The Hilbert space of de Sitter quantum gravity

A Thesis

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgment of collaborative research and discussions.

The work was done under the guidance of Prof. Suvrat Raju at the International Centre for Theoretical Sciences of the Tata Institute of Fundamental Research, Bengaluru, India.

Priyadarshi Paul.

Priyadarshi Paul

In my capacity as the formal supervisor of record of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Suvrat Raju

Prof. Suvrat Raju

Date: 10 December 2024

Preface

This thesis is based on the publication [1]. This work was performed in collaboration with the co-authors of this article. Sections 1.2, 1.3 and chapter 5 of the thesis are slightly rewritten versions of sections 1, 2 & 6 of [1], respectively. Chapters 2, 3, 4 and appendices B & C of this thesis are entirely based on sections 3, 4 & 5 and appendices A & B of [1] in the respective order. For illustration purposes, a few additional figures (Figures 2.1, 3.1, and 4.1) have been incorporated into chapters 2, 3 & 4, which were not included in [1].

Section 1.1 and appendices A & D are written by the author. Section 1.1 offers a comprehensive background for the thesis. Appendix A presents a concise overview of the technical prerequisites required for this work.

Appendix D provides a brief summary of a related work [2]. While it is not included in the thesis, this provides useful resources complimentary to the work presented here.

List of Publications

Publication which is included in the thesis:

1. T. Chakraborty, J. Chakravarty, V. Godet, P. Paul and S. Raju, *The Hilbert space of de Sitter quantum gravity*, JHEP 01 (2024) 132, [arXiv:2303.16315].

Other publications:

1. T. Chakraborty, J. Chakravarty, V. Godet, P. Paul and S. Raju, *Holography of information in de Sitter space*, JHEP 12 (2023) 120, [arXiv:2303.16316].

2. T. Chakraborty, J. Chakravarty and P. Paul, *Monogamy paradox in empty flat space*, Phys. Rev. D 106 (2022) 086002, [arXiv:2107.06919].

3. Anupam A. H , P.V. Athira , P. Paul and S. Raju, *Interacting Fields at Spatial Infinity*, [arXiv:2405.20326].

List of Figures

To Baba, Ma

&

Rituparna.

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I owe the completion of this work to the incredible individuals I have met throughout my life and during my time at ICTS.

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Contents

"... You have begun to find your answers. Although it will seem difficult, the rewards will be great. Exercise your human mind as fully as possible, knowing that it is only an exercise. Build beautiful artifacts, solve problems, explore the secrets of the physical universe, savor the input from all the senses, filled with joy and sorrow and laughter, empathy, compassion, and tote the emotional memory in your travel bag. ..."

— from the movie 'Waking Life'.

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Abstract

In this thesis, we obtain solutions of the Wheeler-DeWitt equation with positive cosmological constant for a closed universe in the large-volume limit. We argue that this space of solutions provides a complete basis for the Hilbert space of quantum gravity in an asymptotically de Sitter spacetime. Our solutions take the form of a universal phase factor multiplied by distinct diffeomorphism invariant functionals, with simple Weyl transformation properties, that obey the same Ward identities as a CFT partition function. The Euclidean vacuum corresponds to a specific choice of such a functional but other choices are equally valid. Each functional can be thought of as specifying a "theory" and, in this sense, the space of solutions is like "theory space".

We describe another basis for the Hilbert space where all states are represented as excitations of the vacuum that have a specific constrained structure. This gives the finite G_N generalization of the basis proposed by Higuchi in terms of group averaging, which we recover in the nongravitational limit.

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Chapter 1

Introduction

1.1 Background

Almost a century ago, the advent of quantum mechanics changed our understanding of reality. It was smoothly gelled with the special theory of relativity, leading to relativistic quantum field theories. These theories successfully describe the fundamental forces of nature such as electromagnetism and the weak nuclear force. However, the unification of quantum mechanics and gravity, another fundamental force, remains an enduring challenge in theoretical physics. Although string theory is considered the most promising paradigm, we still lack a complete theory capable of answering questions about the real world. Exploring various other approaches, particularly low-energy canonical analysis of gravity, without delving into the theory's ultraviolet completion, can offer valuable insights.

One of the novel features of quantum gravity that challenges our comprehension of the physical universe is its holographic nature [3, 4]. The notion of holography in this context was first concretely realized through Maldacena's groundbreaking AdS-CFT correspondence [5], a conjecture rooted in string theory frameworks. It says that the bulk gravitational dynamics in $(d+1)$-dimensional spacetime can be described by a d-dimensional non-gravitational theory at its boundary.

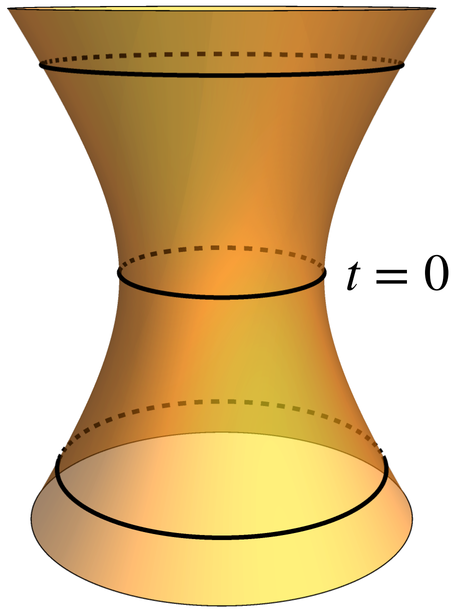

Figure 1.1: *We visualize* $(d + 1)$*-dimensional de Sitter spacetime, namely* dS_{d+1} *as a hyperboloid embedded inside one higher dimensional Minkowski space* $R^{1,d+1}$*. We choose the global de Sitter patch. The black circles correspond to constant time slices (Cauchy slices or spatial slices) which are d-dimensional sphere* S^d*. The figure shows that at positive time, the volume of the spatial slice is increasing, corresponding to an expanding universe.*

However, recent studies via canonical gravity [6–12] have shown that holography already manifests at low energy scales. We do not need intricate details of string theory (whose characteristic energy scale is the string scale) to illustrate holographic properties. In this series of works, a new principle, known as *holography of information* [8] has been proposed. For asymptotically flat spacetime (massless fields) & asymptotically anti-de Sitter spacetime, this principle is stated as follows:

> "*In a theory of quantum gravity, a copy of all the information available on a Cauchy slice is also available near the boundary of the Cauchy slice.*"

The anti-de Sitter space and flat space both have Cauchy slice with boundary. On the other hand, the de Sitter Cauchy slice is compact (a schematic figure of de Sitter spacetime is given in Figure 1.1). It is intriguing to unravel how the holographic principle manifests in asymptotically de Sitter space, despite its lack of a spatial boundary. The de Sitter spacetime is the solution to the Einstein equation with the positive cosmological constant[1]. In quantum gravity, where the metric itself is fluctuating, we consider the notion of the asymptotically de Sitter spacetime, where it is understood that the metric approaches the de Sitter metric at very late and very early times. The de Sitter space is important because it offers a model for an expanding universe.

From the experiments measuring cosmic microwave background radiation [13, 14], we have evidence that our universe had an exponential expansion in its early stage. During this expansion period, which we call "inflation", small quantum fluctuations were inflated to cosmic scale, giving rise to primordial correlation functions. These correlation functions live on the future boundary of the inflationary era. Furthermore, observational evidences [15, 16] indicate that our universe is currently undergoing an accelerated expansionary phase. Studying the late-time correlation functions in de Sitter quantum gravity can give us profound insights about inflationary physics, unveiling the mysteries of the universe's very beginning.

The theory of quantum gravity distinguishes itself from any other quantum field theory because of the existence of gravitational constraints [17]. These constraints arise from the local diffeomorphism invariance of gravity. They can be classified as the Hamiltonian constraint $\mathcal{H}$ and the momentum constraint $\mathcal{H}_i$ (i runs over spatial coordinate variables $1, \ldots, d$). Although $\mathcal{H}$ & $\mathcal{H}_i$ are particular components of Einstein equations[2], they act as constraints on the initial data, not as equations of motion. In a quantum theory of gravity, not all states in the Hilbert space are physical. A physical state should not change under a local diffeomorphism, which is just a gauge-redundancy. This can be ensured by demanding invariance of the physical states under the constraints as follows,

$$\mathcal{H}\,|\Psi\rangle_{\text{physical}} = 0 \quad , \quad \mathcal{H}_i\,|\Psi\rangle_{\text{physical}} = 0\ . \tag{1.1}$$

[1] We have briefly reviewed asymptotically de Sitter spacetime in section A.1

[2] The exact relation is given in the appendix A.3.

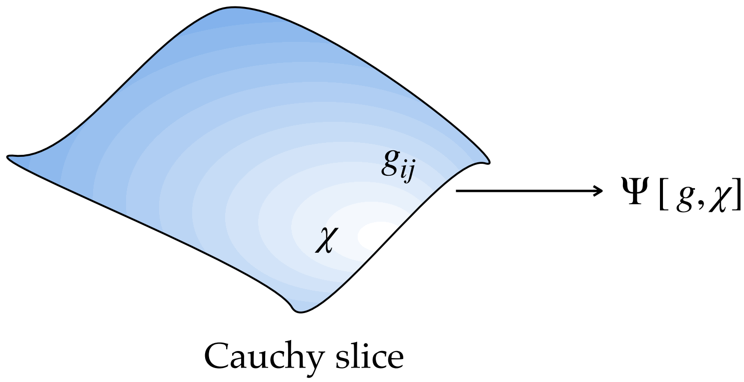

Figure 1.2: *We represent the state $|\Psi\rangle$ as a wavefunctional $\Psi[g_{ij},\chi,\ldots]$, which is functional of the field variables like metric g_{ij}, scalar field χ etc. on a Cauchy slice.*

When analyzing constraints in field theory, it is convenient to describe the states in terms of wavefunctionals, akin to wavefunctions in quantum mechanics. In quantum mechanics, we represent the states in the eigenbasis of suitable variables. For example, we can work with the position operator's eigenstates $\{|x\rangle\}$, where $\hat{x}|x\rangle = x|x\rangle$. So the state $|\psi\rangle$ is represented as a wavefunction $\psi(x) = \langle x|\psi\rangle$, a function over position space. In a quantum field theory we could use the same formalism. This is often referred to as the Schrödinger formalism in field theory[3]. Here the states can be thought as wavefunctionals, which are functionals of the field variables on a Cauchy slice (here constant time slice). For example, we can take the eigenbasis $\{|\chi\rangle\}$ corresponding to field $\hat{\chi}$ on a Cauchy slice, where $\hat{\chi}|\chi\rangle = \chi|\chi\rangle$. Then a state $|\Psi\rangle$ in the Hilbert space is written in this basis as a wavefunctional $\Psi[\chi] = \langle\chi|\Psi\rangle$. In this Schrödinger formalism, the physical state condition (1.1) reads as,

$$\mathcal{H}\,\Psi[g,\chi,\ldots] = 0 \quad ; \quad \mathcal{H}_i\,\Psi[g,\chi,\ldots] = 0\ . \tag{1.2}$$

The left equation is also known as the Wheeler-DeWitt equation [17]. The wavefunctional Ψ could be functional of existing fields like metric g_{ij}, scalar field χ, etc. In

[3]For more details, the reader can check [18].

this formalism, the constraint equations appear as functional differential equations at each point on the Cauchy slice. These constraints are similar in origin to the Gauss law in gauge theories like electromagnetism. While the momentum constraint is linear in metric canonical momentum, similar to the electromagnetic Gauss law, the Hamiltonian constraint is quadratic in metric canonical momentum, making it more difficult to solve.

As the momentum constraint arises from the d dimensional spatial-diffeomorphism invariance, any spatial-diffeomorphism invariant wavefunctional would be a valid solution to the momentum constraint. However, solving the Hamiltonian constraint is not straightforward, as this is a quadratic functional[4] differential equation at each spacetime point on the Cauchy slice. Kuchar [19] examined the solutions of the Wheeler-DeWitt equation in the context of flat space within the free limit. In a recent study [9], it was shown by Raju et al. that in asymptotically anti-de Sitter spacetime, the constraints can be solved perturbatively in orders of G_N. In this work, it was also found that if the states obeying the WDW equation agree at the boundary of AdS, then they agree everywhere in the bulk, thus manifesting holography of information in AdS.

This establishes the background for the thesis. A natural question that arises is: what are the solutions to the Wheeler-DeWitt (WDW) equation in asymptotically de Sitter spacetime, and do these solutions exhibit holography of information? We will now delve into the motivations behind this inquiry in greater technical detail.

[4]This is a quadratic equation in terms of the metric canonical momentum, which can be thought as functional derivative with respect to the spatial metric.

1.2 Motivation

In this work [1], we seek to address a basic question about quantum gravity in asymptotically de Sitter space: what is the space of states in such a theory?

This question has received surprisingly little attention. Considerable attention has been devoted to the Hartle-Hawking state, or the Euclidean vacuum, which is obtained by performing the path integral on a Euclidean space with only one boundary [20] and can be obtained by analytic continuation from AdS [21]. It is sometimes erroneously believed that other states in the Hilbert space can be obtained, as in a nongravitational quantum field theory, by simply acting with arbitrary field operators on the vacuum.

Higuchi [22, 23] pointed out that a naive Fock-space construction does not lead to the correct Hilbert space, even for weakly-coupled gravity. Even as the gravitational coupling is taken to zero, it is necessary to impose the constraints of the gravitational Gauss law on the Fock space. Since the Cauchy slices in de Sitter space are compact, the Gauss law implies that states must have zero charges under the de Sitter isometries [24–26]. At first sight, it would appear that this constraint excludes all states except for the Euclidean vacuum.

Higuchi proposed an ingenious construction, where one starts with a "seed state" and then averages it over the de Sitter-isometry group so as to produce invariant states. These states are not normalizable in the original norm, but Higuchi also proposed a modified norm, which amounts to dividing the QFT norm of these states by the infinite volume of the de Sitter-isometry group. It was later checked, that in some examples, the above prescription leads to a well-defined norm [27, 28].

In this thesis, we aim to understand the states in the Hilbert space for a theory where gravity is coupled to matter in an asymptotically de Sitter spacetime. These states will be determined by solving the WDW equation [17]. As discussed in Section 1.1, solving the WDW equation is a difficult task. However, we will show that in a particular limit, the WDW equation simplifies, allowing us to explicitly write down the solutions.

Under this specific limit, the cosmological constant Λ dominates over the intrinsic curvature R of the Cauchy slice and other local "potential energy" densities. For this condition to hold at every point on the Cauchy slice, the volume of the slice must be large. Therefore, we also call this regime as the "large-volume limit". The large-volume limit stands apart from the perturbative limit explored in [9], as our results hold consistently at all orders of perturbation theory.

Physically, this limit is very easy to understand. An asymptotically de Sitter spacetime attains the large-volume limit at asymptotically late times or early times. Therefore our solution to the WDW equation can be thought of as a form of "asymptotic quantization" — a program [29–31] that can be applied to the full nonlinear theory and has been fruitful in understanding the structure of the Hilbert space in asymptotically flat spacetimes.

We will show below that the solutions to the WDW equation in this limit can be characterized by diffeomorphism invariant wavefunctionals that have a simple specified behaviour under Weyl transformations. The Euclidean vacuum is also described by such a wavefunctional and is known to have these properties. But the new result in this work is that *all* valid wavefunctionals have the same simple behaviour under diffeomorphisms and Weyl transformations.

These wavefunctionals can be expanded in terms of the fluctuations of the metric and other degrees of freedom. The coefficient functions that appear in this expansion obey the same Ward identities as CFT correlators. So, one way to understand our result is that the space of solutions to the WDW equation in asymptotically-de Sitter space is described as "theory space" *i.e.* if one is given a set of correlation functions that obey the Ward identities imposed by conformal invariance (but not necessarily the constraints of unitarity or locality) then they can be used to construct a solution to the constraints.

Moreover, we will show that Higuchi's prescription for group averaging emerges naturally as the weak-coupling limit of such solutions. Therefore our analysis validates Higuchi's ansatz in the limit of weak coupling but also explains how it must be generalized in the interacting theory.

This thesis defines a basis for the Hilbert space in asymptotic de Sitter quantum gravity. However, the description of the Hilbert space is incomplete without a norm or inner product. The norm is defined in an accompanying work [32]. In that paper, we also explore the definition of cosmological correlators in the presence of quantum-gravity effects and find the principle of holography of information in this spacetime.

Relation to previous work. Our results are entirely consistent with the observation that the wavefunctional of the Euclidean vacuum can be computed in terms of the partition function of an appropriate CFT after dressing it with an appropriate phase factor [21, 33–38]; and that the functional derivatives of the wavefunctional obey the conformal Ward identities [39]. This is a useful observation. However, the Euclidean vacuum is a single wavefunctional.[5] It does satisfy the WDW equation but it is not the unique wavefunctional that does so [40]. Our objective in this work is to systematically consider the space of all solutions to the constraints.

An interesting proposal for the Hilbert space was made from a top-down perspective in [41] (building on [42]) for a specific theory with a low-energy description as Vasiliev gravity [43]. Our approach is complementary since it is "bottom up" and starts from the bulk. Since the answer in [41] is provided in terms of an auxiliary set of scalars on the late-time boundary, we cannot immediately compare our proposed answer with [41] but it is an interesting open problem to perform this comparison.

The WDW equation was also recently studied in AdS [44, 45]. It would be interesting to apply these techniques to dS. See [34, 38, 46] for earlier analyses of the constraints, [47, 48] for recent progress in this direction and [49] for related discussion.

There have also been suggestions [50–57] that the Hilbert space in de Sitter space should be finite dimensional. This might happen due to nonperturbative effects that constrain the allowed form of states, but our analysis does not shed light on this issue.

[5] As we discuss in [32], naively, this state does not appear to be normalizable therefore it might not even be part of the Hilbert space.

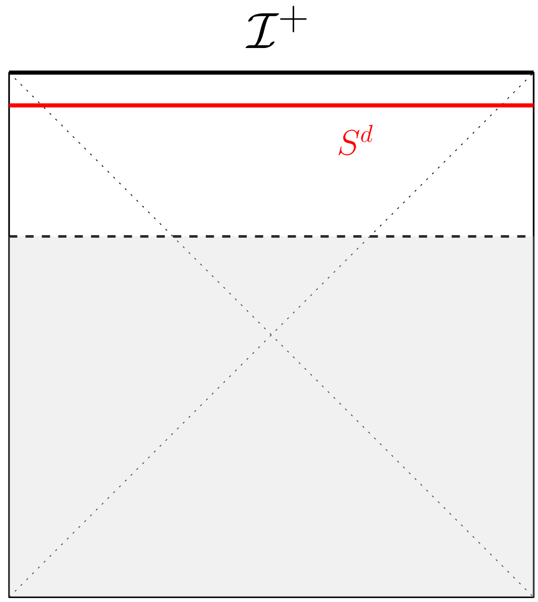

Figure 1.3: *We are considering a late time slice (in red) with topology S^d in an asymptotically de Sitter spacetime. In the late time expansion, the Wheeler-DeWitt equation can be solved and, up to a universal phase factor, the space of solutions is the space of functionals that transform under diff $\times$ Weyl in the same way as CFT$_d$ partition functions.*

1.3 Summary of results

We study the WDW equation in the regime where the cosmological constant dominates pointwise over the Ricci scalar and over the matter potential. In this limit the Cauchy slices, which are topologically S^d, grow to a very large volume. This can be thought of as a late time slice in an asymptotically dS_{d+1} spacetime, see Figure 1.3. It is convenient to work in a coordinate system where this limit corresponds to a large conformal factor $\Omega(x) = \det(g)^{1/2d}$ for the metric.

In chapter 2, we find the solutions to the WDW equation take the following form when expressed as functionals of the metric fluctuations and matter field, χ.

$$\Psi[g,\chi] \underset{\Omega\to\infty}{\longrightarrow} e^{iS[g,\chi]} Z[g,\chi] \ , \tag{1.3}$$

where $S[g,\chi]$ is a universal phase factor obtained by integrating local densities. $Z[g,\chi]$ is, in general, a nonlocal functional of g and χ that is diffeomorphism invariant and transforms in a simple way under Weyl transformations

$$\Omega\frac{\delta Z[g,\chi]}{\delta\Omega(x)} = \mathcal{A}_d[g]Z[g,\chi]\ , \tag{1.4}$$

where the anomaly polynomial $\mathcal{A}_d[g]$ can be computed explicitly. The anomaly polynomial vanishes for d odd and is imaginary for d even and therefore $|Z[g,\chi]|^2$ is a diff $\times$ Weyl invariant functional. The explicit form of $\mathcal{A}_d$ depends on a choice of normal ordering and may be corrected at higher orders in κ. However, we argue that the structural form (1.3) is valid at all orders in perturbation theory.

In chapter 3, we perform a Weyl transformation to study $Z[g,\chi]$ in the vicinity of the flat metric $g_{ij} = \delta_{ij} + \kappa h_{ij}$. This can be done at the expense of introducing an additional phase in the wavefunctionals in even dimensions. Then, provided $\log\, Z[g,\chi]$ is well behaved in the limit $g_{ij} \to \delta_{ij}, \chi \to 0$, we can expand it as

$$\log Z[g,\chi] = \sum_{n,m} \kappa^n \mathcal{G}_{n,m}\ , \tag{1.5}$$

where $\mathcal{G}_{n,m}$ are multilinear functionals of the metric fluctuation, h_{ij} and matter fluctuations of order n and m respectively. In chapter 3 we derive a set of Ward identities that constrain the form of $\mathcal{G}_{n,m}$. If one writes

$$\mathcal{G}_{n,m} = \frac{1}{n!m!}\int d\vec{y}d\vec{z}\, G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})h_{i_1j_1}(z_1)\ldots h_{i_nj_n}(z_n)\chi(y_1)\ldots\chi(y_m)\ , \tag{1.6}$$

then the coefficient functions $G^{\vec{i}\vec{j}}_{n,m}$ obey the same Ward identities as those obeyed by a connected correlator of n stress tensors and m operators of dimension $d-\Delta$, where Δ is related to the mass of the scalar field by (2.50).

The Euclidean vacuum is a particular state of the form above. Our new result is that *all* solutions are spanned by functionals of this form and related in a simple manner to diff $\times$ Weyl invariant functionals.

Since these identities relate functions with different values of n, the different coefficient functions $G_{n,m}^{\vec{i}\vec{j}}$ are not independent of each other. A complete set of such correlation functions satisfying a set of mutually consistent identities can be said to define a "theory". Therefore our space of solutions can be thought of as "theory space". Of course, we emphasize that this theory is not a unitary, or even a local, CFT.

In chapter 4, we show that a convenient basis for the space of states is given by wavefunctionals of the form

$$\Psi[g, \chi] = e^{iS[g,\chi]} \sum_{n,m} \kappa^n \delta\mathcal{G}_{n,m} Z_0[g, \chi] \ , \tag{1.7}$$

where the Euclidean vacuum is represented by $e^{iS[g,\chi]} Z_0[g, \chi]$ and $\delta\mathcal{G}_{n,m}$ is the *difference* of two sets of multilinear functionals, each of which is of the form (1.6).

The differences $\delta\mathcal{G}_{n,m}$ also obey Ward identities that relate $\delta\mathcal{G}_{n+1,m}$ to $\delta\mathcal{G}_{n,m}$ and therefore the series in (1.7) is infinite. However, in the limit $\kappa \to 0$, it is possible to focus on a single term in (1.7). We show that the nongravitational states so obtained correspond *precisely* to the group-averaged states found by Higuchi. For each state, we also explicitly find the corresponding "seed state".

Therefore our analysis justifies Higuchi's proposal in the nongravitational limit. However, it also reveals how Higuchi's prescription must be generalized away from zero coupling. At nonzero κ, one must add the terms required by the Ward identities to complete the series (1.7).

In chapter 5, we draw concluding remarks along with some future directions.

In appendix A, we provide a concise overview of the Wheeler-DeWitt equation's derivation from first principles. While this is available in existing literature, we present a simplified derivation adapted for the framework of de Sitter space.

In appendix B, we present the details about the asymptotic expansion of the Wheeler-DeWitt equation, along with its solution, complementing the discussion in chapter 2.

In appendix C, we include the derivation of the Ward identities for the coefficient functions $G^{\vec{i}\vec{j}}_{n,m}$, which are introduced in chapter 3.

In appendix D, we summarize our related research [2], which establishes a norm in the space of states defined in this thesis. We also briefly mention how one can define cosmological correlators in quantum gravity and how holography of information manifests in de Sitter spacetime.

Chapter 2

Asymptotic solutions to the Wheeler-DeWitt equation

In this chapter, we will show that solutions to the Wheeler-DeWitt equation take on the asymptotic form displayed in (1.3).

We consider Einstein gravity in $d+1$ dimensions with a positive cosmological constant in units where

$$\Lambda = \frac{d(d-1)}{2} \ . \tag{2.1}$$

In a $d+1$ split, the spacetime metric can be written in the form

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \ , \tag{2.2}$$

where N and N^i are the lapse and shift functions [58, 59]. The spatial slices are taken to be compact with metric g_{ij}. In addition, we might have matter degrees of freedom in the theory. As an illustration, we consider a massive scalar χ although we do not expect that our results will depend on the choice of matter.

In the canonical formalism, a state in such a theory is represented by a wavefunctional $\Psi[g_{ij}, \chi]$ that assigns an amplitude to a particular configuration of the metric and the matter fields on a spatial slice. This wavefunctional must obey the Hamiltonian and momentum constraints that arise simply by imposing diffeomorphism invariance on the theory [17]

$$\mathcal{H}\,\Psi[g_{ij},\chi] = 0 \ , \qquad \mathcal{H}_i\,\Psi[g_{ij},\chi] = 0 \ . \tag{2.3}$$

The Hamiltonian constraint is

$$\mathcal{H} = 2\kappa^2 g^{-1}\left(g_{ik}g_{jl}\pi^{kl}\pi^{ij} - \frac{1}{d-1}(g_{ij}\pi^{ij})^2\right) - \frac{1}{2\kappa^2}(R-2\Lambda) + \mathcal{H}_{\rm matter} + \mathcal{H}_{\rm int} \ , \tag{2.4}$$

and the equation in (2.3) setting it to annihilate the wavefunctional is called the Wheeler-DeWitt (WDW) equation. The momentum constraint is

$$\mathcal{H}_i = -2g_{ij}\nabla_k\frac{\pi^{jk}}{\sqrt{g}} + \mathcal{H}_{i,\rm matter} \ . \tag{2.5}$$

The gravitational coupling is $\kappa^2 = 8\pi G_N$. The momentum operator acts on the wavefunctional as

$$\pi^{ij} = -i\frac{\delta}{\delta g_{ij}} \ . \tag{2.6}$$

We take the matter energy density to be of the form

$$\mathcal{H}_{\rm matter} = \frac{1}{2}g^{-1}\pi_\chi^2 + V_{\rm matter}; \qquad V_{\rm matter} = \frac{1}{2}g^{ij}\partial_i\chi\partial_j\chi + \frac{1}{2}m^2\chi^2 \ , \tag{2.7}$$

and $\mathcal{H}_{i,\rm matter} = \frac{1}{\sqrt{g}}\pi_\chi\partial_i\chi$ is the matter momentum density.

The self-interactions of matter, its interaction with gravity and also potentially higher-order interactions have all been included in $\mathcal{H}_{\rm int}$. The analysis that follows will be largely insensitive to the details of $\mathcal{H}_{\rm int}$.

2.1 Asymptotic expansion

The Hamiltonian constraint (2.4) has terms that involve functional derivatives, which we can call "kinetic terms", and terms without functional derivatives that we can call "potential terms". Here, we will study the equation in the regime where the cosmological constant dominates over all other potential terms everywhere on the Cauchy slice. In particular, this means that the Ricci scalar and the matter potential are very small compared to the cosmological constant,

$$R \ll \Lambda; \qquad V_{\text{matter}} \ll \Lambda \,. \tag{2.8}$$

In some cases below, we will encounter higher curvature invariants and we will work in the regime where these are also small in cosmological units. We will find that the WDW equation simplifies in this regime. Since these conditions must apply everywhere on the spatial slice, our analysis does not apply to geometries that have singularities on the Cauchy slice under consideration.

An intrinsic notion of time. We will present solutions to (2.3), which are valid when the assumption (2.8) is met. Our physical interpretation is that these solutions describe "late times" in an asymptotically de Sitter universe. This includes states that might have very complicated features at finite times but settle down asymptotically to de Sitter space.

However, the equations (2.3) do not make any reference to time. Nor, in the case of de Sitter space, do we have an asymptotic boundary that can be used to set up an external clock. It was pointed out long ago by DeWitt [17], that this problem can be addressed by using an intrinsic observable as a clock. Correlators of other observables with this intrinsic clock then provide a notion of how the state varies with "time".

Here, we note that the assumption that the curvature is small everywhere on the

Cauchy slice suggests that the volume of the Cauchy slice

$$\log \int d^d x \sqrt{g}$$

becomes large.[1] So we can use the logarithm of the volume, which is a dimensionless quantity in the units chosen above, as a clock. This provides an operational meaning to the phrase "late time". Our discussion is similar in spirit to [61], where dynamical variables were used to define a clock, although we note that the measure used above is distinct from the "York time."

If one studies a spacetime that contracts from an infinitely large volume in the asymptotic past, then our analysis also applies to asymptotically early times when (2.8) is met. But to ask questions about "finite times", one must necessarily go beyond the assumption (2.8) somewhere on the slice. We will not address this regime here.

Intermediate variables. To facilitate our analysis, we will introduce *intermediate variables*, Ω and γ_{ij} and write the metric on the spatial slice as

$$g_{ij} = \Omega^2 \gamma_{ij} \ , \tag{2.9}$$

where $\det(\gamma_{ij}) = 1$. In terms of the original degrees of freedom, we define

$$\Omega = (\det(g_{ij}))^{\frac{1}{2d}}; \qquad \gamma_{ij} = g_{ij} \det(g_{ij})^{-\frac{1}{d}} \ . \tag{2.10}$$

Subject to the assumption (2.8) and the assumption that the volume of the Cauchy slice is large, it is possible to find a coordinate system where Ω is everywhere large and we will assume that such coordinates have been chosen.

It is also expected on physical grounds that the density of matter fields will get

[1] For instance, in $d = 2$ combining the Gauss-Bonnet theorem, $\int \sqrt{g} R d^2 x = 8\pi$, with (2.1) and (2.8) implies $\int \sqrt{g} d^2 x \gg 1$. However, strictly speaking, it is an independent physical assumption that the volume is large since a small Ricci scalar is insufficient to guarantee this for $d \geq 3$ (see theorem 3 of [60]).

"diluted" as the scale factor increases. This leads us to define a set of intermediate variables O for the matter fields according to

$$\chi = \Omega^{-\Delta} O \ . \tag{2.11}$$

In the analysis below, we will fix Δ in terms of the mass of the field and the cosmological scale.

We emphasize that (2.9) and (2.11) correspond to an exact change of variables, and so the content of the equations (2.3) is preserved. Second, we note that the split of the original metric into a Weyl factor and a Weyl-invariant part is coordinate dependent. Nonetheless, we will be careful to write all our final answers in a diffeomorphism-invariant form in terms of the original variables g and χ. So the reader should think of the change of variables simply as an intermediate technical trick.

2.2 Solution algorithm

In this section we outline and implement an algorithm to find solutions to the constraints in the limit (2.8). This section is somewhat technical and the reader who is interested just in the results can jump ahead to section (2.3).

Our procedure to find a solution to the constraints has three steps.

1. We rewrite the Hamiltonian constraint in terms of the conformal variables, Ω and γ.

2. We then seek a solution where the wavefunctional can be represented as the exponential of a functional, $\mathcal{F}$, that can be expanded in a series of terms that have a distinct scaling at large Ω. With suitable assumptions about normal ordering, the Hamiltonian constraint can be written in terms of $\mathcal{F}$.

3. The functional $\mathcal{F}$ involves two distinct series expansions — one that is present

even for pure gravity and another that involves the matter fields. We first solve for the gravitational part and then for the matter part.

The solution that we present below can be thought of as an elaboration of the solution to the radial WDW equation given in AdS by Freidel [62].

Rewriting the constraints. It is shown in Appendix B.1 that, in terms of these new variables, the Hamiltonian constraint can be rewritten as

$$\mathcal{H} = \frac{2\kappa^2}{\Omega^{2d}}\left[\frac{1}{4d(d-1)}\left(\Omega\frac{\delta}{\delta\Omega} + \Delta O\frac{\delta}{\delta O}\right)^2 - \left(\gamma_{ik}\gamma_{j\ell} - \frac{1}{d}\gamma_{ij}\gamma_{k\ell}\right)\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma_{k\ell}}\right] + \frac{\Lambda}{\kappa^2} - \frac{1}{2\kappa^2}R[\Omega^2\gamma] - \frac{1}{2}\Omega^{2(\Delta-d)}\frac{\delta^2}{\delta O^2} + \frac{1}{2}m^2\Omega^{-2\Delta}O^2 + \frac{1}{2}\Omega^2\gamma^{ij}\partial_i(\Omega^{-\Delta}O)\partial_j(\Omega^{-\Delta}O) + \mathcal{H}_{\text{int}}\ . \tag{2.12}$$

We do *not* seek to rewrite the momentum constraint. This is because the momentum constraint simply imposes that the wavefunctional is invariant under d-dimensional diffeomorphisms. This can be seen more easily in terms of the original variables g and χ that transform as tensors rather than tensor densities.

Series expansion. We expand the solution to (2.3) as

$$\Psi = e^{i\mathcal{F}}; \qquad \mathcal{F} = \sum_{n=0}^{\alpha} X_{\alpha-n} + \sum_{m=0}^{m_\beta} Y_{\beta-m} + \mathrm{O}\left(\frac{1}{\Omega}\right)\ . \tag{2.13}$$

Here, the functionals X_k and Y_k are undetermined functionals that grow as Ω^k at large Ω. More precisely we have

$$X_k, Y_k \sim \Omega^k, \qquad \Omega \to +\infty\ , \tag{2.14}$$

except for $k = 0$ in even d where we find an anomalous term that can be thought of as scaling with $\log(\Omega)$. For now, we keep α and β as undetermined parameters. It will turn out below that the solution to the WDW equation will require a series that grows with integer powers of Ω, corresponding to $\alpha = d$, and a series that grows with non-integer powers, $\beta = d - 2\Delta$. The maximum value of m is $m_\beta = \lfloor \mathrm{Re}\, \beta \rfloor$.

It is justified to separate the two series since β can be varied by varying parameters in the theory, and therefore, the two kinds of terms can be distinguished at a generic point in parameter space. The entire expression (2.13) has an undetermined remainder denoted by $\mathrm{O}\left(\frac{1}{\Omega}\right)$ which corresponds to terms that decay at large Ω. We will not work out the specific form of this remainder in this thesis. It is the term that is necessary to understand "finite-time" physics.

We do not assume that the functionals $X_{\alpha-n}$ and $Y_{\beta-m}$ are local functionals of g_{ij} and χ. It will turn out however that the leading terms in the series $X_{\alpha-n}$ will depend on local functionals of the metric. These correspond to the "gravitational part" of the solution. However, X_0 will be, in general, a nonlocal functional that depends both on g_{ij} and χ. The series $Y_{\beta-m}$ corresponds to the "matter part" of the solution and will comprise local functions of g_{ij} and χ.

Normal ordering and simplification. Acting with the rewritten constraint (2.12) on the ansatz (2.13) we find that the WDW equation can be written as

$$\begin{aligned}0 = &-2\kappa^2 \frac{1}{\Omega^{2d}} \left[\frac{1}{4d(d-1)} \left(\Omega \frac{\delta \mathcal{F}}{\delta \Omega} + \Delta O \frac{\delta \mathcal{F}}{\delta O} \right)^2 - \left(\gamma_{ik}\gamma_{j\ell} - \frac{1}{d}\gamma_{ij}\gamma_{k\ell} \right) \frac{\delta \mathcal{F}}{\delta \gamma_{ij}} \frac{\delta \mathcal{F}}{\delta \gamma_{k\ell}} \right] + \mathcal{D}_{\mathcal{F}} \\ &+ \frac{\Lambda}{\kappa^2} - \frac{1}{2\kappa^2} R[g] + \frac{1}{2}\Omega^{2(\Delta-d)} \left(\frac{\delta \mathcal{F}}{\delta O} \right)^2 + \frac{1}{2} m^2 \Omega^{-2\Delta} O^2 + \frac{1}{2}\Omega^2 \gamma^{ij} \partial_i (\Omega^{-\Delta} O) \partial_j (\Omega^{-\Delta} O) + \mathcal{H}_{\text{int}},\end{aligned} \tag{2.15}$$

Here, we have substituted the form (2.13) into the Hamiltonian constraint (2.12). We have explicitly displayed bilinear combinations of terms where a single functional derivative acts on $\mathcal{F}$. Indices are still raised and lowered using g_{ij} and g^{ij} and therefore $g^{ij} = \Omega^2 \gamma^{ij}$. We have used $\mathcal{D}_{\mathcal{F}}$ to indicate the action of second-order functional derivatives on $\mathcal{F}$ and the form of $\mathcal{D}_{\mathcal{F}}$ can be read off from (2.12).

The action of the second-order functional derivatives is subtle. This is because the action of a double functional derivative on a local term in $\mathcal{F}$ can generate a divergent $\delta(0)$ term. The precise form of these terms depends on the normal ordering prescription used to define the Hamiltonian constraint and it is reasonable to believe that the $\delta(0)$ terms can be removed by a judicious choice of normal ordering. (See Appendix B.2 for more discussion.)

Fortuitously these terms do not enter the leading-order analysis. This is because the terms in $\mathcal{D}_{\mathcal{F}}$ are linear in $\mathcal{F}$ whereas the first-order functional derivative terms displayed in (2.15) are quadratic. Since all the terms in (2.13) (except for X_0) grow with Ω the terms in $\mathcal{D}_{\mathcal{F}}$ always contribute with a lower power of Ω. We will see below that the leading contribution from $\mathcal{D}_{\mathcal{F}}$ can, at most, change the form of the anomaly polynomial at subleading order in G_N but does not change any of the structural features of the answer.

The expression above involves the Ricci scalar of the metric g. In terms of the variables, Ω and γ, this can be written as

$$R[g] = \frac{1}{\Omega^2}\left[R[\gamma] - 2(d-1)\Omega^2\nabla^i\nabla_i\log(\Omega) + (d-1)(d-2)\Omega^2\nabla^i\log(\Omega)\nabla_i\log(\Omega)\right] . \tag{2.16}$$

Note that since indices are raised by g^{ij}, the terms inside the bracket involving derivatives of $\log(\Omega)$ are O(1). Therefore we expect that R is of order Ω^{-2}. In fact, the magnitude of R in cosmological units can be used as an estimate of Ω^{-2} that does not rely on a specific choice of coordinate system.

2.2.1 Gravitational part

We now solve the WDW equation order by order in the large-Ω expansion, focusing first on the gravitational part.

Leading term. The largest term that appears without a derivative in (2.15) is the cosmological constant term. This immediately leads to the conclusion that

$$\alpha = d$$

since any larger term in the expansion cannot be cancelled in (2.15). The leading term in the WDW equation gives

$$\frac{2\kappa^2}{\Omega^{2d}}\frac{1}{4d(d-1)}\left(\Omega\frac{\delta X_d}{\delta\Omega}\right)^2 = \frac{\Lambda}{\kappa^2} \ . \tag{2.17}$$

Using (2.1), this leads to the equation

$$\Omega\frac{\delta X_d}{\delta\Omega} = \pm\frac{d(d-1)}{\kappa^2}\Omega^d \ . \tag{2.18}$$

This yields

$$X_d = -\frac{d-1}{\kappa^2}\int d^dx\sqrt{g} \ , \tag{2.19}$$

where we have chosen the negative sign for physical reasons explained below, and rewritten the expression in terms of the original variables to make manifest its diffeomorphism invariant form.

We would like to make a few comments.

1. We neglected a possible contribution from terms that involve $\frac{\delta}{\delta\gamma_{ij}}$ in going from (2.15) to (2.17). It may be checked that no diffeomorphism invariant γ-dependent term can be added to (2.19) while keeping the right hand side of (2.17), which is independent of γ, unchanged.

2. The choice of negative sign in (2.19) corresponds to the fact that we wish to study an expanding de Sitter universe at late times [63]. With this sign, the leading part

of the wavefunctional satisfies

$$\pi^{ij}\Psi \xrightarrow[\Omega\to\infty]{} -i\frac{\delta}{\delta g_{ij}}e^{iX_d} = -\frac{d-1}{2\kappa^2}\sqrt{g}g^{ij}\,e^{iX_d}\ . \tag{2.20}$$

This is precisely the relation between the canonical momentum and the metric at late times in an expanding universe. A choice of positive sign in (2.19) is allowed but would correspond to the part of the wavefunctional that describes a contracting universe.

3. As advertised, X_d is local. This property arises because the right hand side of (2.18) is a number, which does not allow any nonlocal contributions.

4. X_d is real, which corresponds to an oscillatory phase in the wavefunctional. So although X_d has the highest scaling with Ω, it does not contribute when the wavefunctional is squared to study expectation values [32].

At the next order, matching orders in Ω we now find that

$$\frac{\delta X_{d-1}}{\delta\Omega} = 0 \implies X_{d-1} = 0. \tag{2.21}$$

Beyond the universal terms, X_d and X_{d-1}, the form of the solution varies slightly in different dimensions. We explain the different cases for low dimensions and the general pattern below. In each case, we will use the following observation. We ignore any term that decays when $\Omega \to +\infty$ in (2.13). When combined with the contribution from X_d, such a term can yield a contribution that decays like Ω^{d-1} in (2.15). Therefore any term in (2.13) that yields a contribution of the same order when inserted in (2.15) will be undetermined in our procedure since we expect that its contribution to (2.15) can be cancelled by an appropriate choice of the remainder term.

d = 2. In $d = 2$, we find that $X_{d-2} = X_0$ must obey the equation

$$\Omega \frac{\delta X_0}{\delta \Omega} = \frac{1}{2\kappa^2} \Omega^2 R \qquad (d = 2) \ . \tag{2.22}$$

When rewritten in terms of the original variables, this takes on the familiar form

$$\left(2 g_{ij} \frac{\delta}{\delta g_{ij}} - \Delta \chi \frac{\delta}{\delta \chi} \right) e^{iX_0} = \mathcal{A}_2 \, e^{iX_0} \qquad (d = 2) \quad , \tag{2.23}$$

with

$$\mathcal{A}_2 = \frac{i}{2\kappa^2} \sqrt{g} R \ . \tag{2.24}$$

We note that (2.23) is analogous to the trace anomaly equation for the partition function of a 2d CFT except that the central charge is *imaginary* with $c = \frac{12\pi i}{\kappa^2}$. This can be thought of as the Brown-Henneaux [64] central charge in AdS but analytically continued to dS as proposed in [21].

We note some interesting features of (2.23) that will carry over to other dimensions.

1. Note that (2.23) does not have a unique solution. The addition of any term that is independent of Ω to an existing solution of (2.22) yields another solution. (In terms of partition functions, this is simply the observation that the trace anomaly equation does not uniquely fix the CFT partition function but only the central charge.) The reader might wonder why this freedom appears for X_0 but not for X_{d-1}. In fact, it is possible to add a nontrivial functional independent of Ω to the solution in (2.21) and obtain another solution, but such a functional is ruled out by the asymptotic scaling requirement (2.14).

2. The form of $\mathcal{H}_{\text{int}}$ does not place any constraints on X_0 through (2.15). This is for the following reason. The $\mathrm{O}(\Omega^{-1})$ remainder in $\mathcal{F}$ yields a contribution to (2.15) that is $\mathrm{O}\left(\Omega^{-d-1}\right)$ (Such a contribution arises when one $\frac{\delta}{\delta\Omega}$ derivative acts on X_d and the other acts on the remainder.) Since we are not keeping track of this remainder, we can consistently neglect terms that decay $\mathrm{O}\left(\Omega^{-(d+1)}\right)$ in (2.15).

But in $d = 2$, higher-derivative terms decay at least as fast as $\frac{1}{\Omega^3}$.

3. The anomaly may receive a possible correction through the action of double derivative terms on X_d. After accounting for the leading $\frac{1}{\Omega^{2d}}$ factor in (2.15), such terms contribute at the same order as X_0. The correction is $\mathrm{O}(\kappa^0)$ and so it is subleading compared to (2.24) and its precise value depends on the choice of normal ordering. In Appendix B.2, we show how this term can be made to vanish by a specific choice of normal ordering.

d = 3. For $d \geq 3$, X_{d-2} is determined from the subleading term in the WDW equation, which takes the form

$$-\frac{2\kappa^2}{\Omega^{2d}} \frac{2}{4d(d-1)} \left(\Omega \frac{\delta X_d}{\delta \Omega}\right) \left(\Omega \frac{\delta X_{d-2}}{\delta \Omega}\right) - \frac{1}{2\kappa^2} R = 0 \ . \tag{2.25}$$

This gives the equation

$$\Omega \frac{\delta X_{d-2}}{\delta \Omega} = \frac{1}{2\kappa^2} \sqrt{g} R \ . \tag{2.26}$$

For $d \geq 3$, this can be integrated to give

$$X_{d-2} = \frac{1}{2(d-2)\kappa^2} \int d^d x \sqrt{g} R \quad , \tag{2.27}$$

which can be checked to be a solution using that $\Omega \frac{\delta}{\delta \Omega} = 2 g_{ij} \frac{\delta}{\delta g_{ij}}$.

The next term X_0 is unconstrained and can be any function of γ that is independent of Ω. The action of the differential operators in (2.15) on such a function does yield non-zero terms. However such terms can be cancelled by an appropriate choice of the remainder term in $\mathcal{F}$. In terms of the original variables, this means that X_0 is a *Weyl invariant function* and we have

$$\Omega \frac{\delta}{\delta \Omega} X_0 = 0 \qquad (d = 3) \quad , \tag{2.28}$$

or, in terms of the original variables,

$$\left(2g_{ij}\frac{\delta}{\delta g_{ij}} - \Delta\frac{\delta}{\delta\chi}\right)e^{iX_0} = 0 \qquad (d=3)\ . \tag{2.29}$$

Once again, it is not necessary to keep track of interactions since they do not contribute to the non-decaying parts of $\mathcal{F}$ by the same power-counting argument that was given for $d = 2$.

$\mathbf{d \geq 3}$. The procedure outlined above can be continued to any dimension. The pattern is that higher order terms are determined by the recursive equation coming from

$$\frac{2\kappa^2}{\Omega^{2d}}\left[\frac{1}{4d(d-1)}\left(\Omega\frac{\delta\mathcal{F}}{\delta\Omega}\right)^2 - g_{ik}g_{j\ell}(\delta_g^{ij}\mathcal{F})(\delta_g^{k\ell}\mathcal{F})\right] = \mathcal{H}_{\rm int}\quad , \tag{2.30}$$

where the power-counting argument above tells us that $\mathcal{H}_{\rm int}$ can contribute to higher-order terms. Here we have rewritten the metric variations in terms of the traceless metric variation

$$\delta_g^{ij} = \frac{\delta}{\delta g_{ij}} - \frac{1}{d}g^{ij}g_{kl}\frac{\delta}{\delta g_{k\ell}}\quad , \tag{2.31}$$

using the identity (B.17). The term of order $d-2n$ is determined from this equation as

$$\begin{aligned}
&\Omega\frac{\delta}{\delta\Omega}X_{d-2n}\\
&= -\frac{2\kappa^2}{\Omega^d}\sum_{k=1}^{n-1}\left(g_{ik}g_{j\ell}(\delta_g^{ij}X_{d-2k})(\delta_g^{k\ell}X_{d-2(n-k)}) - \frac{1}{4d(d-1)}\left(\Omega\frac{\delta X_{d-2k}}{\delta\Omega}\right)\left(\Omega\frac{\delta X_{d-2(n-k)}}{\delta\Omega}\right)\right)\\
&+\Omega^d\mathcal{H}_{\rm int}\ .
\end{aligned} \tag{2.32}$$

This recursive structure determines all the higher order terms from the first two X_d and X_{d-2}.[2]

[2]Note that a similar recursive structure was also observed in [65] in the context of holographic renormalization.

As we have pointed out in $d = 2$ and $d = 3$, since the equation involves functional derivatives with respect to Ω, it never fixes X_0 uniquely. Given one solution, there is always the freedom to add an Ω-independent functional to X_0.

We illustrate the procedure with the next term X_{d-4} in pure Einstein gravity where $\mathcal{H}_{\text{int}} = 0$. The above equation gives

$$\Omega \frac{\delta X_{d-4}}{\delta \Omega} = -\frac{1}{2(d-2)^2\kappa^2}\sqrt{g}\left(R_{ij}R^{ij} - \frac{d}{4(d-1)}R^2\right) \quad , \tag{2.33}$$

as derived in Appendix B.3.

In $d = 4$, we obtain the equation

$$\Omega \frac{\delta X_0}{\delta \Omega} = -\frac{1}{8\kappa^2}\sqrt{g}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) \; , \tag{2.34}$$

which leads to

$$\left(2g_{ij}\frac{\delta}{\delta g_{ij}} - \Delta\chi\frac{\delta}{\delta\chi}\right)e^{iX_0} = \mathcal{A}_4\, e^{iX_0} \; , \tag{2.35}$$

where we have defined

$$\mathcal{A}_4 \equiv -\frac{i}{8\kappa^2}\sqrt{g}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) \; . \tag{2.36}$$

We recognize the trace anomaly equation for a four-dimensional CFT partition function $Z = e^{iX_0}$. The anomaly can be written as

$$\mathcal{A}_4 = \frac{1}{16\pi^2}\sqrt{g}(-aE_4 + cW_{abcd}W^{abcd}) \; , \tag{2.37}$$

using the Euler density and Weyl squared curvature given in (B.31) with the coefficients

$$a = c = -\frac{i\pi^2}{\kappa^2} = -\frac{i\pi}{8G_N} \; . \tag{2.38}$$

This is, up to the factor of $-i$, the anomaly of a holographic CFT$_4$ obtained using holographic renormalization in AdS$_5$ [66]. It may be checked that this anomaly polynomial

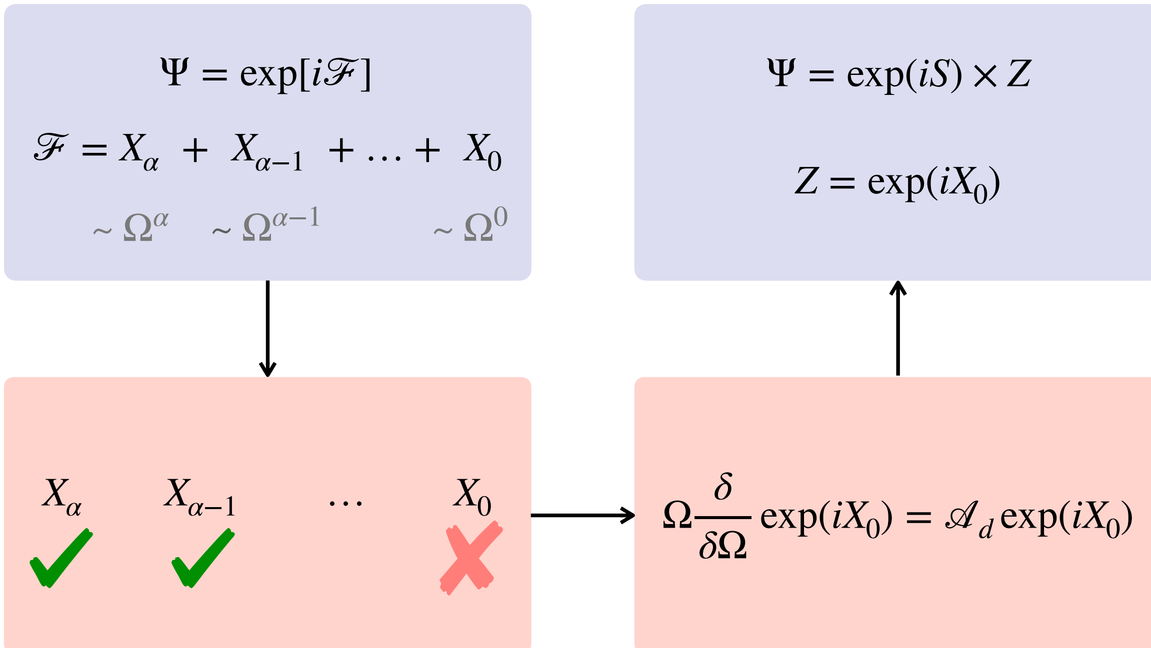

Figure 2.1: *In this flowchart, we demonstrate the algorithm for the gravity part of the solution. We take ansatz* $\Psi = \exp(i\mathcal{F})$*, where a generic term* X_k *of* $\mathcal{F}$ *corresponds to a functional of* Ω *and grows as* Ω^k *at large* Ω*. We have recursively determined all the* X_k *terms as local functionals of fields, except* X_0*. The term* X_0 *satisfies an anomaly equation and cannot be uniquely determined. We have combined all other terms except* X_0 *into* $\exp(iS)$ *and defined* $Z = \exp(iX_0)$*. Hence the solution is written as* $\Psi = \exp(iS)Z$*.*

receives corrections at subleading order in κ from higher-derivative terms and also from a choice of normal ordering in the WDW equation.

In $d \geq 5$, this equation can be integrated to give

$$X_{d-4} = -\frac{1}{2(d-2)^2(d-4)\kappa^2} \int d^dx \sqrt{g} \left(R_{ij}R^{ij} - \frac{d}{4(d-1)} R^2 \right) . \tag{2.39}$$

For $d = 5$, the formula above completes the gravitational part of the series. X_0 can be any functional independent of Ω. For higher d, one must continue the expansion above until X_0.

2.2.2 Matter part

We now solve for the matter part.

Leading term. The leading term Y_β is determined from the equation

$$-\frac{2\kappa^2}{\Omega^{2d}}\frac{2}{4d(d-1)}\left(\Omega\frac{\delta X_d}{\delta\Omega}\right)\left(\Omega\frac{\delta Y_\beta}{\delta\Omega}+\Delta O\frac{\delta Y_\beta}{\delta O}\right)+\frac{1}{2}\Omega^{2(\Delta-d)}\left(\frac{\delta Y_\beta}{\delta O}\right)^2+\frac{1}{2}m^2\Omega^{-2\Delta}O^2=0\ . \tag{2.40}$$

A solution is only possible if these terms compete which requires

$$\beta = d-2\Delta \quad , \tag{2.41}$$

and leads to the equation

$$\frac{2}{\sqrt{g}}g_{ij}\frac{\delta Y_\beta}{\delta g_{ij}}+\frac{1}{2g}\left(\frac{\delta Y_\beta}{\delta\chi}\right)^2+\frac{1}{2}m^2\chi^2=0\ , \tag{2.42}$$

which we have written in terms of the original variables. The solution takes the form

$$Y_\beta = b_\beta\int d^dx\,\sqrt{g}\chi^2\ , \tag{2.43}$$

where b_β satisfies

$$4b_\beta^2+2db_\beta+m^2=0\ . \tag{2.44}$$

Mass formula. We can determine the relation between b_β and Δ by examining the classical limit. In the classical theory, in an asymptotically de Sitter spacetime we have

$$\pi = g_{ij}\frac{\delta\mathcal{S}}{\delta\dot{g}_{ij}} = -\frac{d(d-1)}{2N\kappa^2}\Omega^{d-1}\dot{\Omega}, \qquad \pi_\chi = \frac{\delta\mathcal{S}}{\delta\dot{\chi}} = \frac{1}{N}\Omega^d\dot{\chi}\quad , \tag{2.45}$$

where N is the lapse function, and $\mathcal{S}$ is the Einstein-Hilbert action with possible interactions. From the definition of Δ we expect that

$$\frac{\dot{\chi}}{\chi} = -\Delta \frac{\dot{\Omega}}{\Omega} \ , \tag{2.46}$$

up to terms that vanish at large Ω. Here we have not used the equations of motion but simply the kinematic definition of Δ in (2.11) which determines the scaling behaviour of the field at large volume. In terms of the corresponding canonical momenta, this equation can be written as

$$\frac{1}{\chi}\pi_\chi = \frac{2\kappa^2 \Delta}{d(d-1)}\pi \ , \qquad \text{(classical expectation)} \ . \tag{2.47}$$

We can compare this classical expectation with the relation obtained from our wavefunctional. Using the leading order solution for the matter and metric sector, and the relations $\pi = -i g_{ij} \frac{\delta}{\delta g_{ij}}$ and $\pi_\chi = -i \frac{\delta}{\delta \chi}$ on the wavefunctional, we find

$$\begin{aligned} &\frac{1}{\chi}\pi_\chi \Psi = 2 b_\beta \Omega^d \, \Psi \ , \\ &\pi \Psi = -\frac{d(d-1)}{2\kappa^2} \Omega^d \, \Psi \ , \end{aligned} \tag{2.48}$$

up to subleading terms in the $\Omega \to +\infty$ limit. By matching this with (2.47), we see that we must have

$$b_\beta = -\frac{\Delta}{2} \ . \tag{2.49}$$

Substituting this in (2.44) we find that Δ must be related to the mass through

$$\Delta(d - \Delta) = m^2 \ . \tag{2.50}$$

With these substitutions, the leading term in the matter series becomes

$$Y_\beta = -\frac{\Delta}{2} \int d^d x \sqrt{g} \chi^2 \ . \tag{2.51}$$

Our derivation involved a correspondence with the classical limit, and the choice of specific orderings, such as the ordering of $\frac{1}{\chi}\pi_\chi$ in (2.48). Therefore the result (2.50) can be thought of as being valid to leading order in κ.

There is a class of solutions to (2.50) corresponding to $m > \frac{d}{2}$ of the form

$$\Delta = \frac{d}{2} \pm i\nu, \qquad \nu = \sqrt{m^2 - \frac{d^2}{4}} \ . \tag{2.52}$$

If one studies a nongravitational quantum field theory with this mass, then the single-particle states in such a theory lie in the principal series of representations of the conformal group $\mathrm{SO}(1, d+1)$ that is the isometry group of dS_d [67]. The complementary series correspond to masses in the range $0 < m < \frac{d}{2}$ and we can restrict to the range $0 < \Delta < d/2$. For a nice recent review, we refer the reader to [68].

Subleading term. In the principal series, the leading matter term is the only term we have since $\mathrm{Re}\,\beta = 0$ so that the subleading term decays as $\Omega \to +\infty$.

In the complementary series, the subleading term $Y_{\beta-2}$ contributes when

$$\Delta \leq \frac{d-2}{2} \ . \tag{2.53}$$

It is determined by the equation

$$-\frac{2\kappa^2}{\Omega^{2d}} \frac{1}{4d(d-1)} \left[2\left(\Omega \frac{\delta X_d}{\delta \Omega}\right)\left(\Omega \frac{\delta Y_{\beta-2}}{\delta \Omega} + \Delta O \frac{\delta Y_{\beta-2}}{\delta O}\right) + 2\left(\Omega \frac{\delta X_{d-2}}{\delta \Omega}\right)\left(\Omega \frac{\delta Y_\beta}{\delta \Omega} + \Delta O \frac{\delta Y_\beta}{\delta O}\right)\right]$$
$$+ \frac{\Omega^{2\Delta}}{2\Omega^{2d}} 2 \left(\frac{\delta Y_\beta}{\delta O}\right)\left(\frac{\delta Y_{\beta-2}}{\delta O}\right) + \frac{1}{2}\Omega^2 \gamma^{ij} \partial_i(\Omega^{-\Delta} O) \partial_j (\Omega^{-\Delta} O) = 0. \tag{2.54}$$

Restoring the variables χ and g_{ij} and substituting in the known higher order functionals gives the simpler looking form

$$\frac{2}{\sqrt{g}}\left(g_{ij}\frac{\delta}{\delta g_{ij}}+b_\beta\chi\frac{\delta}{\delta\chi}\right)Y_{\beta-2}-\frac{b_\beta}{2(d-1)}R\chi^2+\frac{1}{2}g^{ij}\partial_i\chi\partial_j\chi=0. \tag{2.55}$$

The solution (derived in Appendix B.4) is

$$Y_{\beta-2}=-\frac{1}{2(d-2-2\Delta)}\int d^dx\sqrt{g}\left(g^{ij}\partial_i\chi\partial_j\chi+\frac{\Delta}{2(d-1)}R[g]\chi^2\right)\ , \tag{2.56}$$

where we have substituted b_β from (2.49).

When $\Delta<\frac{d-4}{2}$, additional terms appear in the matter series and these terms can be worked out recursively using the WDW equation.

2.3 Asymptotic solution

We now give the general form of the asymptotic solution in the limit $\Omega\to\infty$. What we showed is that the constraints imply that it takes the form

$$\Psi \underset{\Omega\to+\infty}{\longrightarrow} e^{iS[g,\chi]}Z[g,\chi]\ . \tag{2.57}$$

Here, S is a universal phase factor that comprises integrals of *local* densities. It takes the form

$$S=\sum_{n=0}^{d-1}X_{d-n}+\sum_{m=0}^{\lfloor \mathrm{Re}\,\beta\rfloor}Y_{\beta-m}\ , \tag{2.58}$$

where $\beta=d-2\Delta$ and Δ is related to the mass of the field by (2.50). Terms corresponding to odd values of m and n vanish in (2.58). Explicit expressions for $n=0,2,4$ are given in (2.19), (2.27) and (2.39) respectively and for $m=0,2$ in (2.51) and (2.56) respectively. All terms in S are subject to the momentum constraint and so they are invariant under d-dimensional diffeomorphisms.

The factor $Z[g,\chi]=e^{iX_0}$ is a diffeomorphism invariant functional involving possibly nonlocal terms in g and χ and it has simple Weyl transformation properties.

$$\Omega\frac{\delta}{\delta\Omega(x)}Z[g,\chi] = \mathcal{A}_d[g]Z[g,\chi] \ . \tag{2.59}$$

The anomaly $\mathcal{A}_d[g]$ vanishes for d odd. For even d, it can be expressed in terms of curvature invariants and explicit expressions for $d = 2$ and $d = 4$ are provided in (2.24) and (2.36) respectively. Since S is real and the anomaly is imaginary, the absolute value of the wavefunctional $|\Psi[g,\chi]|^2$ is always diffeomorphism and Weyl invariant.

$Z[g,\chi]$ is not uniquely fixed by the WDW equation. Structurally, this is because the WDW, at large Ω, relates functional derivatives of a term with a given scaling in Ω to a source term that arises from terms with higher scaling in Ω. Therefore an existing solution for X_0 can be modified by the addition of an Ω-independent term (i.e. a Weyl invariant term) to yield another solution. This can also be seen from the fact that the solution to (2.59) is not unique.

Once $Z[g,\chi]$ has been chosen it is possible to use the equation (2.3) to continue the series expansion in Ω. The choice made for $Z[g,\chi]$ then controls the terms in the wavefunctional that decay with Ω. Physically, this can be thought of as follows. We specify a state at late times ($\Omega \to \infty$) by specifying the arbitrary functional in $Z[g,\chi]$. If we wish to ask questions about finite-time physics, then we must determine the full dependence of the wavefunctional on Ω. This dependence is sensitive to the interaction terms that appear in the Hamiltonian constraint, and we do not investigate it in this work.

The precise numerical values that we have found for the anomaly polynomials, and for X_{d-n} and $Y_{\beta-m}$ rely on a choice of normal ordering and, in some cases, can be affected by higher-order terms in the interactions. However, the structural properties of the WDW equation at large Ω — which are that the higher-order terms are fixed by a recursive set of functional equations, e^{iX_0} is left undetermined up to its Weyl transformation properties, and the momentum constraint imposes diffeomorphism invariance on each term — are robust. Therefore we conjecture that the *form* of the solution (2.57) is valid to all orders in the κ-expansion.

Discussion and comparison to AdS/CFT. The phase factor $S[g, \chi]$ is closely related to the counterterm-action that arises in holographic renormalization [66, 69–73]. The reason for this can be understood as follows.

In Euclidean AdS, it is possible to study the action of the bulk theory, with boundary conditions imposed on radial slices of the spacetime. The procedure of holographic renormalization identifies the divergences in this on-shell action. The wavefunctional in dS in the Euclidean vacuum can be obtained by analytically continuing this action [21]. Since the phase factor is universal, it is sufficient to determine it in a single state, as can be done using this procedure.

However, the analytic continuation introduces factors of i in the anomaly. Also, it makes the phase factor oscillatory in the dS case, whereas the counterterm action is real in AdS. The terms that appear in the matter sector are also slightly different in dS. This is because the dimension of operators dual to matter fields is always real in AdS but it can be complex in dS.

Second, the on-shell action on radial slices also obeys the constraints of diffeomorphism-invariance that lead to a WDW-type equation. As one approaches the boundary of AdS, it is possible to solve this equation asymptotically [62] (See also [74]) and this procedure also yields the correct divergent terms.

However, there is an important conceptual difference between the “radial WDW” equation and the one that we are studying. The radial wavefunctional is subject to regularity at $r = 0$. This fixes its asymptotic form at $r = \infty$ to be a phase factor times the partition function of a specific CFT — the CFT that is dual to the bulk AdS theory by the AdS/CFT correspondence [5, 75, 76]. These constraints cannot be seen from the asymptotic analysis in AdS.

The rules of quantum mechanics suggest that such constraints should *not* apply to our solutions. The specification of $Z[g, \chi]$ is equivalent to the specification of a state on a late-time spatial slice. But, in quantum mechanics, states can be specified on any time slice. Moreover, this specification is not subject to constraints that come from the time evolution of the state.

A comparison with asymptotically flat space might help to clarify this point. The "in" and "out" states that appear in the flat-space S-matrix are specified on Cauchy slices that are taken to be in the infinite past and infinite future, respectively. We do not restrict the set of "in" or "out" states by placing constraints on how these states evolve at finite times. In contrast, "radial wavefunctionals" in AdS specify data on a timelike boundary. They do not correspond to states, and no principle of quantum mechanics tells us that such data can be specified freely.

Some forms of $Z[g, \chi]$ are ruled out by normalizability [32] and it is possible that there are additional constraints that restrict the allowed set of $Z[g, \chi]$ [77] but they are not evident in our analysis.

Chapter 3

Solution space as theory space

In the previous chapter, we have argued that, in the large-volume limit, all solutions of the WDW equation take the form given in (2.57) — a universal phase factor multiplied with a diffeomorphism invariant function, $Z[g, \chi]$ with simple Weyl transformation properties.

Since the phase factor is universal, each distinct choice of $Z[g, \chi]$ leads to a distinct solution of the WDW equation. Second, in quantum mechanics, states can be read off from any time slice — even if that time slice is at arbitrarily late times. So, we expect that the large-volume behaviour of the wavefunctional completely specifies its form everywhere. Therefore each distinct state leads to a distinct choice of $Z[g, \chi]$. The two observations above lead to the conclusion that there is a one-to-one map between the space of allowed $Z[g, \chi]$ and the space of states in the theory.

We now investigate the properties of $Z[g, \chi]$ more carefully and argue that the space of allowed functionals can be thought of as "theory space".

3.1 Z as a CFT partition function

Now, let us examine the equation (2.59) together with the momentum constraint as written in terms of the original variables, g and χ.

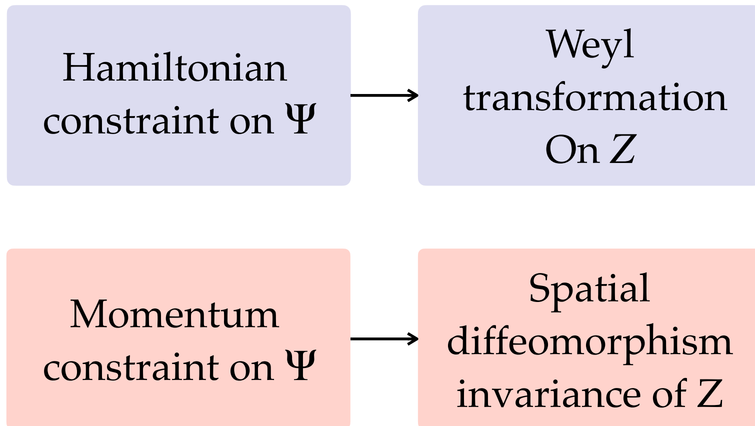

Figure 3.1: *The flowchart above illustrates how the constraints acting on Ψ manifest as Weyl transformation and spatial diffeomorphism invariance of Z.*

The relations that we find on Z are the following

$$\begin{aligned} &\left(2\sqrt{g}\nabla_i \frac{1}{\sqrt{g}}\frac{\delta}{\delta g_{ij}} - g^{ij}\partial_i\chi\frac{\delta}{\delta\chi}\right) Z[g,\chi] = 0; \qquad \text{(diffeomorphism invariance)} \\ &\left(2g_{ij}\frac{\delta}{\delta g_{ij}} - \Delta\chi\frac{\delta}{\delta\chi}\right) Z[g,\chi] = \mathcal{A}_d Z[g,\chi]. \qquad \text{(Weyl transformation)} \end{aligned} \tag{3.1}$$

The first equation comes from the momentum constraint and expresses diffeomorphism invariance while the second equation comes from the Hamiltonian constraint and expresses the anomalous Weyl transformation.

This can be made more explicit by writing the action of an infinitesimal diff $\times$ Weyl transformation on the metric and the scalar field

$$\delta_{(\xi,\varphi)} g_{ij} = \mathcal{L}_\xi g_{ij} + 2\varphi g_{ij}, \qquad \delta_{(\xi,\varphi)}\chi = \xi^i\partial_i\chi - \Delta\varphi\chi\ . \tag{3.2}$$

Then the equations (3.1) are equivalent to

$$\delta_{(\xi,\varphi)} \log Z[g,\chi] = \int d^dx\, \varphi(x)\mathcal{A}_d(x)\ , \tag{3.3}$$

which can be proven by taking the functional derivatives with respect to $\varphi(x)$ and $\xi_k(x)$.

The equations (3.1) are also obeyed by the partition function of d-dimensional CFT with a source χ turned on for an operator ϕ of dimension $\bar{\Delta} = d - \Delta$ on a Euclidean spacetime with metric g_{ij}

$$Z_{\mathrm{CFT}}[g, \chi] = \langle e^{-\int d^d x \, \chi \phi} \rangle_{g_{ij}} \ , \tag{3.4}$$

which obeys (3.1) with an appropriate choice of $\mathcal{A}_d$.

However, several cautionary remarks are in order.

1. First, the anomaly polynomial that appears for even d in (3.1) is imaginary. Second, the dimension of ϕ can be complex for sufficiently large mass. This can be seen from the mass-dimension relation (2.50).

2. Second, it is possible to obtain correlation function of the stress-tensor and of the operator ϕ by differentiating Z_{CFT}. In a local CFT, such correlators obey various constraints, including the constraints of cluster decomposition that follow from locality. Our analysis does not provide any reason to believe that the quantities obtained by functional differentiation of Z with respect to the metric or χ should obey such constraints.

3. Relatedly, the space of allowed Z's has a natural vector-space structure since this space is the space of states for a quantum-mechanical system. But a vector-space structure is unnatural in the space of CFT partition functions since the linear combinations of two partition functions of local CFTs does not, in general, correspond to the partition function of any other local CFT.

Therefore, although $Z[g, \chi]$ obeys the same equations that are obeyed by a CFT partition function, it does not necessarily correspond to the partition function of a unitary or local CFT.

3.2 Coefficient functions as CFT correlators

We will now expand log $Z[g,\chi]$ in the metric and matter fluctuations. This will give a basis of functionals for the solution space, comprising those $Z[g,\chi]$ that do not vanish in the limit where $g_{ij} \to \delta_{ij}$ and $\chi \to 0$.[1]

3.2.1 Weyl transformation of the variables

So far we have considered the wavefunctional in the limit where the volume of the spatial slice becomes arbitrarily large. Physically, we are interested in studying fluctuations about an asymptotically de Sitter spacetime, where the metric on a spatial slice takes on the form

$$g_{ij}^{\text{phys}} = \frac{4\omega^2}{(1+|x|^2)^2}(\delta_{ij} + \kappa h_{ij}) \ , \tag{3.5}$$

which is a perturbation of the round metric on S^d in coordinates x^i rescaled with a large Weyl factor ω.

It may be seen from (3.1) that $|\Psi[g,\chi]|^2 = |Z[g,\chi]|^2$ is diffeomorphism and Weyl invariant. The Weyl anomaly is imaginary so we have

$$\left[2g_{ij}\frac{\delta}{\delta g_{ij}} - \Delta\chi\frac{\delta}{\delta\chi}\right](Z[g,\chi]Z[g,\chi]^*) = \Big(\mathcal{A}_d Z[g,\chi]\Big)Z[g,\chi]^* + \text{c.c} = 0 \ , \tag{3.6}$$

using that $\mathcal{A}_d^* = -\mathcal{A}_d$.

Therefore, at the cost of an additional phase in the wavefunctional in even dimensions, we can make a Weyl transformation of the physical fields and study the behaviour of $Z[g,\chi]$ in the regime where

$$g_{ij} = \delta_{ij} + \kappa h_{ij} \ . \tag{3.7}$$

[1] When we introduce a norm on solutions [32], it will turn out that such solutions do not yield normalizable states. For the present analysis, this issue is not relevant. In the next chapter, we will study a different basis corresponding to functionals that vanish in the limit $g_{ij} \to \delta_{ij}$ and $\chi \to 0$. Those functionals are linear combinations of the functionals studied here, and provide a normalizable basis for the Hilbert space.

In converting the physical metric (3.5) to the form above, we have not only removed the large factor $\omega(x)$ but also made use of the fact that the round metric is related by a Weyl transformation to the flat metric. This does not change the fact that the spatial slices are topologically S^d and in [32], we will utilize this when we place boundary conditions at $|x| \to \infty$.

The Weyl transformation that takes the physical metric g_{ij}^{phys} to g_{ij} also rescales the matter fields according to

$$\chi = \left(\frac{2\omega}{1+|x|^2}\right)^{\Delta} \chi^{\text{phys}} \, . \tag{3.8}$$

We will now study an expansion of $Z[g,\chi]$ in powers of χ and h as it appears in (3.7). This is a convenient regime in which to study Z. If the value of Z is required in the physical regime, the phase factor in the wavefunctional can always be worked out using the anomaly equation and undoing the transformation from (3.7) to the original physical metric (3.5).

3.2.2 Expansion of Z

In what follows, we will assume that the perturbation in (3.7) is small enough that it makes sense to study a series expansion of Z about $h_{ij} = 0$. This focuses us on states whose wavefunctionals are concentrated on metrics that are close to the round sphere up to a Weyl transformation. The assumption in the analysis below is that these are the states of physical interest. Note that the analysis in the previous sections remains valid even if the wavefunctional does not have a good series expansion in h_{ij}.

It is convenient to introduce some notation. We write

$$Z[g,\chi] = \exp\left[\sum_{m,n} \kappa^n \mathcal{G}_{n,m}[h,\ldots h,\chi,\ldots\chi]\right] \, , \tag{3.9}$$

where we have defined multi-linear functionals, $\mathcal{G}_{n,m}$ that take n tensor fields and m

scalar fields as input and return a c-number.

$$\begin{aligned} &\mathcal{G}_{n,m}[h^{(1)},\dots h^{(n)},\chi^{(1)},\dots \chi^{(m)}] \\ &\equiv \frac{1}{n!m!}\int d\vec{y}d\vec{z}\, G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})h^{(1)}_{i_1j_1}(y_1)\dots h^{(n)}_{i_nj_n}(y_n)\chi^{(1)}(z_1)\dots\chi^{(m)}(z_m)\ . \end{aligned} \tag{3.10}$$

The "coefficient functions" $G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})$ depend on $\vec{y}=(y_1,\dots,y_n),\vec{z}=(z_1,\dots,z_m)$ and are tensors with multi-indices $\vec{i}=(i_1,\dots,i_n),\vec{j}=(j_1,\dots,j_n)$ that are symmetric in the (i_a,j_a) indices. Here $\vec{x}=(\vec{y},\vec{z})$ is a collective symbol for all the coordinates in the equation. We demand that these functionals $\mathcal{G}_{n,m}$ be symmetric under the interchange of any two of the $h^{(k)}$ or any two of the $\chi^{(k)}$, which means that $G^{\vec{i}\vec{j}}_{n,m}$ are symmetric under interchange of any two z coordinates and the simultaneous interchange of any two y-coordinates and the associated tensor indices.

We will now use the relations (3.1) to derive constraints on the functions $\mathcal{G}_{n,m}$. A similar analysis was performed in [39] for the Euclidean vacuum. (See also [46].) The general strategy that we adopt will be the following. Under an infinitesimal diff $\times$ Weyl transformation (3.2), h_{ij} transforms as

$$\delta_{\xi,\varphi}h_{ij}=H_{ij}+\frac{1}{\kappa}I_{ij}\ , \tag{3.11}$$

where $H_{ij}=\mathcal{L}_\xi h_{ij}+2\varphi h_{ij}$ is a piece linear in h_{ij} and $I_{ij}=\partial_i\xi^k\delta_{jk}+\partial_j\xi^k\delta_{ik}+2\varphi\delta_{ij}$ is an inhomogeneous piece that comes from the transformation of the background metric.

We then have the variation

$$\delta_{\xi,\varphi}\log Z=\sum_{n,m}\kappa^n(\delta_{\xi,\varphi}\log Z)_{n,m}, \tag{3.12}$$

where we have collected terms according to the expansion in κ:

$$\begin{aligned} (\delta_{\xi,\varphi}\log Z)_{n,m}\equiv&(n+1)\mathcal{G}_{n+1,m}[I,h,\dots h,\chi,\dots\chi]+n\mathcal{G}_{n,m}[H,h,\dots h,\chi,\dots\chi] \\ &+m\mathcal{G}_{n,m}[h,\dots h,\delta\chi,\dots\chi]\ . \end{aligned} \tag{3.13}$$

The constraint (3.3) then leads to identities that relate $\mathcal{G}_{n+1,m}$ to $\mathcal{G}_{n,m}$. These identities are derived in Appendix C. As a consequence of the anomalous Weyl transformation, we obtain a "trace identity"

$$2\delta_{ij}G^{ij\vec{i}\vec{j}}_{n+1,m}(u,\vec{y},\vec{z}) = \left(-2\sum_{a=1}^{n}\delta^{(d)}(u-y_a) + \Delta\sum_{b=1}^{m}\delta^{(d)}(u-z_b)\right)G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z}) + \delta_{m,0}\mathcal{A}^{\vec{i}\vec{j}}_d(u,\vec{y}) \ , \tag{3.14}$$

where

$$\mathcal{A}^{\vec{i}\vec{j}}_d(u,\vec{y}) = \frac{1}{\kappa^n}\frac{\delta^n}{\delta h_{i_1j_1}(y_1)\dots\delta h_{i_nj_n}(y_n)}\mathcal{A}_d(u) \ , \tag{3.15}$$

is an ultra-local term obtained from the expansion of the anomaly $\mathcal{A}_d$ in the fluctuation, which only appears for even d.

The invariance under diffeomorphisms leads to a "divergence identity".

$$\begin{aligned}2\delta_{jk}\partial_i G^{ij\vec{i}\vec{j}}_{n+1,m}(u,\vec{y},\vec{z}) &= -\sum_{b=1}^{m}\frac{\partial}{\partial z_b^k}\left[\delta^{(d)}(u-z_b)G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})\right] \\ &+ \sum_{a=1}^{n}\Bigg[-\frac{\partial}{\partial y_a^k}\left[\delta^{(d)}(u-y_a)G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})\right] + G^{\vec{i}'\vec{j}'}_{n,m}(\vec{y},\vec{z})\left(\delta^{i_a}_{i'_a}\delta^{j_a}_{k}\frac{\partial}{\partial y^{j'_a}} + \delta^{j_a}_{j'_a}\delta^{i_a}_{k}\frac{\partial}{\partial y^{i'_a}}\right)\delta^{(d)}(u-y_a)\Bigg] \ ,\end{aligned} \tag{3.16}$$

where in the bracketed expression, we use $\vec{i}'$ and $\vec{j}'$ to denote the multi-index where (i_a, j_a) has been replaced by (i'_a, j'_a) for the current a in the sum. (See (C.27).)

3.2.3 Conformal symmetry of the coefficient functions

A special role is played by the combinations of diffeomorphism and Weyl transformations that leave the background flat metric invariant. These are conformal transformations. Under these transformations, the inhomogeneous piece in (3.11) vanishes:

$$I_{ij} = \partial_i\xi^k\delta_{jk} + \partial_j\xi^k\delta_{ik} - \frac{2}{d}\delta_{ij}\partial_k\xi^k = 0 \ , \tag{3.17}$$

and corresponds to taking ξ to be a conformal Killing vector.

This imposes conformal invariance on the functions G. More specifically under a conformal transformation $\vec{y} \to \vec{y}', \vec{z} \to \vec{z}'$, we have

$$G^{\vec{i}\vec{j}}_{m,n}(\vec{y}', \vec{z}') = \Big(\prod_{a=1}^{n} R^{i_a}_{i'_a}(y_a) R^{j_a}_{j'_a}(y_a) \Lambda(y_a)^d\Big)\Big(\prod_{b=1}^{m} \Lambda(z_b)^{d-\Delta}\Big) G^{\vec{i}'\vec{j}'}_{m,n}(\vec{y}, \vec{z}) \ , \tag{3.18}$$

where

$$R^i_{i'}(x) = \Lambda(x) J^i_{i'}(x), \qquad J^i_{i'}(x) = \frac{\partial x^i}{\partial x^{i'}}, \qquad \Lambda(x) = |\det J(x)|^{-1/d} \quad , \tag{3.19}$$

see Appendix C.3 for the derivation.

This shows that the coefficient functions $G^{\vec{i}\vec{j}}_{n,m}$ obey the same identities as connected CFT correlators. We can write

$$G^{\vec{i}\vec{j}}_{n,m}(\vec{y}, \vec{z}) \sim \langle T^{i_1 j_1}(y_1) \dots T^{i_n j_n}(y_n) \phi(z_1) \dots \phi(z_m) \rangle^{\text{connected}}_{\text{CFT}} \ , \tag{3.20}$$

where T^{ij} is an operator of spin 2 and dimension d and ϕ is an operator of dimension $d - \Delta$.[2] The reason we put the subscript "connected" is because $G_{n,m}$ are obtained by functional differentiation of the logarithm of Z. The reason we write $\sim$ rather than equality is to indicate that the similarity between the two sides of (3.20) is only restricted to the fact that both sides obey the same Ward identities. We reiterate that (3.20) should be interpreted cautiously beyond this shared property.

3.3 Solution space and theory space

We have therefore reached the following conclusion. Say that we are given a set of functions,

$$\{G^{\vec{i}\vec{j}}_{n,m}(\vec{y}, \vec{z})\} \ , \tag{3.21}$$

[2] From (3.9) and (3.10) it may be seen that the "correlators" differ from the conventional correlators, $\langle \dots T^{ij} \dots \phi \dots \rangle_{\text{conv}} = [\dots \frac{1}{\sqrt{g}} \frac{\delta}{\delta g_{ij}} \dots \frac{1}{\sqrt{g}} \frac{\delta}{\delta \chi} \dots] \log(Z)$, since they are defined without a factor of $\frac{1}{\sqrt{g}}$. So the contact terms that appear in our Ward identities are slightly different even when $g_{ij} = \delta_{ij}$.

for all values of n and m, which satisfy the Ward identities (3.14) and (3.16) and transform under conformal transformations as (3.18). Such a list of functions uniquely specifies a valid solution to the WDW equation when assembled together through (3.9).

Such a list can also be thought of as defining a "theory" with the caveats mentioned above: this theory is a CFT but need not be unitary or local. Moreover, the list of correlators (3.21) does not make reference to other operators in the theory beyond those that correspond to fields in the physical spacetime. In this generalized sense, the space of solutions to the WDW equation is like "theory space".

3.3.1 Relation to the set of Hartle-Hawking wavefunctionals

The Hartle-Hawking no boundary proposal [20] provides a recipe of computing the wavefunctionals that constitute the solution space. Hartle and Hawking proposed that the vacuum wavefunctional should be computed by performing the Euclidean path integral on a manifold with a single boundary. An alternative technique is to compute the path-integral with boundary sources turned on for the same bulk theory in AdS and then continue the answer to dS [21]. The latter technique allows for the easy inclusion of perturbative corrections to the wavefunctional through the computation of AdS correlators.[3]

This computation produces a wavefunctional that we can denote by $\Psi_0[g, \chi]$ and which satisfies the Wheeler-DeWitt equation. It explicitly has the general form we have deduced above; the phase S is the analytic continuation of the divergent part of the on-shell action in AdS and the wavefunctional is obtained by multiplying the phase factor with $Z_0[g, \chi]$ — the analytic continuation of the partition function of the boundary CFT. Since the details of the coefficient functions that enter the partition function depend on

[3] As emphasized in [78], AdS correlators continue to the coefficient functions $G^{\vec{i}\vec{j}}_{n,m}$ in (3.10) and not to correlators on the late-time boundary of dS. These latter correlators are called cosmological correlators and are discussed further in [32]. They must be computed by further squaring and integrating the wavefunctional (see [79] for an example) or by means of the in-in formalism [80].

the bulk Lagrangian, L_{bulk}, we can represent this entire process schematically as

$$L_{\text{bulk}} \longrightarrow \Psi_0[g, \chi] \ . \tag{3.22}$$

The prescription (3.22) leads to an interesting observation. Consider a different bulk Lagrangian, $\widetilde{L}_{\text{bulk}}$ but one which gives rise to the same phase factor and therefore has the same holographic anomaly. It is possible to compute a second wavefunctional using this Lagrangian:

$$\widetilde{L}_{\text{bulk}} \longrightarrow \widetilde{\Psi}_0[g, \chi] \ . \tag{3.23}$$

But since the coefficient functions inside $\Psi_0[g, \chi]$ and $\widetilde{\Psi}_0[g, \chi]$ satisfy the same Ward identities, both wavefunctionals are valid states in either bulk theory. The Hartle-Hawking wavefunctional computed for the Lagrangian $\widetilde{L}_{\text{bulk}}$ can be thought of as an "excited state" in the theory where the bulk interactions are specified by L_{bulk}. Conversely, if one thinks of the bulk theory with the Lagrangian $\widetilde{L}_{\text{bulk}}$ then it is $\widetilde{\Psi}_0[g, \chi]$ that is the vacuum wavefunctional and $\Psi_0[g, \chi]$ that is an excited state.

Therefore the space of states contains the set of Hartle-Hawking wavefunctionals for all possible bulk interactions that give rise to the same holographic anomaly. If one considers a specific bulk theory, then this picks out a specific vector in this space as the one corresponding to the vacuum. But the wavefunctionals for other bulk interactions are still in the state space; they just correspond to non-vacuum states.

Chapter 4

The "small fluctuations" basis for the Hilbert space

In this chapter, we describe an alternate basis for solutions to the WDW equation at late times that is particularly convenient in the limit where $G_N \to 0$. Although we use the adjective "small fluctuations", this basis spans the entire Hilbert space. We show, using this basis, that as $G_N \to 0$, the space of states we have constructed coincides precisely with the Hilbert space constructed by Higuchi [23, 27, 28]. However, our construction also provides a procedure to systematically correct Higuchi's construction at nonzero G_N.

The basis we introduce in this chapter has the additional advantage that it will yield normalizable states in the Hilbert space [32].

4.1 Basis of "small fluctuations"

In chapter 2 and 3 it has been shown that a general solution to the WDW equation is spanned by wavefunctionals of the form

$$\Psi[g, \chi] = e^{iS} \exp\Big[\sum_{n,m} \kappa^n \mathcal{G}_{n,m}\Big] , \tag{4.1}$$

where, as above, each $\mathcal{G}_{n,m}$ takes the form of a conformally invariant “coefficient function” integrated with the fluctuations of the metric and the matter fields. The coefficient functions must obey a set of Ward identities and can be identified as correlation functions of n-insertions of the “stress tensor” and m-insertions of a scalar operator with dimension $d-\Delta$ in a non-unitary conformal-field theory.

Here, and in what follows, we do not display the arguments of $\mathcal{G}_{n,m}$ to condense the notation. It is understood that all n tensor arguments correspond to the metric fluctuation h_{ij} and all m scalar arguments correspond to the matter fluctuation χ.

Consider a state with a specific choice of functionals, $\mathcal{G}_{n,m}$. Now consider another set of functionals $\widetilde{\mathcal{G}}_{n,m}$, which also satisfy the Ward identities of chapter 3. Then the combination

$$\mathcal{G}^{\lambda}_{n,m} = (1-\lambda)\mathcal{G}_{n,m} + \lambda\widetilde{\mathcal{G}}_{n,m} \ , \tag{4.2}$$

also satisfies the identities of chapter 3. The linear combination chosen above ensures that $\mathcal{G}^{\lambda}_{n,m}$ satisfies the Ward identities with the same trace anomaly term. Therefore, the wavefunctional

$$\Psi_{\lambda}[g,\chi] = e^{iS}\exp\Big[\sum_{n,m}\kappa^n\mathcal{G}^{\lambda}_{n,m}\Big] \ , \tag{4.3}$$

also satisfies the WDW equation asymptotically. Since the solution space is linear this means that

$$\frac{\partial\Psi_{\lambda}[g,\chi]}{\partial\lambda}\bigg|_{\lambda=0} = \sum_{n,m}\kappa^n\Big(\widetilde{\mathcal{G}}_{n,m} - \mathcal{G}_{n,m}\Big)\Psi[g,\chi] \ , \tag{4.4}$$

is also a valid state. The combination above will appear frequently and so we define the notation

$$\begin{aligned} \delta\mathcal{G}_{n,m} &\equiv \widetilde{\mathcal{G}}_{n,m} - \mathcal{G}_{n,m} \\ &= \frac{1}{n!m!}\int d\vec{y}d\vec{z}\,\delta G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})h_{i_1j_1}(y_1)\ldots h_{i_nj_n}(y_n)\chi(z_1)\ldots\chi(z_m) \ . \end{aligned} \tag{4.5}$$

We can think of the states (4.4) as corresponding to “small fluctuations” about the base state $\Psi[g,\chi]$. Nevertheless, states of the form (4.4) provide a complete basis for

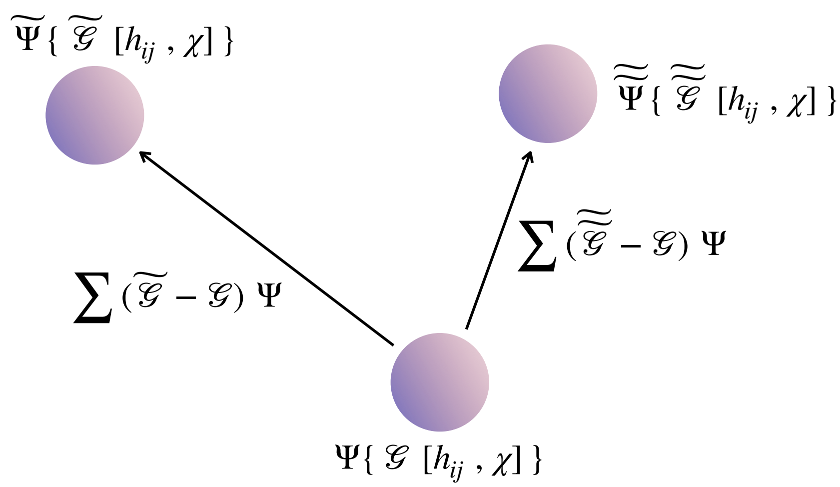

Figure 4.1: *We schematically represent the "small fluctuation" states around a base state* Ψ.

the Hilbert space provided we consider all possible changes $\delta\mathcal{G}_{n,m}$. We can refer to this as the "small fluctuations" basis for the Hilbert space.

The construction above can be performed about any base state but to make contact with the existing literature we will, henceforth, choose the base state in (4.1) to be the Hartle-Hawking state, $\Psi_0[g,\chi]$. The basis above then naturally corresponds to the basis of fluctuations about the Euclidean vacuum.

A few comments are in order.

1. Naively, it might appear possible to take the functionals $\mathcal{G}_{n,m}$ and $\widetilde{\mathcal{G}}_{n,m}$ to coincide for all value of n, m except for some particular values of $n = n_0, m = m_0$. However, this is not possible as both sets of functionals must satisfy the Ward identities. This relates the longitudinal components and the trace of $\widetilde{\mathcal{G}}_{n+1,m}$ to $\widetilde{\mathcal{G}}_{n,m}$ for each n, m by equations (3.14) and (3.16). Therefore

$$\delta\mathcal{G}_{n,m} \neq 0 \quad \Rightarrow \quad \delta\mathcal{G}_{n+1,m} \neq 0\ . \tag{4.6}$$

2. Nevertheless, note that the right hand side of (4.6), $\delta\mathcal{G}_{n+1,m}$ is not completely fixed by the left hand side, $\delta\mathcal{G}_{n,m}$ This is because the Ward identities only fix the

longitudinal components and the trace in $\widetilde{\mathcal{G}}_{n+1,m}$. Except for $n + m \leq 2$ there are an infinite number of possible ways to satisfy the Ward identities.

3. On the other hand, the Ward identities do not prevent the possibility that $\delta\mathcal{G}_{n,m} = 0$ for $n < n_0, m < m_0$ for some choice of n_0, m_0 but that $\delta\mathcal{G}_{n,m} \neq 0$ for other values of n, m. This might appear slightly puzzling since, in a CFT, all higher-point functions are fixed by three-point functions. However, the coefficient functions can be thought of as correlators of "stress-tensor" and the operators dual to the matter fields. The wavefunctional does not directly contain terms that correspond to correlators of other primary operators. Within this restricted class of correlators, it is usually possible to change higher-point correlation functions without changing lower-point functions although some correlators such as the three-point function of a stress-tensor and two scalars are completely fixed in terms of lower-point functions [81].

4.2 The nongravitational limit

The basis above is particularly convenient in the nongravitational limit. When $\kappa \to 0$, the constraints imposed by the Ward identities become trivial. As we show below, this allows us to obtain a precise correspondence with Higuchi's basis [23] of dS-invariant states.

Consider two sets of functionals $\mathcal{G}_{n,m}$ and $\widetilde{\mathcal{G}}_{n,m}$ which differ for some particular $m = m_0$ at $n = n_0$ but coincides for all lower-point correlators:

$$\delta\mathcal{G}_{n,m} = 0, \quad \forall n < n_0; \qquad \delta\mathcal{G}_{n_0,m} = 0, \quad \forall m \neq m_0 \ , \qquad \delta\mathcal{G}_{n_0,m_0} \neq 0 \ . \tag{4.7}$$

The Ward identities imply that the higher-point functionals cannot coincide. However, all of these come with a higher power of κ. Therefore we can consider the state

$$\frac{1}{\kappa^{n_0}} \left. \frac{\partial \Psi_\lambda[g,\chi]}{\partial \lambda} \right|_{\lambda=0} = \left(\delta \mathcal{G}_{n_0,m_0} + \mathrm{O}(\kappa)\right) \Psi[g,\chi] \ , \tag{4.8}$$

which clearly has a good limit as $\kappa \to 0$. The notable feature above is that all the higher-order terms in (4.4) have disappeared.

It is useful to recast this in slightly different notation. Representing the Euclidean vacuum by $|0\rangle$, and choosing the wavefunctional $\Psi[g,\chi]$ to correspond to this state, we see that the set of states that satisfy the constraints in the $\kappa \to 0$ limit can be written as

$$|\Psi_{\rm ng}\rangle = \int d\vec{y} d\vec{z}\, \delta G^{\vec{i}\vec{j}}_{n_0,m_0}(\vec{y},\vec{z}) h_{i_1 j_1}(y_1) \ldots h_{i_{n_0} j_{n_0}}(y_{n_0}) \chi(z_1) \ldots \chi(z_{m_0}) |0\rangle \ . \tag{4.9}$$

Our conclusion can be summarized in the following two points.

1. The set of valid states in the nongravitational limit can be obtained by studying gravitational and matter fluctuations at late times, integrating them with a conformally invariant function on the late-time slice and acting on the Euclidean vacuum. This function is the difference of any two functions that obey the Ward identities of subsection 3.2.2. So it also obeys the Ward identities but without an inhomogeneous term.

2. The smearing function in (4.9) is not arbitrary and is constrained by conformal invariance. This means that, even in the nongravitational limit, the effect of the constraints does not trivialize. This is consistent with the idea that when one takes the zero-coupling limit of a gauge theory, it is still necessary to impose the Gauss law on the Hilbert space.

Examples of states in the nongravitational limit: We now provide a few examples to help elucidate the idea above. To lighten the notation, we provide examples of states obtained by the action of matter-sector operators. It is simple to generalize this to consider states of gravitons, which exist for $d > 2$.

Conformal symmetry sharply constrains the Hilbert space at small “particle number.” Here, by particle number, we refer to the number of fields that must act on the Euclidean vacuum to produce the state. There is a unique two-particle state with gravitons or with matter excitations corresponding to the fact that the two-point coefficient function is fixed up to an overall constant. Similarly, there is a unique three-particle state with scalar excitations and two possible three-particle states with graviton excitations. There exist an infinite number of four-particle states parameterized by functions of conformally invariant cross ratios.

1. **Two-particle states.** The unique two-particle matter state has the form

$$|\chi\chi\rangle = \int d^d x_1 d^d x_2 \frac{1}{|x_1 - x_2|^{2(d-\Delta)}} \chi(x_1)\chi(x_2)|0\rangle \ . \tag{4.10}$$

   Similarly, for $d > 2$, one can construct nontrivial two-particle states of free gravitons.

2. **Three-particle states.** The three-point function of scalar operators is also fixed uniquely by conformal invariance up to an overall normalization. Therefore, we find the unique three-particle state

$$|\chi\chi\chi\rangle = \int d^d x_1 d^d x_2 d^d x_3 \frac{1}{|x_1 - x_2|^{d-\Delta}|x_2 - x_3|^{d-\Delta}|x_1 - x_3|^{d-\Delta}} \chi(x_1)\chi(x_2)\chi(x_3)|0\rangle \ . \tag{4.11}$$

3. **Four-particle states.** Four-point functions that satisfy the Ward identities are undetermined up to a function of the conformal cross ratios. There is an infinite

number of four-particle states that can be written in the form

$$|\chi\chi\chi\chi\rangle = \int d^d x_1 \ldots d^d x_4 \, Q\Big(\frac{x_{12}x_{34}}{x_{13}x_{24}}, \frac{x_{12}x_{34}}{x_{23}x_{14}}\Big) \prod_{i<j} |x_{ij}|^{-\frac{2(d-\Delta)}{3}} \chi(x_1)\chi(x_2)\chi(x_3)\chi(x_4)|0\rangle \,, \tag{4.12}$$

where Q is an arbitrary function.

Apart from graviton excitations, it is also possible to consider states with both graviton and matter excitations. These can be constructed using a procedure similar to the one above. There is no state with one graviton and two matter particles. This is because, as noted above, the corresponding correlation function, including its normalization, is completely fixed by the Ward identities and the two-point matter correlation function [81].

4.3 Correspondence with Higuchi's construction

We now show that the $\kappa \to 0$ limit of our construction described above corresponds precisely to Higuchi's construction of the Fock space for weakly-coupled gravity in de Sitter space. For simplicity, we discuss Higuchi's construction for a scalar field.

4.3.1 Review of Higuchi's proposal

So far, we have been careful to discuss the metric and fields only on a single Cauchy slice. To make contact with Higuchi's construction we will briefly discuss the properties of fields in *spacetime*.

Consider a quantum field theory with a scalar field of mass m, χ, propagating in a background de Sitter geometry with spacetime metric

$$ds^2 = -dt^2 + \cosh^2 t \, d\Omega_d^2; \qquad d\Omega_d^2 = \frac{4dx^2}{(1+|x|^2)^2}. \tag{4.13}$$

Then χ can be expanded in terms of solutions to the equations of motion that, at late times, have the asymptotic behaviour [67, 82]

$$\chi^{\text{phys}}(t,x) \underset{t\to\infty}{\longrightarrow} e^{-\Delta t}\left(\frac{1+|x|^2}{2}\right)^{\Delta}\chi(x) + e^{-\bar{\Delta} t}\left(\frac{1+|x|^2}{2}\right)^{\bar{\Delta}}\bar{\chi}(x)\ , \tag{4.14}$$

where Δ and $\bar{\Delta}$ are the two solutions of the equation (2.50). We are interested in the operator

$$\chi(x) = \lim_{t\to\infty} e^{\Delta t}\left(\frac{1+|x|^2}{2}\right)^{-\Delta}\chi^{\text{phys}}(t,x)\ , \tag{4.15}$$

which is well defined even when Δ has an imaginary part since even in that case the expression $e^{(\Delta-\bar{\Delta})t}\left(\frac{1+|x|^2}{2}\right)^{\bar{\Delta}-\Delta}\bar{\chi}(x)$ can be neglected by the Riemann-Lebesgue lemma. These rescaled late-time operators are precisely the ones that we have been studying in the previous chapters. This can be seen by comparing (4.15) with (3.8).

Starting with the Euclidean vacuum we see that states of the form

$$|\Psi_{\text{seed}}\rangle = \int d\vec{x}\, \psi(\vec{x})\chi(x_1)\ldots\chi(x_n)|0\rangle \tag{4.16}$$

span the Hilbert space in a nongravitational QFT where ψ is a square-integrable smearing function. More details and an oscillator construction can be found in [27, 28, 67].

When the theory is coupled to gravity, it is necessary to impose the gravitational Gauss law even in the limit of arbitrarily weak coupling. The Gauss law requires that states be invariant under the de Sitter-isometry group $\mathrm{SO}(1, d+1)$ [22, 25, 26]. This constraint can also be derived by integrating the Hamiltonian constraint (2.4) with the Killing vectors of dS. But, except for the vacuum, no state of the form (4.16) satisfies this constraint.

Higuchi's proposal [23] was to consider the space of states obtained by “averaging” such seed states over the de Sitter isometry group

$$|\Psi\rangle = \int dU U|\Psi_{\text{seed}}\rangle\ , \tag{4.17}$$

where U is the unitary operator that implements the action of the de Sitter isometries in the quantum field theory and dU is the Haar measure on this unitary group. By construction we now have

$$U|\Psi\rangle = |\Psi\rangle \tag{4.18}$$

for the action of any unitary element of the symmetry group.

The states $|\Psi\rangle$ are not normalizable in the original Hilbert space but Higuchi proposed a modified norm

$$(\Psi, \Psi) = \frac{1}{\mathrm{vol}(\mathrm{SO}(1, d+1))}\langle\Psi|\Psi\rangle = \int dU \langle\Psi_{\text{seed}}|U|\Psi_{\text{seed}}\rangle. \tag{4.19}$$

In [83], it was shown that this procedure can be understood in terms of imposing the equivalence relation $|\Psi_{\text{seed}}\rangle \sim U|\Psi_{\text{seed}}\rangle$ on the original Hilbert space. The final Hilbert space of equivalence classes is the same as the Hilbert space obtained by Higuchi's construction.

4.3.2 Invariance of states under $\mathrm{SO}(1, d+1)$

We now show that the states (4.9) that we have found in the nongravitational limit are invariant under the de Sitter isometries. To simplify the notation, we restrict to scalar states of the form

$$|\Psi\rangle = \int d\vec{x}\, \delta G_{0,m}(x_1, \ldots, x_m)\chi(x_1)\ldots\chi(x_m)|0\rangle\ . \tag{4.20}$$

The inclusion of graviton states is simple but just requires us to keep track of some additional rotation matrices below.

The de Sitter isometries map the late-time boundary back to itself and act as conformal Killing vectors on it. (See [84] for a pedagogical explanation.) Their finite action at late times can be read off from (4.13) and is

$$\begin{aligned}
\text{translations}: &\quad \tilde{x}^i = x^i + c^i, && \tilde{t} = t + \log\frac{(1+|\tilde{x}|^2)}{(1+|x|^2)}; \\
\text{rotations}: &\quad \tilde{x}^i = R^i_j x^j, && \tilde{t} = t; \\
\text{dilatations}: &\quad \tilde{x}^i = \lambda x^i, && \tilde{t} = t + \log\frac{(1+|\tilde{x}|^2)}{\lambda(1+|x|^2)}; \\
\text{SCTs}: &\quad \tilde{x}^i = \frac{x^i - \beta^i |x|^2}{1 - 2(\beta\cdot x) + |\beta|^2|x|^2}, && \tilde{t} = t + \log\frac{(1+|\tilde{x}|^2)}{(1+|x|^2)(1-2(\beta\cdot x)+|\beta|^2|x|^2)},
\end{aligned} \tag{4.21}$$

where c^i and β^j are constant vectors, R^i_j is a constant rotation matrix and λ is a real number. Here we have neglected terms that vanish exponentially in t since such terms are unimportant on the late-time boundary. We note that the transformations above satisfy

$$e^{\tilde{t}-t} = \Lambda(x)\,\frac{1+|\tilde{x}|^2}{1+|x|^2}\ , \tag{4.22}$$

where

$$\Lambda(x) = \left|\det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right)\right|^{-1/d}\ . \tag{4.23}$$

Using (4.15), we see that the unitary operator that implements this transformation on the fields acts as

$$U\chi(\tilde{x})U^\dagger = \Lambda(x)^\Delta \chi(x)\ . \tag{4.24}$$

Therefore, using the transformation of δG under conformal transformations as given in (3.18), the state above transforms as

$$\begin{aligned}
U|\Psi\rangle &= \int d^d\tilde{x}_1 \dots d^d\tilde{x}_m\, \delta G(\tilde{x}_1,\dots,\tilde{x}_m) U\chi(\tilde{x}_1)U^\dagger \dots U\chi(\tilde{x}_m)U^\dagger|0\rangle \\
&= \int d^d\tilde{x}_1 \dots d^d\tilde{x}_m \Big(\prod_{i=1}^m \Lambda(x_i)^{d-\Delta}\Big)\delta G(x_1,\dots,x_m)\Big(\prod_{i=1}^m \Lambda(x_i)^\Delta\Big)\chi(x_1)\dots\chi(x_m)|0\rangle \\
&= \int d^d x_1 \dots d^d x_m\, \delta G(x_1,\dots,x_m)\chi(x_1)\dots\chi(x_m)|0\rangle = |\Psi\rangle,
\end{aligned} \tag{4.25}$$

and is therefore invariant. In the equalities above, we have used that $d^d\tilde{x} = \Lambda(x)^{-d} d^d x$ which ensures that the eventual expression has no factor of Λ.

4.3.3 Lifting seed states

It is also possible to obtain the seed states corresponding to (4.9). The intuition is that the expression (4.9) has an implicit integral over the conformal group that can be pulled out to yield the seed state. This relies on the geometric observation that, in any number of dimensions, the conformal group can be used to fix three points leaving behind an unfixed $\mathrm{SO}(d-1)$ that leaves those three points invariant.

This can be made precise by considering the quantity

$$f = \int_{\mathrm{SO}(1,d+1)} d\gamma\, \delta^{(d)}(\hat{x}_1 - \gamma x_1)\delta^{(d)}(\hat{x}_2 - \gamma x_2)\delta^{(d)}(\hat{x}_3 - \gamma x_3)\ . \tag{4.26}$$

f must be invariant under an arbitrary conformal transformation $x_k \to \gamma x_k$ (as $\hat{x}_k$ are kept unchanged) by invariance of the Haar measure $d\gamma$. Since the only conformally invariant function of three points is a constant, f cannot depend on x_k. Under transformations $\hat{x}_k \to \gamma\hat{x}_k$, we have $f \to \Big(\Lambda(\hat{x}_1)\Lambda(\hat{x}_2)\Lambda(\hat{x}_3)\Big)^d f$. This fixes the dependence of f on $\hat{x}_k$ and, by normalizing the measure appropriately, we can set

$$f = \mathrm{vol}(\mathrm{SO}(d-1))\Big(|\hat{x}_1 - \hat{x}_2||\hat{x}_2 - \hat{x}_3||\hat{x}_3 - \hat{x}_1|\Big)^{-d} \tag{4.27}$$

From this we obtain the identity

$$\begin{aligned} 1 \;\; = \;\; & \frac{1}{f}\int_{\mathrm{SO}(1,d+1)} d\gamma\, \delta^{(d)}(x_1 - \gamma^{-1}\hat{x}_1)\delta^{(d)}(x_2 - \gamma^{-1}\hat{x}_2)\delta^{(d)}(x_3 - \gamma^{-1}\hat{x}_3) \\ & \times(\Lambda(x_1)\Lambda(x_2)\Lambda(x_3))^d, \end{aligned} \tag{4.28}$$

using $\delta^{(d)}(\tilde{x} - \gamma x) = \delta^{(d)}(x - \gamma^{-1}\tilde{x})\Lambda(x)^d$.

Inserting this in the expression (4.20) removes the integrals over x_1, x_2, x_3 and gives

$$\begin{aligned}
|\Psi\rangle &= \frac{1}{f}\int_{\mathrm{SO}(1,d+1)} d\gamma \int d^d x_4 \dots d^d x_m \, \delta G_m(\gamma^{-1}\hat{x}_1, \gamma^{-1}\hat{x}_2, \gamma^{-1}\hat{x}_3, x_4, \dots, x_m) \\
&\qquad \times (\Lambda(\gamma^{-1}\hat{x}_1)\Lambda(\gamma^{-1}\hat{x}_2)\Lambda(\gamma^{-1}\hat{x}_3))^d \chi(\gamma^{-1}\hat{x}_1)\chi(\gamma^{-1}\hat{x}_2)\chi(\gamma^{-1}\hat{x}_3)\chi(x_4)\dots\chi(x_m)|0\rangle \\
&= \frac{1}{f}\int_{\mathrm{SO}(1,d+1)} d\gamma \int d^d x_4 \dots d^d x_m \Big(\prod_{i=4}^{m} \Lambda(x_i)^{-d}\Big)\delta G_m(\hat{x}_1, \hat{x}_2, \hat{x}_3, \tilde{x}_4, \dots, \tilde{x}_m) \\
&\qquad \times U\chi(\hat{x}_1)U^{-1}U\chi(\hat{x}_2)U^{-1}U\chi(\hat{x}_3)U^{-1}U\chi(\tilde{x}_4)U^{-1}\dots U\chi(\tilde{x}_m)U^{-1}|0\rangle \\
&= \frac{1}{f}\int_{\mathrm{SO}(1,d+1)} d\gamma \int d^d \tilde{x}_4 \dots d^d \tilde{x}_m \, \delta G_m(\hat{x}_1, \hat{x}_2, \hat{x}_3, \tilde{x}_4, \dots, \tilde{x}_m) \\
&\qquad\qquad \times U\chi(\hat{x}_1)\chi(\hat{x}_2)\chi(\hat{x}_3)\chi(\tilde{x}_4)\dots\chi(\tilde{x}_m)|0\rangle \ ,
\end{aligned} \tag{4.29}$$

where in the second step we have applied γ to all the variables and used the conformal transformation properties of δG_m and χ to simplify the answer.

We recognize that this takes the form of a group average

$$|\Psi\rangle = \int dU \, U|\Psi_{\mathrm{seed}}\rangle \quad , \tag{4.30}$$

where the seed state is given by

$$\begin{aligned}
|\Psi_{\mathrm{seed}}\rangle \quad &= \quad \frac{1}{f}\int d^d x_4 \dots d^d x_m \delta G_m(\hat{x}_1, \hat{x}_2, \hat{x}_3, x_4, \dots, x_m) \\
&\qquad \times\chi(\hat{x}_1)\chi(\hat{x}_2)\chi(\hat{x}_3)\chi(x_4)\dots\chi(x_m)|0\rangle \ .
\end{aligned} \tag{4.31}$$

As a result, up to normalization, we obtain a valid seed state by simply dropping the integral over three points in (4.9) and fixing those points to arbitrary positions.

Chapter 5

Conclusion and future directions

In this thesis, we studied solutions to the WDW equation in asymptotically de Sitter space. A natural clock in de Sitter space is provided by the volume of the Cauchy slices. We found that, in the limit of large volume, all solutions could be written as the product of a universal phase factor multiplied by a diffeomorphism-invariant functional with simple Weyl transformation properties. This result is derived in chapter 2 and we argued that the structural form of the solution is valid at all orders in perturbation theory. The Euclidean vacuum is well known to have these properties but the new result is that all states in the theory have these properties.

In chapter 3, we showed that a solution could be specified by providing a list of coefficient functions that obey the same constraints as correlation functions of a CFT. These functions are related to one another by Ward identities. A specification of these functions provides a complete description of the state but it can also be said to specify a "theory". In this sense, the space of solutions to the WDW equation is similar to theory space.

In chapter 4, we rewrote these solutions in a basis of "excitations" about the Euclidean vacuum. Here, each solution is written as a series of multilinear functionals of the metric and other fields multiplied with the wavefunctional for the Euclidean vacuum. These excitations must again obey the constraints of conformal invariance and the Ward identities. It is shown in chapter 4 that, in the nongravitational limit, these states reduce to those constructed by Higuchi through "group averaging". Therefore, our procedure not only provides a systematic justification for Higuchi's result, it specifies how the result should be generalized away from zero gravitational coupling.

In our analysis, we have assumed that the Cauchy slice has the topology of the sphere S^d. From a technical perspective, there does not appear to be any immediate obstruction to generalizing our asymptotic solution to alternate topologies but some of the interesting physics might require us to go beyond perturbation theory. It would be interesting to examine the effects of change in topology and to understand whether a nonperturbative analysis reveals additional restrictions on the Hilbert space.

In an accompanying work [32] [1], we describe a norm on the space of solutions to the WDW equation. The norm we propose is to simply average the square of the absolute value of the wavefunctional over the space of all possible metrics and matter fluctuations. In [32], we show that the states examined in chapter 4 yield a normalizable basis for the Hilbert space. We also show that this prescription for the norm reduces, in the nongravitational limit, to the group-averaged norm proposed by Higuchi but differs at finite κ.

In [32], we define and study "cosmological correlators" in a gravitational theory. We find that these correlators display a remarkable property: knowledge of cosmological correlators in an arbitrary small region of the late-time spatial slice suffices to fix them everywhere, even in an arbitrary state. This result relies on the observation that all states, and not just the Euclidean vacuum, are covariant under scale transformations and translations. This provides a generalization of the principle of "holography of information" — previously explored in AdS and in flat space [6–12] — to asymptotically de Sitter space.

One interesting implication of our analysis is that all states in the Hilbert space share the symmetries of the Hartle-Hawking state. The inflationary era was presumably

[1] We give a brief summary of this work in appendix D.

described by a state from this Hilbert space. On the one hand, this strengthens arguments like [85] that are based only on symmetries. But it makes the effort to extract early-universe physics from inflationary correlators [86] more interesting since one must contend not only with inflationary physics but also the possible states of the system.

In our approach, we quantized the system before imposing the constraints. However, the reverse order is also possible - the constraints could be applied prior to quantization [87–89]. In particular, for dS_3 we could write pure gravity action in terms of Chern-Simons theory with gauge group $SO(3,1)$. We can solve for the constraints of the Chern-Simons theory first, then quantize. It would be nice to verify whether the results obtained through this method match with our existing solutions.

In [1] we have solved the WDW equation at the leading order of the large volume of the Cauchy slice. At the sub-leading order, a specific double derivative term in the metric canonical momenta will also contribute. This term exactly looks like $T\overline{T}$ term [90] in $(2+1)$ dimension or T^2 term [91] in higher dimension [92–95]. It would be interesting if one finds that, in finite volume, the solution to the WDW equation can be understood as $T\overline{T}$ or T^2 deformed "CFT" partition functions.

As previously noted, the Wheeler-DeWitt (WDW) equation in de Sitter spacetime does not inherently reference time. While we have used the volume of the Cauchy slice, alternative definitions of a clock are also viable [96–98]. In [99], it was demonstrated that when the tachyon effective field theory is coupled with gravity, the WDW equation simplifies to the Schrödinger equation which contains time evolution with respect to the tachyon field. It would be intriguing to explore the quantum version of this model and see whether it is consistent with our results.

Appendix A

Review of Wheeler-DeWitt equation

In this appendix, we revisit the derivation of the Wheeler-DeWitt equation [17]. While it is commonly available in the literature, we present it using simplified notation specifically tailored to the context of de Sitter spacetime. For various reference, we have followed parts of [58], [96], [100], [101].

A.1 Asymptotically de Sitter space

We can represent de Sitter space of $(d+1)$ dimension, namely dS_{d+1} as a hypersurface embedded inside a Minkowski space $R^{1,d+1}$ (with coordinate $\{X^\mu\}; \mu = 0, 1, \ldots, d+1$) as following

$$-(X^0)^2 + (X^1)^2 + ... + (X^d)^2 + (X^{d+1})^2 = 1 \ . \tag{A.1}$$

The de Sitter spacetime can be parametrized using various coordinate patches. For our analysis, we select the global patch, defined as

$$\begin{aligned} X^0 &= \sinh t \\ X^i &= \cosh t \; \Omega_i \quad ; \quad i = 1, \ldots, d+1 \; . \end{aligned} \tag{A.2}$$

Here, Ω_i denotes the coordinates on a d-dimensional sphere embedded inside $(d+1)$-dimensional Euclidean space. The metric in the global patch is given by

$$ds^2 = -dt^2 + \cosh^2 t \; d\Omega_d^2 \; . \tag{A.3}$$

In this scenario, the spatial slice is a d-dimensional sphere with a volume of $\cosh^d t$. It is evident that, at both large positive and negative times, the volume of the spatial slice becomes exponentially large.

When we incorporate gravity, we consider the notion of asymptotic de Sitter spacetime. This implies that while the metric may fluctuate within the bulk, it converges to the global de Sitter metric described by equation (A.3) as $t \to \infty$ or $t \to -\infty$.

We take massive scalar matter minimally coupled with standard Einstein gravity. The total action is

$$S = S_{\text{gravity}} + S_{\text{matter}} + S_{\text{int}} \; . \tag{A.4}$$

There might be more interaction terms present, which are denoted by S_{int}. However the details of interaction will not be relevant for the results discussed in this thesis. The gravity action is given by

$$S_{\text{gravity}} = \frac{1}{2\kappa^2} \int d^{d+1}\hat{x} \sqrt{-\hat{g}} (\hat{R} - 2\Lambda) + S_{\text{GHY}} \; . \tag{A.5}$$

The cosmological constant Λ, which is positive, is chosen to be $\frac{d(d-1)}{2}$ in units where the de Sitter radius is normalized to unity. The Gibbons-Hawking-York term is set to cancel any boundary contribution (involving the extrinsic curvature) resulting from the variation of the Ricci scalar part of the action. These boundary contributions are only at the spatial surfaces at very early and very late time. Since the spatial slices are compact, there is no boundary contribution from the boundary of the spatial slices.

The action of minimally coupled scalar field is

$$S_{\text{matter}} = -\frac{1}{2}\int d^{d+1}\hat{x}\sqrt{-\hat{g}}\Big(\hat{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + m^2\chi^2\Big)\ . \tag{A.6}$$

Einstein equation

Taking variation with respect to the metric, we yield the Einstein equation in asymptotically de Sitter spacetime as[1]

$$\hat{G}_{\mu\nu} + \hat{g}_{\mu\nu}\Lambda = \kappa^2\hat{T}_{\mu\nu}\ , \tag{A.7}$$

where, the matter stress tensor $\hat{T}_{\mu\nu}$ is defined as follows

$$\hat{T}_{\mu\nu} \equiv -\frac{2}{\sqrt{-\hat{g}}}\frac{\delta S_{\text{matter}}}{\delta\hat{g}^{\mu\nu}} = \partial_\mu\chi\partial_\nu\chi - \frac{1}{2}\hat{g}_{\mu\nu}\Big(\hat{g}^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi + m^2\chi^2\Big)\ . \tag{A.8}$$

A.2 ADM split of metric

The formulation of general relativity is independent of the choice of coordinates. However, there exists a particular slicing [58, 59] of spacetime which makes the analysis of constraints manifest.

Foliation of spacetime

We take a $(d+1)$-dimensional asymptotically de Sitter spacetime, initially parametrized by arbitrary coordinates $\{\hat{x}^\mu\}$. This spacetime is then foliated with a family of d-dimensional spacelike hypersurfaces, denoted as Σ_t. Each hypersurface is defined by the

[1]We have excluded the S_{int} part here.

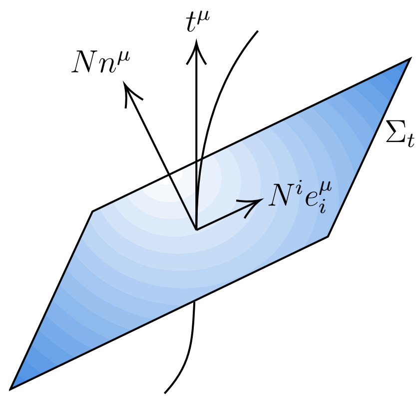

Figure A.1: *We consider a generic foliation of spacetime with spaclike hypersurfaces labeled with scalar function t. The tangent t^μ to the curves with affine parameter t is not necessarily orthogonal to the hypersurface. Its orthogonal component is specified by Nn^μ, whereas its component tangent to the hypersurface is given by $N^i e_i^\mu$.*

value of some scalar function $t(\hat{x}^\mu)$ over the spacetime as follows.

$$\Sigma_t = \{\hat{x}^\mu : t(\hat{x}^\mu) = t\} \tag{A.9}$$

This foliation allows us to write a coordinate system $\{t, x^i\}$, where $\{x^i\}$ are the coordinates on the hypersurface. We can consider the curves with affine parameter t (defined as $\hat{x}^\mu_{x^i}(t) \equiv \hat{x}^\mu(t, x^i)$) connecting different hypersurfaces with same value of $\{x^i\}$. Moving from one hypersurface to another, the t parameter changes, providing a notion of time evolution. The initial set of arbitrary coordinates $\{\hat{x}^\mu\}$ can be expressed in terms of $\{t, x^i\}$.

We can write tangent vectors to the surface as

$$e_i^\mu = \left.\frac{\partial \hat{x}^\mu(t, x^i)}{\partial x^i}\right|_t , \tag{A.10}$$

whereas, the vector tangent to the t curve is given by,

$$t^\mu = \left.\frac{\partial \hat{x}^\mu(t, x^i)}{\partial t}\right|_{x^i} . \tag{A.11}$$

Normal. We introduce the normal n_μ to the constant t hypersurface. Since the hypersurface is spacelike, the normal is timelike. We normalize it as $n^\mu n_\mu = -1$. As a normal to the surface it satisfies

$$n_\mu \propto \partial_\mu t \quad ; \quad n_\mu e^\mu_i = 0 \ . \tag{A.12}$$

We take the normalization factor N which is also referred as the "lapse" function as

$$n_\mu = -N \partial_\mu t \ . \tag{A.13}$$

In general, t curves may not be always orthogonal to the hypersurfaces. Then the vector t^μ would have components along the surface. So in principle, we could have

$$t^\mu = N n^\mu + N^i e^\mu_i \ , \tag{A.14}$$

where, we have introduced another set of proportionality factors N^i, also named as "shift" functions. Hence, we can get

$$d\hat{x}^\mu = t^\mu dt + e^\mu_i dx^i = (N dt) n^\mu + (dx^i + N^i dt) e^\mu_i \ . \tag{A.15}$$

Metric on the hypersurface. The metric $g_{\mu\nu}$ on the spatial hypersurface can be written in terms of the original metric $\hat{g}_{\mu\nu}$ and the normal n_μ as

$$\hat{g}_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \ . \tag{A.16}$$

It satisfies the orthogonality relation[2],

$$n^\mu g_{\mu\nu} = 0 \ . \tag{A.17}$$

[2]all $\{\mu.\nu...\}$ contractions are done with respect to $\hat{g}_{\mu\nu}$ metric.

We can use g^{μ}_{ν} to project objects onto the hypersurface. On the hypersurface we have already defined the local coordinates $\{x^i\}$ with $e^{\mu}_i = \frac{\partial \hat{x}^{\mu}}{\partial x^i}$. In terms of corrdinates on the hypersurface we write

$$g_{ij} = \hat{g}_{\mu\nu} e^{\mu}_i e^{\nu}_j = g_{\mu\nu} e^{\mu}_i e^{\nu}_j \ . \tag{A.18}$$

ADM Metric

With this new coordinate system $\{t, x^i\}$, we can express the line element as

$$ds^2 = \hat{g}_{\mu\nu} d\hat{x}^{\mu} d\hat{x}^{\nu} = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \ . \tag{A.19}$$

This is the ADM metric. In matrix notation, we write it

$$\begin{bmatrix} \hat{g}_{00} & \hat{g}_{0j} \\ \hat{g}_{i0} & \hat{g}_{ij} \end{bmatrix} = \begin{bmatrix} -N^2 + N_k N^k & N_j \\ N_i & g_{ij} \end{bmatrix} \ . \tag{A.20}$$

The inverse metric can be found by solving $\hat{g}^{\mu\nu}\hat{g}_{\nu\rho} = \delta^{\mu}_{\rho}$, which is given by

$$\begin{bmatrix} \hat{g}^{00} & \hat{g}^{0j} \\ \hat{g}^{i0} & \hat{g}^{ij} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{N^2} & \dfrac{N^j}{N^2} \\ \dfrac{N^i}{N^2} & g^{ij} - \dfrac{N^i N^j}{N^2} \end{bmatrix} \ . \tag{A.21}$$

In the above expression, we have used the notation

$$g^{ik} g_{jk} = \delta^i_k \quad ; \quad N_i = g_{ij} N^j \ . \tag{A.22}$$

The determinant of the $(d+1)$-dimensional metric, $\hat{g}$ can be related to the determinant of the d-dimensional metric, g as

$$\sqrt{-\hat{g}} = N\sqrt{g} \ . \tag{A.23}$$

Extrinsic Curvature

Besides the intrinsic curvature (Ricci tensor), the hypersurface has extrinsic curvature, which describes how it is embedded inside the full spacetime. We define it as[3]

$$K_{\mu\nu} = -\frac{1}{2} g^{\alpha}_{\mu} g^{\beta}_{\nu} \mathcal{L}_n g_{\alpha\beta} \ . \tag{A.24}$$

Here, the Lie derivative is taken along the direction of n_μ, the normal to the hypersurface. In terms of hypersurface indices, we can also write

$$K_{ij} = K_{\mu\nu} e^{\mu}_{i} e^{\nu}_{j} \ . \tag{A.25}$$

The time derivative of the spatial metric g_{ij} can be evaluated in terms of Lie derivative w.r.t t^μ vector as

$$\dot{g}_{ij} = \mathcal{L}_t(g_{ij}) = -2NK_{ij} + \nabla_i N_j + \nabla_j N_i \ , \tag{A.26}$$

where, ∇_i is covariant derivative w.r.t g_{ij}. We shall use this relation later to write the metric canonical momentum in terms of the extrinsic curvature.

Gauss-Codazzi equation

The Gauss-Codazzi equation relates $(d+1)$-dimensional Ricci scalar to d-dimensional Ricci scalar and extrinsic curvature terms. To get this relation we need to project all the curvature tensors on the hypersurface. For spacelike hypersurface, this equation reads as

$$\sqrt{-\hat{g}}\hat{R} = \sqrt{g} N \Big(R + (K_{ij}K^{ij} - K^2) \Big) + \text{derivative terms} \ . \tag{A.27}$$

[3] This convention is according to ADM [58] or Isham [96]; whereas Poisson [100] and Blau [101] use the positive sign.

With this the gravity action is

$$S_{\text{gravity}} \equiv \int dt\, d^d x \sqrt{-\hat{g}} \mathcal{L}_{\text{gravity}} = \frac{1}{2\kappa^2} \int dt\, d^d x \sqrt{g} N \Big(R - 2\Lambda + (K_{ij} K^{ij} - K^2) \Big) \,. \quad \text{(A.28)}$$

We have written $(d+1)$-dimensional metric $\hat{g}_{\mu\nu}$ in terms of $\{g_{ij}, N, N^i\}$. So in principle, we could have canonical momenta corresponding to each variables $\{\pi_N, \pi_{N^i}, \pi_{ij}\}$. However, given the above form of the Lagrangian, we get

$$\pi_N = \frac{\delta S_{\text{gravity}}}{\delta \dot{N}} = 0 \quad ; \quad \pi_{N^i} = \frac{\delta S_{\text{gravity}}}{\delta \dot{N}^i} = 0 \,. \quad \text{(A.29)}$$

We find that the variables N & N^i are not dynamical variables. Their canonical momenta are constrained on a given Cauchy slice. The canonical momentum corresponding to the spatial metric g_{ij} is given by

$$\pi^{ij} = \frac{\delta S_{\text{gravity}}}{\delta \dot{g}_{ij}} = -\frac{1}{2\kappa^2 N} \frac{\delta S_{\text{gravity}}}{\delta K_{ij}} = \frac{1}{2\kappa^2} \sqrt{g} \Big(K g^{ij} - K^{ij} \Big) \,. \quad \text{(A.30)}$$

Or, we could invert the relation

$$K^{ij} = -\frac{2\kappa^2}{\sqrt{g}} \Big(\pi^{ij} - \frac{\pi}{d-1} g^{ij} \Big) \,. \quad \text{(A.31)}$$

The trace of the momentum is related to trace of extrinsic curvature as

$$\pi = \frac{(d-1)}{2\kappa^2} \sqrt{g} K \,. \quad \text{(A.32)}$$

In classical de Sitter spacetime, we find $K, \pi < 0$ for an expanding universe; whereas, $K, \pi > 0$ corresponds to a contracting universe.

A.3 Hamiltonian and constraints

After we have identified the dynamical and non-dynamical degrees of freedom of the metric, we are at a stage to define the Hamiltonian for gravity. We shall find that for gravity, the Hamiltonian is always a boundary term at the boundary of its Cauchy slice. In absence of any Cauchy slice boundary, the de Sitter (in global patch) Hamiltonian is always zero. It has two parts: one for the gravity action and the other for the matter action, which in this case is a minimally coupled scalar field action.

Gravity part

We can write the Hamiltonian by taking a Legendre transformation of the Lagrangian

$$H_{\text{gravity}} = \int d^dx \Big(\pi^{ij}\dot{g}_{ij} - \sqrt{-\hat{g}}\mathcal{L}_{\text{gravity}}\Big) \ . \tag{A.33}$$

We evaluate this with substituting the values of $\dot{g}_{ij}$ (see equation (A.26)), which gives

$$H_{\text{gravity}} = \int d^dx \sqrt{g}\Big[N\mathcal{H}_{\text{gravity}} + N^i\mathcal{H}_{i,\text{gravity}}\Big] \ . \tag{A.34}$$

In this step we have defined Hamiltonian and momentum constraint for pure Einstein gravity as

$$\begin{aligned}\mathcal{H}_{\text{gravity}} =& \frac{2\kappa^2}{g}(\pi_{ij}\pi^{ij} - \frac{1}{d-1}\pi^2) - \frac{1}{2\kappa^2}(R-2\Lambda) \ , \\ \mathcal{H}_{i,\text{gravity}} =& -2g_{ij}\nabla_k\Big(\frac{\pi^{jk}}{\sqrt{g}}\Big) \ .\end{aligned} \tag{A.35}$$

Matter part

If we include minimally coupled scalar field, we get additional terms in the Hamiltonian and momentum constraints. Incorporating the ADM split in (A.6), we find

$$\begin{aligned} S_{\text{matter}} &\equiv \int dt\ d^d x \sqrt{-\hat{g}} \mathcal{L}_{\text{matter}} \\ &= -\frac{1}{2} \int dt\ d^d x \sqrt{g} N \Big(-\frac{1}{N^2} \dot{\chi}\dot{\chi} + \frac{2N^i}{N^2} \dot{\chi} \partial_i \chi + (g^{ij} - \frac{N^i N^j}{N^2}) \partial_i \chi \partial_j \chi + m^2 \chi^2 \Big) \ . \end{aligned} \tag{A.36}$$

In similar way, we define canonical momenta for scalar field

$$\pi_\chi = \frac{\delta S_{\text{matter}}}{\delta(\dot{\chi})} = \frac{\sqrt{g}}{N} \Big(\dot{\chi} - N^i \partial_i \chi \Big) \ . \tag{A.37}$$

Writing $\dot{\chi}$ in terms of π_χ, the matter Hamiltonian is evaluated as

$$H_{\text{matter}} = \int d^d x \Big(\pi_\chi \dot{\chi} - \sqrt{-\hat{g}} \mathcal{L}_{\text{matter}} \Big) = \int d^d x \sqrt{g} \Big(N \mathcal{H}_{\text{matter}} + N^i \mathcal{H}_{i,\text{matter}} \Big) \ . \tag{A.38}$$

We have defined the matter constraints

$$\mathcal{H}_{\text{matter}} = \frac{1}{2} \Big(\frac{1}{g} \pi_\chi^2 + g^{ij} \partial_i \chi \partial_j \chi + m^2 \chi^2 \Big) \quad ; \quad \mathcal{H}_{i,\text{matter}} = \frac{1}{\sqrt{g}} \pi_\chi \partial_i \chi \ . \tag{A.39}$$

Constraints

Combining both the gravity and matter part we write,

$$H = \int d^d x \sqrt{g} (N \mathcal{H} + N^i \mathcal{H}_i) \ , \tag{A.40}$$

with

$$\mathcal{H} = \mathcal{H}_{\text{gravity}} + \mathcal{H}_{\text{matter}} \quad ; \quad \mathcal{H}_i = \mathcal{H}_{i,\text{gravity}} + \mathcal{H}_{i,\text{matter}} \ . \tag{A.41}$$

As we have previously noted, the fields N & N^i are not dynamical fields. This is because the Lagrangian does not contain their derivatives. Hence we have primary constraints

$$\pi_N = 0 \quad ; \quad \pi_{N^i} = 0 \ . \tag{A.42}$$

We also want the primary constraints to be preserved under the Hamiltonian flow. Hence we get the secondary constraints

$$\{H, \pi_N\} = \mathcal{H} = 0 \quad ; \quad \{H, \pi_{N^i}\} = \mathcal{H}_i = 0 \ . \tag{A.43}$$

This makes the Hamiltonian itself to be zero

$$H = 0 \ . \tag{A.44}$$

We have found that in the absence of any boundary of the Cauchy slice (as we have in the global de Sitter patch), the Hamiltonian is zero as it consists of only the bulk constraints. Hence the canonical time t is not a good clock for de Sitter spacetime.

Relation between Einstein equation and Constraints

We note that the constraints $\mathcal{H}$ & $\mathcal{H}_i$ are specific components of the Einstein equations. By projecting the Einstein equations in the following way, we recover the Hamiltonian and momentum constraints:

$$\mathcal{E}_{\mu\nu} \equiv -\frac{1}{\kappa^2}\left(\hat{G}_{\mu\nu} + \hat{g}_{\mu\nu}\Lambda - \kappa^2 \hat{T}_{\mu\nu}\right) \implies \mathcal{H} = \mathcal{E}_{\mu\nu} n^\mu n^\nu \quad ; \quad \mathcal{H}_i = \mathcal{E}_{\mu\nu} e^\mu_i n^\nu \ . \tag{A.45}$$

A.4 Pointwise constraints: WDW equation

Following Dirac [102], the physical states of the theory should be annihilated by the constraints. This leads to the following set of equations

$$\begin{aligned} \Pi_N \Psi[N, N^i, g, \chi] = 0 \quad &; \quad \Pi_{N^i} \Psi[N, N^i, g, \chi] = 0 \; ; \\ \mathcal{H}\Psi[N, N^i, g, \chi] = 0 \quad &; \quad \mathcal{H}_i \Psi[N, N^i, g, \chi] = 0 \; . \end{aligned} \tag{A.46}$$

In wavefunctional notation we have

$$\begin{aligned} \Pi_N \Psi[N, N^i, g, \chi] = 0 &\implies \frac{\delta}{\delta N}\Psi[N, N^i, g, \chi] = 0 \; ; \\ \Pi_{N^i} \Psi[N, N^i, g, \chi] = 0 &\implies \frac{\delta}{\delta N^i}\Psi[N, N^i, g, \chi] = 0 \; . \end{aligned} \tag{A.47}$$

The primary constraints can be implemented by restricting the wavefunctional to depend solely on the spatial metric on the Cauchy slice and the matter field. Then we are left with the secondary constraints

$$\mathcal{H}\Psi[g, \chi] = 0 \quad ; \quad \mathcal{H}_i \Psi[g, \chi] = 0 \; . \tag{A.48}$$

The Hamiltonian constraint equation is also referred as the Wheeler-DeWitt equation as we have mentioned earlier. We want to emphasize that these constraints restrict the wavefunctional at each point on the Cauchy slice, thus justifying their name. In the main part of thesis (chapter 2, equation (2.57)) we have derived the solutions to the WDW equation.

A.5 Integrated constraints

In the context of asymptotically de Sitter spacetimes, the Killing vectors $\{\xi^\mu\}$ can be defined in an asymptotic sense. We project the pointwise constraints in the direction of the killing vectors to define the integrated constraints (isometry charges) on the Cauchy surface, which takes the following form:

$$\Phi_\xi[g,\chi] = \int d\Sigma n^\mu \xi^\nu \mathcal{E}_{\mu\nu} = \int d\Sigma \Big(\xi^0 (N\mathcal{H} + N^i \mathcal{H}_i) + \xi^i \mathcal{H}_i \Big) \ . \tag{A.49}$$

Here $d\Sigma = \sqrt{g} d^d x$, which is the measure on the constant t hypersurface, and n_μ is the normal to the hypersurface. The Killing vectors of the asymptotically de Sitter spacetime form an $SO(1, d+1)$ algebra. Consequently, the corresponding isometry charges inherit this algebraic structure.

We can also find the solution to the integrated constraint as follows

$$\Phi_\xi[g,\chi]\Psi[g,\chi] = 0 \ . \tag{A.50}$$

As outlined in the main part of the thesis (section 4.2 & 4.3), the solutions to these equations are the non-gravitational states, $\{|\Psi_{ng}\rangle\}$ or the Higuchi states.

Appendix B

Wheeler-DeWitt expansion

In this Appendix we include technical details about the asymptotic expansion of the WDW equation and its solutions described in chapter 2.

B.1 Rewriting the constraints

We first explain how to rewrite the constraints in terms of intermediate variables to make the asymptotic expansion manifest. The original Hamiltonian constraint is

$$\mathcal{H} \;=\; \frac{2\kappa^2}{g}\left(\pi_{ij}\pi^{ij}-\frac{1}{d-1}\pi^2\right)-\frac{1}{2\kappa^2}(R-2\Lambda)+\mathcal{H}_{\text{matter}}+\mathcal{H}_{\text{int}}\ . \tag{B.1}$$

We define new variables Ω, γ_{ij} and χ by the relations

$$g_{ij}=\Omega^2\gamma_{ij}, \qquad \chi=\Omega^{-\Delta}\mathcal{O}, \qquad \det\gamma_{ij}=1\ \ , \tag{B.2}$$

and they can be written in terms of the original variables as

$$\Omega=\det(g)^{1/2d}, \qquad \gamma_{ij}=\frac{1}{\det(g)^{1/d}}g_{ij}, \qquad O=\det(g)^{\Delta/2d}\chi\ . \tag{B.3}$$

Their variations are then given by

$$\delta\Omega = \frac{1}{2d}\Omega g^{ij}\delta g_{ij} \tag{B.4}$$

$$\delta\gamma_{ij} = \Omega^{-2}\left(\delta g_{ij} - \frac{1}{d}g_{ij}g^{k\ell}\delta g_{k\ell}\right) \tag{B.5}$$

$$\delta O = \frac{1}{2d}\Delta O g^{ij}\delta g_{ij} + \Omega^{\Delta}\delta\chi \ . \tag{B.6}$$

From the identification

$$\delta\Psi = \frac{\delta\Psi}{\delta g_{ij}}\delta g_{ij} + \frac{\delta\Psi}{\delta\chi}\delta\chi = \frac{\delta\Psi}{\delta\Omega}\delta\Omega + \frac{\delta\Psi}{\delta\gamma_{ij}}\delta\gamma_{ij} + \frac{\delta\Psi}{\delta O}\delta O \ , \tag{B.7}$$

we obtain the differential operators

$$i\pi^{ij} = \frac{\delta}{\delta g_{ij}} = \frac{1}{2d}g^{ij}\left(\Omega\frac{\delta}{\delta\Omega} + \Delta O\frac{\delta}{\delta O}\right) + \Omega^{-2}\left(\frac{\delta}{\delta\gamma_{ij}} - \frac{1}{d}\gamma^{ij}\gamma_{k\ell}\frac{\delta}{\delta\gamma_{k\ell}}\right) \ , \tag{B.8}$$

$$i\pi = g_{ij}\frac{\delta}{\delta g_{ij}} = \frac{1}{2}\Omega\frac{\delta}{\delta\Omega} + \frac{1}{2}\Delta O\frac{\delta}{\delta O} \ , \tag{B.9}$$

$$i\pi_{\chi} = \frac{\delta}{\delta\chi} = \Omega^{\Delta}\frac{\delta}{\delta O} \ . \tag{B.10}$$

We see that what appears is the traceless differential

$$\delta_g^{ij} \equiv \Omega^{-2}\left(\frac{\delta}{\delta\gamma_{ij}} - \frac{1}{d}\gamma^{ij}\gamma_{k\ell}\frac{\delta}{\delta\gamma_{k\ell}}\right) \ . \tag{B.11}$$

A useful fact is that it can be written in terms of the original metric g as

$$\delta_g^{ij} = \frac{\delta}{\delta g_{ij}} - \frac{1}{d}g^{ij}g_{k\ell}\frac{\delta}{\delta g_{k\ell}} \ . \tag{B.12}$$

The momentum then takes the form

$$i\pi^{ij} = \frac{1}{2d}g^{ij}\left(\Omega\frac{\delta}{\delta\Omega} + \Delta O\frac{\delta}{\delta O}\right) + \delta_g^{ij} \ , \tag{B.13}$$

and so the kinetic piece is

$$\pi_{ij}\pi^{ij} - \frac{1}{d-1}\pi^2 = \frac{1}{4d(d-1)}\left(\Omega\frac{\delta}{\delta\Omega} + \Delta O\frac{\delta}{\delta O}\right)^2 - g_{ik}g_{j\ell}\delta_g^{ij}\delta_g^{k\ell}\ , \tag{B.14}$$

using the tracelessness condition $g_{ij}\delta_g^{ij} = 0$ to cancel off-diagonal terms.

Now note that the Hamiltonian constraint, (B.1), involves a composition of two such differential operators. This yields terms where the second differential operator acts on the variable coefficients that appear in (B.8) and produces the divergent expression, $\delta(0)$. Such terms can also arise if the second-order functional derivative acts on a local expression. We discuss these terms further in Appendix B.2 but we drop these terms for now. As explained in section 2.2, this issue does not affect our leading-order analysis.

The Hamiltonian constraint is then

$$\mathcal{H} = \frac{2\kappa^2}{\Omega^{2d}}\left[\frac{1}{4d(d-1)}\left(\Omega\frac{\delta}{\delta\Omega} + \Delta O\frac{\delta}{\delta O}\right)^2 - g_{ik}g_{j\ell}\delta_g^{ij}\delta_g^{k\ell}\right] - \frac{1}{2\kappa^2}(R - 2\Lambda) + \mathcal{H}_{\text{matter}} + \mathcal{H}_{\text{int}} \tag{B.15}$$

and for a scalar field we have

$$\mathcal{H}_{\text{matter}} = -\frac{1}{2}g^{-1}\left(\frac{\delta}{\delta\chi}\right)^2 + \frac{1}{2}(g^{ij}\partial_i\chi\partial_j\chi + m^2\chi^2)\ . \tag{B.16}$$

We obtain the form given in (2.15) after rewriting the second term in the bracket in terms of γ_{ij} using that

$$\begin{aligned} g_{ik}g_{j\ell}\delta_g^{ij}\delta_g^{k\ell} &= \gamma_{ik}\gamma_{j\ell}\left(\frac{\delta}{\delta\gamma_{ij}} - \frac{1}{d}\gamma^{ij}\gamma_{ab}\frac{\delta}{\delta\gamma_{ab}}\right)\left(\frac{\delta}{\delta\gamma_{k\ell}} - \frac{1}{d}\gamma^{k\ell}\gamma_{cd}\frac{\delta}{\delta\gamma_{cd}}\right) \\ &= \left(\gamma_{ik}\gamma_{j\ell} - \frac{1}{d}\gamma_{ij}\gamma_{k\ell}\right)\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma_{k\ell}}\ . \end{aligned} \tag{B.17}$$

B.2 Normal ordering prescription

We describe a natural choice of normal ordering prescription that gets rid of the $\delta(0)$ terms appearing at leading order when acting with the Hamiltonian constraint on an asymptotic wavefunctional

$$\Psi = e^{i\mathcal{F}}, \qquad \mathcal{F} = \int d^dx\sqrt{g}\left(-\frac{(d-1)}{\kappa^2} + b_\beta \chi^2 + \dots\right) , \tag{B.18}$$

where $\dots$ corresponds to subleading pieces in our asymptotic expansion.

The leading contributions come from the derivatives with respect to the Weyl factor. We can choose the normal ordering prescription

$$\begin{aligned} :\mathcal{H}: \;\; &= \;\; \frac{2\kappa^2}{4d(d-1)}\frac{1}{\Omega^d}\left(\Omega\frac{\delta}{\delta\Omega} + \Delta : O\frac{\delta}{\delta O}:\right)\frac{1}{\Omega^d}\left(\Omega\frac{\delta}{\delta\Omega} + \Delta : O\frac{\delta}{\delta O}:\right) \\ &\quad + \frac{\Lambda}{\kappa^2} - \frac{1}{2}\Omega^{2(\Delta-d)}\frac{\delta^2}{\delta O^2} + \frac{1}{2}m^2\Omega^{-2\Delta}O^2 + \mathcal{H}_{\text{sub}} , \end{aligned} \tag{B.19}$$

where $\mathcal{H}_{\text{sub}}$ corresponds to terms that are subleading when acting on Ψ. For the matter we choose the normal ordering

$$: O\frac{\delta}{\delta O} := \frac{1}{2}\left(O\frac{\delta}{\delta O} + \frac{\delta}{\delta O}O\right) = O\frac{\delta}{\delta O} + \frac{1}{2}\delta(0) . \tag{B.20}$$

Recalling that $\sqrt{g} = \Omega^d$, it is clear that the choice of ordering cancels the $\delta(0)$ appearing in the leading gravity piece. At leading order in the matter sector, we have

$$\begin{aligned} \frac{:\mathcal{H}:\Psi}{\Psi} \;\; &= \;\; \frac{2\kappa^2}{4d(d-1)}\frac{2}{\Omega^d}\left(i\Omega\frac{\delta X_d}{\delta\Omega}\right)\frac{\Delta}{2}\delta(0) - ib_\beta\Omega^{-d}\delta(0) \\ &= \;\; -\frac{i}{\Omega^d}\frac{\Delta}{2}\delta(0) - \frac{ib_\beta}{\Omega^d}\delta(0) , \end{aligned} \tag{B.21}$$

which vanishes since $b_\beta = -\frac{\Delta}{2}$.

The contribution of second order derivatives on subleading terms in the gravitational and matter part of the solution produces terms that compete with the remainder term in (2.13). Therefore, these terms are important for understanding finite-time physics but not for the asymptotic form of the solution. This is to be expected since finite-time physics should depend on the details of the UV-completion whereas the form of the Hilbert space can be determined more easily.

B.3 Anomaly in $d = 4$

For pure Einstein gravity, the term X_{d-4} satisfies the equation

$$\Omega\frac{\delta X_{d-4}}{\delta\Omega} = -\frac{2\kappa^2}{\Omega^d}\left(g_{ik}g_{j\ell}(\delta_g^{ij}X_{d-2})(\delta_g^{k\ell}X_{d-2}) - \frac{1}{4d(d-1)}\left(\Omega\frac{\delta X_{d-2}}{\delta\Omega}\right)^2\right) . \tag{B.22}$$

Using that δ_g^{ij} it the traceless part of the variation with respect to g_{ij}, we see that

$$\delta_g^{ij}X_{d-2} = -\frac{1}{2(d-2)\kappa^2}\sqrt{g}\left(R^{ij} - \frac{1}{d}g^{ij}R\right) , \tag{B.23}$$

so that we have

$$g_{ik}g_{j\ell}(\delta_g^{ij}X_{d-2})(\delta_g^{k\ell}X_{d-2}) = \frac{\Omega^{2d}}{4(d-2)^2\kappa^4}\left(R_{ij}R^{ij} - \frac{1}{d}R^2\right) . \tag{B.24}$$

The second contribution takes the form

$$\frac{1}{4d(d-1)}\left(\Omega\frac{\delta X_{d-2}}{\delta\Omega}\right)^2 = \frac{1}{16d(d-1)\kappa^4}\Omega^{2d}R^2 , \tag{B.25}$$

and so the equation becomes

$$\Omega\frac{\delta X_{d-4}}{\delta\Omega} = -\frac{1}{2(d-2)^2\kappa^2}\sqrt{g}\left(R_{ij}R^{ij} - \frac{d}{4(d-1)}R^2\right) . \tag{B.26}$$

In $d \neq 4$, this equation can be integrated to give

$$X_{d-4} = -\frac{1}{2(d-2)^2(d-4)\kappa^2} \int d^d x \sqrt{g} \left(R_{ij}R^{ij} - \frac{d}{4(d-1)} R^2 \right) , \tag{B.27}$$

which matches with the holographic renormalization results, see *e.g.* (B.4) in [72].

In $d = 4$, we obtain the equation

$$\Omega \frac{\delta X_0}{\delta \Omega} = -\frac{1}{8\kappa^2} \sqrt{g} \left(R_{ij}R^{ij} - \frac{1}{3} R^2 \right) , \tag{B.28}$$

which leads to

$$\Omega \frac{\delta}{\delta \Omega} e^{iX_0} = \mathcal{A}_4 e^{iX_0}, \qquad \mathcal{A}_4 \equiv -\frac{i}{8\kappa^2} \sqrt{g} \left(R_{ij}R^{ij} - \frac{1}{3} R^2 \right) . \tag{B.29}$$

We recognize the trace anomaly equation for the CFT partition function $Z = e^{iX_0}$. The anomaly can be written as

$$\mathcal{A}_4 = \frac{1}{16\pi^2} \sqrt{g}(-aE_4 + cW_{abcd}W^{abcd}) \tag{B.30}$$

using the Euler density and Weyl squared curvature

$$\begin{aligned} E_4 &= R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 , \\ W_{abcd}W^{abcd} &= R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + \frac{1}{3} R^2 , \end{aligned} \tag{B.31}$$

with the anomaly coefficients

$$a = c = -\frac{i\pi^2}{\kappa^2} = -\frac{i\pi}{8G_N} . \tag{B.32}$$

This is, up to the factor of $-i$, the anomaly of a holographic CFT_4 obtained using holographic renormalization in AdS_5 [66].

B.4 Subleading matter term

The subleading matter term $Y_{\beta-2}$ is determined by the equation

$$\frac{2}{\sqrt{g}}\left(g_{ij}\frac{\delta}{\delta g_{ij}}+b_\beta\chi\frac{\delta}{\delta\chi}\right)Y_{\beta-2}-\frac{b_\beta}{2(d-1)}R\chi^2+\frac{1}{2}g^{ij}\partial_i\chi\partial_j\chi=0. \tag{B.33}$$

An ansatz for the solution is to take the local and diffeomorphism invariant functional

$$Y_{\beta-2}=c_1\underbrace{\int d^dx\,\sqrt{g}R\chi^2}_{\mathrm{I}}+c_2\underbrace{\int d^dx\,\sqrt{g}g^{ij}\partial_i\chi\partial_j\chi}_{\mathrm{II}}, \tag{B.34}$$

where $c_{1,2}$ are undetermined coefficients. These functionals I and II have been chosen due to their $\Omega^{\beta-2}=\Omega^{d-2\Delta-2}$ scaling. Defining

$$\delta_1=\frac{g_{ij}}{\sqrt{g}}\frac{\delta}{\delta g_{ij}},\qquad \delta_2=\frac{\chi}{\sqrt{g}}\frac{\delta}{\delta\chi}\ , \tag{B.35}$$

we have the formulae

$$\delta_1\mathrm{I}=\frac{g_{ij}}{\sqrt{g}}\frac{\delta}{\delta g_{ij}}\int d^dx\,\sqrt{g}R\chi^2=\left(\frac{d}{2}-1\right)R\chi^2-2(d-1)\left((\nabla\chi)^2+\chi\Box\chi\right), \tag{B.36}$$

$$\delta_1\mathrm{II}=\frac{g_{ij}}{\sqrt{g}}\frac{\delta}{\delta g_{ij}}\int d^dx\,\sqrt{g}g^{ij}\partial_i\chi\partial_j\chi=\left(\frac{d}{2}-1\right)(\nabla\chi)^2, \tag{B.37}$$

$$\delta_2\mathrm{I}=\frac{\chi}{\sqrt{g}}\frac{\delta}{\delta\chi}\int d^dx\,\sqrt{g}R\chi^2=2R\chi^2, \tag{B.38}$$

$$\delta_2\mathrm{II}=\frac{\chi}{\sqrt{g}}\frac{\delta}{\delta\chi}\int d^dx\,\sqrt{g}g^{ij}\partial_i\chi\partial_j\chi=-2\chi\Box\chi\ . \tag{B.39}$$

Our equation now becomes

$$(\delta_1+b_\beta\delta_2)(c_1\mathrm{I}+c_2\mathrm{II})=-\frac{1}{4}(\nabla\chi)^2+\frac{b_\beta}{4(d-1)}R\chi^2\ . \tag{B.40}$$

Requiring that the $\chi\Box\chi$ term cancels out from the left side gives

$$-2(d-1)c_1 - 2b_\beta c_2 = 0 \qquad \Longrightarrow \qquad c_2 = -\frac{(d-1)}{b_\beta}c_1 \ . \tag{B.41}$$

Matching the coefficients of $R\chi^2$ on both sides then gives

$$c_1\left(\frac{d-2}{2} + 2b_\beta\right) = \frac{b_\beta}{4(d-1)} \ . \tag{B.42}$$

Solution to above equation gives c_1 and the proportionality gives c_2 as

$$c_1 = \frac{b_\beta}{2(d-1)(d-2+4b_\beta)}, \qquad c_2 = -\frac{1}{2(d-2+4b_\beta)}. \tag{B.43}$$

We can see that this choice also matches the coefficient of $(\nabla\chi)^2$ on both sides, meaning the system of equations was overdetermined, albeit with a solution. Substituting $b_\beta = -\Delta/2$ from (2.44) we have,

$$Y_{\beta-2} = -\frac{1}{2(d-2-2\Delta)}\int d^dx\,\sqrt{g}\left(g^{ij}\partial_i\chi\partial_j\chi + \frac{\Delta}{2(d-1)}R\chi^2\right). \tag{B.44}$$

Appendix C

Derivation of the Ward identities

In this Appendix, we derive the Ward identities for the coefficient functions. We have

$$Z[g,\chi] = \exp\left[\sum_{m,n} \kappa^n \mathcal{G}_{n,m}[h,\dots h,\chi,\dots \chi]\right] \ , \qquad \text{(C.1)}$$

where we remind the reader of the definition of the multi-linear functionals, $\mathcal{G}_{n,m}$ that take n tensor fields and m scalar fields as input and return a c-number. They are defined as

$$\begin{aligned} &\mathcal{G}_{n,m}[h^{(1)},\dots h^{(n)},\chi^{(1)},\dots \chi^{(m)}] \qquad \text{(C.2)}\\ \equiv\ &\frac{1}{n!m!}\int d^dy_1\dots d^dy_n d^dz_1\dots d^dz_m\, G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z}) h^{(1)}_{i_1j_1}(y_1)\dots h^{(n)}_{i_nj_n}(y_n)\chi^{(1)}(z_1)\dots\chi^{(m)}(z_m)\ , \end{aligned}$$

so that

$$G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z}) = \frac{\delta^n}{\delta h_{i_1j_1}(y_1)\dots\delta h_{i_nj_n}(y_n)}\frac{\delta^m}{\delta\chi(z_1)\dots\chi(z_m)}\mathcal{G}_{n,m}[h,\dots,h,\chi,\dots,\chi]\ . \qquad \text{(C.3)}$$

Under a diffeomorphism and Weyl transformation, we have

$$\delta_{(\xi,\varphi)}g_{ij} = \nabla_i\xi_j + \nabla_j\xi_i + 2\varphi g_{ij}, \qquad \delta_{(\xi,\varphi)}\chi = \xi^i\partial_i\chi - \Delta\varphi\chi \qquad \text{(C.4)}$$

so that h_{ij} transforms as

$$\kappa\delta h_{ij} = \kappa H_{ij} + I_{ij} \ , \tag{C.5}$$

where

$$H_{ij} = \mathcal{L}_\xi h_{ij} + 2\varphi h_{ij}, \qquad I_{ij} = \partial_i \xi^k \delta_{jk} + \partial_j \xi^k \delta_{ik} + 2\varphi\delta_{ij} \ , \tag{C.6}$$

and the definition of the Lie derivative gives

$$\mathcal{L}_\xi h_{ij} = \xi^k \partial_k h_{ij} + \partial_i \xi^k h_{kj} + \partial_j \xi^k h_{ik} \ . \tag{C.7}$$

We then have the variation

$$\delta_{(\xi,\varphi)} \log Z = \sum_{n,m} \kappa^n (\delta_{(\xi,\varphi)} \log Z)_{n,m} \ , \tag{C.8}$$

where we have collected terms according to the expansion in κ:

$$\begin{aligned} (\delta_{(\xi,\varphi)} \log Z)_{n,m} \ &= \ (n+1)\mathcal{G}_{n+1,m}[I, h, \dots h, \chi, \dots \chi] + n\mathcal{G}_{n,m}[H, h, \dots h, \chi, \dots \chi] \\ &\quad + m\mathcal{G}_{n,m}[h, \dots h, \delta\chi, \dots \chi] \ . \end{aligned} \tag{C.9}$$

The Weyl and diffeomorphism invariance can be summarized by the identity

$$\delta_{(\xi,\varphi)} \log Z = \int d^d x \, \varphi(x) \mathcal{A}_d(x) \ . \tag{C.10}$$

That this is equivalent to the equations (3.1) can be proven by taking the functional derivatives with respect to $\varphi(x)$ and $\xi_k(x)$. This follows from the fact that

$$\begin{aligned} \frac{\delta}{\delta\varphi(x)} \delta_{(\xi,\varphi)} \log Z \ &= \ \left(2g_{ij} \frac{\delta}{\delta g_{ij}} - \Delta\chi \frac{\delta}{\delta\chi} \right) \log Z \ , \\ \frac{\delta}{\delta\xi_j(x)} \delta_{(\xi,\varphi)} \log Z \ &= \ \left(-2\sqrt{g} \nabla_i \frac{1}{\sqrt{g}} \frac{\delta}{\delta g_{ij}} + g^{ij} \partial_i \chi \frac{\delta}{\delta\chi} \right) \log Z \ . \end{aligned} \tag{C.11}$$

C.1 Trace identity

The trace identity is obtained by considering only a Weyl transformation. Under this we have

$$H_{ij} = 2\varphi h_{ij}, \qquad I_{ij} = 2\varphi\delta_{ij}, \qquad \delta_\varphi\chi = -\Delta\varphi\chi\ . \tag{C.12}$$

As a result, we get

$$\begin{aligned}(\delta_\varphi \log Z)_{n,m} &= 2(n+1)\mathcal{G}_{n+1,m}[\varphi\delta, h, \ldots, h, \chi, \ldots, \chi] + 2n\mathcal{G}_{n,m}[\varphi h, h, \ldots, h, \chi, \ldots, \chi] \\ &\quad -m\Delta\mathcal{G}_{n,m}[h, \ldots, h, \varphi\chi, \chi, \ldots, \chi]\ . \end{aligned} \tag{C.13}$$

We can expand the anomaly in the κ to get

$$\mathcal{A}_d(x) = \sum_n \frac{\kappa^n}{n!}\int d^dy_1 \ldots d^dy_n\, \mathcal{A}_d^{\vec{i}\vec{j}}(x,\vec{y})h_{i_1j_1}(y_1)\ldots h_{i_nj_n}(y_n)\ . \tag{C.14}$$

This defines the coefficients $\mathcal{A}_d^{\vec{i}\vec{j}}(x,\vec{y})$ which can be recovered as the functional derivative

$$\kappa^n\mathcal{A}_d^{\vec{i}\vec{j}}(x,\vec{y}) = \frac{\delta^n}{\delta h_{i_1j_1}(y_1)\ldots\delta h_{i_nj_n}(y_n)}\mathcal{A}_d(x)\ . \tag{C.15}$$

As $\mathcal{A}_d(x)$ is local, these coefficients are ultralocal, in the sense that

$$\mathcal{A}_d^{\vec{i}\vec{j}}(x,\vec{y}) = A_d^{\vec{i}\vec{j}}(x)\prod_{a=1}^{n}\delta^{(d)}(y_a - x)\ , \tag{C.16}$$

where $A_d^{\vec{i}\vec{j}}(x)$ are the coefficients appearing in expanding $\mathcal{A}_d(x)$ in the metric.

The Weyl transformation equation then becomes

$$(\delta_\varphi \log Z)_{n,m} = \delta_{m,0}\frac{1}{n!}\int d^dxd^dy_1\ldots d^dy_n\,\varphi(x)\mathcal{A}_d^{\vec{i}\vec{j}}(x,\vec{y})h_{i_1j_1}(y_1)\ldots h_{i_nj_n}(y_n)\ . \tag{C.17}$$

To obtain the relations satisfied by the coefficient functions $G_{n,m}^{\vec{i}\vec{j}}(\vec{y},\vec{z})$, we take n functional derivatives with respect to the metric and m functional derivatives with respect to the matter.

For the LHS, we get

$$\frac{\delta^n}{\delta h_{i_1j_1}(y_1)\dots\delta h_{i_nj_n}(y_n)}\frac{\delta^m}{\delta\chi(z_1)\dots\chi(z_m)}(\delta_\varphi\log Z)_{n,m} = 2\int d^dx\,\varphi(x)\delta_{ij}G^{ij\vec{i}\vec{j}}_{n+1,m}(x,\vec{y},\vec{z}) + \left[2(\varphi(y_1)+\dots+\varphi(y_n)) - \Delta(\varphi(z_1)+\dots+\varphi(z_m))\right]G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})\ . \quad (C.18)$$

The equality with the RHS implies that

$$\frac{\delta^n}{\delta h_{i_1j_1}(y_1)\dots\delta h_{i_nj_n}(y_n)}\frac{\delta^m}{\delta\chi(z_1)\dots\chi(z_m)}(\delta_\varphi\log Z)_{n,m} = \delta_{m,0}\int d^dx\,\varphi(x)\mathcal{A}^{\vec{i}\vec{j}}_d(x,\vec{y})\ . \quad (C.19)$$

Equating these two expressions and taking the functional derivative with respect to $\varphi(x)$ gives the trace identity

$$2\delta_{ij}G^{ij\vec{i}\vec{j}}_{n+1,m}(x,\vec{y},\vec{z}) = \left(-2\sum_{a=1}^{n}\delta^{(d)}(x-y_a) + \Delta\sum_{b=1}^{m}\delta^{(d)}(x-z_b)\right)G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z}) + \delta_{m,0}\mathcal{A}^{\vec{i}\vec{j}}_d(x,\vec{y})\ . \quad (C.20)$$

C.2 Divergence identity

Under a diffeomorphism we have

$$H_{ij} = \mathcal{L}_\xi h_{ij} = \xi^k\partial_k h_{ij} + \partial_i\xi^k h_{kj} + \partial_j\xi^k h_{ik}\ , \quad (C.21)$$

$$I_{ij} = \mathcal{L}_\xi\delta_{ij} = \partial_i\xi^k\delta_{jk} + \partial_j\xi^k\delta_{ki}\ , \quad (C.22)$$

$$\delta_\xi\chi = \mathcal{L}_\xi\chi = \xi^k\partial_k\chi\ . \quad (C.23)$$

The Ward identity is

$$\delta_\xi\log Z = 0\ , \quad (C.24)$$

which we expand as

$$
\begin{aligned}
0=(\delta_\xi \log Z)_{n,m} &= (n+1)\mathcal{G}_{n+1,m}[\mathcal{L}_\xi\delta, h,\ldots,h,\chi,\ldots,\chi] + n\mathcal{G}_{n,m}[\mathcal{L}_\xi h, h,\ldots,h,\chi,\ldots,\chi] \\
&\quad + m\mathcal{G}_{n,m}[h,\ldots,h,\mathcal{L}_\xi\chi,\chi,\ldots,\chi]\ . \qquad \text{(C.25)}
\end{aligned}
$$

As above we take functional derivatives and obtain

$$
\begin{aligned}
0 &= \frac{\delta^n}{\delta h_{i_1j_1}(y_1)\ldots\delta h_{i_nj_n}(y_n)}\frac{\delta^m}{\delta\chi(z_1)\ldots\chi(z_m)}(\delta_\xi \log Z)_{n,m} \qquad \text{(C.26)} \\
&= 2\int d^dx\,\delta_{jk}\partial_i\xi^k(x)G^{ij\vec{i}\vec{j}}_{n+1,m}(x,\vec{y},\vec{z}) - \sum_{b=1}^m \frac{\partial}{\partial z_b^k}(\xi^k(z_b)G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})) \\
&\quad + \sum_{a=1}^n \left[-\frac{\partial}{\partial y_a^k}(\xi^k(y_a)G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})) + (\delta^{i_a}_{i'_a}\partial_{j'_a}\xi^{j_a}(y_a) + \delta^{j_a}_{j'_a}\partial_{i'_a}\xi^{i_a}(y_a))G^{\vec{i}'\vec{j}'}_{n,m}(\vec{y},\vec{z})\right]\ ,
\end{aligned}
$$

where in the bracketed expression, we use the notation

$$
(\vec{i}'\vec{j}') = (i_1, j_1\ldots i_{a-1}, j_{a-1}, i'_a, j'_a, i_{a+1}, j_{a+1}\ldots i_n, j_n) \qquad \text{(C.27)}
$$

for the current a in the sum. We now take the functional derivative with respect to $\xi^k(x)$ which can be done by simply replacing $\xi^\ell(x')$ by $\delta^\ell_k\delta^{(d)}(x-x')$. This leads to the divergence identity

$$
\begin{aligned}
2\delta_{jk}\partial_i G^{ij\vec{i}\vec{j}}_{n+1,m}(x,\vec{y},\vec{z}) &= -\sum_{b=1}^m \frac{\partial}{\partial z_b^k}\Big[\delta^{(d)}(x-z_b)G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})\Big] \qquad \text{(C.28)} \\
+\sum_{a=1}^n \Big[-\frac{\partial}{\partial y_a^k}\Big[\delta^{(d)}(x-y_a)G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})\Big] &+ G^{\vec{i}'\vec{j}'}_{n,m}(\vec{y},\vec{z})\Big(\delta^{i_a}_{i'_a}\delta^{j_a}_k\frac{\partial}{\partial y^{j'_a}} + \delta^{j_a}_{j'_a}\delta^{i_a}_k\frac{\partial}{\partial y^{i'_a}}\Big)\delta^{(d)}(x-y_a)\Big]\ .
\end{aligned}
$$

C.3 Conformal symmetry

The conformal symmetry is obtained by combining a diffeomorphism ξ and a Weyl transformation with $\varphi = -\partial_k \xi^k / d$ so that

$$I_{ij} = \partial_i \xi^k \delta_{jk} + \partial_j \xi^k \delta_{ik} - \frac{2}{d} \delta_{ij} \partial_k \xi^k = 0 \; . \tag{C.29}$$

This is achieved by taking ξ to be a conformal Killing vector on the sphere. We then have

$$\delta h_{ij} = H_{ij} = \xi^k \partial_k h_{ij} + \partial_i \xi^k h_{kj} + \partial_j \xi^k h_{ik} - \frac{2}{d} \partial_k \xi^k h_{ij}, \qquad \delta\chi = \xi^i \partial_i \chi + \frac{\Delta}{d} \partial_k \xi^k \chi \; . \tag{C.30}$$

The variation of $\log Z$ doesn't mix different terms in the expansion as $I = 0$. So we get

$$(\delta \log Z)_{n,m} = n \mathcal{G}_{n,m}[H, h, \ldots h, \chi, \ldots \chi] + m \mathcal{G}_{n,m}[h, \ldots h, \delta\chi, \ldots \chi] \; , \tag{C.31}$$

and the constraint gives

$$(\delta \log Z)_{n,m} = -\delta_{m,0} \frac{1}{dn!} \int d^d x d^d y_1 \ldots d^d y_n \, \partial_k \xi^k(x) \mathcal{A}_d^{\vec{i}\vec{j}}(x, \vec{y}) h_{i_1 j_1}(y_1) \ldots h_{i_n j_n}(y_n) \; . \tag{C.32}$$

Taking successive functional derivatives gives

$$\begin{aligned}
&\frac{\delta^n}{\delta h_{i_1 j_1}(y_1) \ldots \delta h_{i_n j_n}(y_n)} \frac{\delta^m}{\delta \chi(z_1) \ldots \chi(z_m)} (\delta \log Z)_{n,m} \\
&= \sum_{a=1}^{n} \Big[\Big(\delta^{i_a}_{i'_a} \partial_{j'_a} \xi^{j_a}(y_a) + \delta^{j_a}_{j'_a} \partial_{i'_a} \xi^{i_a}(y_a) \Big) G^{\vec{i}'\vec{j}'}_{n,m}(\vec{y}, \vec{z}) - \frac{\partial}{\partial y^k_a} \Big[\xi^k(y_a) G^{\vec{i}\vec{j}}_{n,m}(\vec{y}, \vec{z}) \Big] - \frac{2}{d} \partial_k \xi^k(y_a) G^{\vec{i}\vec{j}}_{n,m}(\vec{y}, \vec{z}) \Big] \\
&+ \sum_{b=1}^{m} \Big[- \frac{\partial}{\partial z^k_b} \Big[\xi^k(z_b) G^{\vec{i}\vec{j}}_{n,m}(\vec{y}, \vec{z}) \Big] + \frac{\Delta}{d} \partial_k \xi^k(z_b) G^{\vec{i}\vec{j}}_{n,m}(\vec{y}, \vec{z}) \Big] \; , \\
&= \sum_{a=1}^{n} \Big[- \xi^k(y_a) \frac{\partial}{\partial y^k_a} G^{\vec{i}\vec{j}}_{n,m}(\vec{y}, \vec{z}) - \frac{2+d}{d} \partial_k \xi^k(y_a) G^{\vec{i}\vec{j}}_{n,m}(\vec{y}, \vec{z}) + \Big(\delta^{i_a}_{i'_a} \partial_{j'_a} \xi^{j_a}(y_a) + \delta^{j_a}_{j'_a} \partial_{i'_a} \xi^{i_a}(y_a) \Big) G^{\vec{i}'\vec{j}'}_{n,m}(\vec{y}, \vec{z}) \Big] \\
&+ \sum_{b=1}^{m} \Big[- \xi^k(z_b) \frac{\partial}{\partial y^k_b} G^{\vec{i}\vec{j}}_{n,m}(\vec{y}, \vec{z}) + \frac{\Delta - d}{d} \partial_k \xi^k(z_b) G^{\vec{i}\vec{j}}_{n,m}(\vec{y}, \vec{z}) \Big] \; .
\end{aligned} \tag{C.33}$$

Ignoring the anomaly which is ultralocal, we get the conformal Ward identity

$$0 = \sum_{a=1}^{n} \left[-\xi^k(y_a) \frac{\partial}{\partial y_a^k} G_{n,m}^{\vec{i}\vec{j}}(\vec{y}, \vec{z}) - \frac{2+d}{d} \partial_k \xi^k(y_a) G_{n,m}^{\vec{i}\vec{j}}(\vec{y}, \vec{z}) + \left(\delta_{i'_a}^{i_a} \partial_{j'_a} \xi^{j_a}(y_a) + \delta_{j'_a}^{j_a} \partial_{i'_a} \xi^{i_a}(y_a) \right) G_{n,m}^{\vec{i}'\vec{j}'}(\vec{y}, \vec{z}) \right]$$
$$+ \sum_{b=1}^{m} \left[-\xi^k(z_b) \frac{\partial}{\partial z_b^k} G_{n,m}^{\vec{i}\vec{j}}(\vec{y}, \vec{z}) + \frac{\Delta - d}{d} \partial_k \xi^k(z_b) G_{n,m}^{\vec{i}\vec{j}}(\vec{y}, \vec{z}) \right] \tag{C.34}$$

Under a finite conformal transformation $x \to x'$, the Jacobian is

$$J_{i'}^{i}(x) = \frac{\partial x'^{i}}{\partial x^{i'}} \quad , \tag{C.35}$$

and the scale factor is defined as

$$\Lambda(x) = |\det J(x)|^{-1/d} \ . \tag{C.36}$$

The rotation matrix is defined as

$$R_{i'}^{i}(x) = \Lambda(x) J_{i'}^{i}(x) \ , \tag{C.37}$$

so that it satisfies $\det R = 1$.

The finite conformal transformation takes the form

$$G_{n,m}^{\vec{i}\vec{j}}(\vec{y}', \vec{z}') = \left(\prod_{a=1}^{n} R_{i'_a}^{i_a}(y_a) R_{j'_a}^{j_a}(y_a) \Lambda(y_a)^d \right) \left(\prod_{b=1}^{m} \Lambda(z_b)^{d-\Delta} \right) G_{n,m}^{\vec{i}'\vec{j}'}(\vec{y}, \vec{z}) \ . \tag{C.38}$$

We can check that this is the integrated version of the conformal Ward identity by expanding infinitesimally. Under an infinitesimal conformal transformation, we have

$$x'_a = x_a + \xi(x_a) + \ldots, \qquad J_{i'}^{i}(x) = \delta_{i'}^{i} + \partial_{i'} \xi^i(x) + \ldots, \qquad \Lambda(x) = 1 - \frac{1}{d} \partial_k \xi^k(x) + \ldots \ , \tag{C.39}$$

so that we get

$$\begin{aligned} G^{\vec{i}\vec{j}}_{n,m}(\vec{y}',\vec{z}') &= \prod_{a=1}^{n}(\delta^{i_a}_{i'_a}+\partial_{i'_a}\xi^{i_a}(y_a))(\delta^{j_a}_{j'_a}+\partial_{j'_a}\xi^{j_a}(y_a))\left(1-\frac{2+d}{d}\partial_k\xi^k(y_a)\right) \\ &\times\prod_{b=1}^{m}\left(1+\frac{\Delta-d}{d}\partial_k\xi^k(z_b)\right)G^{\vec{i}'\vec{j}'}_{n,m}(\vec{y},\vec{z})\ , \end{aligned} \tag{C.40}$$

which reproduces at linear order the conformal Ward identity (C.34). This shows that (C.38) is the finite conformal transformation properties of the coefficient functions.

We note that the coefficient functions have the same symmetries of a CFT correlator:

$$G^{\vec{i}\vec{j}}_{n,m}(\vec{y},\vec{z})\sim\langle T^{i_1j_1}(y_1)\dots T^{i_nj_n}(y_n)\phi(z_1)\dots\phi(z_m)\rangle_{\rm CFT}\ , \tag{C.41}$$

where ϕ is an operator of dimension $d-\Delta$ and T^{ij} is an operator of spin 2 and dimension d.

Appendix D

Holography of information in de Sitter quantum gravity

This appendix provides a brief summary of a related work [2] [1], complementary to but separate from the main thesis.

In this work, we give a proposal of a natural norm for the space of states, which we derived in the main part of this thesis (chapters 2, 3, and 4). We write the expectation value of a diffeomorphism and Weyl invariant operator A in the state Ψ is defined as follows:

$$(\Psi, A\Psi) = \frac{\mathcal{N}_1}{\text{vol(diff}\times\text{Weyl)}} \int Dg D\chi \sum_{n,m,n',m'} \kappa^{n+n'} \delta\mathcal{G}^*_{n,m} \delta\mathcal{G}_{n',m'} |Z_0[g,\chi]|^2 A[g,\chi] \ . \quad \text{(D.1)}$$

When $A = I$, we get the norm of the state Ψ. In the above expression, we have squared the wave-functionals and integrated over all configurations of the fields on the Cauchy slice. As the integrand is invariant under the spatial-diffeomorphism and Weyl transformation, we have divided by the corresponding gauge group volume. The factor $\mathcal{N}_1$ is an unimportant normalization constant.

We explicitly evaluate the inner product using the standard Faddeev-Popov trick.

[1] This publication is included in Tuneer Chakraborty's thesis.

For this purpose, the following gauge condition is chosen for the metric:

$$\partial_i g_{ij} = 0; \qquad \delta^{ij} g_{ij} = d \ . \tag{D.2}$$

However, some residual gauge transformations remain unfixed by the aforementioned conditions. These residual transformations manifest as zero modes of the ghost action. We find that the zero modes actually correspond to the generators of the conformal group of d-dimensional Euclidean space (or de Sitter isometry group of $(d+1)$-dimensions). This group (namely $SO(1, d+1)$) comprises translation, rotation, dilatation and special conformal transformation. Although the algebra is same as $SO(1, d+1)$, the action of special conformal transformation is modified to a field-dependent transformation (depends on the metric fluctuation [79]).

We can eliminate the residual gauge freedom by imposing the following gauge-fixing conditions

$$x_1 = 0; \qquad x_2 = 1; \qquad x_3 = \infty \ . \tag{D.3}$$

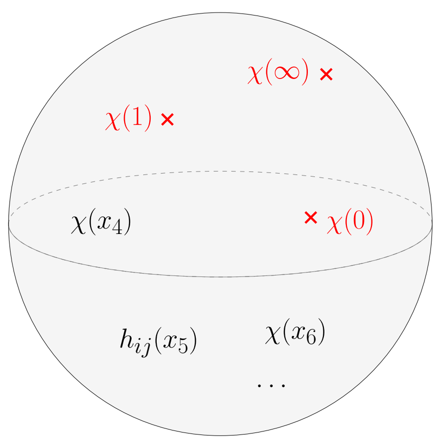

Figure D.1: *We find that the residual gauge group is the Euclidean conformal group in d dimensions, denoted as* $\mathrm{SO}(1, d+1)$*. This group can be fixed, up to a compact subgroup, by specifying the positions of three points.*

Although this does not fix compact degrees of freedom corresponding to $SO(d-1)$

transformations, we could either exclude or integrate them out. The above choice looks very similar to the zero mode fixing of the perturbative string theory.

Let X be an integrated operator on a Cauchy slice, defined as

$$X = \int d\vec{x}\mathcal{X}(x_1, x_2, x_3, \ldots, x_{n+m}) \ . \tag{D.4}$$

We introduce a new notation $\overline{X}$, where we fix 3 points (as prescribed in equation (D.3)) as follows

$$\overline{X} \equiv \int d\vec{x}\, \delta(x_1)\delta(x_2 - 1)\delta(\tilde{x}_3)\mathcal{X}(x_1, x_2, x_3, \ldots, x_{n+m}) \ . \tag{D.5}$$

Here we have set $\tilde{x}_3 = \frac{x_3}{|x_3|^2} = 0$, which fixes the point $x_3 = \infty$. With this we can write the expectation value of operator A in a condensed expression

$$(\Psi, A\Psi) = \sum_{n,m,n',m'} \kappa^{n+n'} \langle\langle \overline{\delta\mathcal{G}[n,m]^* A[g,\chi]\delta\mathcal{G}[n',m']} \rangle\rangle \ . \tag{D.6}$$

The symbol $\langle\langle\, .\, \rangle\rangle$ stands for

$$\langle\langle Q \rangle\rangle \equiv \mathcal{N}_1\mathcal{N}_2 \int DgD\chi\, \delta(g_{ii} - d)\delta(\partial_i g_{ij})\Delta'_{\text{FP}}\, |Z_0[g,\chi]|^2 Q \ . \tag{D.7}$$

The constant $\mathcal{N}_2$ is another physically irrelevant constant, whereas Δ'_{FP} is a restricted Faddeev-Popov determinant which we obtain by integrating out the ghosts excluding for the zero modes.

One remarkable feature of our norm is that we get back the norm prescribed by Higuchi [23] in the non-gravitational limit. In this $\kappa \to 0$ limit, we find the norm is same as the QFT norm divided by the volume of the de Sitter isometry group (upto finite volume factor of $SO(d-1)$). The norm in this limit becomes

$$(\Psi_{\text{ng}}, \Psi_{\text{ng}}) = \frac{\text{vol}(SO(d-1))}{\text{vol}(SO(1,d+1))} \lim_{\kappa\to 0} \langle\langle \delta\mathcal{G}[n,m]^* \delta\mathcal{G}[n,m] \rangle\rangle \ . \tag{D.8}$$

We can also write cosmological correlators in our formalism. These correlators are QFT expectation value of local operators. On the other hand in gravity, there is no local gauge-invariant observable. Hence, we write cosmological correlators as the following gauge-fixed correlators in quantum gravity

$$\langle\langle\Psi|\mathcal{C}^{p,q}_{\vec{i}\vec{j}}(\vec{x})|\Psi\rangle\rangle_{\rm CC} \equiv \sum_{n,m,n',m'} \kappa^{n+n'}\langle\langle\delta\mathcal{G}[n,m]^*\delta\mathcal{G}[n,m]\mathcal{C}^{p,q}_{\vec{i}\vec{j}}(\vec{x})\rangle\rangle \ . \tag{D.9}$$

Here $\mathcal{C}^{p,q}_{\vec{i}\vec{j}}(\vec{x})$ is a product of local operators

$$\mathcal{C}^{p,q}_{\vec{i}\vec{j}}(\vec{x}) = h_{i_1j_1}(z_1)\dots h_{i_pj_p}(z_p)\chi(y_1)\dots\chi(y_q) \ . \tag{D.10}$$

We remind the reader that unlike the norm (equation (D.6)) we do not fix the residual gauge transformation in the equation (D.9). This reflects to the fact that the cosmological correlators exhibits certain symmetries with the algebra same as $SO(1,d+1)$. The correlators transforms covariantly under translation, rotation and scaling. As the action of special conformal transformation includes metric fluctuation, this symmetry relates higher point cosmological correaltors to lower point correlators. Under a translation and a scaling the cosmological correlators change as follows,

$$\langle\langle\Psi|\mathcal{C}^{p,q}_{\vec{i}\vec{j}}(\lambda\vec{x}+\zeta)|\Psi\rangle\rangle{\rm CC} = \lambda^{-q\Delta}\langle\langle\Psi|\mathcal{C}^{p,q}_{\vec{i}\vec{j}}(\vec{x})|\Psi\rangle\rangle_{\rm CC} \ . \tag{D.11}$$

The parameter λ is a number denoting dilatation (scaling). Using the symmetry mentioned above, we can draw an important conclusion about cosmological correlators.

"*The set of all cosmological correlators in any open region $\mathcal{R}$ in a state Ψ determines all observables in the state.*"

More precisely, if all the cosmological correlators inside region $\mathcal{R}$ are same for two different states Ψ_1 and Ψ_2, then correlators involving any other observable A (on the

Cauchy slice) are also the same.

$$\langle\langle\Psi_1|\mathcal{C}^{p,q}_{\vec{i}\vec{j}}(\vec{x})|\Psi_1\rangle\rangle_{\mathrm{CC}} = \langle\langle\Psi_2|\mathcal{C}^{p,q}_{\vec{i}\vec{j}}(\vec{x})|\Psi_2\rangle\rangle_{\mathrm{CC}},\ \ \forall\vec{x}\in\mathcal{R}\ \text{and}\ \forall p,q \implies (\Psi_1, A\Psi_1) = (\Psi_2, A\Psi_2)\,. \tag{D.12}$$

Basically, the states are identified using only the information inside $\mathcal{R}$. This generalizes the principle of holography of information [6–12] to de Sitter space.

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List of Publications

Publication which is included in the thesis:

1. T. Chakraborty, J. Chakravarty, V. Godet, P. Paul and S. Raju, *The Hilbert space of de Sitter quantum gravity*, JHEP 01 (2024) 132, [arXiv:2303.16315].

Other publications:

1. T. Chakraborty, J. Chakravarty, V. Godet, P. Paul and S. Raju, *Holography of information in de Sitter space*, JHEP 12 (2023) 120, [arXiv:2303.16316].
2. T. Chakraborty, J. Chakravarty and P. Paul, *Monogamy paradox in empty flat space*, Phys. Rev. D 106 (2022) 086002, [arXiv:2107.06919].
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