eta) \otimes_k \underline{\mathrm{Map}}_{\mathcal{C}}(E^\eta, E^\eta) \arrow[r] & \underline{\mathrm{Map}}_{\mathcal{C}}(E^\eta \oplus E^\eta[n+2], E^\eta \oplus E^\eta[n+2]) \arrow[d] \\
 & \underline{\mathrm{Map}}_{\mathcal{C}}(E^\eta, E^\eta)
\end{tikzcd}$$

which is compatible with the $\mathbb{E}_1$-algebra structure on $\underline{\mathrm{Map}}_{\mathcal{C}}(E^\eta, E^\eta)$ endowed by the composition on $\mathcal{C}$. We conclude that $B^\eta := \underline{\mathrm{Map}}_{\mathcal{C}}(E^\eta, E^\eta)$ has the structure of an $\mathbb{E}_{n+1}$-algebra. By lemma 3.5.5, $\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}} \simeq \mathrm{RMod}^{n+1}_{\mathrm{pr},B^\eta}$. In turn, $\mathfrak{Z}_n(B) \simeq \mathfrak{Z}(\mathrm{RMod}^{n+1}_{\mathrm{pr},B^\eta})$ is an equivalence of $\mathbb{E}_{n+2}$-algebras. Upon viewing $\mathrm{RMod}^n_{\mathrm{pr},B^\eta}$ as an object of $\mathrm{RMod}^{n+1}_{\mathrm{pr},B^\eta}$, we find that $\mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},B^\eta}) \simeq B^\eta$ is an equivalence of $\mathbb{E}_{n+1}$-algebras, and there is a map $\mathfrak{Z}_n(B) \longrightarrow B^\eta$ of $\mathbb{E}_{n+1}$-algebras (lemma 3.4.8).

From the naturality of η, it follows that the map of $\mathbb{E}_{n+1}$-algebras $k[\beta] \xrightarrow{\eta} \mathfrak{Z}_n(B) \longrightarrow B^\eta$, factors via the natural augmentation $k[\beta] \longrightarrow k$. This implies that E^η is a k-module object of the $k[\beta]$-linear structure on $\mathcal{C}$. In other words, we have a factorization

$$\begin{tikzcd}
 & k[\beta] \arrow[dl, dashed, "\tau"'] \arrow[d, "\eta"] \arrow[dr, "0"] & \\
\mathfrak{Z}(\mathrm{RMod}^{n+1}_{\mathrm{pr},B^\eta}, \mathrm{RMod}^n_{\mathrm{pr},B^\eta}) \arrow[r] & \mathfrak{Z}_n(B) \arrow[r] & B^\eta
\end{tikzcd}$$

where by the universal property of the endomorphism object of the pair $(\mathrm{RMod}^{n+1}_{\mathrm{pr},B^\eta}, \mathrm{RMod}^n_{\mathrm{pr},B^\eta})$ (remark 3.4.7), the τ is a map of $\mathbb{E}_{n+2}$-algebras.

Let $B^\eta := \underline{\mathrm{End}}_{\mathcal{C}}(E^\eta)$, then there exists $\mathrm{RMod}_{B^\eta} \simeq \mathcal{C}$ a Morita equivalence of $\mathbb{E}_n$-monoidal k-linear ∞-categories. In other words, $\mathrm{RMod}^{n+1}_{B^\eta} \simeq \mathrm{RMod}^n_{\mathcal{C}}$ is an equivalence of k-linear $(\infty, n+1)$-categories. We see that η may be viewed as the data of a $k[\beta]$-linear structure on $\mathrm{RMod}^{n+1}_{\mathrm{pr},B^\eta}$ such that the induced action on $\mathrm{RMod}^{n}_{\mathrm{pr},B^\eta}$ is trivial. In other words, $\mathrm{RMod}^{n}_{\mathrm{pr},B^\eta}$ is a $\mathrm{Mod}^{n}_{\mathrm{pr},k}$-module object of $\mathrm{RMod}^{n+1}_{\mathrm{pr},B^\eta}$.

Following [BKP18, §4] (also see remark 3.5.6), we have a commutative diagram (see next page). Note that as B^η is m-connective for some negative integer m, AlgDef_{B^η} is a formal moduli problem (proposition 3.1.3).

$$\begin{tikzcd}
& \mathrm{SimDef}^{\wedge}_{(\mathrm{RMod}^{n+1}_{\mathrm{pr},B^{\eta}},\mathrm{RMod}^{n}_{\mathrm{pr},B^{\eta}})}(k[[t]]) \arrow[r, "\simeq"] \arrow[d] & \mathrm{LMod}_{\mathrm{LMod}^{n+1}_{\mathrm{pr},k[\beta]}}(((n+1)\mathrm{Pr}^{L}_{k})_{\mathrm{Mod}^{n+1}_{\mathrm{pr},k}/})^{\simeq}_{(\mathrm{RMod}^{n+1}_{\mathrm{pr},B^{\eta}},\mathrm{RMod}^{n}_{\mathrm{pr},B^{\eta}})} \arrow[dd, "\pi"] \\
\mathrm{AlgDef}_{B^{\eta}}(k[[t]]) \arrow[r, "\simeq"] \arrow[d, "\mathcal{M}"] & \mathrm{AlgDef}^{\wedge}_{B^{\eta}}(k[[t]]) \arrow[d] & \\
\mathrm{CatDef}_{\mathrm{RMod}^{n}_{\mathrm{pr},\mathcal{C}}}(k[[t]]) \arrow[r, "\theta(k[[\mathrm{t}]])"] & \mathrm{CatDef}^{\wedge}_{\mathrm{RMod}^{n}_{\mathrm{pr},\mathcal{C}}}(k[[t]]) \arrow[r, "\simeq"] & \mathrm{LMod}_{\mathrm{LMod}^{n+1}_{\mathrm{pr},k[\beta]}}((n+1)\mathrm{Pr}^{L}_{k})^{\simeq}_{\mathrm{RMod}^{n+1}_{\mathrm{pr},B^{\eta}}}
\end{tikzcd}$$

Here ζ is a lift of η across π. The image of ζ in $\mathrm{CatDef}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[[t]])$ provides a lift of η across $\theta(k[[t]])$. This establishes that $\theta(k[[t]])$ is a surjection on connected components. The injectivity also follows from this proof: the method of proof shows that there can only be a unique (upto homotopy) lift of η across the map $\theta(k[[t]])$. □

The following is one of the main results of the thesis.

Theorem 3.5.10. *Let $n \geq 0$ be an integer, k be a field, $\mathcal{C}$ be a tamely compactly generated $\mathbb{E}_n$-monoidal k-linear ∞-category admitting a single compact generator E. Assume that $E \otimes E \simeq E$ when $n \geq 1$. Then*

$$\mathrm{CatDef}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[[t]]) \xrightarrow{\theta(k[[t]])} \mathrm{CatDef}^{\wedge}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[[t]])$$

is a homotopy equivalence.

Proof. Let $B := \underline{\mathrm{Map}}_{\mathcal{C}}(E, E)$ be the $\mathbb{E}_{n+1}$-algebra of endomorphisms of E. Following the notation of lemma 3.5.9, $\mathfrak{Z}_n(B)$ is the k-linear center of $\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}$.

We will inductively prove that $\theta(k[[t]])$ induces a bijection on homotopy groups. The base step is that $\theta(k[[t]])$ induces a bijection of π_0, i.e. connected components (lemma 3.5.9). This already proves the theorem when $n = 0$.

Assume that $n \geq 1$ and $\theta(k[[t]])$ induces an equivalence of $(i-1)^{th}$-homotopy groups for some $i \geq 1$. In what follows, a good mental aid it to set $i = 1$ and verify that the statements make sense. We assume that $i - 1 < n$.

Let $\Omega^{i-1}\mathrm{CatDef}^{\wedge}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[[t]])$ denote the $(i-1)$-fold loop space for arbitrary choices of base points. Assume that given an arbitrary point

$$\eta \in \Omega^{i-1}\mathrm{CatDef}^{\wedge}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[[t]]) \simeq \Omega^{i-1}\mathrm{Map}_{\mathrm{Alg}^{n+2}_k}(k[\beta], \mathfrak{Z}_n(B))$$

the image of η under the map $\Omega^{i-1}\mathrm{Map}_{\mathrm{Alg}^{n+2}_k}(k[\beta], \mathfrak{Z}_n(B)) \longrightarrow \mathrm{Map}_{\mathrm{Alg}^{n+1}_k}(k[\beta], B[-i+1])$ is contained in the connected component of the base point: in case $i = 1$, the proof of lemma 3.5.9 says that we can always find a suitable compact generator such that this statement holds. Denote this compact generator by E_i^η, and $B_i^\eta := \underline{\mathrm{Map}}_{\mathcal{C}}(E_i^\eta, E_i^\eta)$. When $i = 1$, there is the compact generator $E_1^\eta := E^\eta$ constructed in the proof of lemma 3.5.9. This guarantees a lift of η to a point

$$\tau \in \Omega^{i-1}\mathrm{SimDef}_{\mathrm{RMod}^{n+1}_{\mathrm{pr},B_i^\eta},\mathrm{RMod}^{n}_{\mathrm{pr},B_i^\eta}}(k[[t]]) \simeq \Omega^{i-1}\mathrm{Map}_{\mathrm{Alg}_k^{n+2}}(k[\beta], \mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}, \mathrm{RMod}^{n-1}_{\mathrm{pr},\mathcal{C}}))$$

The proof of lemma 3.5.9 shows that this lift exists when $i = 1$.

Consider an arbitrary point $\varepsilon \in \Omega_\eta\Omega^{i-1}\mathrm{Map}_{\mathrm{Alg}_k^{n+2}}(k[\beta], \mathfrak{Z}_n(B))$. As the image of η in $\mathrm{Map}_{\mathrm{Alg}_k^{n+1}}(k[\beta], B_i^\eta[-i+1])$ is homotopic to the base point, ε induces a map of $\mathbb{E}_{n+1}$-algebras $k[\beta] \xrightarrow{\varepsilon(E_i^\eta)} B_i^\eta[-i]$.

Let us denote $\eta_i(E_i^\eta) := \varepsilon(E_i^\eta)$. If the map $\eta_i(E_i^\eta)$ were to factor via the augmentation $k[\beta] \longrightarrow k$, then by the universal property of the k-linear center of the pair $(\mathrm{RMod}^{n+1}_{\mathrm{pr},B_i^\eta}, \mathrm{RMod}^{n}_{\mathrm{pr},B_i^\eta})$, we would have a lift of ε to the space $\Omega_\tau\Omega^{i-1}\mathrm{SimDef}_{(\mathrm{RMod}^{n+1}_{\mathrm{pr},B_i^\eta},\mathrm{RMod}^{n}_{\mathrm{pr},B_i^\eta})}(k[[t]])$. Unfortunately, this is not guaranteed. We need to construct a suitable compact generator for this purpose.

Note that the map $k[\beta] \xrightarrow{\eta_i(E_i^\eta)} B_i^\eta[-i]$ corresponds to a map $E_i^\eta \xrightarrow{\phi_i(E_i^\eta)} E_i^\eta[n+2-i]$. As in the proof of lemma 3.5.9, form the fiber sequence

$$E_{i+1}^\eta := E^\varepsilon \longrightarrow E_i^\eta \xrightarrow{\phi_i(E_i^\eta)} E_i^\eta[n+2+i]$$

It follows that E_{i+1}^η is a compact generator. Denote $B_{i+1}^\eta := B^\varepsilon := \underline{\mathrm{Map}}_{\mathcal{C}}(E^\varepsilon, E^\varepsilon)$. By construction, the image of η under the map

$$\Omega^{i-1}\mathrm{Map}_{\mathrm{Alg}_k^{n+2}}(k[\beta], \mathfrak{Z}_n(B)) \longrightarrow \mathrm{Map}_{\mathrm{Alg}_k^{n+1}}(k[\beta], B^\varepsilon[-i+1])$$

is contained in the connected component of the base point (note that this is true for $i = 1$). By construction, the map $k[\beta] \xrightarrow{\eta_{i+1}(E_{i+1}^\eta)} B^\varepsilon[-i]$ factors via the augmentation $k[\beta]$.

To finish the proof it is enough to show that B^{η}_{i+1} is an $\mathbb{E}_{n+1}$-algebra. To show this we use the $\mathcal{C}$-linearity of ϕ_i, as done in the proof of lemma 3.5.9: $\phi_i(E^{\eta}_i) \otimes E^{\eta}_i \simeq \phi_i(E^{\eta}_i \otimes E^{\eta}_i)$.

We claim that

$$E^{\eta}_{i+1} \otimes E^{\eta}_i \simeq E^{\eta}_{i+1} \oplus N_{i+1}$$

for some $N_i \in \mathcal{C}$. Convention: $E^{\eta}_0 := E$. One can check that $N_1 \simeq 0$, $N_2 \simeq E^{\eta}_2[n+2]$.

$$E^{\eta}_i \otimes E^{\eta}_i \simeq E^{\eta}_i \oplus M_i$$

for some object $M_i \in \mathcal{C}$. One can check that $M_0 \simeq 0$, $M_1 \simeq E^{\eta}_1[n+2]$, $M_2 \simeq E^{\eta}_2[n+2] \oplus E^{\eta}_2[n+2-1] \oplus E^{\eta}_2[2n+4-1]$.

Because of $\mathcal{C}$-linearity of ϕ_i: $\phi_i(E^{\eta}_i) \otimes E^{\eta}_i \simeq \phi_i(E^{\eta}_i \otimes E^{\eta}_i)$ are equivalent morphisms. Therefore, the fibers of these morphisms should be equivalent. By definition, $\mathrm{fib}(\phi_i(E^{\eta}_i)) = E^{\eta}_{i+1}$ for every $i \geq 0$. According to our convention, $\mathrm{fib}(\phi_0(E^{\eta}_0)) = \mathrm{fib}(\phi(E)) = E^{\eta}_1$ as constructed in the proof of lemma 3.5.9. So,

$$\mathrm{fib}(\phi_i(E^{\eta}_i \otimes E^{\eta}_i)) \simeq \mathrm{fib}(\phi_i(E^{\eta}_i) \otimes E^{\eta}_i) \simeq \mathrm{fib}(\phi_i(E^{\eta}_i)) \otimes E^{\eta}_i \simeq E^{\eta}_{i+1} \otimes E^{\eta}_i \tag{3.1}$$

must be equivalent to

$$\begin{aligned}\mathrm{fib}(\phi_i(E^{\eta}_i \otimes E^{\eta}_i)) &\simeq \mathrm{fib}(\phi_i(E^{\eta}_i \oplus M_i)) \\ &\simeq \mathrm{fib}(\phi_i(E^{\eta}_i)) \oplus \mathrm{fib}(\phi_i(M_i)) \\ &\simeq E^{\eta}_{i+1} \oplus N_{i+1} \end{aligned} \tag{3.2}$$

where $N_{i+1} := \mathrm{fib}(\phi_i(M_i))$. The above equivalences hold because ϕ distributes over finite sums and $\mathrm{fib}(-)$ commutes with direct sums.

Note that, by $\mathcal{C}$-linearity of ϕ_i, we have the following equivalences

$$\mathrm{fib}(\phi_i(E^{\eta}_i \otimes E^{\eta}_{i+1})) \simeq \mathrm{fib}(E^{\eta}_i \otimes \phi_i(E^{\eta}_{i+1})) \simeq \mathrm{fib}(\phi_i(E^{\eta}_i) \otimes E^{\eta}_{i+1})$$

The right-most space is equivalent to $E^{\eta}_{i+1} \otimes E^{\eta}_{i+1}$, because $E^{\eta}_{i+1} = \mathrm{fib}(\phi_i(E^{\eta}_i))$. While the middle space is equivalent to $E^{\eta}_i \otimes (E^{\eta}_{i+1} \oplus E^{\eta}_{i+1}[n+2-i])$, because by construction the map $\phi_i(E^{\eta}_i)$ factors via the zero object.

Using equations 3.1 & 3.2, we conclude that

$$\begin{aligned} E^{\eta}_{i+1} \otimes E^{\eta}_{i+1} &\simeq (E^{\eta}_i \otimes E^{\eta}_{i+1}) \oplus (E^{\eta}_i \otimes E^{\eta}_{i+1}[n+2-i]) \\ &\simeq (E^{\eta}_{i+1} \oplus N_{i+1}) \oplus (E^{\eta}_{i+1} \oplus N_{i+1})[n+2-i] \\ &\simeq E^{\eta}_{i+1} \oplus M_{i+1} \end{aligned}$$

where $M_{i+1} := N_{i+1} \oplus E^{\eta}_{i+1}[n+2-i] \oplus N_{i+1}[n+2-i]$ The $\mathbb{E}_n$-monoidal structure on $\mathcal{C}$ induces a map

$$\begin{tikzcd} \underline{\mathrm{Map}}_{\mathcal{C}}(E^{\eta}_{i+1}, E^{\eta}_{i+1}) \otimes_k \underline{\mathrm{Map}}_{\mathcal{C}}(E^{\eta}_{i+1}, E^{\eta}_{i+1}) \arrow[r] & \underline{\mathrm{Map}}_{\mathcal{C}}(E^{\eta}_{i+1} \oplus M_{i+1}, E^{\eta}_{i+1} \oplus M_{i+1}) \arrow[d] \\ & \underline{\mathrm{Map}}_{\mathcal{C}}(E^{\eta}_{i+1}, E^{\eta}_{i+1}) \end{tikzcd}$$

endowing $B^{\eta}_{i+1} = \underline{\mathrm{Map}}_{\mathcal{C}}(E^{\eta}_{i+1}, E^{\eta}_{i+1})$ with the structure of an $\mathbb{E}_{n+1}$-algebra. We conclude that $\mathrm{RMod}^{n+1}_{\mathrm{pr},B^{\eta}_{i+1}} \xrightarrow{\sim} \mathrm{RMod}^{n}_{\mathrm{pr},\mathcal{C}}$. We have a lift of ϵ to the space

$$\varrho \in \Omega_{\tau}\Omega^{i-1}\mathrm{SimDef}_{(\mathrm{RMod}^{n+1}_{\mathrm{pr},B^{\eta}_{i+1}}, \mathrm{RMod}^{n}_{\mathrm{pr},B^{\eta}_{i+1}})}(k[[t]])$$

Let ϖ be the lift of η across the map $\Omega^{i-1}\theta(k[[t]])$. Using the commutative diagram in the proof of lemma 3.5.9 adapted to B^{η}_{i+1}, the image of ϱ in $\Omega_{\varpi}\Omega^{i-1}\mathrm{CatDef}_{\mathrm{RMod}^{n}_{\mathrm{pr},\mathcal{C}}}(k[[t]])$ provides a unique (up to homotopy) lift of ϵ across the map $\Omega_{\eta}\Omega^{i-1}\theta(k[[t]])$. This establishes a bijection of i^{th} homotopy groups of $\mathrm{CatDef}_{\mathrm{RMod}^{n+1}_{\mathrm{pr},B^{\eta}_{i+1}}}(k[[t]])$ and $\mathrm{CatDef}^{\wedge}_{\mathrm{RMod}^{n+1}_{\mathrm{pr},B^{\eta}_{i+1}}}(k[[t]])$. We conclude the proof by noting that this process is finite because the natural transformation θ is n-truncated (corollary 3.2.4). There would have been an issue if this process was infinite, because for $i = n+2$, our strategy cannot be implemented. □

3.5.2 The fiber sequence of corollary 3.4.5

Assuming that $\mathcal{C}$ is the k-linear ∞-category of right modules over an $\mathbb{E}_{n+1}$-algebra, it is possible to relate the fiber sequence of formal moduli problems from corollary 3.4.5 with the fiber sequence of [Fra13, theorem 1.1]. This fiber sequence's existence was originally suggested in [Kon99]. The goal of this section is to prove proposition 3.5.16. We begin by introducing some preliminaries.

The following definition is standard:

Definition 3.5.11. Let k be a field and B be an $\mathbb{E}_1$-algebra over k. The $\mathbb{E}_1$-*Hochschild cohomology of* B is an $\mathbb{E}_2$-algebra, defined to be

$$\mathrm{HH}^*_{\mathbb{E}_1}(B) \coloneqq \underline{\mathrm{Map}}_{{}_B\mathrm{Bimod}_B}(B, B)$$

Note that $\mathrm{HH}^*_{\mathbb{E}_1}(B) \simeq \mathfrak{Z}(\mathrm{RMod}_B)$ is an equivalence of $\mathbb{E}_2$-algebra ([Lur17, theorems 4.3.2.7 & 4.8.4.1]-we will freely make use of these results in this part, without an explicit reference).

Let B be an $\mathbb{E}_2$-algebra over k. Then RMod_B is an $\mathbb{E}_1$-monoidal k-linear ∞-category. Following definition 3.5.11,

$$\mathrm{HH}^*_{\mathbb{E}_1}(\mathrm{RMod}_B) \coloneqq \underline{\mathrm{Map}}_{{}_{\mathrm{RMod}_B}\mathrm{Bimod}_{\mathrm{RMod}_B}}(\mathrm{RMod}_B, \mathrm{RMod}_B)$$

We have an identification : ${}_{\mathrm{RMod}_B}\mathrm{Bimod}_{\mathrm{RMod}_B} \simeq \underline{\mathrm{Map}}_{2\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}^2_{\mathrm{pr},B}, \mathrm{RMod}^2_{\mathrm{pr},B})$.

$$\mathrm{HH}^*_{\mathbb{E}_1}(\mathrm{RMod}_B) \simeq \underline{\mathrm{Map}}_{2\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}^2_{\mathrm{pr},B}, \mathrm{RMod}^2_{\mathrm{pr},B})(id_{\mathrm{RMod}^2_{\mathrm{pr},B}}, id_{\mathrm{RMod}^2_{\mathrm{pr},B}})$$

Therefore, $\mathfrak{Z}(\mathrm{RMod}^2_{\mathrm{pr},B}) \simeq \mathrm{HH}^*_{\mathbb{E}_1}(\mathrm{RMod}_B)(id, id)$ is an equivalence of $\mathbb{E}_3$-algebras.

The analog of $\mathbb{E}_1$-Hochschild cohomology for $\mathbb{E}_n$-algebras is defined as follows:

Definition 3.5.12. [Fra13, definition 3.1] Let B be an $\mathbb{E}_n$-algebra over k.

$$\mathrm{HH}^*_{\mathbb{E}_n}(B) \coloneqq \underline{\mathrm{Map}}_{\mathrm{Mod}^{\mathbb{E}_n}_B}(B, B)$$

is called the $\mathbb{E}_n$-*Hochschild cohomology of* B. Using [Fra13, proposition 4.37], we define

$$\mathrm{Mod}^{\mathbb{E}_n}_B \coloneqq \mathrm{HH}^*_{\mathbb{E}_{n-1}}(\mathrm{RMod}_B)$$

The definition readily applies to $\mathbb{E}_n$-monoidal k-linear ∞-categories as well.

We conclude that $\mathrm{HH}^*_{\mathbb{E}_2}(B) \simeq \mathfrak{Z}(\mathrm{RMod}^2_{\mathrm{pr},B})$.

Lemma 3.5.13. *Let $n \geq 1$ be an integer, B be an $\mathbb{E}_n$-algebra over k, and $\mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},B})$ be a k-linear center of the presentable k-linear (∞, n)-category $\mathrm{RMod}^n_{\mathrm{pr},B}$ (definition 3.2.5). There is a canonical equivalence of $\mathbb{E}_{n+1}$-algebras over k*

$$\mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},B}) \simeq \mathrm{HH}^*_{\mathbb{E}_n}(B)$$

Proof. The cases $n = 1, 2$ have already been verified above. The proof follows directly from definition 3.5.12. Note that $\underline{\mathrm{Map}}_{n\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}^n_{\mathrm{pr},B}, \mathrm{RMod}^n_{\mathrm{pr},B}) \simeq {}_{\mathrm{RMod}^{n-1}_{\mathrm{pr},B}}\mathrm{Bimod}_{\mathrm{RMod}^{n-1}_{\mathrm{pr},B}}$. It follows from definitions that

$$\mathrm{Mod}^{\mathbb{E}_2}_{\mathrm{RMod}^{n-2}_{\mathrm{pr},B}} \simeq \mathrm{HH}^*_{\mathbb{E}_1}(\mathrm{RMod}^{n-1}_{\mathrm{pr},B}) \simeq \underline{\mathrm{Map}}_{n\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}^n_{\mathrm{pr},B}, \mathrm{RMod}^n_{\mathrm{pr},B})(id^{(1)}, id^{(1)})$$

proceeding forward, we find

$$\mathrm{Mod}^{\mathbb{E}_3}_{\mathrm{RMod}^{n-3}_{\mathrm{pr},B}} \simeq \mathrm{HH}^*_{\mathbb{E}_2}(\mathrm{RMod}^{n-2}_{\mathrm{pr},B}) \simeq \underline{\mathrm{Map}}_{n\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}^n_{\mathrm{pr},B}, \mathrm{RMod}^n_{\mathrm{pr},B})(id^{(1)}, id^{(1)})(id^{(2)}, id^{(2)})$$

This process eventually gives

$$\mathrm{Mod}^{\mathbb{E}_n}_B \simeq \underline{\mathrm{Map}}_{n\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}^n_{\mathrm{pr},B}, \mathrm{RMod}^n_{\mathrm{pr},B})(id^{(1)}, id^{(1)})(id^{(2)}, id^{(2)}) \ldots (id^{(n-1)}, id^{n-1})$$

Therefore, we see that $\mathrm{HH}^*_{\mathbb{E}_n}(B)$ is equivalent to

$$\underline{\mathrm{Map}}_{n\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}^n_{\mathrm{pr},B}, \mathrm{RMod}^n_{\mathrm{pr},B})(id^{(1)}, id^{(1)})(id^{(2)}, id^{(2)}) \dots (id^{(n-1)}, id^{n-1})(id^{(n)}, id^{(n)})$$

By definition, the preceding $\mathbb{E}_{n+1}$-algebra is $\mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},B})$, a k-linear center of $\mathrm{RMod}^n_{\mathrm{pr},B}$. □

It follows from lemma 3.5.13 and proposition 3.4.14 that the fiber sequence of formal moduli problems from corollary 3.4.5 can canonically be written as

$$B[-1] \longrightarrow \mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},B}\mathrm{RMod}^{n-1}_{\mathrm{pr},B}) \longrightarrow \mathrm{HH}^*_{\mathbb{E}_n}(B)$$

The above is a fiber sequence of non-unital $\mathbb{E}_{n+1}$-algebras.

Now we focus on the middle term in the preceding fiber sequence:

Definition 3.5.14. [Fra13, definition 2.6] Let B be an $\mathbb{E}_n$-algebra over k. We define a functor

$$\mathrm{Der}(B, -) \coloneqq \mathrm{Map}_{(\mathrm{Alg}^n_k)_{/A}}(B, B \oplus -) : \mathrm{Mod}^{\mathbb{E}_n}_B \longrightarrow \mathrm{Mod}_k$$

The *tangent complex of* B, T_B, is defined to be the value of the above functor on B viewed as an object of $\mathrm{Mod}^{\mathbb{E}_n}_B$.[9] The cotangent complex L_B is an object of $\mathrm{Mod}^{\mathbb{E}_n}_B$ which corepresents the functor $\mathrm{Der}(B, -)$.

Here $(\mathrm{Alg}^n_k)_{/A}$ is the ∞-category of $\mathbb{E}_n$-algebras over A. While $A \oplus - : \mathrm{Mod}^{\mathbb{E}_n}_B \longrightarrow (\mathrm{Alg}^n_k)_{/A}$ is the functor giving the *split-square zero extension.*

Let B be an $\mathbb{E}_n$-algebra over k. By [Fra13, proposition 4.43], there is a fiber sequence of k-modules

$$\mathrm{HH}^*_{\mathbb{E}_n}(B) \longrightarrow B \longrightarrow T_B[1-n]$$

---

[9] The tangent complex $T_B \coloneqq \mathrm{Der}(A, A)$ should be viewed as classifying the infinitesimal deformations of the identity morphism $B \longrightarrow B$.

where the left arrow is induced by the forgetful functor:

$$\mathrm{Mod}_B^{\mathbb{E}_n} \longrightarrow \mathrm{RMod}_B \tag{3.3}$$

In turn, there is another fiber sequence of k-modules:

$$T_B[-n] \longrightarrow \mathrm{HH}^*_{\mathbb{E}_n}(B) \longrightarrow B$$

Denote $\mathcal{C} = \mathrm{RMod}^n_{\mathrm{pr},B}$, a presentable k-linear (∞, n)-category, and $E = \mathrm{RMod}^{n-1}_{\mathrm{pr},B}$ an object of $\mathcal{C}$. It follows from lemma 3.5.13 that $\mathrm{Mod}_B^{\mathbb{E}_n}$ is canonically identified with the n-fold endomorphism object of $\mathcal{C}$, while it follows from definitions that RMod_B is canonically the $(n-1)$-fold endomorphism object of E.

When $n = 1$,

$$\mathrm{Mod}_B^{\mathbb{E}_1} \simeq {}_B\mathrm{Bimod}_B \simeq \underline{\mathrm{Map}}_{\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}_B, \mathrm{RMod}_B)$$

and the natural forgetful functor of equation 3.3 coincides with the evaluation functor

$$\underline{\mathrm{Map}}_{\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod}_B, \mathrm{RMod}_B) \xrightarrow{\mathrm{ev}_B} \mathrm{RMod}_B$$

which sends $F \mapsto F(B)$. In turn, the induced map $\mathrm{HH}^*_{\mathbb{E}_1}(B) \longrightarrow B$ is a map of $\mathbb{E}_1$-algebras. Therefore, when $n = 1$, we note that the map $\mathrm{HH}^*_{\mathbb{E}_1}(B) \longrightarrow B$ from [Fra13, proposition 4.43] coincides with the map $\mathrm{HH}^*_{\mathbb{E}_1}(B) \simeq \mathfrak{Z}(\mathrm{RMod}_B) \longrightarrow \mathfrak{Z}(B) \simeq B$ from lemma 3.4.8. We conclude that when $n = 1$: $\mathfrak{Z}(\mathrm{RMod}_B, B) \simeq T_B[-1]$ is a canonical equivalence of $\mathbb{E}_1$-algebras.

In fact, this equivalence canonically lifts to an equivalence of $\mathbb{E}_2$-algebras. Consider the pullback square of remark 3.4.7

$$\begin{tikzcd}
\mathcal{E}_B \arrow[r] \arrow[d] & \underline{\mathrm{Map}}_{\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{RMod_B}, \mathrm{RMod_B}) \simeq {}_B\mathrm{Bimod}_B \arrow[d, "B^*"] \\
\{*\} \arrow[r, "B"] & \underline{\mathrm{Map}}_{\mathrm{Pr}^{\mathrm{L}}_k}(\mathrm{Mod}_k, \mathrm{RMod}_B) \simeq \mathrm{RMod}_B
\end{tikzcd}$$

Following [Fra13, proposition 4.23], this pullback square is identified with the one in [Fra13, corollary 4.22]. In fact, by [Lur17, theorem 4.8.5.5], there is an equivalence of spaces, $(\mathcal{E}_B)^{\simeq} \simeq \mathrm{Map}_{\mathrm{Alg}_k}(B, B)$. It follows that the $\mathbb{E}_2$-algebra structure on $\mathfrak{Z}(\mathrm{RMod_B}, B)$ coincides with the $\mathbb{E}_2$-algebra structure on $T_B[-1]$.

This result holds for $n > 1$ as well:

Lemma 3.5.15. *Let B be an $\mathbb{E}_n$-algebra over k. There exists a canonical equivalence of $\mathbb{E}_{n+1}$-algebras*

$$\mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},B}, \mathrm{RMod}^{n-1}_{\mathrm{pr},B}) \simeq T_B[-n]$$

Proof. The case $n = 1$ has been established above. Let $n > 1$. The map $\mathrm{HH}^*_{\mathbb{E}_n}(B) \longrightarrow B$ of Francis is induced by a functor

$$\mathrm{HH}^*_{\mathbb{E}_{n-1}}(\mathrm{RMod}_B) := \mathrm{Mod}^{\mathbb{E}_n}_B \longrightarrow \mathrm{RMod}_B$$

By the same definition, the preceding map must be induced by a functor

$$\mathrm{HH}^*_{\mathbb{E}_{n-2}}(\mathrm{RMod}^2_B) := \mathrm{Mod}^{\mathbb{E}_{n-1}}_{\mathrm{RMod}_B} \longrightarrow \mathrm{RMod}^2_B$$

proceeding in this fashion, we arrive at the following map

$$\mathrm{Mod}^{\mathbb{E}_1}_{\mathrm{RMod}^{n-1}_B} \longrightarrow \mathrm{RMod}^n_B$$

We have come back to the case $n = 1$, where we know that Francis's map coincides with the map of lemma 3.4.8.

Given a pair $(\mathcal{C}, E)$ such that $\mathcal{C} = \mathrm{RMod}^n_B$ and $E = \mathrm{RMod}^{n-1}_B$, Francis' map $\mathrm{HH}^*_{\mathbb{E}_n}(B) \longrightarrow B$ canonically coincides with the map $\mathrm{HH}^*_{\mathbb{E}_n}(B) \simeq \mathfrak{Z}(\mathcal{C}) \longrightarrow \mathfrak{Z}(E) \simeq B$

of lemma 3.4.8. These two maps are maps of $\mathbb{E}_n$-algebras. It follows that the fibers of these two maps must also coincide as $\mathbb{E}_n$-algebras: $\mathfrak{Z}(\mathcal{C}, E) \simeq T_B[-n]$. We have already seen that there is a commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}_k(\mathrm{RMod}^n_{\mathrm{pr},B}, \mathrm{RMod}^n_{\mathrm{pr},B}) \arrow[r, "\sim"] \arrow[d, "\mathrm{ev}_{\mathrm{RMod}^{n-1}_{\mathrm{pr},B}}"'] & \mathrm{Mod}^{\mathbb{E}_1}_{\mathrm{RMod}^{n-1}_B} \arrow[d] \\
\mathrm{Fun}_k(\mathrm{Mod}^n_{\mathrm{pr},k}, \mathrm{RMod}^n_{\mathrm{pr},B}) \arrow[r, "\sim"] & \mathrm{RMod}^n_B
\end{tikzcd}$$

Recall that the horizontal arrows are canonical equivalences. The left vertical arrow is the evaluation functor, while the right vertical arrows is the forgetful functor from bimodules to right modules. The fiber of the left vertical arrow over $\mathrm{RMod}^{n-1}_{\mathrm{pr},B}$ gives rise to $\mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},B}, \mathrm{RMod}^{n-1}_{\mathrm{pr},B})$, while the fiber of the right arrow over the same object gives rise to $T_B[-n]$. The $\mathbb{E}_{n+1}$-structure on both these objects has its origins in this diagram. We conclude that there is a canonical equivalence of $\mathbb{E}_{n+1}$-algebras: $\mathfrak{Z}(\mathrm{RMod}^n_{\mathrm{pr},B}, \mathrm{RMod}^{n-1}_{\mathrm{pr},B}) \simeq T_B[-n]$. □

Proposition 3.5.16. *Let $n \geq 1$ be an integer, k be a field, $B \in \mathrm{Alg}^n_k$ an $\mathbb{E}_n$-algebra. By [Fra13, theorem 1.1], there is a fiber sequence of non-unital $\mathbb{E}_{n+1}$-algebras:*

$$B[-1] \longrightarrow T_B[-n] \longrightarrow \mathrm{HH}^*_{\mathbb{E}_n}(B)$$

Let $\mathcal{C} = \mathrm{RMod}^n_{\mathrm{pr},B}$, a presentable k-linear (∞, n)-category, and $E = \mathrm{RMod}^{n-1}_{\mathrm{pr},B}$ be an object of $\mathcal{C}$. By corollary 3.2.10, corollary 3.4.5, proposition 3.4.11, lemma 3.4.13, and proposition 3.4.14 there is a fiber sequence of non-unital $\mathbb{E}_{n+1}$-algebras:

$$\mathfrak{Z}(E)[-1] \longrightarrow \mathfrak{Z}(\mathcal{C}, E) \longrightarrow \mathfrak{Z}(\mathcal{C})$$

These two fiber sequences fit into a commutative diagram of $\mathbb{E}_{n+1}$-algebras:

$$\begin{tikzcd}
B[-1] \arrow[r] \arrow[d, "\simeq"] & T_B[-n] \arrow[r] \arrow[d, "\simeq"] & \mathrm{HH}^*_{\mathbb{E}_n}(B) \arrow[d, "\simeq"] \\
\mathfrak{Z}(E)[-1] \arrow[r] & \mathfrak{Z}(\mathcal{C}, E) \arrow[r] & \mathfrak{Z}(\mathcal{C})
\end{tikzcd}$$

where the vertical arrows are equivalences of $\mathbb{E}_{n+1}$*-algebras.*

Proof. The middle and the right vertical arrows are canonical equivalences, as a consequence of lemmas 3.5.15 & 3.5.13 respectively. The proof of lemma 3.5.15 shows that the right square is commutative. By the universal property of fiber sequences, the left vertical arrow is a canonical equivalence, which makes the diagram commute. ☐

3.6 Summary

We have undertaken a detailed study of deformations of $\mathbb{E}_n$-structures. We collect all relevant results below for convenient reference:

1. (Proposition 3.1.3) Let $n \geq 1$, $m \leq 0$ be integers, k be a field, B be an $\mathbb{E}_n$-algebra over k such the the underlying spectrum is m-connective. Then the deformation functor $\mathrm{AlgDef}_B : \mathrm{Alg}_k^{n+1,\mathrm{small}} \to \mathcal{S}$ is a formal moduli problem.

   This says that deformations of B up to equivalences of $\mathbb{E}_n$-algebras form a formal moduli problem, i.e. are described by an $\mathbb{E}_{n+1}$-algebra.

2. (Corollary 3.4.5) Let $n \geq 1$ be an integer, k be a field, $\mathcal{C}$ be a presentable k-linear (∞, n)-category, $E \in \mathcal{C}$ be an object. There is a commutative diagram of functors

$$\begin{tikzcd}
\mathrm{ObjDef}_E^{\mathbb{E}_{n+1}} \arrow[r] \arrow[d] & \mathrm{SimDef}_{(\mathcal{C},E)} \arrow[r, "\tau(\mathcal{C},E)"] \arrow[d] & \mathrm{CatDef}_{\mathcal{C}} \arrow[d] \\
\mathrm{ObjDef}_E^{\mathbb{E}_{n+1},\wedge} \arrow[r] & \mathrm{SimDef}_{(\mathcal{C},E)}^{\wedge} \arrow[r] & \mathrm{CatDef}_{\mathcal{C}}^{\wedge}
\end{tikzcd}$$

   where each row is a fiber sequence, and the functors in bottom row are formal moduli problems.

   One way to think of this fiber sequence is the following: when $\mathcal{C}$ is the higher category of n-fold iterated modules over an $\mathbb{E}_n$-algebra B, and the object E is the higher category of $n-1$-fold iterated modules, then this fiber sequence characterizes the difference between deformations of B up to Morita equivalences (right term) and deformations of B up to equivalences of $\mathbb{E}_n$-algebras (middle term).

3. (Theorem 3.5.10) Let $n \geq 0$ be an integer, k be a field, $\mathcal{C}$ be a tamely compactly generated $\mathbb{E}_n$-monoidal k linear ∞-category. Assume that $\mathcal{C}$ admits a single compact generator E such that $E \otimes E \simeq E$. Then the space $\mathrm{CatDef}_{\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}}(k[[t]])$ is equivalent to the space of $k[\beta]$-linear structures on $\mathrm{RMod}^n_{\mathrm{pr},\mathcal{C}}$.

We have seen that one cannot guarantee that the deformations of $\mathbb{E}_n$-structures form a formal moduli problem in full generality. Indeed, it is possible to construct counter examples, see [KL09, example 3.14]. A 'boundedness condition' along with a 'compact generator' enable us to achieve improved control over these deformations. One can understand this as the statement that 'solutions to the Maurer-Cartan equation over $k[[t]]$' describe the $\mathrm{Spf}(k[t]])$ neighbourhood of such objects.

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