Open Quantum Mechanics for Cosmological Observers

A Thesis

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of **Professor R Loganayagam** at the International Centre for Theoretical Sciences, Tata Institute of Fundamental Research (ICTS-TIFR), Bengaluru.

Omkar Sanjay Shetye

In my capacity as the formal supervisor of record of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

R Loganayagam

Date: 25-08-2025

To my parents

Acknowledgements

Many people have helped me throughout my PhD journey, and it would be impossible to list every single one of them. But I would like to begin with thanking the excellent support system of non-academic staff present at ICTS: housekeeping staff who keep our rooms and offices clean, the cafeteria employees who provide the 6-7 doses of caffeine I go through daily, the people who keep the lights on in the hostels, those who keep our gardens beautiful, and many more who have a silent but vital role in providing the environment in which I could do my research without having to worry about food, shelter and security; which ICTS arguably does at least as well as any of the best institutes on the planet. This is overseen by some of the nicest administrative staff members you could find, and with whom it was always a pleasure to interact.

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What is now proved was once only imagined.

—William Blake, Proverbs of Hell.

Abstract

Inspired by real-time computations in AdS black holes, this thesis proposes a geometric method to obtain the influence phase of a cosmological observer by calculating the on-shell action on a doubled spacetime geometry. The influence phase is the effective action for an open system: for a dS static patch observer coupled to scalar/electromagnetic fields, it incorporates the radiation reaction due to the bulk fields and their dS Hawking radiation. For a general extended source in dS, we describe how to account for finite-size effects. In the long-time limit, we get a Markovian open quantum system susceptible to cosmological fluctuations, whereas the short-time limit reproduces the worldline theory of flat-space radiation reaction. We also present a fully covariantised form for the cubic corrections to the radiation reaction in even spacetime dimensions, including Hubble contributions, and find an intriguing recursive structure across dimensions for the scalar radiation reaction. The self-force is rendered finite through holographic renormalisation applied to the dS static patch connecting two otherwise unrelated regularisation procedures. We also review and extend many properties of vector spherical harmonics(VSHs) in arbitrary dimensions, explain the relation between spherical and cartesian VSH, and derive an addition theorem for VSHs, in order to facilitate the discussion on arbitrary dimensional electromagnetism in both dS as well as flat spacetime.

Chapter 1

Introduction

Over the last few decades, many independent lines of evidence have converged on the fact that our universe has a positive cosmological constant [1–4]. This has presented a difficult conundrum for those who want to think about the relation between gravity and quantum mechanics [5,6]. Among the most fruitful ideas coming out of research in quantum gravity has been holography, i.e. the statement that a gravitational theory is equivalent to a quantum system living on its boundary. However, spacetimes with a positive cosmological constant do not have any time-like boundaries for a dual quantum system to live in. Thus, it seems that gravity in such spacetimes cannot have a holographic dual theory (or at least they cannot have a dual which is a conventional quantum dynamical system).

One attempt to overcome this obstacle is as follows [7, 8]: imagine a lone observer probing such a spacetime. The worldline of such an observer can then be thought of as a time-like boundary where a possible holographic description might reside. This is the idea of *solipsistic holography*, which posits that a quantum system[1] living on such a worldline encodes the quantum theory of gravity that describes the universe. To claim that the information about the entire universe can be gleaned from a single worldline within it might seem speculative, but it is pertinent to remember that all existing knowledge about our universe can be traced ultimately to measurements around the earth. Thus, we might want to assess the viability of such a proposal by examining it further.[2]

[1]Perhaps a large N matrix model as in BFSS duality [9] (See [10, 11] for a review).

[2]Another alternative approach is to focus not on the observables but rather on meta-observables like the global wave-function of the universe [12–19]. As has been emphasised in these references, this approach is especially suited to model the physics of inflation, with us serving as meta-observers to some extent. How the spacetime dynamics get encoded in the proposed dual is not yet entirely clear, though much progress has been achieved over the past few years [20–23].

Any object in a dynamical spacetime influences and is influenced by its surroundings. In this sense, any gravitational observer should be thought of as an open quantum system constantly interacting with the rest of the universe. On the quantum mechanical side, we hope, a similar distinction emerges between the observer's degrees of freedom(which acts as a probe) from the other degrees of freedom, in some appropriate limits. The open system then emerges by integrating out everything except the observer's degrees of freedom. This is the cosmological analogue of the fluid-gravity correspondence [24], as we will demonstrate in this thesis by establishing this analogy further[3]. In the context of the Anti-de Sitter/Conformal Field Theory(AdS/CFT) correspondence, on the gravity side, fluid dynamics emerges by integrating out the physics in the radial direction, whereas on the gauge theory side, it is a consequence of coarse-graining quarks and gluons. There is, by now, non-trivial evidence supporting this statement, including precise matching of anomalous effects on both sides [26, 27]. In a similar vein, one might ask how we could go about checking the cosmological version of this statement.

The central challenge in answering this question is twofold: first, to derive an open quantum system on the worldline from the ambient dynamics. As we shall see, a precise definition of this first step already involves some work.[4] More precisely, what we need is a cosmological analogue of the Gubser-Klebanov-Polyakov-Witten(GKPW) prescription [32, 33] in AdS/CFT that will allow us to derive the open system for an observer. This thesis is aimed at addressing this issue.

The second step would be to construct a dual unitary quantum system that, after integrating out appropriate degrees of freedom, leads to the same open theory as gravity. This might be a hard undertaking: after all, even in the fluid-gravity correspondence, to derive the fluid dynamics from a strongly coupled gauge theory is practically impossible. But, since we are dealing with a quantum mechanical system here, there is reason for hope. One immediate goal would be to check whether the putative open quantum system derived in the first step shows the right structural features to admit a solipsistic interpretation. We will postpone further thoughts on this issue to the discussion section.

[3] An earlier work interpreting worldline holography in terms of incompressible Navier-Stokes can be found in [25].

[4] Systematic description of observers in the middle of a spacetime (as opposed to asymptotic observers) is well-known to be a hard problem. Some of the approaches to the AdS version of this question, starting from the CFT side, can be found in [28–31]. It would be interesting to extend these ideas to take into account the open nature of the observer, as we do here.

Let us return now to the issue of constructing the open system on the gravitational side. Imagine a universe described by a dynamical spacetime along with a variety of fields living on it. A local observer in such a theory may be modelled as a source for these fields: a source that emits/radiates as well as a source that absorbs/detects. Any autonomous motion of the observer is then accompanied by an outgoing radiation and an associated *radiation reaction*. This results in the dissipation of the observer's energy, and we seek an open quantum system that describes this physics. The open quantum theory on the world line should also describe the influence of the incoming radiation from the rest of the universe. As we shall elaborate on later, this incoming radiation also includes the Hawking radiation from the Hubble horizon.[5]

Quite independently of such holographic quests, the worldline open quantum theory under question shows up in a variety of concrete physical questions. As an example, worldline effective field theory has emerged as a useful way to organise the post-newtonian expansion of a binary system radiating gravitational waves [39–43]. The basic idea in such approaches is to systematically integrate out the short-distance gravitational physics that binds the binaries to get an effective theory that describes the inspiral process. Due to the radiation of gravitational waves, ultimately such a binary is also an open system of the type described above. These ideas can be generalised into a cosmological setting where, for example, a *worldline* effective field theory(EFT) which takes also the expansion of the universe into account might be useful in studying the dynamics of galactic formation, cooling and mergers.[6] A motivation of this thesis is to describe an approach that might help us systematically derive such an EFT.

In section §1.1, we begin by describing the basic geometric set-up used in deriving the open quantum mechanics associated with the cosmological observer. The prescription we propose is inspired by the recent developments in real-time AdS/CFT [45–54] that have led to systematic derivation of open quantum systems by integrating out a thermal holographic CFT bath. The essential idea here is a real-time version of Gibbons-Hawking procedure [55]: one proposes an appropriate semi-classical geometry only containing the

[5]See [34] for an analysis of the dS observer from the point of view of von Neumann algebras. It would be interesting to link such an analysis to the ideas discussed in this thesis, e.g., one may ask how the physics of radiation reaction is encoded within von Neumann algebras. Another algebraic statement of potential interest is the 'time-like tube theorem' [35–38], but, again, it is unclear to us how such formal statements relate to the description of dS observer as an open system.

[6]See [44] for the role played by worldline methods in the effective field theory(EFT) of large scale structure(LSS).

relevant region (BH exterior for Gibbons-Hawking, dS static patch in the current problem), and computes the path integral in a saddle-point approximation by evaluating on-shell action. We will argue that such a prescription leads to an answer which correctly encodes both the radiation reaction and Hawking radiation from the Hubble horizon.

The problem of cosmological observer exhibits broad structural similarities to the AdS case, which we exploit. But we also find significant differences: for one, much of the standard holographic machinery (e.g. GKPW prescription, counter-term procedure) available on the AdS side is simply absent. We outline a regularisation procedure that gives finite answers.

1.1 The cosmological influence phase S_{CIP}

Our goal is to describe the experience of an observer in an expanding spacetime. This, in turn, will help us in understanding the spacetime itself. In particular, we want to ask how to construct the open quantum system that describes the cosmological observer. In its full generality, this is a difficult problem, but we can start with a simple model for the observer. We can think of the observer as a single worldline undergoing absorption and emission processes. So the observer is privy to 3 kinds of data:

- Outgoing radiation: Emission data, along with the outgoing propagator, tells us the field values at a later time in the spacetime.
- Incoming radiation: The fields in the past can be reconstructed by using an incoming propagator, given the absorption data.
- Fluctuations: The observer will also be sensitive to *cosmic noise*, which shows up in the absorption data.

This is reminiscent of the motion of a Brownian particle in Langevin theory. A pollen grain in water is sensitive not only to coarse-grained currents in the water (analogous to the incoming radiation) but also to fluctuations arising from the motion of water molecules. Finally, the motion of the Brownian particle can influence the dynamics of water as well (analogous to outgoing radiation).

The dynamics of such open quantum systems can be derived by the path integral prescription of Feynman and Vernon [56] describing the density matrix evolution. Ac-

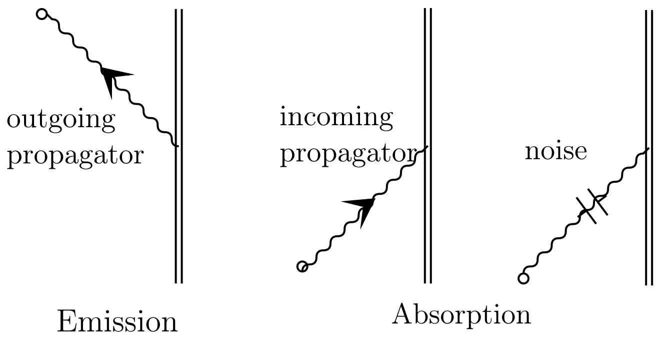

Figure 1.1: A cosmological observer can access 3 kinds of data: radiation due to its own emissions, incoming radiation from sources in the environment and noise.

cording to the authors of [56], the effective description of the open system can be derived starting from two non-interacting copies of each of the system as well as the environment (describing the combined density matrix). Integrating out two copies of the environment then induces new interactions between the copies of the system, resulting in a non-unitary evolution of the system state. These terms constitute the *influence phase*, which encodes completely the effect of the environment on the system. Applying this insight to the question at hand, we conclude that all cosmological effects on an observer(the system) are succinctly summarised in a *cosmological influence phase* S_{CIP}.

What does S_{CIP} depend on? It should depend on how effective the observer is at emitting/absorbing radiation of a given frequency ω and a given multipole type $\mathbb{L}$[7]. Say we have two sets of functions $\mathcal{J}_A(\omega, \mathbb{L})$ and $\mathcal{J}_D(\omega, \mathbb{L})$ characterising the emission/absorption efficiency of the observer. From the Feynman-Vernon viewpoint, $\mathcal{J}_A(\omega, \mathbb{L})$ and $\mathcal{J}_D(\omega, \mathbb{L})$ have the following interpretation: to begin with, we have two copies of the observer (left/right), each probing their copy of the universe via their respective multipole moments $\mathcal{J}_L(\omega, \mathbb{L})$ and $\mathcal{J}_R(\omega, \mathbb{L})$ respectively. The influence phase, which results from integrating out the universe, then depends on the average

$$\mathcal{J}_A(\omega, \mathbb{L}) \equiv \frac{1}{2}[\mathcal{J}_R(\omega, \mathbb{L}) + \mathcal{J}_L(\omega, \mathbb{L})] \ ,$$

as well as the difference

$$\mathcal{J}_D(\omega, \mathbb{L}) \equiv \mathcal{J}_R(\omega, \mathbb{L}) - \mathcal{J}_L(\omega, \mathbb{L}) \ ,$$

[7]The symbol $\mathbb{L}$ is used as a placeholder for various kinds of multipole moments corresponding to scalar, vector, and tensor harmonics on the sphere. For the usual scalar case, $\mathbb{L}$ takes the form $\{\ell, \vec{m}\}$ corresponding to scalar spherical harmonics. Vector and tensor spherical harmonics on arbitrary-dimensional spheres come with additional labels.

of these two multipole moments. The fact that the average/difference sources characterise its emissive/absorptive properties is a well-known feature of the Feynman-Vernon formalism [57–60]: this fact can ultimately be traced to the past/future boundary conditions on the two copies imposed within this formalism. To conclude, the cosmology as seen by an observer with multipole moments $\mathcal{J}_A(\omega, \mathbb{L})$ and $\mathcal{J}_D(\omega, \mathbb{L})$ is encoded in a single influence functional $S_{\text{CIP}}\left[\mathcal{J}_A(\omega, \mathbb{L}), \mathcal{J}_D(\omega, \mathbb{L})\right]$. In terms of the Schwinger-Keldysh path integral of quantum gravity, we can write

$$e^{iS_{\text{CIP}}} \equiv \int [d\varphi_R][d\varphi_L]\ e^{iS_g[\varphi_R,\mathcal{J}_R]-iS_g[\varphi_L,\mathcal{J}_L]}\ , \tag{1.1}$$

where $\varphi_{L,R}$ denote the bra/ket copy of the bulk quantum fields in cosmology (including the spacetime metric) and $S_g[\varphi, \mathcal{J}]$ is the full gravitational action in the background of an observer with multipole moments $\mathcal{J}$. The above path integral should then be interpreted in a wilsonian sense: we want to integrate out the fast modes of quantum gravitational theory, while freezing the slow degrees of freedom of the observer, and obtain an effective action which describes the open dynamics of such an observer.

The cosmological influence phase S_{CIP} is a direct observable. Given an expanding universe, assuming we have a sufficiently long-lived observer with arbitrary multipole moments in some region, the force on an observer due to radiation reaction as well as radiation reception can be directly measured. This force serves to determine all terms in the 'effective action' S_{CIP} that encodes the influence of the ambient universe. All the *real* observables of astrophysics and cosmology, e.g. the sky maps at different frequencies, can be incorporated this way into the absorptive part of S_{CIP}.

From this viewpoint, all cosmological calculations should, in principle, be recast in terms of S_{CIP} to connect them with observations. This is already implicit in the existing approaches to cosmology: for example, the final step in cosmic microwave background(CMB) power spectrum computation is to expand it in spherical harmonics centred around us. Phrasing observables in terms of S_{CIP} makes explicit this observer-dependence (which is probably essential for defining observables *within* a quantum spacetime). Talking in terms of a single functional S_{CIP} may also be convenient for effective field theory (EFT) based approaches to cosmology based on direct observables (e.g. those based on classifying sources in the red-shift space [61–63]). More ambitiously, one may conceive of a bootstrap program based on the cosmological influence phase that complements existing proposals for cosmological bootstrap [17, 19–21, 23, 64].

What are the general principles that constrain S_{CIP}? First of all, when $\mathcal{J}_D(\omega, \mathbb{L})$ is set to zero, S_{CIP} should vanish. This statement arises from the microscopic unitarity of the environment: if the two copies of the observer in Feynman-Vernon formalism introduce identical perturbations into the environment, their effect cancels out of all correlators [59]. From the viewpoint of the observer, the above condition is equivalent to the conservation of the observer density matrix's trace. Apart from this, there are also constraints on S_{CIP} coming from causality. For example, causality implies that the coefficient of $\mathcal{J}_D^*(\omega, \mathbb{L})\mathcal{J}_A(\omega, \mathbb{L})$ is analytic in the upper half plane of complex ω : this coefficient is the retarded correlator on the worldline of the observer [57–60, 65, 66]. A similar statement holds for the coefficients of any term of the form $\mathcal{J}_D^*(\omega, \mathbb{L}) \prod_k \mathcal{J}_A(\omega_k, \mathbb{L}_k)$.

Evaluation of the influence phase requires us to know the real-time or Schwinger-Keldysh (SK) propagators of the environment. It is unclear how to perform such computations in generic cosmological spacetimes, especially if gravity is also to be quantised. We will show that for an observer in dS, this computation can be geometrised roughly akin to recent implementations of SK path integrals in case of AdS black holes [45–54]. The hope then is that one can later generalise it beyond dS to incorporate full FRW cosmology.

Specifically, in the case of dS, we conjecture that the computation of cosmological influence phase S_{CIP} is dominated by a geometric saddle point built out of two copies of the static patch stitched together at their horizons. We will call this doubled geometry, the dS Schwinger-Keldysh(dS-SK) spacetime. In the rest of this subsection, we will describe this geometry in more detail before moving to the evidence for our conjecture in the subsequent sections.

Let us begin by setting up the basic notation required: consider a $(d+1)$-dimensional de Sitter spacetime dS_{d+1} whose Penrose diagram is shown in Fig.1.2. A horizontal slice (i.e., a constant time slice) in this diagram denotes the prime-meridian on a spatial sphere S^d, with the two ends denoting the poles. Each point in the horizontal slice corresponds to a sphere S^{d-1}, which shrinks to a point near the poles. The first example we will consider is a co-moving dS observer whom we place at the south pole. Our focus will be on the static patch of such an observer, i.e., the patch between the past and future cosmological horizons of the observer. We will later describe a more general class of

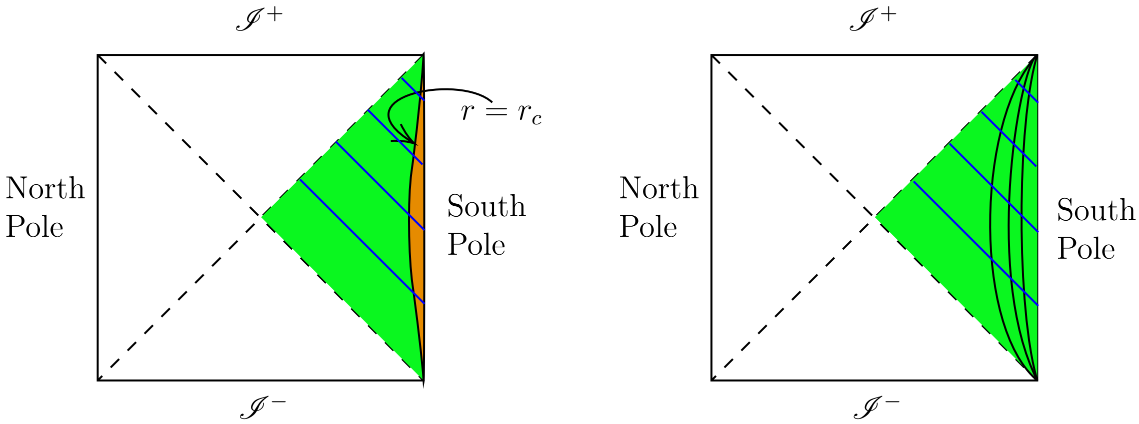

Figure 1.2: Penrose diagrams of dS with the static patch of the south pole observer shown in green. Constant u slices are shown in blue. **Left :** a localised observer at the south pole whose worldline is thickened to a world-tube(orange) of radius r_c. **Right :** an extended observer modelled as a sequence of spherical shells of radius r_i with $i = 1, \ldots, N$. [67] shows how the Penrose diagram is obtained through conformal compactification and how various slicings look like on the diagram.

observers spread arbitrarily over this static patch, modelled as a sequence of spherical shells around south pole (Fig.1.2).

We will find it convenient to work with *outgoing* Eddington-Finkelstein coordinates on the static patch. The metric in this coordinate system takes the form

$$ds^2 = -(1 - r^2 H^2)\, du^2 - 2du dr + r^2 d\Omega_{d-1}^2 \ . \tag{1.2}$$

Here H is the Hubble constant of dS spacetime, r is the radial distance from the observer, u denotes the outgoing time labelling the outgoing waves and $d\Omega_{d-1}^2$ is the line element on a unit S^{d-1} sphere. The south-pole observer sitting at $r = 0$ sees a future horizon at $r = 1/H$ where the outgoing coordinates are well-behaved. In most of what follows, we will set $H = 1$ for convenience and restore it later when we examine the flat space (i.e. $H \to 0$) limit.

We now turn to the model of the observer: Conceptually, the simplest model is that of a point particle with specified multipole moments sitting at $r = 0$. However, such a model needs to be regulated with appropriate counter-terms to allow the computation of radiation reaction effects. To this end, we will take the observer to be a small sphere of radius r_c and thicken its worldline into a time-like 'world-tube'. The point particle limit then corresponds to taking $r_c \to 0$ limit *after* the addition of counter-terms: both the Green functions and required counter-terms can be determined exactly for a dS observer coupled to generalised free scalar fields. The radius r_c then acts like a UV regulator for the problem.

Apart from the formal requirements of regularisation, we are also interested in the actual problem of an extended observer of a finite size. In such a case, there are no divergences. Nevertheless, finite counter-terms are needed to renormalise the bare parameters into the physically measured properties of the observer. As mentioned before, a simple model of the extended observer is a sequence of spherical shells of radius r_i with $i = 1, \ldots, N$: their complement in the static patch is then the rest of the universe to be integrated out. We can define multipole moments for such extended observers and still write down a cosmological influence phase as a function of those multipole moments. The locality in radial direction gets obscured in such a description: this is however natural in the solipsistic viewpoint where radial locality is an approximate/emergent property of the dual quantum mechanics.

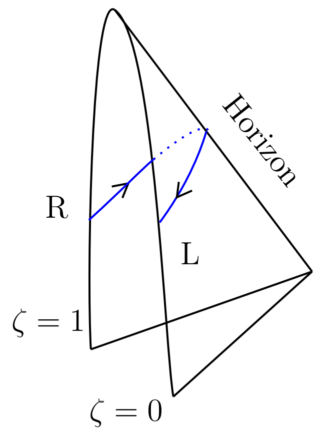

Figure 1.3: The two sheeted complex dS-SK geometry can be thought of as two static patches smoothly connected at the future horizon. The radial contour along an outgoing Eddington-Finkelstein slice (i.e., a constant u slice) is shown in blue. The radial contour has an outgoing R branch and an incoming L branch.

We now turn to our conjecture for the de Sitter-Schwinger Keldysh(dS-SK) geometry, i.e., the semi-classical saddle point that dominates the quantum gravity path integral for S_{CIP}. What we seek is a real-time analogue of the Gibbons-Hawking construction [68] as well as gr-SK construction in AdS [45–49, 52], which would compute for us the cosmological influence phase. Here is the geometry we propose: take two copies of the static patch and stitch them together smoothly at the future horizon (Fig.1.3). To parametrise this geometry, we complexify the radial coordinate and think of dS-SK as a co-dimension one contour in the complex r plane (Fig.1.4). To make this precise, let us define a *mock tortoise coordinate* ζ as follows:

$$\zeta(r) = \frac{1}{i\pi} \int_{r}^{0-i\epsilon} \frac{dr'}{1-r'^2} = \frac{1}{2\pi i} \ln\left(\frac{1-r}{1+r}\right) . \tag{1.3}$$

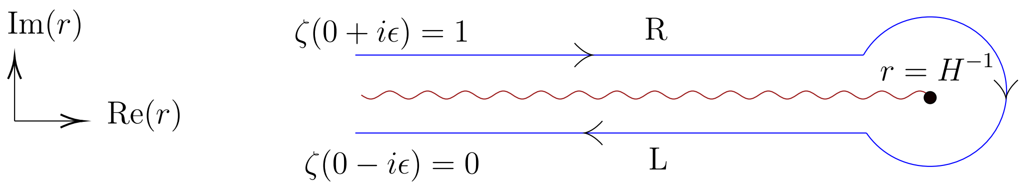

Figure 1.4: The branch cut structure of $\zeta(r)$ in the complex r plane at fixed u: branch-cut shown as a wiggly line. We also show the *clockwise* dS-SK radial contour running from $\zeta = 1$ to $\zeta = 0$ (the blue curve in this figure and in Fig.1.3). The Im $r > 0$ branch is the time-ordered/right branch, whereas the Im $r < 0$ branch is the anti-time-ordered/left branch.

This integral has logarithmic branch points at $r = \pm 1$ and we choose its branch-cut to be over the interval $r \in [-1, 1]$ on the real line. As shown in Fig.1.4, our normalisation is such that, if we begin from $0 + i\epsilon$ (i.e., just above the midpoint of the branch-cut) and then go clockwise around the branch cut to $0 - i\epsilon$ (i.e., just above the midpoint of the branch-cut), we pick up a discontinuity in ζ equal to negative unity. The choice of the overall constant in (1.3) is such that the real part is 1 on the R static patch (the $r + i\epsilon$ contour), and the real part falls to zero as we move clockwise and turn to traverse the L boundary(the $r - i\epsilon$ contour)[8]. The horizon in this geometry then becomes the entire circle around $r = H^{-1}$ of the contour, sometimes referred to as the 'horizon cap'.

We are now ready to state our prescription:

$$\textbf{Cosmological influence phase} = \\ \textbf{On-shell gravitational action of the dS-SK geometry} . \tag{1.4}$$

[8]The reader should note the use of clockwise contours in the complex r plane for dS, in contrast to the counter-clockwise contours used in the AdS black-brane case. This fact means that we need to be careful to add appropriate minus signs whenever we use the residue theorem, but this inconvenience seems unavoidable given the standard time orientations of the Schwinger-Keldysh contour.

To be clear, on both sides of this equality, we treat observer(s) as prescribed sources, viz., we take it off-shell by freezing its dynamics. Both sides can then be thought of as functionals of the observer multipole moments that emit/detect fields. In the dual quantum mechanics, these multipole moments should be thought of as the 'slow macroscopic degrees of freedom' whose influence phase is computed by integrating out the 'fast microscopic degrees of freedom'. The solipsistic holography would then imply that we can replace the LHS in the above equality with such an influence phase computed in the dual quantum mechanics. The above statement can then be thought of as giving a GKPW-like prescription [32, 33] for solipsistic holography. The primary aim of this note is to exhibit simple example systems where we can show that the above prescription yields sensible answers.

Before we turn to examples, we would like to comment on an interesting philosophical point: In this geometric picture, the cosmology reduces entirely to the static patch accessible to the observer, bypassing questions about the rest of the universe (or multiverse as the case may be). We think of this focus on actual observables as a desirable feature of our proposal, in contrast to traditional descriptions of quantum gravity in dS spacetime phrased in terms of global questions. In the AdS black-brane case, gravitational Schwinger-Keldysh geometry (and its Gibbons-Hawking predecessor) divorces the phenomenology of the exterior from speculations about singularity and BH interior. In a similar vein, our geometric proposal aims at isolating the physics of the static patch from speculations about super-horizon modes, side-stepping the measure problem in cosmology. Our saddle point geometry can be thought of as a way to implement the causal-diamond-based cosmological measures ala Bousso [69, 70].

1.2 Radiation reaction in de Sitter

The experimental detection of gravitational waves has brought renewed attention to the problem of radiation reaction and self-force in classical field theories. The standard puzzle is easily stated: given a point charge in arbitrary motion, the self-force due to its own electromagnetic fields seems naively infinite. Both experimental evidence as well as the momentum flux at infinity suggest that this conclusion is plainly wrong! There is a finite electromagnetic force on the particle due to a *renormalised* field.

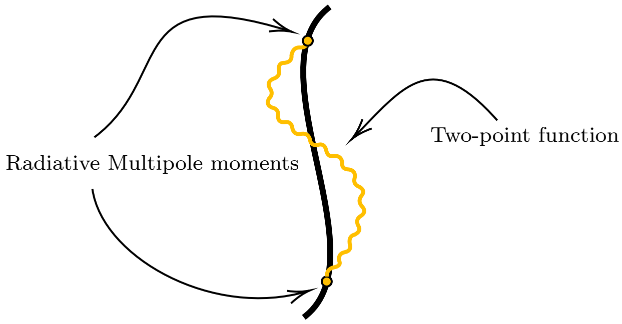

Figure 1.5: Radiation reaction computation has two main ingredients: radiative multipole moments and a two-point function describing how one multipole moment affects another. The black solid line denotes the trajectory of the source.

Over the past century, many successful proposals have been made to address the issue mentioned above. The main idea is twofold: we first solve for the field produced by a charge with *outgoing* boundary conditions far away. Next, we identify and remove the divergent pieces from this field near the charge to get a finite answer.

It is instructive to contrast this procedure against the holographic prescription in AdS/CFT to compute thermal CFT correlators. As described by Son and Starinets [71], the AdS/CFT computation proceeds again in two steps: first, we take a bulk black hole and impose *infalling* boundary conditions at its horizon. Next, we look at this solution near the conformal boundary of AdS, put counter-terms and read off the renormalised CFT correlators [33, 72, 73]. This similarity suggests that we can think of *holography as a kind of radiation reaction*[9]. Taking such a slogan seriously might give us a way to generalise holographic insights to spacetimes other than AdS.

The EM radiation reaction problem, of course, has a long and rich history[10], but we shall see that our 'dS holographic perspective' adds new elements to this story. We will see how the counterterm procedure in the radiation reaction(RR) problem mirrors the one used in AdS holography. We would also like to point out how the RR problem in de Sitter is somewhat better behaved than one in flat spacetime. We will show how almost all memory/tail effects in the scalar RR problem go away at cosmologically long times. This is true even in the case of odd-dimensional spacetimes, where the flat spacetime RR problem has serious memory/tail effects. We will see that the dS version of the EM RR problem is also better behaved in this sense.

[9]See [74, 75] for how *absorption* processes also have holographic features.

[10]See, e.g., references [76–83] for the $3 + 1$ dimensional version. For higher dimensions, we refer the reader to [84–91].

In essence, the charge always forgets its past at cosmological time scales, and the long-time physics is always that of a Langevin particle executing Brownian motion within dS thermal bath. For fast motions/short time scales, the dS RR problem should approach the flat spacetime answers. Thus, de Sitter provides a nice infrared cut-off for the RR problem.[11] This statement can be made mathematically precise at the level of Fuchsian ODEs that control free theory radial functions. In flat spacetime, these are Bessel-like functions with one regular singularity at $r = 0$ and another irregular singularity at $r = \infty$. Once we move to de Sitter, the irregular singularity at $r = \infty$ splits into two *regular* singularities. The consequence is that radial functions in dS are hypergeometric functions with three regular singularities. Our goal in this work is to generalise all these above statements to the case of scalar and electromagnetic fields.

1.3 Outline

We will conclude the introduction with a brief outline of the thesis.

Chapter §2 computes the influence phase for an observer interacting with a generic class of scalar fields. First, we will describe the outgoing solutions for these scalars in the de Sitter static patch in detail (§2.1). We will then use the outgoing solutions and their time reversal to obtain the solutions on the doubled dS-SK geometry §2.2. These solutions are substituted back into the scalar action. The on-shell action is regulated, which provides the correct influence phase. Finally, we will verify this claim by computing the correct radiation reaction experienced by the observer by reproducing the known flat space results for scalar radiation reaction as well as the cosmological corrections to it in §2.3.

Chapter §3 describes the electromagnetic observer's influence phase. There, we will follow a similar path to that of the scalar case: retarded solutions, dS-SK solutions, influence phase and its regularisation (§3.1) and finally, the cosmological corrections to the Abraham-Lorentz-Dirac force obtained from our influence phase(§3.2). This further establishes that ideas of holographic renormalisation can be fruitfully adapted to the self-force regularisation problem.

[11] We are used to thinking of AdS as a good IR cutoff, but the fact that AdS is a confining spacetime makes the AdS RR problem non-markovian (i.e., it is expected to have even worse memory problems than the flat spacetime version). Consequently, one does not expect a local description of the self-force at long times, unlike what happens in de Sitter.

We end the main chapters of the thesis with a summary and discussion in chapter §4.

We develop most of our notation in de Sitter in a manner that can be easily compared with flat space results in the zero curvature limit. To facilitate such a comparison, we review the flat space radiation reaction problem extensively in the appendices. Appendix A discusses scalar multipole radiation and radiation reaction in flat spacetime. The additional technical complication in electromagnetism is the appearance of vector spherical harmonics (VSHs) both in their spherical and cartesian avatars. We found that the existing literature on VSHs had many gaps that need to be addressed to solve our problem. We face this challenge head-on in our appendix B, where the reader can find many new results about VSHs in arbitrary dimensions. They include a VSH version of the addition theorem, the relation between cartesian and spherical VSHs, and a set of toroidal operators in higher dimensions, which generalise the famous $-\vec{r}\times\vec{\nabla}$ operator in $\mathbb{R}^3$. Apart from these new results, we also review existing approaches to VSHs based on symmetric trace-free(STF) tensors as well as weight-shifting operators.

Once the technical machinery of VSHs is in place, the next step is to take the flat space EM multipole expansion and then see how it extends to de Sitter. Here we encounter our next obstacle. The existing literature on EM multipole expansions can be divided into two disjoint sets, one focused on spherical harmonic methods and the other on cartesian STF tensor methods. We found that the de Sitter problem requires an efficient mix of *both* these methods. Even in flat spacetime, the conversion between these two methods is not clearly explained in the current literature. We address this lacuna in appendix C, which consolidate our knowledge about EM multipole expansions in flat spacetime.

Chapter 2

Designer scalar in dS

In this chapter, we will evaluate the on-shell action on the dS-SK geometry described above and show that we get meaningful semi-classical results for the cosmological influence phase S_{CIP}. We will do this in three parts: First, in the following section, we will describe a class of systems where observers act as sources for scalar fields. We will describe how the on-shell action can be computed for these systems to yield S_{CIP}. Next, in section§2.3, we will argue how S_{CIP} does indeed capture the physics of radiation reaction for a moving dS observer. Finally, in section §2.4, we will describe how field interactions could be taken into account.

The general class of scalar fields we will analyse in this section, deemed 'designer scalars', are a two parameter $\{\mathcal{N}, \mu\}$ family that extends the usual Klein-Gordon field. These designer scalars, for specific values of $\mathcal{N}$ and μ, encode the physics of electromagnetic fields as well as linearised gravitational perturbations [92, 93] (see table 2.1). As such, the gauge field and gravity problems have their own peculiarities that should be addressed when computing the corresponding S_{CIP}. Yet, several results from the designer scalar analysis can be directly borrowed when analysing electromagnetic fields and linearised gravity, and hence we analyse the full class of such scalars.

2.1 Green functions, regularisation and renormalisation

We begin with the description of the scalar Green functions in dS spacetime in some amount of detail. Our focus will be on a point-like observer sitting on the south pole, and our Green functions are all hence 'boundary-to-bulk' with the boundary being the world line at the south pole. The point-like nature necessitates a careful discussion of regularisation, counter-terms etc.: our discussion will closely parallel the flat spacetime discussion in the appendix A.3 as well as the dS discussion in [7, 94]. We will also confine ourselves to a single copy of the static patch in this section, relegating the applications to dS-SK to the next section.

We will work with outgoing Eddington Finkelstein(EF) coordinates [67] describing the static patch of dS spacetime dS_{d+1}. This spacetime is a solution of the Einstein equations with a positive cosmological constant

$$\Lambda = \frac{1}{2}d(d-1)\ . \tag{2.1}$$

We have chosen units where the Hubble constant is unity. The spacetime metric is

$$ds^2 = -2\ du\ dr - (1-r^2)\ du^2 + r^2 d\Omega_{d-1}^2\ . \tag{2.2}$$

Here $d\Omega_{d-1}^2$ denotes the metric on a unit $\mathbb{S}^{d-1}$. The outgoing Eddington Finkelstein time u is related to the more commonly used time t via $u = t - r_*$ where r_* is the tortoise coordinate defined via

$$r_* \equiv -i\pi\zeta \equiv \int_0^r \frac{d\rho}{1-\rho^2} = \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right)\ . \tag{2.3}$$

The radial coordinate r is centred around a static observer sitting at $r = 0$. We will mostly work with the frequency domain where the time dependence of fields[1] is taken to be $\sim e^{-i\omega u}$. Further, we will decompose everything into appropriate spherical harmonics on $\mathbb{S}^{d-1}$. The spherical harmonics are labelled by the eigenvalue of the sphere Laplacian $\nabla^2_{\mathbb{S}^{d-1}}$ which is $-\ell(\ell+d-2)$.

As described in the main text, we will consider a class of *designer* scalar systems in

[1]We note a slight inconsistency in our definitions when compared to definitions in the appendix A. In appendix A, we fourier-transformed with respect to standard time slices, whereas here in dS we are fourier-transforming with respect to outgoing EF time u. Since in flat spacetime $u = t - r$, this means that all the flat space radial functions in appendix A should be multiplied with a pre-factor of $e^{-i\omega r}$ before they can be compared against the dS results described here.

Table 2.1: $\mathcal{N}, \mu$ values for different massless fields: massless Klein Gordon scalar, electric and magnetic parity sectors of electromagnetism and gravity as well as the additional tensor sector that arises for $d > 3$ linearised gravity.

	KG Sca	EM Mag	EM Elec	Grav Tens	Grav Mag	Grav Elec
$\mathcal{N}$	$d-1$	$d-3$	$3-d$	$d-1$	$1-d$	$3-d$
μ	$\frac{d}{2}$	$\frac{d}{2}-1$	$\frac{d}{2}-2$	$\frac{d}{2}$	$\frac{d}{2}-1$	$\frac{d}{2}-2$

dS with an action

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\, r^{\mathcal{N}+1-d}\Big\{\partial^\mu\Phi_\mathcal{N}\,\partial_\mu\Phi_\mathcal{N} \\ + \frac{\Phi_\mathcal{N}^2}{4r^2}\left[(d+\mathcal{N}-3)(d-\mathcal{N}-1) - r^2\left(4\mu^2-(\mathcal{N}+1)^2\right)\right]\Big\}\ . \tag{2.4}$$

After we strip out the harmonic dependence in time/angles, the above action results in a radial ODE of the form

$$\frac{1}{r^\mathcal{N}}D_+[r^\mathcal{N}D_+\varphi_\mathcal{N}] + \omega^2\varphi_\mathcal{N} \\ + \frac{1-r^2}{4r^2}\Big\{(\mathcal{N}-1)^2 - (d+2\ell-2)^2 + [4\mu^2-(\mathcal{N}+1)^2]r^2\Big\}\varphi_\mathcal{N} = 0\ . \tag{2.5}$$

Here $\varphi_\mathcal{N}(r,\omega,\ell,\vec{m})$ is the radial part of the field, the derivative operators $D_\pm \equiv (1-r^2)\partial_r \pm i\omega$, and the equation depends on the parameters $\{\mu,\mathcal{N},\ell\}$ whose physical interpretation will be clear momentarily.

The combination $(\mathcal{N}+1)^2-4\mu^2$ can be interpreted as a mass term $4m^2$ for the scalar in Hubble units. The exponent $\mathcal{N}$ describes the auxiliary radial varying dilaton. The index ℓ is associated with the eigenvalue of the sphere laplacian. The expressions involved simplify considerably if we use, instead of ℓ, the following parameter:

$$\nu \equiv \frac{d}{2} + \ell - 1\ . \tag{2.6}$$

For example, in terms of ν, the eigenvalue of the sphere laplacian becomes $(\frac{d}{2}-1)^2-\nu^2$. Since we will be concerned with the cases where $d > 2$ and $\ell \geq 0$, ν is a positive number. We can then rewrite the above ODE as

$$\frac{1}{r^\mathcal{N}}D_+[r^\mathcal{N}D_+\varphi_\mathcal{N}] + \omega^2\varphi_\mathcal{N} \\ + \frac{1-r^2}{4r^2}\Big\{(\mathcal{N}-1)^2 - 4\nu^2 + [4\mu^2-(\mathcal{N}+1)^2]r^2\Big\}\varphi_\mathcal{N} = 0\ . \tag{2.7}$$

It is instructive to rewrite the above ODE in terms of a new field $\psi \equiv r^{\frac{\mathcal{N}}{2}}\varphi_{\mathcal{N}}$ as

$$(D_+^2 + \omega^2)\psi + \frac{1-r^2}{4r^2}\Big\{1 - 4\nu^2 + [4\mu^2 - 1]r^2\Big\}\psi = 0 \ . \tag{2.8}$$

The absence of $\mathcal{N}$ in this ODE shows that $\mathcal{N}$ merely controls the overall pre-factor. We also note a symmetry under $\nu \mapsto -\nu$ and $\mu \mapsto -\mu$: either of these sign changes should map one solution to the other.

2.1.1 Outgoing Green function

The above second-order radial ODE can be exactly solved in terms of hypergeometric functions. The worldline to bulk outgoing Green function is given by [7, 94, 95]

$$\begin{aligned} G^{\text{Out}}_{\mathcal{N}}(r,\omega,\ell) = r^{\nu-\frac{\mathcal{N}}{2}}(1+r)^{-i\omega}\frac{\Gamma\left(\frac{1+\nu-\mu-i\omega}{2}\right)\Gamma\left(\frac{1+\nu+\mu-i\omega}{2}\right)}{\Gamma(1-i\omega)\Gamma\left(1+\nu\right)} \\ \times\ {}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1-i\omega; 1-r^2\right] \ . \end{aligned} \tag{2.9}$$

Here we have fixed the overall normalisation by an appropriate boundary condition to be described below. We will devote this subsection to a detailed study of the above Green function.

We remind the reader that the hypergeometric function always has a nice series expansion around the point where its last argument vanishes. It then follows that the above solution is manifestly regular at the future horizon $r = 1$ without any branch cuts or poles. An alternate form for the same function that emphasises the small r behaviour near the observer's worldline is

$$\begin{aligned} G^{\text{Out}}_{\mathcal{N}} &= r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}(1+r)^{-i\omega} \\ &\times \Big\{{}_2F_1\left[\frac{1-\nu+\mu-i\omega}{2}, \frac{1-\nu-\mu-i\omega}{2}; 1-\nu; r^2\right] \\ &\quad -(1+i\cot\nu\pi)\widehat{K}_{\text{Out}}\frac{r^{2\nu}}{2\nu}\ {}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1+\nu; r^2\right]\Big\} \ . \end{aligned} \tag{2.10}$$

Here $\widehat{K}_{\text{Out}}$ is the worldline retarded Green function given by the expression [7,94].

$$\begin{aligned}\widehat{K}_{\text{Out}}(\omega,\ell) &\equiv 2\frac{\Gamma\left(\frac{1+\nu-\mu-i\omega}{2}\right)\Gamma\left(\frac{1+\nu+\mu-i\omega}{2}\right)\Gamma\left(1-\nu\right)}{\Gamma\left(\frac{1-\nu+\mu-i\omega}{2}\right)\Gamma\left(\frac{1-\nu-\mu-i\omega}{2}\right)\Gamma\left(\nu\right)\left(1+i\cot\nu\pi\right)} \\ &= -e^{i\nu\pi}\frac{2\pi i}{\Gamma(\nu)^2}\frac{\Gamma\left(\frac{1+\nu-\mu-i\omega}{2}\right)\Gamma\left(\frac{1+\nu+\mu-i\omega}{2}\right)}{\Gamma\left(\frac{1-\nu+\mu-i\omega}{2}\right)\Gamma\left(\frac{1-\nu-\mu-i\omega}{2}\right)}\ .\end{aligned} \tag{2.11}$$

The reason for choosing the normalisation of $\widehat{K}_{\text{Out}}$ this way will become clear eventually. The above equation is the dS analogue of the Hankel function decomposition into Neumann and Bessel functions. As in Eq.(A.63), when d is even and ν is an integer, the above expression should be understood as a limit, with the $\cot\nu\pi$ divergence exactly cancelling the divergence in the first term of Eq.(2.10).

The hypergeometric identity used for the above decomposition is

$$\begin{aligned}&\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c)}z^{a+b-c}{}_2F_1(a,b;c;1-z) \\ &= {}_2F_1(c-a,c-b;1+c-a-b;z) \\ &\quad + z^{a+b-c}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c-a-b)}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b-c+1;z)\ ,\end{aligned} \tag{2.12}$$

where we have taken

$$a=\frac{1+\nu-\mu-i\omega}{2}\ ,\ b=\frac{1+\nu+\mu-i\omega}{2}\ ,\ c=1-i\omega\ ,\ z=r^2\ . \tag{2.13}$$

In all these identities, we take the branch cuts of hypergeometric functions as well as $(1+r)^{-i\omega}$ to be outside the open unit disk in the complex r plane. Thus, all these functions are analytic within the open static patch and in turn, on the dS-SK contour.

With this new form for the outgoing Green function, it is straightforward to obtain a near-origin expansion to all orders. The explicit expressions are given by

$$\begin{aligned}&{}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2},\frac{1+\nu+\mu-i\omega}{2};1+\nu;r^2\right] \\ &=\sum_{k=0}^{\infty}\frac{r^{2k}}{(2k)!}\frac{(\nu-\mu-i\omega-1+2k)!!}{(\nu-\mu-i\omega+1)!!}\frac{(\nu+\mu-i\omega-1+2k)!!}{(\nu+\mu-i\omega-1)!!}\frac{(2\nu)!!(2k-1)!!}{(2\nu+2k)!!}\ ,\end{aligned} \tag{2.14}$$

as well as

$$
\begin{aligned}
&{}_2F_1\left[\frac{1-\nu+\mu-i\omega}{2},\frac{1-\nu-\mu-i\omega}{2};1-\nu;r^2\right]\\
&=\sum_{k=0}^{\infty}\frac{(-r^2)^k}{(2k)!}\frac{(\nu-\mu+i\omega-1)!!}{(\nu-\mu+i\omega-1-2k)!!}\frac{(\nu+\mu+i\omega-1)!!}{(\nu+\mu+i\omega-1-2k)!!}\frac{(2\nu-2-2k)!!(2k-1)!!}{(2\nu-2)!!}\ .
\end{aligned}
\tag{2.15}
$$

The second expansion can be interpreted literally only for d odd (i.e., when $\nu \in \mathbb{Z}+\frac{1}{2}$). For d even, the above expansion (and most of the discussion below) should be understood in a dimensionally regularised sense.

The near-origin form of the outgoing Green function shows the normalisation

$$
\lim_{r\to 0} r^{\nu+\frac{\mathbb{N}-1}{2}} G^{\text{Out}}_{\mathbb{N}} = 1\ . \tag{2.16}
$$

This condition can be thought of as the dS analogue of the condition on the AdS boundary-to-bulk Green function. As in that case, the above condition along with outgoing property/analyticity at the future horizon uniquely determines $G^{\text{Out}}_{\mathbb{N}}$. Extending this analogy to AdS, we can roughly read off the retarded worldline Green function $\widehat{K}_{\text{Out}}$ by looking at the ratio of coefficients of the sub-dominant solution to the dominant solution in the outgoing solution $G^{\text{Out}}_{\mathbb{N}}$. This is essentially the Son-Starinets prescription [71] of AdS/CFT adapted to the present dS context. Such analogies have been noted before in [7]: our aim here is to give a more systematic derivation of these statements, taking into account the subtleties associated with divergences, regularisation, finite size effects, etc.

To this end, let us begin with a physical interpretation of the outgoing Green function $G^{\text{Out}}_{\mathbb{N}}$. If we are given that $\varphi_{\mathbb{N}}$ behaves at small r near the worldline as

$$
\varphi_{\mathbb{N}}(r,\omega,\ell,\vec{m}) = \frac{\mathfrak{J}(\omega,\ell,\vec{m})}{r^{\nu+\frac{1}{2}(\mathbb{N}-1)}} + \dots\ , \tag{2.17}
$$

where $\mathbb{S}_{\mathbb{L}}$ is a spherical harmonic on $\mathbb{S}^{d-2}$, we then have a unique outgoing solution

$$
\varphi_{\mathbb{N}}(r,\omega,\ell,\vec{m}) = G^{\text{Out}}_{\mathbb{N}}(r,\omega,\ell)\mathfrak{J}(\omega,\ell,\vec{m})\ ,
$$

describing the field that is radiated out of the worldline. This is the dS analogue of the outgoing Hankel Green function in flat space.[2]

The alternate form we have written down above in Eq.(2.10) is then the dS version of the familiar statement[3] that the outgoing Hankel Green function can be written as the sum of a Neumann Green function (which diverges near the origin) and a Bessel J function (which is regular at the origin). Such a decomposition of the outgoing Green function into a singular Green function and a regular solution is a first step in Dirac's approach to the self-force [79] (the curved space version is sometimes also termed as the Detweiler-Whiting decomposition [96]). We will later show in section §E.2 that our answer matches in dS_4 with the regular part quoted in [97, 98] using the rules of Deitweiler-Whiting decomposition.

2.1.2 Renormalised conjugate field and K_{Out}

We will now turn to the question of deriving the worldline Green function K_{Out} from the outgoing Green function $G^{\text{Out}}_{\mathcal{N}}$. As we will describe in detail below, the physics here is that of radiation reaction and the main subtlety is how to deal with divergences. Our main strategy here will be to define a renormalised conjugate field which reduces to K_{Out} near the source worldline. The idea here is philosophically similar to other radiation reaction computations in the literature [39–43, 96] as well as the counter-term subtraction in AdS/CFT [72]. The implementation is however sufficiently different that we provide a detailed analysis below.

The radial ODE Eq.(2.7) can be derived by extremising the action

$$\begin{aligned} S = -\frac{1}{2}\sum_{\mathbb{L}}\int\frac{d\omega}{2\pi}\oint\frac{r^{\mathcal{N}}dr}{1-r^2}\Big[&(D_+\varphi_{_\mathcal{N}})^*D_+\varphi_{_\mathcal{N}}-\omega^2\varphi^*_{_\mathcal{N}}\varphi_{_\mathcal{N}} \\ &-\frac{1-r^2}{4r^2}\Big\{(\mathcal{N}-1)^2-4\nu^2+[4\mu^2-(\mathcal{N}+1)^2]r^2\Big\}\varphi^*_{_\mathcal{N}}\varphi_{_\mathcal{N}}\Big]+S_{ct}\ . \end{aligned} \tag{2.18}$$

Here S_{ct} denotes the counter-term action to be determined later. The integration over r ranges over the regulated dS-SK contour (clockwise from the right static patch to the left static patch) and, in addition, we have indicated an integration over all frequencies and a sum over spherical harmonics. The reality condition in the Fourier domain takes

[2]More precisely, in outgoing EF coordinates the corresponding Green functions in flat space are the outgoing Hankel Green functions given in Eq.(A.63) multiplied with a prefactor of $e^{-i\omega r}$. See footnote 1 for an explanation for this pre-factor.

[3]We review this statement, for the benefit of the reader, around Eq.(A.63).

the form

$$\varphi^*_{\mathcal{N}}(r,w,\ell,\vec{m}) = \varphi_{\mathcal{N}}(r,-\omega,\ell,-\vec{m}) \ . \tag{2.19}$$

Here $\vec{m}$ denotes the additional labels appearing in the spherical harmonic decomposition.

The canonical conjugate field for radial evolution is obtained by varying the above action with respect to $\partial_r\varphi^*_{\mathcal{N}}$ which yields $-r^{\mathcal{N}}D_+\varphi_{\mathcal{N}}$ after we take into account the fact that $\varphi^*_{\mathcal{N}}$ and $\varphi_{\mathcal{N}}$ are related by the reality condition quoted above. The minus sign in the canonical conjugate is because we are looking at evolution along a space-like direction.

Taking into account the powers of r multiplying the multipole moment $\mathcal{J}$ in Eq.(2.17), the canonical conjugate of $\mathcal{J}$ should be defined with the opposite power, viz., we should consider instead

$$-r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}\left[r^{\mathcal{N}}D_+\varphi_{\mathcal{N}}\right] \ . \tag{2.20}$$

The canonical conjugate field of the radial evolution at the two regulated boundaries is given by evaluating the above expressions at $r=r_c\pm i\varepsilon$. Naively the $r_c\to 0$ limit should then yield the required canonical conjugate that couples to the right/left point multipole source. This limit however does not work: on a generic solution, the $r_c\to 0$ limit is beset with divergences. Appropriate counter-terms need to be added to the above bare expression before a sensible $r_c\to 0$ limit can be taken. The counter-terms arise from adding in a worldline counter-term action

$$S_{ct} = -\frac{1}{2}\sum_{\mathbb{L}}\int\frac{d\omega}{2\pi}r^{\mathcal{N}-1}\mathscr{C}_{\mathcal{N}}(r,\omega,\ell)\varphi^*_{\mathcal{N}}\varphi_{\mathcal{N}}|_{\text{Bnd}} \ . \tag{2.21}$$

Here $|_{\text{Bnd}}$ refers to the fact that we add such a contribution at every boundary. Being a boundary contribution, this addition does not change the equations of motion for the scalar field. If the original variational principle was defined with a Dirichlet boundary condition $\delta\varphi_{\mathcal{N}}|_{\text{Bnd}}=0$, the counterterm above does not change that boundary condition.

In the above expression, we should take $\mathscr{C}_{\mathcal{N}}(r,\omega,\mathbb{L})$ to be a real and even function of ω to get a real counter-term action. Addition of this worldline action modifies the canonical conjugate evaluated at the radial boundaries to

$$r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}\pi_{\mathcal{N}} \equiv -r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}\left[r^{\mathcal{N}}D_{+} + r^{\mathcal{N}-1}\mathscr{C}_{\mathcal{N}}\right]\varphi_{_{\mathcal{N}}} \ . \tag{2.22}$$

The $\mathscr{C}_{\mathcal{N}}$ should then be chosen such that this object evaluated at $r = r_c \pm i\varepsilon$ has a well-defined $r_c \to 0$ limit.

We will now determine $\mathscr{C}_{\mathcal{N}}$ by studying the outgoing Green function (the counter-terms determined using a generic enough solution should work for every other solution). As we shall see, the boundary value of the *renormalised* conjugate field in this case is the boundary Green function K_{Out}. Before going into the details of the computation, it might be useful to situate it in a familiar physical context.

In the case of electromagnetism, the worldline Green function K_{Out} for a charged particle encodes the radiation reaction or self-force due to the particle's EM fields acting on itself. While this statement is broadly true, it is clear that this idea has to be interpreted with some care. If we take the bare electric field produced by the point charge and try to compute the self-force on it naively, the calculation will be dominated by the Coulomb divergence at the origin yielding an infinite answer.

A little bit of thought however reveals that these divergences merely serve to relate the bare properties (e.g., mass) of the fictitious charge-free particle to the properties of the actual physical particle. What we should do instead is to compute the renormalised electric field felt by the particle after adding counter-terms which shift the mass to the experimentally measured value. This renormalised field associated with the radiation is determined from the near field by imposing the outgoing boundary condition and can then be used to compute the self-force of the particle.

With this physical example in mind, we can interpret the first term in Eq.(2.10) as analogous to the Coulomb field in the near region whose divergent contributions need to be removed by using counter-terms. It is only after this is done that we can extract $\widehat{K}_{\text{Out}}$ as the renormalised worldline Green function.

We will now demand that the renormalised conjugate field computed over the first term in Eq.(2.10) vanish. This fixes the counter-term function $\mathscr{C}_{\mathcal{N}}$ to be

$$\frac{\mathscr{C}_{\mathcal{N}}}{1-r^2} \equiv -r\frac{d}{dr}\ln\left\{r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}(1-r^2)^{-\frac{i\omega}{2}} \right.$$
$$\left. \times\, {}_2F_1\left[\frac{1-\nu+\mu-i\omega}{2},\frac{1-\nu-\mu-i\omega}{2};1-\nu;r^2\right]\right\} . \tag{2.23}$$

Here we take the branch cut of $(1-r^2)^{-\frac{i\omega}{2}}$ to be away from the open unit disc $|r|<1$ in the complex r plane and, with this choice, $\mathscr{C}_{\mathcal{N}}$ is analytic everywhere inside each copy of the static patch, and has no discontinuity across the dS-SK branch-cut. While it is not obvious from the expression above, we can invoke the Euler transformation formula for the hypergeometric function which states that

$$\begin{aligned} &{}_2F_1\left[\frac{1\pm\nu+\mu+i\omega}{2},\frac{1\pm\nu-\mu+i\omega}{2};1\pm\nu;r^2\right] \\ &\quad= (1-r^2)^{-i\omega}{}_2F_1\left[\frac{1\pm\nu+\mu-i\omega}{2},\frac{1\pm\nu-\mu-i\omega}{2};1\pm\nu;r^2\right] \ , \end{aligned} \tag{2.24}$$

to conclude that $\mathscr{C}_{\mathcal{N}}$ is a real and even function of ω. Here we have taken the function to be analytic in static patch again and hence $\mathscr{C}_{\mathcal{N}}$ has a well-behaved small r expansion. The first few terms in this expansion are given by

$$\begin{aligned} \mathscr{C}_{\mathcal{N}} &= (1-r^2)\left(\nu+\frac{1}{2}(\mathcal{N}-1)\right) + r^2\frac{(\nu-\mu-1)(\nu+\mu-1)-\omega^2}{2\nu-2} \\ &\quad + r^4\frac{[(\nu-\mu-1)^2+\omega^2][(\nu+\mu-1)^2+\omega^2]}{(2\nu-2)^2(2\nu-4)} \\ &\quad + r^6\frac{[(\nu-\mu-1)^2+\omega^2][(\nu+\mu-1)^2+\omega^2]}{(2\nu-2)^3(2\nu-4)(2\nu-6)} \\ &\qquad \times\left[(2\nu-2)(2\nu-4)-2(\nu-\mu-1)(\nu+\mu-1)+2\omega^2\right] + \dots \ . \end{aligned} \tag{2.25}$$

Note that all terms in the above expansion are indeed real and even functions of ω as claimed. Note that all the r and ω factors appear in the numerator implying that this counter-term is local in time/radial direction.

Now that we have the expression for the counter-term, it is straightforward to compute the renormalised conjugate field evaluated over the outgoing Green function. We obtain the following answer

$$\begin{aligned}\pi^{\text{Out}}_{\mathcal{N}} &\equiv -\left[r^{\mathcal{N}} D_+ + r^{\mathcal{N}-1}\mathscr{C}_{\mathcal{N}}\right] G^{\text{Out}}_{\mathcal{N}} \\ &= (1+i\cot\pi\nu)\widehat{K}_{\text{Out}}\mathscr{Z}_{\mathcal{N}}(r,\omega) r^{\nu+\frac{1}{2}(\mathcal{N}-1)}(1+r)^{-i\omega} \\ &\qquad {}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1+\nu; r^2\right] ,\end{aligned} \tag{2.26}$$

where $\mathscr{Z}_{\mathcal{N}}(r,\omega)$ is a function given by the expression

$$\frac{\mathscr{Z}_{\mathcal{N}}}{1-r^2} \equiv 1 - \frac{r}{2\nu}\frac{d}{dr}\ln\left\{\frac{{}_2F_1\left[\frac{1-\nu-\mu-i\omega}{2}, \frac{1-\nu+\mu-i\omega}{2}; 1-\nu; r^2\right]}{{}_2F_1\left[\frac{1+\nu+\mu-i\omega}{2}, \frac{1+\nu-\mu-i\omega}{2}; 1+\nu; r^2\right]}\right\} . \tag{2.27}$$

This is also a real and even function of ω with a well-behaved series expansion near the origin. We thus see that the renormalised conjugate field of the outgoing wave is essentially its regular part, obtained after dropping its singular part and then renormalised by a factor of $\mathscr{Z}_{\mathcal{N}}$. Taking the $r \to 0$ limit yields

$$\lim_{r\to 0} r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}\pi^{\text{Out}}_{\mathcal{N}} \equiv -\lim_{r\to 0} r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}\left[r^{\mathcal{N}} D_+ + r^{\mathcal{N}-1}\mathscr{C}_{\mathcal{N}}\right] G^{\text{Out}}_{\mathcal{N}} = (1+i\cot\pi\nu)\widehat{K}_{\text{Out}} . \tag{2.28}$$

This then justifies our original definition for $\widehat{K}_{\text{Out}}$.

If d is odd and $\nu \equiv \frac{d}{2} + \ell - 1 \in \mathbb{Z} + \frac{1}{2}$, we can set ν to its actual value everywhere (i.e., remove dim-reg.) in our result: the value of the renormalised conjugate field at the world line (which we shall henceforth refer to by the symbol K_{Out}) is then finite. We can then write

$$K_{\text{Out}}|_{\text{Odd d}} = (1+i\cot\pi\nu)\widehat{K}_{\text{Out}}|_{\text{Odd d}} = -e^{i\nu\pi}\frac{2\pi i}{\Gamma(\nu)^2}\frac{\Gamma\left(\frac{1+\nu-\mu-i\omega}{2}\right)\Gamma\left(\frac{1+\nu+\mu-i\omega}{2}\right)}{\Gamma\left(\frac{1-\nu+\mu-i\omega}{2}\right)\Gamma\left(\frac{1-\nu-\mu-i\omega}{2}\right)} . \tag{2.29}$$

For the massless case in odd d, we have $\mu, \nu \in \mathbb{Z} + \frac{1}{2}$ for all values of interest given in table 2.1. If we further assume that $\mu \neq 1 + \nu \equiv \frac{d}{2} + \ell$, the above expression is, in fact, an odd polynomial of $i\omega$ with degree 2ν (see table 2.2 for an illustration). An interesting example is that of a conformally coupled scalar in odd d, where we have a closed-form expression

$$K_{\text{Out}}\Big|_{\mu=\frac{1}{2}} = \frac{(-1)^{\nu-\frac{1}{2}}}{(2\nu-2)!!^2}\prod_{k=1}^{2\nu}\left[\nu+\frac{1}{2}-k-i\omega\right] . \tag{2.30}$$

In all such cases, for every multipole moment, the Hubble corrections for the radiation correction terminate. Hence, we get a completely markovian influence phase with no memory/tail terms. Further, as we shall explain in detail in the section 2.3, for an arbitrarily moving point-like source, all the multipole contributions add up nicely into a local generally covariant expression for the radiation reaction force.

Table 2.2: $\frac{K_{\text{Out}}}{-i\omega}$ for $\mu \in \left\{\frac{d}{2}-1, \frac{d}{2}-2\right\}$ (gauge/gravity scalar/vector sectors)

$\mu = \frac{d}{2}-1$	$\ell = 0$	$\ell = 1$	$\ell = 2$
$d=3$	1	ω^2+1	$\frac{\omega^4}{9}+\frac{5\omega^2}{9}+\frac{4}{9}$
$d=5$	ω^2+4	$\frac{\omega^4}{9}+\frac{10\omega^2}{9}+1$	$\frac{\omega^6}{225}+\frac{7\omega^4}{75}+\frac{28\omega^2}{75}+\frac{64}{225}$
$d=7$	$\frac{\omega^4}{9}+\frac{20\omega^2}{9}+\frac{64}{9}$	$\frac{\omega^6}{225}+\frac{7\omega^4}{45}+\frac{259\omega^2}{225}+1$	$\frac{\omega^8}{11025}+\frac{19\omega^6}{3675}+\frac{8\omega^4}{105}+\frac{3088\omega^2}{11025}+\frac{256}{1225}$
$\mu = \frac{d}{2}-2$	$\ell = 0$	$\ell = 1$	$\ell = 2$
$d=3$	1	ω^2+1	$\frac{\omega^4}{9}+\frac{5\omega^2}{9}+\frac{4}{9}$
$d=5$	ω^2+1	$\frac{\omega^4}{9}+\frac{5\omega^2}{9}+\frac{4}{9}$	$\frac{\omega^6}{225}+\frac{14\omega^4}{225}+\frac{49\omega^2}{225}+\frac{4}{25}$
$d=7$	$\frac{\omega^4}{9}+\frac{10\omega^2}{9}+1$	$\frac{\omega^6}{225}+\frac{7\omega^4}{75}+\frac{28\omega^2}{75}+\frac{64}{225}$	$\frac{\omega^8}{11025}+\frac{13\omega^6}{3675}+\frac{19\omega^4}{525}+\frac{1261\omega^2}{11025}+\frac{4}{49}$

For the minimally coupled massless scalar ($\mu = \frac{d}{2}$), we still obtain a polynomial K_{Out} for all multipoles except the monopole ($\ell = 0$) contribution. The monopole has an extra $1/\omega$ correction in addition to the polynomial terms odd in ω (See tables 2.3 and 2.4). An explicit expression for $\ell = 0$ contribution is given by

$$K_{\text{Out}}\big|_{\mu=1+\nu=\frac{d}{2}} = \frac{(d-2)^2}{i\omega}\cosh\frac{\pi\omega}{2}\,\frac{\Gamma\left(\frac{d-i\omega}{2}\right)\Gamma\left(\frac{d+i\omega}{2}\right)}{\Gamma\left(\frac{d}{2}\right)^2} \tag{2.31}$$

The inverse omega that appears in the front of this expression suggests that the correct variable for a low-frequency expansion in this case is the time integral of the scalar source rather than the source itself. Such a mild non-markovianity for minimally coupled scalars in dS has been noted before [97,99], and we will review its physical interpretation in section 2.3.

Table 2.3: $i\omega K_{\text{Out}}$

$\mu = \frac{d}{2}$	$\ell = 0$
$d = 3$	$\omega^2 + 1$
$d = 5$	$\omega^4 + 10\omega^2 + 9$
$d = 7$	$\frac{\omega^6}{9} + \frac{35\omega^4}{9} + \frac{259\omega^2}{9} + 25$

Table 2.4: $\frac{K_{\text{Out}}}{-i\omega}$

$\mu = \frac{d}{2}$	$\ell = 1$	$\ell = 2$
$d = 3$	$\omega^2 + 4$	$\frac{\omega^4}{9} + \frac{10\omega^2}{9} + 1$
$d = 5$	$\frac{\omega^4}{9} + \frac{20\omega^2}{9} + \frac{64}{9}$	$\frac{\omega^6}{225} + \frac{7\omega^4}{45} + \frac{259\omega^2}{225} + 1$
$d = 7$	$\frac{\omega^6}{225} + \frac{56\omega^4}{225} + \frac{784\omega^2}{225} + \frac{2561}{25}$	$\frac{\omega^8}{11025} + \frac{4\omega^6}{525} + \frac{94\omega^4}{525} + \frac{12916\omega^2}{11025} + 1$

For generic values of (μ, ν), a small ω expansion of K_{Out} is easy to obtain by expanding out the gamma functions in terms of polygamma functions. We get

$$\begin{aligned} K_{\text{Out}}\big|_{\text{Odd } d} &= -e^{i\nu\pi} \frac{2\pi i}{\Gamma(\nu)^2} \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right)\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma\left(\frac{1-\nu+\mu}{2}\right)\Gamma\left(\frac{1-\nu-\mu}{2}\right)} \\ &\times \exp\left\{\sum_{k=0}^{\infty} \frac{\left(\frac{-i\omega}{2}\right)^{k+1}}{(k+1)!} \left[\psi^{(k)}\left(\frac{1+\nu-\mu}{2}\right) + \psi^{(k)}\left(\frac{1+\nu+\mu}{2}\right)\right.\right. \\ &\qquad \left.\left. -\psi^{(k)}\left(\frac{1-\nu-\mu}{2}\right) - \psi^{(k)}\left(\frac{1-\nu+\mu}{2}\right)\right]\right\} , \end{aligned} \tag{2.32}$$

where $\psi^{(k)}(z) \equiv \frac{d^{k+1}}{dz^{k+1}} \ln\Gamma(z)$ is the polygamma function. When both $\nu + \mu$ or $\nu - \mu$ are non-negative integers, the terms in the above expressions become indeterminate and should instead be interpreted as a limit. In such cases, explicit computations show that the above exponential terminates, yielding an odd polynomial in ω when ν is a half-integer.

We will now comment on the even d/integer ν case. The $\cot \pi\nu$ diverges in this limit, and we need the analogue of Eq.(A.79) to figure out the counter-terms needed to remove this divergence. The analogous expansion is given by

$$
\begin{aligned}
(1+i\cot(\pi\nu))\widehat{K}_{\text{Out}} = \frac{(-)^n}{\Gamma(n)^2}\frac{\Gamma\left(\frac{1+n-\mu-i\omega}{2}\right)\Gamma\left(\frac{1+n+\mu-i\omega}{2}\right)}{\Gamma\left(\frac{1-n+\mu-i\omega}{2}\right)\Gamma\left(\frac{1-n-\mu-i\omega}{2}\right)}\Bigg[\frac{2}{\nu-n} \\
+\psi^{(0)}\left(\frac{1+n-\mu-i\omega}{2}\right)+\psi^{(0)}\left(\frac{1+n+\mu-i\omega}{2}\right) \\
+\psi^{(0)}\left(\frac{1-n-\mu-i\omega}{2}\right)+\psi^{(0)}\left(\frac{1-n+\mu-i\omega}{2}\right) \\
-4\psi^{(0)}(n)+O(\nu-n)\Bigg] \ .
\end{aligned}
\tag{2.33}
$$

As in the flat spacetime, we can counter-term away the first two terms, and change the n back to ν. This yields the renormalised worldline Green function as [7, 94]

$$
\begin{aligned}
K_{\text{Out}}|_{\text{Even } d} = \Delta_{\mathbb{N}}(\nu,\mu,\omega)\Bigg[\psi^{(0)}\left(\frac{1+\nu-\mu-i\omega}{2}\right)+\psi^{(0)}\left(\frac{1+\nu+\mu-i\omega}{2}\right) \\
+\psi^{(0)}\left(\frac{1-\nu-\mu-i\omega}{2}\right)+\psi^{(0)}\left(\frac{1-\nu+\mu-i\omega}{2}\right)-4\psi^{(0)}(\nu)\Bigg] \ ,
\end{aligned}
\tag{2.34}
$$

where the function $\Delta_{\mathbb{N}}$ is defined below in Eq.(2.36). To get this answer, we add to the counterterm in Eq.(2.21), further terms of the form

$$
S_{ct,\text{Even}} = \sum_{\mathbb{L}} \frac{1}{\nu-n}\int \frac{d\omega}{2\pi} r^{\mathbb{N}-1+2n}\Delta_{\mathbb{N}}(n,\mu,\omega)\varphi^*_{\mathbb{N}}\varphi_{\mathbb{N}}|_{r_c} \ , \tag{2.35}
$$

where $n = \ell + \frac{d}{2} - 1$ and we have defined

$$
\begin{aligned}
\Delta_{\mathbb{N}}(n,\mu,\omega) &\equiv \frac{(-)^n}{\Gamma(n)^2}\frac{\Gamma\left(\frac{1+n-\mu-i\omega}{2}\right)\Gamma\left(\frac{1+n+\mu-i\omega}{2}\right)}{\Gamma\left(\frac{1-n+\mu-i\omega}{2}\right)\Gamma\left(\frac{1-n-\mu-i\omega}{2}\right)} \\
&= \frac{1}{\Gamma(n)^2}\prod_{k=1}^{n}\left[\frac{\omega^2}{4}+\frac{1}{4}(\mu-n+2k-1)^2\right] = \Delta^*_{\mathbb{N}}(n,\mu,\omega) \ .
\end{aligned}
\tag{2.36}
$$

Note that the explicit product form we give above is valid for $n \in \mathbb{Z}_+$. This form shows that $\Delta_{\mathbb{N}}$ is a real and even function of ω, which is an essential condition for such a counterterm to be admissible. With this counterterm, Eq.(2.34) is the dS generalisation of the radiation reaction influence phase in flat spacetime described by Eq.(A.80). The simple logarithmic running in flat spacetime is now replaced by a more complicated RGE with the Hubble constant playing the role of the IR cutoff. A low-frequency expansion

Table 2.5: Residues of $K_{\rm Out}$ in even d for $\mu \in \left\{\frac{d}{2}, \frac{d}{2}-1, \frac{d}{2}-2\right\}$at $\omega = -i(\mu+\nu+1)$.

$\mu = \frac{d}{2}$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$24i$	$-192i$	$720i$	$-1920i$	$4200i$
$d = 6$	$-320i$	$1440i$	$-4480i$	$11200i$	$-24192i$
$d = 8$	$2520i$	$-8960i$	$25200i$	$-60480i$	$129360i$
$\mu = \frac{d}{2}-1$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$16i$	$-96i$	$288i$	$-640i$	$1200i$
$d = 6$	$-192i$	$720i$	$-1920i$	$4200i$	$-8064i$
$d = 8$	$1440i$	$-4480i$	$11200i$	$-24192i$	$47040i$
$\mu = \frac{d}{2}-2$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$8i$	$-32i$	$72i$	$-128i$	$200i$
$d = 6$	$-96i$	$288i$	$-640i$	$-1200i$	$-2016i$
$d = 8$	$720i$	$-1920i$	$4200i$	$-8064i$	$14112i$

$K_{\rm Out}$ can be obtained by using the polygamma series expansion

$$\begin{aligned}
&\psi^{(0)}\left(\frac{1+\nu-\mu-i\omega}{2}\right)+\psi^{(0)}\left(\frac{1+\nu+\mu-i\omega}{2}\right)\\
&\qquad+\psi^{(0)}\left(\frac{1-\nu-\mu-i\omega}{2}\right)+\psi^{(0)}\left(\frac{1-\nu+\mu-i\omega}{2}\right)\\
&=\sum_{k=0}^{\infty}\frac{\left(\frac{-i\omega}{2}\right)^{k+1}}{(k+1)!}\left[\psi^{(k)}\left(\frac{1+\nu-\mu}{2}\right)+\psi^{(k)}\left(\frac{1+\nu-\mu}{2}\right)\right.\\
&\qquad\left.+\psi^{(k)}\left(\frac{1-\nu-\mu}{2}\right)+\psi^{(k)}\left(\frac{1-\nu+\mu}{2}\right)\right]\ ,
\end{aligned} \tag{2.37}$$

which is well-defined except when any one of the polygamma arguments is a negative integer. These results agree with the dS expressions derived in [7].

We will now add a remark about the poles of $K_{\rm Out}$ in the complex frequency plane. These are sometimes termed 'de-Sitter quasi-normal modes' although we think this is a misleading terminology for the following reason. The adjective 'quasi-normal' is usually applied to poles of Green functions that have a real as well as an imaginary part: as the name suggests, these are 'almost' normal modes that characterise the physics of ring-down. The dS horizon does not ring down and has no quasi-normal modes in this sense.

The poles of $K_{\rm Out}$, when present, are more akin to Matsubara modes of thermal Green functions in that they lie along the imaginary axis in the complex frequency plane. As is evident from Tables 2.2,2.3 and 2.4, for d odd and $\mu \in \left\{\frac{d}{2}, \frac{d}{2}-1, \frac{d}{2}-2\right\}$, $K_{\rm Out}$ is

Table 2.6: Residues of K_{Out} in even d for $\mu \in \left\{\frac{d}{2}, \frac{d}{2}-1, \frac{d}{2}-2\right\}$at $\omega = -i(5+\nu-\mu)$.

$\mu = \frac{d}{2}$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$24i$	$-192i$	$720i$	$-1920i$	$4200i$
$d = 6$	0	0	0	0	0
$d = 8$	0	0	0	0	0

$\mu = \frac{d}{2} - 1$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$48i$	$-576i$	$2880i$	$-9600i$	$25200i$
$d = 6$	$-192i$	$720i$	$-1920i$	$4200i$	$-8064i$
$d = 8$	0	0	0	0	0

$\mu = \frac{d}{2} - 2$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$72i$	$-1152i$	$7200i$	$-28800i$	$88200i$
$d = 6$	$-576i$	$2880i$	$-9600i$	$25200i$	$-56448i$
$d = 8$	$720i$	$-1920i$	$4200i$	$-8064i$	$14112i$

Table 2.7: Residues of K_{Out} in even d for $\mu \in \left\{\frac{d}{2}, \frac{d}{2}-1, \frac{d}{2}-2\right\}$at $\omega = -i(7-\nu+\mu)$.

$\mu = \frac{d}{2}$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$120i$	$-960i$	$720i$	0	0
$d = 6$	$-1440i$	$1440i$	0	0	0
$d = 8$	$2520i$	0	0	0	0

$\mu = \frac{d}{2} - 1$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$96i$	$-576i$	$288i$	0	0
$d = 6$	$-960i$	$720i$	0	0	0
$d = 8$	$1440i$	0	0	0	0

$\mu = \frac{d}{2} - 2$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$72i$	$-288i$	$72i$	0	0
$d = 6$	$-576i$	$288i$	0	0	0
$d = 8$	$720i$	0	0	0	0

Table 2.8: Residues of K_{Out} in even d for $\mu \in \left\{\frac{d}{2}, \frac{d}{2}-1, \frac{d}{2}-2\right\}$at $\omega = -i(11-\nu-\mu)$.

$\mu = \frac{d}{2}$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$120i$	$-960i$	$720i$	0	0
$d = 6$	$-320i$	0	0	0	0
$d = 8$	0	0	0	0	0
$\mu = \frac{d}{2} - 1$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$160i$	$-1920i$	$2880i$	$-640i$	0
$d = 6$	$-960i$	$720i$	0	0	0
$d = 8$	0	0	0	0	0
$\mu = \frac{d}{2} - 2$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 4$	$200i$	$-3200i$	$7200i$	$-3200i$	$200i$
$d = 6$	$-1920i$	$2880i$	$-640i$	0	0
$d = 8$	$720i$	0	0	0	0

a polynomial function of ω and has no poles whatsoever. For d even, the polygamma functions appearing in Eq.(2.34) have simple poles when their arguments become negative integers (this happens only along the negative imaginary axis in the complex frequency plane). The kernel K_{Out} inherits these poles except when they get cancelled by the zeroes of $\Delta_{\mathcal{N}}(n, \mu, \omega)$ given in Eq.(2.36). We exhibit the residues of some of these poles in the table 2.5,2.6,2.7 and 2.8: the ones with zero residues correspond to cancelled poles. The presence of these poles indicates that the small-frequency Langevin description might fail beyond a certain cut-off frequency.

We will conclude this section with a comment on the flat spacetime limit of the expressions derived in this section. Intuitively, we expect that the high-frequency modes with $\omega \gg 1$ would be insensitive to the cosmological constant, and would behave like Minkowski modes. This intuition can indeed be made precise by examining the high-frequency expansion of K_{Out}. Using Stirling approximation for the Gamma functions, we can indeed check the following statement valid for $\omega \gg 1$:

$$K_{\text{Out}} \approx \begin{cases} \frac{2\pi i}{\Gamma(\nu)^2}\left(\frac{\omega}{2}\right)^{2\nu} & \text{for } d \text{ odd} , \\ \frac{1}{\Gamma(\nu)^2}\left(\frac{\omega}{2}\right)^{2\nu} \ln\left(\frac{\omega^4}{H^4}\right) & \text{for } d \text{ even} . \end{cases} \tag{2.38}$$

Comparing these limits with Eqs.(A.78) and (A.80), we conclude K_{Out} is indeed the dS generalisation of the radiation reaction kernel.

2.2 SK Green functions and the cosmological influence phase

We now turn to the problem of constructing the solution on the dS-SK spacetime contour. The construction here closely parallels corresponding derivation in AdS [45–47,49,50] and we include a concise summary here mainly for completeness. The reader is encouraged to see these references for a more extensive discussion and interpretation of the expressions quoted below.

Our discussion in this section is structured as follows: we begin by extending our discussion of counter-terms, etc., to the *incoming* Green functions. Physically, such Green functions are relevant while describing the effect of a distant source on the past of the observer. As will be derived below, even if there are no sources present, an observer in dS spacetime sees cosmic background radiation at the dS temperature. We will need the incoming Green function to describe these waves.

2.2.1 Time reversal, incoming waves and their branch-cut

We would now like to argue that the renormalised conjugate field continues to be finite for the Green function describing incoming waves. The incoming Green function can be computed from the answers we already have by using the time reversal isometry of the dS spacetime. The only non-trivial step involved is to realise how the time reversal isometry acts on EF coordinates.

The action of time reversal is achieved by the diffeomorphism

$$u \mapsto 2\pi i\zeta - u \ , \ \omega \mapsto -\omega \ , \tag{2.39}$$

where ζ is the mock tortoise coordinate introduced in Eq.(1.3). One can check that this diffeomorphism preserves the metric in Eq.(2.2) and is hence an isometry. The map $\omega \mapsto -\omega$ is necessary maintain the $\sim e^{-i\omega u}$ factor in Fourier domain. The time reversal is hence achieved by reversing ω and then multiplying all Fourier domain functions by a factor $e^{-2\pi\omega\zeta}$.

Using the time reversal isometry, the bulk to worldline Green function with incoming boundary condition takes the form

$$G_{\mathcal{N}}^{\text{In}}(r,\omega,\ell) = e^{-2\pi\omega\zeta} G_{\mathcal{N}}^{\text{Out}*}(r,\omega,\ell) \tag{2.40}$$

Unlike $G_{\mathcal{N}}^{\text{Out}}$, the Green function $G_{\mathcal{N}}^{\text{In}}$ has a branch-cut on the dS-SK contour, taking different values in the left vs. right static patches. The near origin expansion of $G_{\mathcal{N}}^{\text{In}}$ can be obtained by using the Euler transformation in Eq.(2.24):

$$\begin{split} G_{\mathcal{N}}^{\text{In}}(r,\omega,\ell) \equiv e^{-2\pi\omega\zeta} G_{\mathcal{N}}^{\text{Out}*} = e^{-2\pi\omega\zeta}\left(\frac{1-r}{1+r}\right)^{-i\omega} \times r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}(1+r)^{-i\omega} \\ \times \left\{ {}_2F_1\left[\frac{1-\nu+\mu-i\omega}{2}, \frac{1-\nu-\mu-i\omega}{2}; 1-\nu; r^2\right] \right. \\ \left. -(1-i\cot\pi\nu)\widehat{K}_{\text{In}}\frac{r^{2\nu}}{2\nu}\ {}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1+\nu; r^2\right]\right\} . \end{split} \tag{2.41}$$

The branch cuts of the explicit $(1\pm r)^{-i\omega}$ are chosen to lie outside the open unit disc in the complex r plane, and a careful evaluation of the pre-factor above yields

$$e^{-2\pi\omega\zeta}\left(\frac{1-r}{1+r}\right)^{-i\omega} = \begin{cases} 1 & \text{L contour} \\ e^{-2\pi\omega} & \text{R contour .} \end{cases} \tag{2.42}$$

This shows explicitly the branch-cut and jump in the incoming Green function. In the above equation, the symbol $\widehat{K}_{\text{In}}$ denotes the worldline advanced Green function given by the expression

$$\begin{split} \widehat{K}_{\text{In}}(\omega,\ell) \equiv [\widehat{K}_{\text{Out}}(\omega,\ell)]^* = -e^{-2\pi i\nu}\widehat{K}^{\text{Out}}(-\omega,\ell) \\ = e^{-i\nu\pi}\frac{2\pi i}{\Gamma(\nu)^2}\frac{\Gamma\left(\frac{1+\nu-\mu+i\omega}{2}\right)\Gamma\left(\frac{1+\nu+\mu+i\omega}{2}\right)}{\Gamma\left(\frac{1-\nu+\mu+i\omega}{2}\right)\Gamma\left(\frac{1-\nu-\mu+i\omega}{2}\right)} . \end{split} \tag{2.43}$$

The comments made in the context of $\widehat{K}_{\text{Out}}$ below Eq.(2.11) apply also in this case. The decomposition in Eq.(2.41), just like the outgoing case, makes explicit the small r behaviour and helps read off the $\widehat{K}_{\text{In}}$ easily.

Given the above definition of $G_{\mathcal{N}}^{\text{In}}$, it is now straightforward to compute the renormalised conjugate field. Since the incoming mode has a branch cut, it behaves differently at the two boundaries. Adding in the counterterm in Eq.(2.23), we get the renormalised conjugate field as

$$\begin{aligned}
\pi^{\text{In}}_{\mathcal{N}} &\equiv -\left[r^{\mathcal{N}}D_+ + r^{\mathcal{N}-1}\mathscr{C}_{\mathcal{N}}\right] G^{\text{In}}_{\mathcal{N}} \\
&= -e^{-2\pi\omega\zeta}\left[r^{\mathcal{N}}D_- + r^{\mathcal{N}-1}\mathscr{C}_{\mathcal{N}}\right] G^{\text{Out}*}_{\mathcal{N}} \\
&= e^{-2\pi\omega\zeta}\pi^{\text{Out}*}_{\mathcal{N}} .
\end{aligned} \tag{2.44}$$

Here we have used $D_\pm \equiv (1-r^2)\partial_r \pm i\omega$ as well as the property that $D_+[e^{-2\pi\omega\zeta}\#] = e^{-2\pi\omega\zeta}D_-[\#]$. Using Eq.(2.26), we obtain

$$\begin{aligned}
\pi^{\text{In}}_{\mathcal{N}} = (1 - i\cot\pi\nu)\widehat{K}_{\text{In}}e^{-2\pi\omega\zeta}\left(\frac{1-r}{1+r}\right)^{-i\omega}\mathscr{Z}_{\mathcal{N}}(r,\omega) \\
\times\, r^{\nu+\frac{1}{2}(\mathcal{N}-1)}(1+r)^{-i\omega}{}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1+\nu; r^2\right] .
\end{aligned} \tag{2.45}$$

As in the case of outgoing waves, we see again that the renormalised conjugate field is the regular part of the incoming waves renormalised with the same factor $\mathscr{Z}_{\mathcal{N}}(r,\omega)$. We can then take the $r \to 0$ limit above and below the branch cut to get

$$\lim_{r\to 0} r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}\pi^{\text{In}}_{\mathcal{N}} = \begin{cases} (1-i\cot\pi\nu)\widehat{K}_{\text{In}} & \text{L boundary} , \\ e^{-2\pi\omega}(1-i\cot\pi\nu)\widehat{K}_{\text{In}} & \text{R boundary} . \end{cases} \tag{2.46}$$

This shows that the counter-term we derived also works for the incoming waves. When d is odd and $\cot\pi\nu = 0$, we can remove the dimensional regularisation without any further counterterms. The analogue of Eq.(2.29) for the incoming waves is

$$\begin{aligned}
K_{\text{In}}|_{\text{Odd d}} &= (1-i\cot\pi\nu)\widehat{K}_{\text{In}}|_{\text{Odd d}} = (K_{\text{Out}})^*|_{\text{Odd d}} \\
&= e^{-i\nu\pi}\frac{2\pi i}{\Gamma(\nu)^2}\frac{\Gamma\left(\frac{1+\nu-\mu+i\omega}{2}\right)\Gamma\left(\frac{1+\nu+\mu+i\omega}{2}\right)}{\Gamma\left(\frac{1-\nu+\mu+i\omega}{2}\right)\Gamma\left(\frac{1-\nu-\mu+i\omega}{2}\right)} .
\end{aligned} \tag{2.47}$$

All our statements about K_{Out} in odd d apply mutatis mutandis to K_{In}.

When d is even and ν approaches an integer, there are additional divergences due to $\cot\pi\nu$. We already encountered such divergences and countertermed them away for outgoing waves. We have to check now that the counterterms in Eq.(2.35) added to cancel such divergences out of outgoing waves, work also for the incoming waves. To see this, we examine the expansion

$$
\begin{aligned}
(1-i\cot\pi\nu)\widehat{K}_{\text{In}}(\nu) = \Delta_{\mathcal{N}}(n,\mu,\omega)\Bigg[&\frac{2}{\nu-n}\\
&+\psi^{(0)}\left(\frac{1+n-\mu+i\omega}{2}\right)+\psi^{(0)}\left(\frac{1+n+\mu+i\omega}{2}\right)\\
&+\psi^{(0)}\left(\frac{1-n-\mu+i\omega}{2}\right)+\psi^{(0)}\left(\frac{1-n+\mu+i\omega}{2}\right)\\
&-4\psi^{(0)}(n)+O(\nu-n)\Bigg]\ ,
\end{aligned}
\tag{2.48}
$$

where $\Delta_{\mathcal{N}}(n,\mu,\omega)$ is given by Eq.(2.36). Here, we have used crucially the fact that $\Delta_{\mathcal{N}}$ is a real, even function of ω.

From the above expression, we can see that the incoming conjugate field in Eq.(2.45) is also rendered finite by the same counterterms as before. Crucially, the monodromy factors of $e^{-2\pi\omega\zeta}$ work out correctly to cancel the divergences near both the left/right world lines. We get the final renormalised advanced worldline Green function as

$$
\begin{aligned}
K_{\text{In}}|_{\text{Even }d} = \Delta_{\mathcal{N}}(\nu,\mu,\omega)&\left[\psi^{(0)}\left(\frac{1+\nu-\mu+i\omega}{2}\right)+\psi^{(0)}\left(\frac{1+\nu+\mu+i\omega}{2}\right)\right.\\
&\left.+\psi^{(0)}\left(\frac{1-\nu-\mu+i\omega}{2}\right)+\psi^{(0)}\left(\frac{1-\nu+\mu+i\omega}{2}\right)-4\psi^{(0)}(\nu)\right]\ ,
\end{aligned}
\tag{2.49}
$$

To conclude, we have demonstrated a set of counterterms which result in finite answers for conjugate fields evaluated over both outgoing as well as incoming waves. The final renormalised conjugate field is given by

$$
\lim_{r\to 0} r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}\pi^{\text{In}}_{\mathcal{N}} = \begin{cases} K_{\text{In}} & \text{L boundary}\ ,\\ e^{-2\pi\omega}K_{\text{In}} & \text{R boundary}\ . \end{cases}
\tag{2.50}
$$

Since the most general solution on the dS-SK geometry is a linear combination of outgoing/incoming waves, it follows that our counterterm prescription will yield a finite answer for the cosmological influence phase.

2.2.2 Point-like sources and Green functions on dS-SK contour

In this subsection, we solve for the unique combination of outgoing and incoming waves corresponding to a point source placed at the centre(s) of left/right static patches in dS-SK geometry. As we will describe subsequently, with some more effort, arbitrary extended sources on the dS-SK background can also be dealt with.

We describe the point source problem first to introduce, within a simpler setting, the ingredients needed for the extended sources. As we shall see, in analogy with AdS, we can think of the problem of point sources placed at the centre of the static patch as one involving boundary-to-bulk Green functions. In contrast, the problem of extended sources is that of bulk-to-bulk Green functions, and it is hence fairly more involved.

We begin with the most general linear combination of outgoing/incoming modes for the radial part

$$\varphi_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m}) = -G^{\rm Out}_{\mathcal{N}}(r,\omega,\ell)\mathcal{J}_{\bar{F}}(\omega,\ell,\vec{m}) + e^{2\pi\omega(1-\zeta)}G^{\rm Out*}_{\mathcal{N}}(r,\omega,\ell)\mathcal{J}_{\bar{P}}(\omega,\ell,\vec{m})\ . \qquad (2.51)$$

Here the subscripts F and P denote the sources that radiate to the future and detectors that absorb from the past respectively. We use ζ to indicate the radial argument of $\varphi_{\mathcal{N}}$ to emphasise that this general linear combination takes two different values in the two branches of dS-SK geometry.

The coefficients $\mathcal{J}_{\bar{F}}, \mathcal{J}_{\bar{P}}$ appearing above can be linked to the left/right sources via the double Dirichlet condition, i.e., at the left/right copy of the worldlines, we impose

$$\begin{aligned}
\mathcal{J}_L(\omega,\ell,\vec{m}) &\equiv \lim_{\zeta\to 0} r^{\nu+\frac{\mathcal{N}-1}{2}}\varphi_{\mathcal{N}} = -\mathcal{J}_{\bar{F}}(\omega,\ell,\vec{m}) + e^{2\pi\omega}\mathcal{J}_{\bar{P}}(\omega,\ell,\vec{m})\ ,\\
\mathcal{J}_R(\omega,\ell,\vec{m}) &\equiv \lim_{\zeta\to 1} r^{\nu+\frac{\mathcal{N}-1}{2}}\varphi_{\mathcal{N}} = -\mathcal{J}_{\bar{F}}(\omega,\ell,\vec{m}) + \mathcal{J}_{\bar{P}}(\omega,\ell,\vec{m})\ .
\end{aligned} \qquad (2.52)$$

Our use of the symbol $\mathcal{J}$ here is a deliberate allusion to the observer's multipole moments. Inverting the above relations, we obtain

$$\begin{aligned}
\mathcal{J}_{\bar{F}}(\omega,\ell,\vec{m}) &\equiv -\Big\{(1+n_\omega)\mathcal{J}_R(\omega,\ell,\vec{m}) - n_\omega\mathcal{J}_L(\omega,\ell,\vec{m})\Big\}\\
&= -\mathcal{J}_A(\omega,\ell,\vec{m}) - \left(n_\omega + \frac{1}{2}\right)\mathcal{J}_D(\omega,\ell,\vec{m})\\
\mathcal{J}_{\bar{P}}(\omega,\ell,\vec{m}) &\equiv -n_\omega\Big\{\mathcal{J}_R(\omega,\ell,\vec{m}) - \mathcal{J}_L(\omega,\ell,\vec{m})\Big\} = -n_\omega\ \mathcal{J}_D(\omega,\ell,\vec{m})\ .
\end{aligned} \qquad (2.53)$$

Here we have introduced the average/difference sources $J_A \equiv \frac{1}{2}J_R + \frac{1}{2}J_L$ and $J_D \equiv J_R - J_L$. We note here the natural appearance of the Bose-Einstein factor

$$n_\omega \equiv \frac{1}{e^{2\pi\omega} - 1} \ . \tag{2.54}$$

Such a factor arises naturally by solving the detailed-balance constraint $1 + n_\omega = e^{2\pi\omega} n_\omega$ which equates the probability of spontaneous/stimulated emission by the source to the absorption probability. The appearance of such a factor is evidence that dS-SK contour naturally incorporates the thermality of Hawking radiation emitted from the dS horizon [68].

The solution for the bulk field produced by point-like sources is given by Eq.(2.51). Using Eq.(2.53), we then have

$$\varphi_{\mathcal{N}} = g_R J_R - g_L J_L \ , \tag{2.55}$$

where we have defined

$$\begin{aligned} g_L &\equiv n_\omega \Big(G^{\text{Out}}_{\mathcal{N}} - e^{2\pi\omega(1-\zeta)} G^{\text{Out}*}_{\mathcal{N}} \Big) \ , \\ g_R &\equiv (1 + n_\omega) \Big(G^{\text{Out}}_{\mathcal{N}} - e^{-2\pi\omega\zeta} G^{\text{Out}*}_{\mathcal{N}} \Big) \ . \end{aligned} \tag{2.56}$$

These are the dS analogues of the left/right *boundary-to-bulk* Green functions which tell us how left and right sources affect the solution on the dS-SK geometry(see Figure 2.1). They obey the Kubo-Martin-Schwinger (KMS) relation $g_R(\zeta) = e^{2\pi\omega} g_L(1 + \zeta)$ as well as the following boundary conditions on the dS-SK contour:

$$\begin{aligned} \lim_{\zeta\to 0} r^{\nu + \frac{\mathcal{N}-1}{2}} g_L = -1 \ , &\quad \lim_{\zeta\to 0} r^{\nu + \frac{\mathcal{N}-1}{2}} g_R = 0 \ , \\ \lim_{\zeta\to 1} r^{\nu + \frac{\mathcal{N}-1}{2}} g_L = 0 \ , &\quad \lim_{\zeta\to 1} r^{\nu + \frac{\mathcal{N}-1}{2}} g_R = 1. \end{aligned} \tag{2.57}$$

This result can be derived directly from the boundary condition in Eq.(2.16). The above conditions imply that the Green function $g_{L,R}$ are two different smooth interpolations between the homogeneous solution regular at the origin on one side and a Green function with a source singularity on the other side. Thus, g_R is regular near the left boundary whereas g_L is regular near the right boundary.

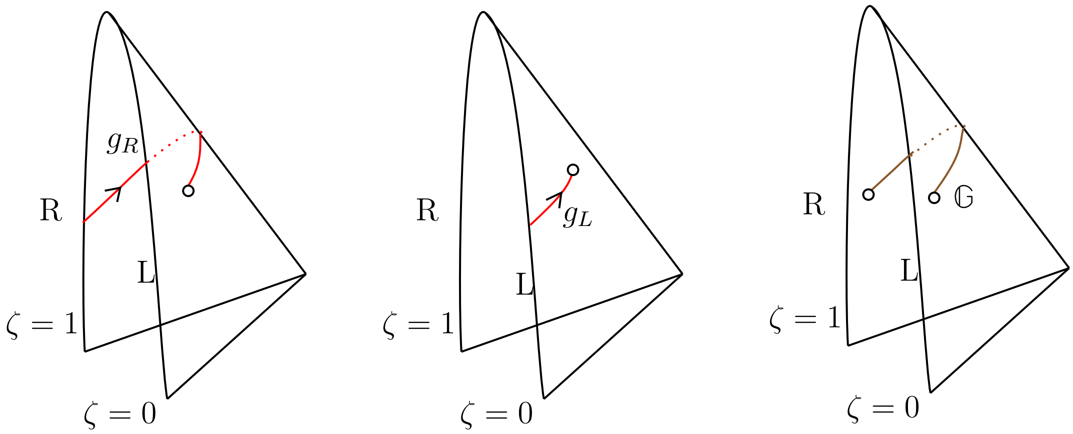

Figure 2.1: Propagators in dS-SK geometry: the boundary to bulk propagators are denoted in red and the bulk to bulk propagator is denoted in brown.

The Green functions $g_{L,R}$ can be written down explicitly. Substituting Eqs.(2.10) and (2.41) into Eq.(2.56), we get the following expressions:

$$\begin{aligned}
g_L = {} & n_\omega r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}(1+r)^{-i\omega} \\
& \times \Bigg\{ \left[1 - e^{2\pi\omega(1-\zeta)} \left(\frac{1-r}{1+r}\right)^{-i\omega} \right] \\
& \qquad\qquad \times {}_2F_1\left[\frac{1-\nu+\mu-i\omega}{2}, \frac{1-\nu-\mu-i\omega}{2}; 1-\nu; r^2\right] \\
& \quad - i\cot\pi\nu \left[\widehat{K}_{\text{Out}} + e^{2\pi\omega(1-\zeta)} \left(\frac{1-r}{1+r}\right)^{-i\omega} \widehat{K}_{\text{In}} \right] \\
& \qquad\qquad \times \frac{r^{2\nu}}{2\nu}\, {}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1+\nu; r^2\right] \\
& \quad - \left[\widehat{K}_{\text{Out}} - e^{2\pi\omega(1-\zeta)} \left(\frac{1-r}{1+r}\right)^{-i\omega} \widehat{K}_{\text{In}} \right] \\
& \qquad\qquad \times \frac{r^{2\nu}}{2\nu}\, {}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1+\nu; r^2\right] \Bigg\} ,
\end{aligned} \tag{2.58}$$

and

$$
\begin{aligned}
g_R = &(1+n_\omega) r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}(1+r)^{-i\omega} \\
&\times \Bigg\{ \left[1 - e^{-2\pi\omega\zeta} \left(\frac{1-r}{1+r}\right)^{-i\omega} \right] \\
&\qquad\qquad \times {}_2F_1\left[\frac{1-\nu+\mu-i\omega}{2}, \frac{1-\nu-\mu-i\omega}{2}; 1-\nu; r^2\right] \\
&\quad - i\cot\pi\nu \left[\widehat{K}_{\text{Out}} + e^{-2\pi\omega\zeta}\left(\frac{1-r}{1+r}\right)^{-i\omega} \widehat{K}_{\text{In}} \right] \\
&\qquad\qquad \times \frac{r^{2\nu}}{2\nu}\, {}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1+\nu; r^2\right] \\
&\quad - \left[\widehat{K}_{\text{Out}} - e^{-2\pi\omega\zeta}\left(\frac{1-r}{1+r}\right)^{-i\omega} \widehat{K}_{\text{In}} \right] \\
&\qquad\qquad \times \frac{r^{2\nu}}{2\nu}\, {}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1+\nu; r^2\right] \Bigg\} .
\end{aligned}
\tag{2.59}
$$

These equations describe the Dirac-Deitweiler-Whiting [79, 96] type decomposition of the left/right Green functions into a singular solution which does not contribute to the radiation reaction, and a regular solution (the terms in the last line of each equation) which contributes to the finite influence phase.

Having said that, the reader should note that the expressions above are fairly complicated, with an elaborate branch cut structure that cannot be easily guessed a priori without the dS-SK prescription. These formulae are more complicated by the fact that we are forced to work with dimensional regularisation for even d. We will simplify the expressions for these dS boundary-to-bulk propagators in the next subsection when we describe extended sources. For present purposes, it is, however, sufficient to note the following: despite the complexity of expressions, given that we have a counterterm procedure that works both for outgoing and incoming waves, we are guaranteed a finite renormalised conjugate field.

To see this explicitly, we construct the corresponding renormalised conjugate field

$$
\begin{aligned}
\pi_{\mathcal{N}}(\zeta,\omega,\ell) &= -\pi_{\mathcal{N}}^{\text{Out}}(r,\omega,\ell)\mathfrak{J}_{\bar{F}} + e^{2\pi\omega(1-\zeta)}\pi_{\mathcal{N}}^{\text{Out}*}(r,\omega,\ell)\mathfrak{J}_{\bar{P}} \\
&= \pi_R(\zeta,\omega,\ell)\mathfrak{J}_R - \pi_L(\zeta,\omega,\ell)\mathfrak{J}_L \ ,
\end{aligned}
\tag{2.60}
$$

with the left/right boundary-to-bulk Green functions for the conjugate field defined by

$$\begin{aligned}\pi_L(\zeta,\omega,\ell) &\equiv -\left[r^{\mathcal{N}}D_+ + r^{\mathcal{N}-1}\mathscr{C}_{\mathcal{N}}\right]g_L(\zeta,\omega,\ell)\\ &= n_\omega\Big(\pi^{\text{Out}}_{\mathcal{N}}(r,\omega,\ell) - e^{2\pi\omega(1-\zeta)}\pi^{\text{Out}*}_{\mathcal{N}}(r,\omega,\ell)\Big)\ ,\\ \pi_R(\zeta,\omega,\ell) &\equiv -\left[r^{\mathcal{N}}D_+ + r^{\mathcal{N}-1}\mathscr{C}_{\mathcal{N}}\right]g_R(\zeta,\omega,\ell)\\ &= (1+n_\omega)\Big(\pi^{\text{Out}}_{\mathcal{N}}(r,\omega,\ell) - e^{-2\pi\omega\zeta}\pi^{\text{Out}*}_{\mathcal{N}}(r,\omega,\ell)\Big)\ .\end{aligned} \tag{2.61}$$

The equality here follows from a logic similar to that used in Eq.(2.44). The explicit forms of $\pi^{\text{Out}}_{\mathcal{N}}$ and $e^{-2\pi\omega\zeta}\pi^{\text{Out}*}_{\mathcal{N}}$ are given in Eqs.(2.26) and (2.45) respectively. Substituting them in, we get

$$\begin{aligned}\pi_L &= n_\omega\left[(1+i\cot\pi\nu)\widehat{K}_{\text{Out}} - e^{2\pi\omega(1-\zeta)}\left(\frac{1-r}{1+r}\right)^{-i\omega}(1-i\cot\pi\nu)\widehat{K}_{\text{In}}\right]\mathscr{Z}_{\mathcal{N}}(\omega,r)\\ &\quad\times r^{\nu+\frac{1}{2}(\mathcal{N}-1)}(1+r)^{-i\omega}{}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2},\frac{1+\nu+\mu-i\omega}{2};1+\nu;r^2\right]\ ,\\ \pi_R &= (1+n_\omega)\left[(1+i\cot\pi\nu)\widehat{K}_{\text{Out}} - e^{-2\pi\omega\zeta}\left(\frac{1-r}{1+r}\right)^{-i\omega}(1-i\cot\pi\nu)\widehat{K}_{\text{In}}\right]\mathscr{Z}_{\mathcal{N}}(\omega,r)\\ &\quad\times r^{\nu+\frac{1}{2}(\mathcal{N}-1)}(1+r)^{-i\omega}{}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2},\frac{1+\nu+\mu-i\omega}{2};1+\nu;r^2\right]\ .\end{aligned} \tag{2.62}$$

This picks out the regular part of the solution on dS-SK contour renormalised by $\mathscr{Z}_{\mathcal{N}}(\omega,r)$, as expected.

For $\nu\in\mathbb{Z}$, we should subtract the $\cot\pi\nu$ divergences using further counterterms in Eq.(2.35): once this is done, we can relax the dimensional regularisation and effectively replace

$$(1+i\cot\pi\nu)\widehat{K}_{\text{Out}}\to K_{\text{Out}}\ ,\quad (1-i\cot\pi\nu)\widehat{K}_{\text{In}}\to K_{\text{In}}\ .$$

After this is done, we can take $r\to 0$ limit on both sides of the dS-SK contour to get

$$\lim_{r\to 0} r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}\pi_{\mathcal{N}} = \begin{cases} K_{LR}\mathcal{J}_R - K_{LL}\mathcal{J}_L & \text{L boundary}\ ,\\ K_{RR}\mathcal{J}_R - K_{RL}\mathcal{J}_L & \text{R boundary}\ ,\end{cases} \tag{2.63}$$

where we have defined the Schwinger-Keldysh worldline Green functions defined via

$$
\begin{aligned}
K_{LL} &\equiv n_\omega K_{\text{Out}} - (1+n_\omega)K_{\text{In}} \ , \quad K_{LR} \equiv (1+n_\omega)\Big(K_{\text{Out}} - K_{\text{In}}\Big) \ , \\
K_{RL} &\equiv n_\omega\Big(K_{\text{Out}} - K_{\text{In}}\Big) \ , \quad K_{RR} \equiv (1+n_\omega)K_{\text{Out}} - n_\omega K_{\text{In}} \ .
\end{aligned}
\tag{2.64}
$$

These are exactly the expressions for the Schwinger-Keldysh two-point functions of a bosonic system coupled to a thermal bath [56, 57, 65].

Now that we have the near origin values of both the generalised free scalar system as well as its renormalised conjugate field, we are ready to compute the influence phase of the observer in the saddle point approximation by evaluating the on-shell action. We want to compute the action given in Eq.(2.18) along with the counter-term in Eqs.(2.21) and (2.35) over the dS-SK solution we found in Eq.(2.55). We begin with the full-action

$$
\begin{aligned}
S = -\frac{1}{2}\sum_{\mathbb{L}}\int\frac{d\omega}{2\pi}\oint\frac{r^{\mathcal{N}}dr}{1-r^2}\Big[(D_+\varphi_{_\mathcal{N}})^*D_+\varphi_{_\mathcal{N}} - \omega^2(1-r^2)\varphi^*_{_\mathcal{N}}\varphi_{_\mathcal{N}} \\
-\frac{1}{4r^2}\Big\{(\mathcal{N}-1)^2 - 4\nu^2 + [4\mu^2 - (\mathcal{N}+1)^2]r^2\Big\}\varphi^*_{_\mathcal{N}}\varphi_{_\mathcal{N}}\Big] + S_{ct} \ ,
\end{aligned}
\tag{2.65}
$$

integrate by parts over the bulk terms and then use the equation of motion in Eq.(2.7). This results in an on-shell action written purely in terms of boundary terms:

$$
\begin{aligned}
S_{\text{On-Shell}} &= \frac{1}{2}\sum_{\mathbb{L}}\int\frac{d\omega}{2\pi}\varphi^*_{_\mathcal{N}}\pi_{\mathcal{N}}|_{\text{Bnd}} \\
&= -\frac{1}{2}\sum_{\mathbb{L}}\int\frac{d\omega}{2\pi}\Big\{\mathcal{J}^*_R[K_{RR}\mathcal{J}_R - K_{RL}\mathcal{J}_L] - \mathcal{J}^*_L[K_{LR}\mathcal{J}_R - K_{LL}\mathcal{J}_L]\Big\} \ ,
\end{aligned}
\tag{2.66}
$$

where $\pi_\mathcal{N}$ is the renormalised conjugate field defined in Eq.(2.22). Here, we have used the fact that the integrand in the first step can be written as a product

$$
[r^{\nu+\frac{1}{2}(\mathcal{N}-1)}\varphi_{_\mathcal{N}}]^* r^{-\nu-\frac{1}{2}(\mathcal{N}-1)}\pi_\mathcal{N} \ ,
\tag{2.67}
$$

and each factor in this product has a finite limit as we remove the regulator at the boundary (i.e. take $r_c \to 0$ limit). The dS-SK contour integral $\oint$ runs clockwise from the right static patch to the left static patch, thus resulting in the sign of the final expression above.

We can further simplify the above expression using the reality properties of the multipole sources as well as $1 + n_\omega + n_{-\omega} = 0$. We will now argue that the fluctuations also admit a small ω expansion. To this end, we use $1 + n_\omega + n_{-\omega} = 0$ to rewrite the cosmological influence phase as

$$S_{\text{CIP}} = -\sum_{\mathbb{L}} \int \frac{d\omega}{2\pi} \left[K_{\text{Out}}(\omega, \ell)\ \mathcal{J}_D^* \mathcal{J}_A + \frac{1}{2}\left(n_\omega + \frac{1}{2}\right) \left[K_{\text{Out}}(\omega, \ell) - K_{\text{Out}}(-\omega, \ell)\right] \mathcal{J}_D^* \mathcal{J}_D \right] . \tag{2.68}$$

Since $\omega\ n_\omega$ has a regular small ω expansion, we conclude from the above expression that S_{CIP} has a regular small frequency expansion provided K_{Out} has such an expansion. Up to 1st order in ω, we have

$$K_{\text{Out}} = K_{\text{Out}}|_{\omega=0} - i\ \omega\ \tau_{dS} + \dots \tag{2.69}$$

where τ_{dS} can be interpreted as the cosmological decay time-scale for slowly varying multipole moments in dS.[4] Due to the dS version of the fluctuation-dissipation theorem, this is also proportional to the variance of the Hubble Hawking noise. This fact can be gleaned from the leading $\mathcal{J}_D^* \mathcal{J}_D$ term in the cosmological influence phase:

$$S_{\text{CIP}} \supset i \sum_{\mathbb{L}} \frac{\tau_{dS}}{2\pi} \int \frac{d\omega}{2\pi} \mathcal{J}_D^* \mathcal{J}_D\ . \tag{2.70}$$

Using the Hubbard-Stratonovich transformation, we can think of this term arising from integrating out a noise field with a time-domain action:

$$\sum_{\mathbb{L}} \int du \left[\frac{i}{2} \frac{\pi}{\tau_{dS}} \mathcal{N}^2(u) + \mathcal{J}_D(u) \mathcal{N}(u) \right] \ . \tag{2.71}$$

The first term here then shows that $\mathcal{N}(u)$ behaves like a Gaussian noise field with variance $\frac{\tau_{dS}}{\pi}$.

The important fact to note about the influence phase is that, for all values of μ appearing in table 2.1 except $\mu = \frac{d}{2}$, we get a nice small ω expansion. For $\mu = \frac{d}{2}$, we still get a small ω expansion for all $\ell > 0$: only the $\ell = 0$ term has a $1/\omega$ behaviour at small ω. The physical interpretation of these statements is this: in all these cases except $\ell = 0, \mu = \frac{d}{2}$, one obtains a Markovian open system at small ω, i.e., a cosmically old

[4]We tabulate τ_{dS} for various cases of interest in tables 2.9, 2.10 and 2.11.

observer in dS does not retain any memory of its past.[5] This is an interesting observation, especially in even d where the corresponding flat spacetime problem has memory terms [86]. This suggests that *the radiation reaction problem in an expanding spacetime is perhaps better behaved than the one in flat spacetime.* In dual quantum mechanics, this predicts that a clean separation of slow/fast degrees of freedom should be possible, at least in the leading large N approximation.

Table 2.9: τ_{dS} for $\mu = \frac{d}{2}$ (Massless KG scalar, Gravity tensor sector)

$\mu = \frac{d}{2}$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$d = 3$	4	1	$\frac{64}{225}$	$\frac{4}{49}$	$\frac{256}{11025}$
$d = 4$	$\frac{9\pi^2}{16}$	1	$\frac{25\pi^2}{1024}$	$\frac{1}{16}$	$\frac{441\pi^2}{262144}$
$d = 5$	$\frac{64}{9}$	1	$\frac{256}{1225}$	$\frac{4}{81}$	$\frac{16384}{1334025}$
$d = 6$	$\frac{225\pi^2}{256}$	1	$\frac{1225\pi^2}{65536}$	$\frac{1}{25}$	$\frac{3969\pi^2}{4194304}$
$d = 7$	$\frac{256}{25}$	1	$\frac{16384}{99225}$	$\frac{4}{121}$	$\frac{65536}{9018009}$
$d = 8$	$\frac{1225\pi^2}{1024}$	1	$\frac{3969\pi^2}{262144}$	$\frac{1}{36}$	$\frac{9801\pi^2}{16777216}$
$d = 9$	$\frac{16384}{1225}$	1	$\frac{65536}{480249}$	$\frac{4}{169}$	$\frac{1048576}{225450225}$
$d = 10$	$\frac{99225\pi^2}{65536}$	1	$\frac{53361\pi^2}{4194304}$	$\frac{1}{49}$	$\frac{1656369\pi^2}{4294967296}$
$d = 11$	$\frac{65536}{3969}$	1	$\frac{1048576}{9018009}$	$\frac{4}{225}$	$\frac{4194304}{1329696225}$

[5] The mild breakdown of small ω expansion in $\mu = \frac{d}{2}$ gives a tail term in the radiation reaction. This has been previously noted in [97]. This tail term can be avoided either by turning off the monopole moment or by giving the scalar a small mass.

Table 2.10: τ_{dS} for $\mu = \frac{d}{2} - 1$ (EM/Gravity vector sector)

$\mu = \frac{d}{2} - 1$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$d = 3$	1	1	$\frac{4}{9}$	$\frac{4}{25}$	$\frac{64}{1225}$	$\frac{64}{3969}$
$d = 4$	$\frac{\pi^2}{4}$	1	$\frac{9\pi^2}{256}$	$\frac{1}{9}$	$\frac{225\pi^2}{65536}$	$\frac{1}{100}$
$d = 5$	4	1	$\frac{64}{225}$	$\frac{4}{49}$	$\frac{256}{11025}$	$\frac{64}{9801}$
$d = 6$	$\frac{9\pi^2}{16}$	1	$\frac{25\pi^2}{1024}$	$\frac{1}{16}$	$\frac{441\pi^2}{262144}$	$\frac{1}{225}$
$d = 7$	$\frac{64}{9}$	1	$\frac{256}{1225}$	$\frac{4}{81}$	$\frac{16384}{1334025}$	$\frac{64}{20449}$
$d = 8$	$\frac{225\pi^2}{2566}$	1	$\frac{1225\pi^2}{65536}$	$\frac{1}{25}$	$\frac{3969\pi^2}{4194304}$	$\frac{1}{441}$
$d = 9$	$\frac{256}{25}$	1	$\frac{16384}{99225}$	$\frac{4}{121}$	$\frac{65536}{9018009}$	$\frac{64}{38025}$
$d = 10$	$\frac{1225\pi^2}{1024}$	1	$\frac{25\pi^2}{1024}$	$\frac{1}{16}$	$\frac{441\pi^2}{262144}$	$\frac{1}{225}$
$d = 11$	$\frac{16384}{1225}$	1	$\frac{65536}{480249}$	$\frac{4}{169}$	$\frac{1048576}{225450225}$	$\frac{64}{65025}$

Table 2.11: τ_{dS} for $\mu = \frac{d}{2} - 2$ (EM/Gravity scalar sector)

$\mu = \frac{d}{2} - 2$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$d = 3$	1	1	$\frac{4}{9}$	$\frac{4}{25}$	$\frac{64}{1225}$	$\frac{64}{3969}$
$d = 4$	1	$\frac{\pi^2}{16}$	$\frac{1}{4}$	$\frac{9\pi^2}{1024}$	$\frac{1}{36}$	$\frac{225\pi^2}{262144}$
$d = 5$	1	$\frac{4}{9}$	$\frac{4}{25}$	$\frac{64}{1225}$	$\frac{64}{3969}$	$\frac{256}{53361}$
$d = 6$	1	$\frac{9\pi^2}{256}$	$\frac{1}{9}$	$\frac{225\pi^2}{65536}$	$\frac{1}{100}$	$\frac{1225\pi^2}{4194304}$
$d = 7$	1	$\frac{64}{225}$	$\frac{4}{49}$	$\frac{256}{11025}$	$\frac{64}{9801}$	$\frac{16384}{9018009}$
$d = 8$	1	$\frac{25\pi^2}{1024}$	$\frac{1}{16}$	$\frac{441\pi^2}{262144}$	$\frac{1}{225}$	$\frac{2025\pi^2}{16777216}$
$d = 9$	1	$\frac{256}{1225}$	$\frac{4}{81}$	$\frac{16384}{1334025}$	$\frac{64}{20449}$	$\frac{65536}{81162081}$
$d = 10$	1	$\frac{1225\pi^2}{65536}$	$\frac{1}{25}$	$\frac{3969\pi^2}{4194304}$	$\frac{1}{441}$	$\frac{245025\pi^2}{4294967296}$
$d = 11$	1	$\frac{16384}{99225}$	$\frac{4}{121}$	$\frac{65536}{9018009}$	$\frac{64}{38025}$	$\frac{1048576}{2606204601}$

2.2.3 Regularisation for Neumann scalars

This section gives a procedure for obtaining the boundary 2-point function K_{Out} for the designer scalar with the scalar satisfying Neumann boundary conditions at the $r = 0$

boundary. It is slightly tangential to the route we were following but becomes useful for the treatment of the electromagnetic problem considered in the next chapter, so this chapter can be safely skipped if the reader without losing continuity.

The action (E.1) describes the dynamics of the designer scalar field with two parameters $\mathcal{N}$ and μ. For the specific case of the electric Debye potential, for which we use the results of this section, these parameters take the values $\mathcal{N} = d - 3$ and $\mu = \frac{d}{2} - 2$.

The conjugate field for radial evolution for $\varphi_{\mathcal{N}}$ is given by $\pi_{\mathcal{N}} = -r^{\mathcal{N}} D_+ \varphi_{\mathcal{N}}$. For a Neumann boundary condition, we will fix the value of $\pi_{\mathcal{N}}$ to some source multipole moment. The boundary 2-point function is then specified by the behaviour of the $\varphi_{\mathcal{N}}$ at $r \to 0$. Naively, this limit yields a divergence similar to the divergence of the Coulomb field of a point charge at $r = 0$. We would like to regulate this divergence by the usual QFT technique of adding appropriate counterterms to our action. To this end, let's first look at the divergent behaviour of the $\varphi_{\mathcal{N}}$ at as we take the limit $r \to 0$.

We require $\varphi_{\mathcal{N}}$ to satisfy outgoing boundary conditions at the horizon, which is equivalent to demanding analyticity at $r = 1$ in the outgoing Eddington-Finkelstein coordinates. The Neumann boundary condition at $r = 0$ is imposed as:

$$\lim_{r\to 0} r^{\nu + \frac{\mathcal{N}+1}{2}} \left\{ -r^{\mathcal{N}} D_+ \varphi_{\mathcal{N}} \right\} = \mathcal{J}_{\ell\vec{m}}(\omega) \tag{2.72}$$

Given these boundary condition, $\varphi_{\mathcal{N}}$ can be written as:

$$\varphi_{\mathcal{N}} = \frac{G^{\mathrm{Out}}_{\mathcal{N}}}{\nu + \frac{\mathcal{N}-1}{2}} \mathcal{J}_{\ell\vec{m}} \tag{2.73}$$

where $G^{\mathrm{Out}}_{\mathcal{N}}$ is given by:

$$\begin{aligned} G^{\mathrm{Out}}_{\mathcal{N}} &= r^{-\nu - \frac{1}{2}(\mathcal{N}-1)} (1+r)^{-i\omega} \\ &\times \left\{ {}_2F_1 \left[\frac{1-\nu+\mu-i\omega}{2}, \frac{1-\nu-\mu-i\omega}{2}; 1-\nu; r^2 \right] \right. \\ &\qquad \left. -(1 + i\cot\nu\pi) \widehat{K}_{\mathrm{Out}} \frac{r^{2\nu}}{2\nu}\, {}_2F_1 \left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1+\nu; r^2 \right] \right\}. \end{aligned} \tag{2.74}$$

As one can see from the above formula, $\varphi_{\mathcal{N}}$ has a term with leading behaviour of $r^{-\nu - \frac{\mathcal{N}-1}{2}}$ as $r \to 0$ that diverges and needs to be countertermed away.

The renormalised field ($\varphi_{\mathcal{N}\text{ren}}$) is given by:

$$\varphi_{\mathcal{N}\text{ren}} = \varphi_{\mathcal{N}} + \frac{r}{\mathscr{C}_{\mathcal{N}}} D_+ \varphi_{\mathcal{N}} \tag{2.75}$$

where $\mathscr{C}_{\mathcal{N}}$ is the same function that appears in the counterterming of the conjugate field in the corresponding Dirichlet problem. In particular,

$$\begin{aligned} \frac{\mathscr{C}_{\mathcal{N}}}{1-r^2} \equiv -r \frac{d}{dr} \ln \Bigg\{ & r^{-\nu-\frac{1}{2}(\mathcal{N}-1)} (1-r^2)^{-\frac{i\omega}{2}} \\ & \times {}_2F_1 \left[\frac{1-\nu+\mu-i\omega}{2}, \frac{1-\nu-\mu-i\omega}{2}; 1-\nu; r^2 \right] \Bigg\} . \end{aligned} \tag{2.76}$$

We showed previously that $\mathscr{C}_{\mathcal{N}}$ is an even function in ω and has a well-behaved small r expansion. Let's see how $\frac{1}{\mathscr{C}_{\mathcal{N}}}$ behaves at small r:

$$\frac{1}{\mathscr{C}_{\mathcal{N}}} = \frac{2}{2\nu + \mathcal{N} - 1} \left\{ 1 + r^2 - \frac{(\nu-1)^2 - \mu^2 - \omega^2}{(\nu-1)(2\nu+\mathcal{N}-1)} r^2 + \dots \right\} \tag{2.77}$$

Even in this case, the counterterm is local in time, which can be verified by further expanding the above function. Given this definition of the renormalised field, its boundary behaviour becomes:

$$\lim_{r\to 0} r^{-\nu + \frac{\mathcal{N}-1}{2}} \varphi_{\mathcal{N}\text{ren}} = -\frac{1 + i \cot \nu\pi}{\left(\nu + \frac{\mathcal{N}-1}{2}\right)^2} \widehat{K}_{\text{Out}} \tag{2.78}$$

Renormalising the $\varphi_{\mathcal{N}}$ in this manner is equivalent to adding the following counterterm to the action:

$$S_{ct} = \frac{1}{2} \sum_{\ell \vec{m}} \int \frac{d\omega}{2\pi} r^{\mathcal{N}+1} \frac{1}{\mathscr{C}_{\mathcal{N}}(r,\omega,\ell,\vec{m})} (D_+\varphi_{\mathcal{N}})^* D_+ \varphi_{\mathcal{N}}|_{\text{Bnd}} \ . \tag{2.79}$$

For the case of even d, one needs an additional counterterm to make the action finite. This counterterm is the same as the one required in the Dirichlet case:

$$S_{ct,\text{Even}} = \sum_{\ell \vec{m}} \frac{1}{\nu - n} \int \frac{d\omega}{2\pi} r^{\mathcal{N}-1+2n} \Delta_{\mathcal{N}}(n,\mu,\omega) \varphi_{\mathcal{N}}^* \varphi_{\mathcal{N}}|_{\text{Bnd}} \ . \tag{2.80}$$

2.2.4 Extended sources on dS-SK contour I : bulk-to-bulk propagator

In this section, we will describe the problem of a finite size observer within dS spacetime. One motivation for such an exercise is to give a more physical version of the regularisation, counter-terms and renormalisation described in the previous sub-sections. We will see that indeed a finite size observer has a renormalised cosmological influence phase, which, as its size is reduced, approaches the result for a point-like observer. Apart from this formal motivation, we are also interested in checking whether the conjectured dS-SK saddle point correctly reproduces the finite size physics in dS. As we shall see, this is also a way to naturally generalise our construction to a non-co-moving observer with a peculiar velocity as well as to describe observers made of multiple worldlines (or equivalently the case of a string or a membrane in dS).

The main physics in all the above cases is that of relative time-delays: for an extended source, its effective radiative multipole moments have to be computed by adding up source strengths at various points with different time-delays. This is necessary because the emitted wave takes a finite amount of time to cross an extended source, and this wave-crossing time has to be accounted for when adding up emissions from two farther ends of the source. For spherical sources in flat space, this translates to modulating the source with an appropriate Bessel J function in frequency domain. We will see below that an analogous statement in dS emerges naturally out of dS-SK saddle-point geometry.

Let us begin by describing our setup. Consider an extended source of the generalised/designer scalar field in dS spacetime. This means modifying the radial ODE in Eq.(2.7) by a source term of the form

$$\begin{aligned}&\frac{1}{r^{\mathcal{N}}}D_+[r^{\mathcal{N}}D_+\varphi_{\mathcal{N}}]+\omega^2\varphi_{\mathcal{N}}\\&\qquad+\frac{1-r^2}{4r^2}\Big\{(\mathcal{N}-1)^2-4\nu^2+[4\mu^2-(\mathcal{N}+1)^2]r^2\Big\}\varphi_{\mathcal{N}}+(1-r^2)\varrho_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m})=0\ .\end{aligned} \tag{2.81}$$

In the context of dS-SK contour, we will let $\varrho_{\mathcal{N}}$ be a general function over the saddle-point geometry, allowing it to even take completely different values in the two copies of the static patch (i.e., as a function of complex r, it is allowed to have a branch-cut along the static patch). The $(\omega,\mathbb{L})$ arguments of $\varrho_{\mathcal{N}}$ imply that we also allow the most general time/angle dependence.

The solution for the above ODE can then be written in terms of an appropriate dS-SK contour-ordered, bulk-to-bulk Green function:

$$\varphi_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m}) = \oint r_0^{\mathcal{N}} dr_0 \ \mathbb{G}(\zeta|\zeta_0,\omega,\ell)\varrho_{\mathcal{N}}(\zeta_0,\omega,\ell,\vec{m}) \ . \tag{2.82}$$

Here $\oint$ refers to the integral over clockwise dS-SK contour and $\mathbb{G}$ is the radial Green function satisfying the appropriate boundary conditions (which we will detail below).

According to our proposal in this note, the influence phase of the extended source can be computed by solving the above ODE everywhere on dS-SK and then substituting the solution into the action corresponding to the above ODE, viz., by evaluating

$$\begin{aligned} S = -\frac{1}{2}\sum_{\mathbb{L}}\int\frac{d\omega}{2\pi}\oint\frac{r^{\mathcal{N}}dr}{1-r^2}\Big[&(D_+\varphi_{\mathcal{N}})^* D_+\varphi_{\mathcal{N}} - \omega^2\varphi^*_{\mathcal{N}}\varphi_{\mathcal{N}} \\ &-\frac{1-r^2}{4r^2}\Big\{(\mathcal{N}-1)^2-4\nu^2+[4\mu^2-(\mathcal{N}+1)^2]r^2\Big\}\varphi^*_{\mathcal{N}}\varphi_{\mathcal{N}}\Big] \\ &+\sum_{\mathbb{L}}\int\frac{d\omega}{2\pi}\oint r^{\mathcal{N}}dr\ \varphi^*_{\mathcal{N}}\varrho_{\mathcal{N}} + S_{ct}[\varrho_{\mathcal{N}}] \end{aligned} \tag{2.83}$$

on the Green function solution above. The last line in the action above gives the source term and the counter-term parts of the action.[6] For a truly extended source, counter-terms are not necessary for finiteness, and their job is to provide the finite renormalisation of the conservative part of the action.

Using the radial ODE above, on-shell action can be reduced to the following simple form

$$\begin{aligned} S|_{\textbf{On-shell}} &= \frac{1}{2}\sum_{\ell\vec{m}}\int\frac{d\omega}{2\pi}\oint r^{\mathcal{N}}dr\ \varrho^*_{\mathcal{N}}\varphi_{\mathcal{N}}|_{\textbf{On-shell}} + S_{ct}[\varrho_{\mathcal{N}}] \\ &= \frac{1}{2}\sum_{\ell\vec{m}}\int\frac{d\omega}{2\pi}\oint r^{\mathcal{N}}dr\ \oint r_0^{\mathcal{N}}dr_0\ [\varrho_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m})]^*\mathbb{G}(\zeta|\zeta_0,\omega,\ell)\varrho_{\mathcal{N}}(\zeta_0,\omega,\ell,\vec{m}) \\ &\qquad + S_{ct}[\varrho_{\mathcal{N}}] \ . \end{aligned} \tag{2.84}$$

Thus, once we solve for the bulk-to-bulk Green function $\mathbb{G}$, we can substitute it into the above expression to obtain the dS-SK saddle point answer for cosmological influence phase

[6] The reader should note that the counterterms used here for extended sources need not (and, indeed, will not) match with the counterterms used for point sources in the previous subsections.

S_{CIP}. While it is not immediately evident, we will demonstrate in the next subsection that *the dissipative part of the influence phase for the extended sources computed from the expression above, when written in terms of appropriate multipole moments, takes a form identical to that for a point source derived before.* In addition to this radiation reaction, for extended sources, we also expect conservative interactions between their different internal parts.

Let us now derive an explicit expression for the bulk-to-bulk Green function $\mathbb{G}$. The construction here is analogous to the one in vacuum AdS [33], as well as the contour-ordered bulk-to-bulk Green function in the SK contour corresponding to planar AdS black holes [54, 100]. We will demand that this Green function be regular at the edges of dS-SK contour, viz., we require that

$$\lim_{\zeta\to 0} r^{\nu+\frac{\mathcal{N}-1}{2}}\,\mathbb{G} = \lim_{\zeta\to 1} r^{\nu+\frac{\mathcal{N}-1}{2}}\,\mathbb{G} = 0. \tag{2.85}$$

Further, to be a Green function, it should obey the ODE

$$\begin{aligned}&\frac{1}{r^{\mathcal{N}}}D_+[r^{\mathcal{N}}D_+\mathbb{G}] + \omega^2\mathbb{G}\\ &\qquad + \frac{1-r^2}{4r^2}\Big\{(\mathcal{N}-1)^2 - 4\nu^2 + [4\mu^2 - (\mathcal{N}+1)^2]r^2\Big\}\mathbb{G} + \frac{1}{r^{\mathcal{N}}}(1-r^2)\delta_c(r-r_0) = 0\ .\end{aligned} \tag{2.86}$$

Here $\delta_c(r-r_0)$ is the contour-ordered delta function on the dS-SK contour. The above ODE implies that $\mathbb{G}$ is a solution of the homogeneous radial ODE for $\zeta \neq \zeta_0$ with a unit discontinuity in the conjugate field at $\zeta = \zeta_0$. We have already solved the homogeneous radial ODE for point sources to construct the left and right boundary-to-bulk Green functions in Eq.(2.56). These are solutions characterised by the boundary conditions specified in Eq.(2.57).

Looking at Eq.(2.57), we conclude that we should take $\mathbb{G} \propto g_R$ near the left boundary and $\mathbb{G} \propto g_L$ near the right boundary since these are the solutions that satisfy the necessary regularity conditions in Eq.(2.85). Demanding continuity, we surmise that

$$\begin{aligned} \mathbb{G}(\zeta|\zeta_0,\omega,\ell) &= \frac{1}{W_{LR}(\zeta_0,\omega,\ell)} g_R(\zeta_\succ,\omega,\ell) g_L(\zeta_\prec,\omega,\ell) \\ &\equiv \frac{1}{W_{LR}(\zeta_0,\omega,\ell)} \begin{cases} g_R(\zeta,\omega,\ell) g_L(\zeta_0,\omega,\ell) & \text{if } \zeta \succ \zeta_0 \\ g_L(\zeta,\omega,\ell) g_R(\zeta_0,\omega,\ell) & \text{if } \zeta \prec \zeta_0 \end{cases} . \end{aligned} \tag{2.87}$$

Here, the symbols $\succ$ and $\prec$ denote comparison using the radial contour ordering of the dS-SK contour. The unit discontinuity condition on the conjugate field fixes the function W_{RL} to be the Wronskian between right and left boundary-to-bulk Green functions, viz.,

$$\begin{aligned} W_{RL}(\zeta,\omega,\ell) &\equiv g_L \pi_R - g_R \pi_L = (1+n_\omega) e^{-2\pi\omega\zeta} \left(G^{\text{Out}}_{\mathbb{N}} \pi^{\text{Out}*}_{\mathbb{N}} - G^{\text{Out}*}_{\mathbb{N}} \pi^{\text{Out}}_{\mathbb{N}} \right) \\ &= (1+n_\omega) e^{-2\pi\omega\zeta} \left[(1 - i\cot\pi\nu) \widehat{K}_{\text{In}} - (1 + i\cot\pi\nu) \widehat{K}_{\text{Out}} \right] . \end{aligned} \tag{2.88}$$

Here, the equality in the first line follows from Eqs.(2.56) and (2.61). The last equality follows by substituting the expressions for $G^{\text{Out}}_{\mathbb{N}}$ and $\pi^{\text{Out}}_{\mathbb{N}}$ from Eqs.(2.10) and (2.26), and then invoking the following hypergeometric Wronskian identity

$$\begin{aligned} &\mathscr{Z}_{\mathbb{N}}(r,\omega) r^{\nu+\frac{1}{2}(\mathbb{N}-1)} (1+r)^{-i\omega} {}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1+\nu; r^2\right] \\ &\quad = \left(\frac{1-r}{1+r}\right)^{i\omega} \frac{r^{\nu+\frac{1}{2}(\mathbb{N}-1)}(1+r)^{i\omega}}{{}_2F_1\left[\frac{1-\nu+\mu-i\omega}{2}, \frac{1-\nu-\mu-i\omega}{2}; 1-\nu; r^2\right]} . \end{aligned} \tag{2.89}$$

This identity expresses a combination of the derivatives of hypergeometric functions in terms of the hypergeometric functions, and such an identity can be derived from a wronskian-like argument associated with the corresponding radial ODE.

The reader should note an important subtlety in the statement above: the Wronskian here is *not* a constant function along radial direction, but rather varies as we traverse the dS-SK contour. A similar subtlety was already noted in the AdS context by [54]. As we shall eventually see, the extra $e^{-2\pi\omega\zeta}$ factor is here for a good physical reason: it ensures that multipole moments that enter into cosmological influence phase are computed using source distributions in standard time-slices, instead of source distributions along Eddington-Finkelstein null time-slices.

To proceed further, we should now substitute the explicit forms of dS-SK boundary-to-bulk propagators given in Eqs.(2.58) and (2.59) into the expression for the bulk-to-bulk propagator in Eq.(2.87), and then perform the dS-SK contour integral in Eq.(2.84). To this end, we first regroup the expressions for g_L and g_R into somewhat more tractable expressions with clear branch-cut structures. For what follows, we will find it convenient to separate out the solutions into a singular (non-normalisable) part Ξ_{nn} vs a regular (normalisable) part Ξ_n, *using the renormalised world line Green functions instead of the bare ones from the start.* The adjectives singular/regular refer here to their behaviour near the worldline (i.e., near $r = 0$). To this end, let us begin by defining two functions Ξ_{nn}, Ξ_n implicitly via

$$\begin{aligned} \left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}} G_{\mathcal{N}}^{\text{Out}}(r,\omega,\ell) &\equiv \Xi_{nn}(r,\omega,\ell) - K_{\text{Out}}\ \Xi_n(r,\omega,\ell)\ , \\ \left(\frac{1-r}{1+r}\right)^{\frac{i\omega}{2}} G_{\mathcal{N}}^{\text{Out}*}(r,\omega,\ell) &\equiv \Xi_{nn}(r,\omega,\ell) - K_{\text{In}}\ \Xi_n(r,\omega,\ell)\ , \end{aligned} \tag{2.90}$$

where K_{Out} and K_{In} are the final renormalised world line Green functions. The above equality should be thought of as defining the functions $\Xi_n(r,\omega,\ell)$ and $\Xi_{nn}(r,\omega,\ell)$ as analytic functions on the open static patch $0 < r < 1$, viz., in the equations above, we align all the potential branch cuts away from the unit disc in complex radius plane. The above equations can be inverted to give a direct definition of these functions

$$\begin{aligned} (K_{\text{In}} - K_{\text{Out}})\ \Xi_n &\equiv \left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}} G_{\mathcal{N}}^{\text{Out}} - \left(\frac{1-r}{1+r}\right)^{\frac{i\omega}{2}} G_{\mathcal{N}}^{\text{Out}*}\ , \\ (K_{\text{In}} - K_{\text{Out}})\ \Xi_{nn} &\equiv \left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}} K_{\text{In}} G_{\mathcal{N}}^{\text{Out}} - \left(\frac{1-r}{1+r}\right)^{\frac{i\omega}{2}} K_{\text{Out}} G_{\mathcal{N}}^{\text{Out}*}\ . \end{aligned} \tag{2.91}$$

Since $K_{\text{In}}(\omega,\ell) = K_{\text{Out}}(-\omega,\ell)$ and $G_{\mathcal{N}}^{\text{Out}*}(\omega,\ell) = G_{\mathcal{N}}^{\text{Out}}(-\omega,\ell)$, the above expressions imply that both Ξ_n and Ξ_{nn} are even functions of ω. Explicit expressions can be written down for these two functions using Eq.(2.10). We have

$$\begin{aligned} \Xi_n \equiv \frac{1}{2\nu} r^{\nu-\frac{1}{2}(\mathcal{N}-1)} (1-r^2)^{-\frac{i\omega}{2}} & \frac{(1+i\cot\nu\pi)\widehat{K}_{\text{Out}} - (1-i\cot\nu\pi)\widehat{K}_{\text{In}}}{K_{\text{Out}} - K_{\text{In}}} \\ \times\, {}_2F_1 & \left[\frac{1+\nu-\mu-i\omega}{2}, \frac{1+\nu+\mu-i\omega}{2}; 1+\nu; r^2\right]\ , \end{aligned} \tag{2.92}$$

for the normalisable/regular mode and

$$
\begin{aligned}
\Xi_{nn} \equiv r^{-\nu-\frac{1}{2}(\mathbb{N}-1)}(1-r^2)^{-\frac{i\omega}{2}}&\left\{{}_2F_1\left[\frac{1-\nu+\mu-i\omega}{2},\frac{1-\nu-\mu-i\omega}{2};1-\nu;r^2\right]\right.\\
-\ &\frac{K_{\text{In}}(1+i\cot\nu\pi)\widehat{K}_{\text{Out}}-K_{\text{Out}}(1-i\cot\nu\pi)\widehat{K}_{\text{In}}}{K_{\text{In}}-K_{\text{Out}}}\\
&\left.\times\frac{r^{2\nu}}{2\nu}{}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2},\frac{1+\nu+\mu-i\omega}{2};1+\nu;r^2\right]\right\}
\end{aligned}
\tag{2.93}
$$

for the non-normalisable/singular mode. One advantage of working with such renormalised functions is that we can safely remove the dimensional regularisation in the above expressions, resulting in a finite limit. When d is odd and $\nu \equiv \ell + \frac{d}{2} - 1 \in \mathbb{Z} + \frac{1}{2}$, we can simply set $\cot\nu\pi = 0$ and take $\widehat{K}_{\text{Out}} \to K_{\text{Out}}$ and $\widehat{K}_{\text{In}} \to K_{\text{In}}$. All the K's then drop out of the above expression, and Ξ_{nn} and Ξ_n become proportional to single hypergeometric functions.

When d is even and $\nu \to n \in \mathbb{Z}$, we can use Eqs.(2.33) and (2.48) to write

$$
\begin{aligned}
(1+i\cot\nu\pi)\widehat{K}_{\text{Out}} &= \frac{2}{\nu-n}\Delta_{\mathbb{N}}(n,\mu,\omega) + K_{\text{Out}} + O(\nu-n)\ ,\\
(1-i\cot\nu\pi)K_{\text{In}} &= \frac{2}{\nu-n}\Delta_{\mathbb{N}}(n,\mu,\omega) + K_{\text{In}} + O(\nu-n)\ .
\end{aligned}
\tag{2.94}
$$

Using these expansions, the Ks cancel out again and we are left with the following limits:

$$
\begin{aligned}
\Xi_n|_{\text{Even d}} &\equiv \lim_{\nu\to n}\frac{1}{2\nu}r^{\nu-\frac{1}{2}(\mathbb{N}-1)}(1-r^2)^{-\frac{i\omega}{2}}{}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2},\frac{1+\nu+\mu-i\omega}{2};1+\nu;r^2\right]\ ,\\
\Xi_{nn}|_{\text{Even d}} &\equiv \lim_{\nu\to n} r^{-\nu-\frac{1}{2}(\mathbb{N}-1)}(1-r^2)^{-\frac{i\omega}{2}}\left\{{}_2F_1\left[\frac{1-\nu+\mu-i\omega}{2},\frac{1-\nu-\mu-i\omega}{2};1-\nu;r^2\right]\right.\\
&\qquad\left.-\frac{r^{2\nu}}{\nu(\nu-n)}\Delta_{\mathbb{N}}(n,\mu,\omega)\ {}_2F_1\left[\frac{1+\nu-\mu-i\omega}{2},\frac{1+\nu+\mu-i\omega}{2};1+\nu;r^2\right]\right\}\ .
\end{aligned}
\tag{2.95}
$$

One can explicitly check that these limits exist and result in finite expressions for both regular/singular modes when d is even. To summarise, Eq.(2.90) decomposes the outgoing/incoming Green functions into renormalised pieces in any d.

We will now rewrite the full bulk-to-bulk propagator in Eq.(2.87) in terms of these renormalised modes. We begin by rewriting the boundary-to-bulk propagators: using

Eq.(2.56), we obtain

$$
\begin{aligned}
g_L &= n_\omega \left(\frac{1-r}{1+r}\right)^{\frac{i\omega}{2}} \left\{\left[1 - e^{2\pi\omega(1-\zeta)}\left(\frac{1-r}{1+r}\right)^{-i\omega}\right]\Xi_{nn}\right. \\
&\qquad \left. - \left[K_{\text{Out}} - K_{\text{In}} e^{2\pi\omega(1-\zeta)}\left(\frac{1-r}{1+r}\right)^{-i\omega}\right]\Xi_n\right\}, \\
g_R &= (1+n_\omega) \left(\frac{1-r}{1+r}\right)^{\frac{i\omega}{2}} \left\{\left[1 - e^{-2\pi\omega\zeta}\left(\frac{1-r}{1+r}\right)^{-i\omega}\right]\Xi_{nn}\right. \\
&\qquad \left. - \left[K_{\text{Out}} - K_{\text{In}} e^{-2\pi\omega\zeta}\left(\frac{1-r}{1+r}\right)^{-i\omega}\right]\Xi_n\right\}.
\end{aligned} \tag{2.96}
$$

Substituting them back into Eq.(2.87), we get an explicit expression for the bulk-to-bulk propagator of the form

$$
\begin{aligned}
\mathbb{G}(\zeta|\zeta_0,\omega,\ell) &= \frac{1}{W_{LR}(\zeta_0,\omega,\ell)} g_R(\zeta_\succ,\omega,\ell) g_L(\zeta_\prec,\omega,\ell) \\
&= \frac{n_\omega e^{2\pi\omega\zeta_0}}{K_{\text{In}} - K_{\text{Out}}} \left(\frac{1-r}{1+r}\right)^{\frac{i\omega}{2}} \left(\frac{1-r_0}{1+r_0}\right)^{\frac{i\omega}{2}} \\
&\times \left\{\left[1 - e^{-2\pi\omega\zeta_\succ}\left(\frac{1-r_\succ}{1+r_\succ}\right)^{-i\omega}\right]\Xi_{nn}(r_\succ)\right. \\
&\qquad \left. - \left[K_{\text{Out}} - K_{\text{In}} e^{-2\pi\omega\zeta_\succ}\left(\frac{1-r_>}{1+r_\succ}\right)^{-i\omega}\right]\Xi_n(r_\succ)\right\} \\
&\times \left\{\left[1 - e^{2\pi\omega(1-\zeta_\prec)}\left(\frac{1-r_\prec}{1+r_\prec}\right)^{-i\omega}\right]\Xi_{nn}(r_\prec)\right. \\
&\qquad \left. - \left[K_{\text{Out}} - K_{\text{In}} e^{2\pi\omega(1-\zeta_\prec)}\left(\frac{1-r_\prec}{1+r_\prec}\right)^{-i\omega}\right]\Xi_n(r_\prec)\right\}.
\end{aligned} \tag{2.97}
$$

Here, since all quantities are already renormalised, we have removed the dimensional regularisation[7] in the Wronskian given in Eq.(2.88). To conclude, given an arbitrary extended source on the dS-SK geometry, the above bulk-to-bulk propagator, we can get the bulk field by substituting the above bulk-to-bulk Green function into Eq.(2.82). Further, we can also compute the on-shell action Eq.(2.84), which, according to our prescription, should yield the influence phase of that extended source.

[7] For odd d, we set $\cot\nu\pi = 0$ and remove the hats on Ks. For even d, we use Eq.(2.94).

2.2.5 Extended sources on dS-SK contour II: Radiative multipoles

In this subsection, we would like to evaluate both the field and the influence of an extended source. We will find it convenient to discretise the source into a set of spherical shells around the centre of the right/left static patches. Let $\zeta = 1 + \zeta_i$ characterise the radial position of the i^{th} spherical shell in the right patch, the same radial position in the left patch is then characterised by $\zeta = \zeta_i$. We will let the i vary over 1 to N_s, where N_s is the number of shells in each copy of the static patch. We will take the strength of the scalar source on these spherical shells to be

$$r^{\mathcal{N}} \varrho_{_\mathcal{N}}(\zeta, \omega, \ell, \vec{m}) = \sum_i \sigma_i^R(\omega, \ell, \vec{m}) \; \delta_c(\zeta|1+\zeta_i) - \sum_i \sigma_i^L(\omega, \ell, \vec{m}) \; \delta_c(\zeta|\zeta_i) \; . \tag{2.98}$$

Here, as before, we work in frequency domain/orthonormal spherical harmonic basis and allow arbitrary time/angle dependence. Any arbitrary source distribution confined within the open static patch can be approximated to any desired accuracy as being built from such spherical shell sources. As we shall see, such a discrete model regularises the divergences associated with the self-interactions.

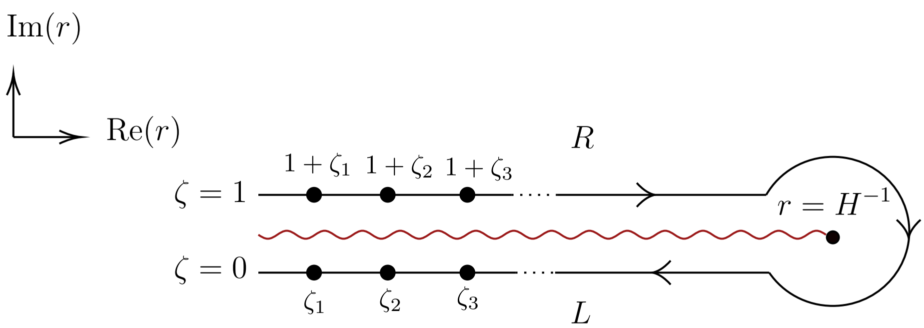

Figure 2.2: Spherical shell sources centred around the right/left static patches shown in the complex r plane. Their positions on the L contour are related to their position on the R contour by the branch cut discontinuity in ζ.

We will begin by writing down the bulk field due to the spherical shell sources described above. We have, using Eq.(2.82), a superposition of fields produced by each shell source, i.e.,

$$
\begin{aligned}
&\varphi_{\rm N}(\zeta,\omega,\ell,\vec{m})\\
&=\oint r_0^{\rm N} dr_0\ \mathbb{G}(\zeta|\zeta_0,\omega,\ell)\varrho_{\rm N}(\zeta_0,\omega,\ell,\vec{m})\\
&=\sum_i \frac{1}{W_{LR}(\zeta_i,\omega,\ell)}\begin{cases} e^{2\pi\omega} g_L(\zeta,\omega,\ell) & \left[g_R(1+\zeta_i,\omega,\ell)\ \sigma_i^R - g_L(1+\zeta_i,\omega,\ell)\ \sigma_i^L\right] \\ & \qquad \text{if } \zeta \prec 1+\zeta_i\ , \\ g_R(\zeta_i,\omega,\ell) & \left[g_R(\zeta,\omega,\ell)\ \sigma_i^R - g_L(\zeta,\omega,\ell)\ \sigma_i^L\right] \\ & \qquad \text{if } 1+\zeta_i \prec \zeta \prec \zeta_i\ , \\ g_R(\zeta,\omega,\ell) & \left[g_R(\zeta_i,\omega,\ell)\ \sigma_i^R - g_L(\zeta_i,\omega,\ell)\ \sigma_i^L\right] \\ & \qquad \text{if } \zeta \succ \zeta_i\ . \end{cases}
\end{aligned}
\tag{2.99}
$$

We remind the reader that $\prec$ and $\succ$ are comparisons using the radial contour ordering of the dS-Sk contour. We also remind the reader that ζ changes from 1 to 0, as we traverse the clockwise dS-SK contour, starting from the right static patch (See Fig.2.2.5). The reader should note that the above superposition of fields is continuous everywhere, but its derivative (and hence the conjugate field) is discontinuous at each spherical shell, with the discontinuity being determined by the strength of the scalar source at that shell. This is expected since the bulk-to-bulk Green function was constructed in the last subsection with precisely these boundary conditions in mind.

Given the above field, computing the on-shell action is straightforward. We use Eq.(2.84) to write

$$
\begin{aligned}
S|_{\textbf{On-shell}} &= \frac{1}{2}\sum_{\mathbb{L}}\int\frac{d\omega}{2\pi}\oint r^{\rm N} dr\ \varrho_{\rm N}^{*}\varphi_{\rm N}|_{\textbf{On-shell}}\\
&= \frac{1}{2}\sum_{ij\mathbb{L}}\int\frac{d\omega}{2\pi}\frac{g_R(\zeta_i,\omega,\ell)}{W_{LR}(\zeta_i,\omega,\ell)}\Big\{\sigma_j^{R*}\ \left[g_R(1+\zeta_j,\omega,\ell)\sigma_i^R - g_L(1+\zeta_j,\omega,\ell)\sigma_i^L\right]\\
&\qquad -\sigma_j^{L*}\ \left[g_R(\zeta_j,\omega,\ell)\sigma_i^R - g_L(\zeta_j,\omega,\ell)\sigma_i^L\right]\Big\}\ .
\end{aligned}
\tag{2.100}
$$

Even though we are working with distributional sources/fields, given the continuity of $\varphi_{\rm N}$, the computation above is unambiguous. Next, we substitute explicit forms of the boundary-to-bulk Green functions as well as the Wronskian in terms of renormalised quantities. We have, using Eqs.(2.96) and (2.42), the following set of equalities:

$$
\begin{aligned}
W_{LR}(\zeta_i,\omega,\ell) &= -(1+n_\omega)\left(\frac{1-r_i}{1+r_i}\right)^{i\omega}[K_{\text{Out}}-K_{\text{In}}]\ ,\\
\frac{g_R(\zeta_i,\omega,\ell)}{W_{LR}(\zeta_i,\omega,\ell)} &= \left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}}\Xi_n(r_i,\omega,\ell)\ ,\\
g_L(\zeta_i,\omega,\ell) &= -\left(\frac{1-r_i}{1+r_i}\right)^{\frac{i\omega}{2}}\left\{\Xi_{nn}(r_i,\omega,\ell)+[n_\omega K_{\text{Out}}-(1+n_\omega)K_{\text{In}}]\,\Xi_n(r_i,\omega,\ell)\right\}\ ,\\
g_L(1+\zeta_i,\omega,\ell) &= -n_\omega\left(\frac{1-r_i}{1+r_i}\right)^{\frac{i\omega}{2}}[K_{\text{Out}}-K_{\text{In}}]\,\Xi_n(r_i,\omega,\ell)\ ,\\
g_R(\zeta_i,\omega,\ell) &= -(1+n_\omega)\left(\frac{1-r_i}{1+r_i}\right)^{\frac{i\omega}{2}}[K_{\text{Out}}-K_{\text{In}}]\,\Xi_n(r_i,\omega,\ell)\ ,\\
g_R(1+\zeta_i,\omega,\ell) &= \left(\frac{1-r_i}{1+r_i}\right)^{\frac{i\omega}{2}}\left\{\Xi_{nn}(r_i,\omega,\ell)-[(1+n_\omega)K_{\text{Out}}-n_\omega K_{\text{In}}]\,\Xi_n(r_i,\omega,\ell)\right\}\ .
\end{aligned}
\tag{2.101}
$$

Substituting these expressions back into the on-shell action yields the following double sum:

$$
\begin{aligned}
S|_{\textbf{On-shell}} = \frac{1}{2}\sum_{ij}\sum_{\ell\vec{m}}\int\frac{d\omega}{2\pi}&\left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}}\left(\frac{1-r_j}{1+r_j}\right)^{\frac{i\omega}{2}}\\
\times\Big\{&\Xi_n(r_i,\omega,\ell)\ \Xi_{nn}(r_j,\omega,\ell)\ [\sigma_j^{R*}\sigma_i^R-\sigma_j^{L*}\sigma_i^L]\\
&-\Xi_n(r_i,\omega,\ell)\ \Xi_n(r_j,\omega,\ell)\ K_{\text{Out}}(\sigma_j^R-\sigma_j^L)^*[(1+n_\omega)\sigma_i^R-n_\omega\sigma_i^L]\\
&-\Xi_n(r_i,\omega,\ell)\ \Xi_n(r_j,\omega,\ell)\ K_{\text{In}}(\sigma_i^R-\sigma_i^L)[(1+n_{-\omega})\sigma_j^{R*}-n_{-\omega}\sigma_j^{L*}]\Big]\ .
\end{aligned}
\tag{2.102}
$$

Let us begin by interpreting the terms in the above double sum. We first note that the last two lines in the above expression are related by the relabelling $\omega\to-\omega$ and are hence equal. The physical meaning of the last two lines is clarified by defining the *radiative multipole moments*:

$$
\begin{aligned}
\mathcal{J}_R(\omega,\ell,\vec{m}) &\equiv \sum_i \left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}} \Xi_n(r_i,\omega,\ell)\ \sigma_i^R \\
&\equiv \int_R dr\ r^{\mathcal{N}}\Xi_n(r,\omega,\ell)\ \left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}} \varrho_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m})\ , \\
\mathcal{J}_L(\omega,\ell,\vec{m}) &\equiv \sum_i \left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}} \Xi_n(r_i,\omega,\ell)\ \sigma_i^L \\
&\equiv -\int_L dr\ r^{\mathcal{N}}\Xi_n(r,\omega,\ell)\ \left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}} \varrho_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m})\ .
\end{aligned}
\tag{2.103}
$$

The integrals here are performed over right/left *half* of the dS-SK contour respectively. We will also find it convenient to define the average/difference multipole moments via

$$\mathcal{J}_A(\omega,\ell,\vec{m}) \equiv \frac{1}{2}[\mathcal{J}_R(\omega,\ell,\vec{m}) + \mathcal{J}_L(\omega,\ell,\vec{m})]\ ,$$

and

$$\mathcal{J}_D(\omega,\ell,\vec{m}) \equiv \mathcal{J}_R(\omega,\ell,\vec{m}) - \mathcal{J}_L(\omega,\ell,\vec{m})$$

Here we deliberately use the same notation as we did for multipole moments in flat spacetime (see Eq.(A.73)) and for point-like dS sources (See Eq.(2.52)). One reason for this is as follows: the last two lines of Eq.(2.102) can be recast in terms of the above definitions, into the cosmological influence phase of a point-source

$$
\begin{aligned}
S^{\text{Pt}}_{\text{CIP}} &\equiv -\sum_{\ell\vec{m}} \int \frac{d\omega}{2\pi} K_{\text{Out}} (\mathcal{J}_R - \mathcal{J}_L)^* [(1+n_\omega)\mathcal{J}_R - n_\omega \mathcal{J}_L] \\
&= -\sum_{\ell\vec{m}} \int \frac{d\omega}{2\pi} K_{\text{Out}}\ \mathcal{J}_D^* \left[\mathcal{J}_A + \left(n_\omega + \frac{1}{2}\right)\mathcal{J}_D\right]\ .
\end{aligned}
\tag{2.104}
$$

We recognise here the exact influence phase derived for a point-like dS observer in Eq.(2.68), using a detailed counterterm procedure. More evidence for this identification will be presented in section 2.3, where we describe how these multipole moments correctly reproduce the flat space answers with Hubble corrections.

For now, we turn our attention to the remaining terms, viz., the first double sum in Eq.(2.102). The presence of the singular Green solution Ξ_{nn}, as well as the right/left factorised form of this sum, indicates that these terms incorporate non-dissipative *self-*

energy corrections of the extended source. The final on-shell action can then be written as $S|_{\textbf{On-shell}} = S^{\rm Pt}_{\rm CIP} + S_{\rm Int}$, where $S_{\rm Int}$ denotes the internal potential energy of the spherical shells:

$$S_{\rm Int} \equiv \frac{1}{2}\sum_{i\neq j\mathbb{L}} \int \frac{d\omega}{2\pi} \left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}} \left(\frac{1-r_j}{1+r_j}\right)^{\frac{i\omega}{2}} \Xi_n(r_i,\omega,\ell)\ \Xi_{nn}(r_j,\omega,\ell)\ [\sigma_j^{R*}\sigma_i^R - \sigma_j^{L*}\sigma_i^L]\ . \tag{2.105}$$

Another instructive way to rewrite this potential energy contribution is to define radially averaged mean fields on the right/left static patch via

$$\begin{aligned}
\overline{\varphi}_{R,\rm Int}(\omega,\ell,\vec{m}) &\equiv \sum_i \left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}} \Xi_{nn}(r_i,\omega,\ell)\ \sigma_i^R \\
&\equiv \int_R dr\ r^{\mathcal{N}}\Xi_{nn}(r,\omega,\ell)\ \left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}} \varrho_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m})\ , \\
\overline{\varphi}_{L,\rm Int}(\omega,\ell,\vec{m}) &\equiv \sum_i \left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}} \Xi_{nn}(r_i,\omega,\ell)\ \sigma_i^L \\
&\equiv -\int_L dr\ r^{\mathcal{N}}\Xi_{nn}(r,\omega,\ell)\ \left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}} \varrho_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m})\ .
\end{aligned} \tag{2.106}$$

We can then rewrite the potential energy as that of multipole moments placed in such an average field, viz.,

$$S_{\rm Int} = \frac{1}{2}\sum_{\mathbb{L}} \int \frac{d\omega}{2\pi} [\mathcal{J}_R^*\overline{\varphi}_{R,\rm Int} - \mathcal{J}_L^*\overline{\varphi}_{L,\rm Int}]\ . \tag{2.107}$$

2.2.6 Neumann designer scalar on the dS-SK Geometry

In paper I [101], we analysed extended sources in de Sitter coupled to designer scalars through Dirichlet boundary conditions. In this section, we will solve for the designer scalar fields in the presence of extended sources but obeying Neumann boundary conditions. This will be relevant to our study of the electric Debye scalar.

For a generic bulk source $\rho_{\mathcal{N}}$, the Neumann boundary condition arises from a coupling of $\rho_{\mathcal{N}}$ to the conjugate field $\pi_{\mathcal{N}} = -r^{\mathcal{N}} D_+ \varphi_{\mathcal{N}}$.

$$
\begin{aligned}
S = -\frac{1}{2}\sum_{\ell\vec{m}} \int \frac{d\omega}{2\pi} \oint \frac{r^{\mathcal{N}} dr}{1-r^2} \Big[(D_+\varphi_{\mathcal{N}})^* D_+\varphi_{\mathcal{N}} - \omega^2 \varphi_{\mathcal{N}}^* \varphi_{\mathcal{N}} \\
- \frac{1-r^2}{4r^2}\Big\{(\mathcal{N}-1)^2 - 4\nu^2 + [4\mu^2 - (\mathcal{N}+1)^2]r^2\Big\}\varphi_{\mathcal{N}}^*\varphi_{\mathcal{N}}\Big] \\
+ \sum_{\mathbb{L}} \int \frac{d\omega}{2\pi} \oint dr\ \pi_{\mathcal{N}}^* \varrho_{\mathcal{N}} + S_{ct}[\varrho_{\mathcal{N}}]
\end{aligned} \tag{2.108}
$$

The inhomogeneous equation of motion satisfied by such a Neumann scalar is given by:

$$
\begin{aligned}
\frac{1}{r^{\mathcal{N}}} D_+[r^{\mathcal{N}} D_+\varphi_{\mathcal{N}}] + \omega^2\varphi_{\mathcal{N}} + \frac{1-r^2}{4r^2}\Big\{(\mathcal{N}-1)^2 - 4\nu^2 + [4\mu^2 - (\mathcal{N}+1)^2]r^2\Big\}\varphi_{\mathcal{N}} \\
+ \frac{1}{r^{\mathcal{N}}} D_+ \left[r^{\mathcal{N}}(1-r^2)\varrho_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m})\right] = 0\ .
\end{aligned} \tag{2.109}
$$

Notice that the equation is the same as the inhomogeneous equation of motion for the Dirichlet scalar with the inhomogeneous term replaced with the radial derivative of the source: $\frac{1}{r^{\mathcal{N}}} D_+ \left[r^{\mathcal{N}}(1-r^2)\varrho_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m})\right]$. In such a case, we can use the same bulk-to-bulk Green function derived in the previous section[8], convolved with the appropriate Neumann source, to write the solution. In particular:

$$
\varphi_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m}) = \oint dr_0\ \frac{\mathbb{G}(\zeta|\zeta_0,\omega,\ell)}{1-r_0^2} D_+^0 \left[r_0^{\mathcal{N}}(1-r_0^2)\varrho_{\mathcal{N}}(\zeta_0,\omega,\ell,\vec{m})\right]\ . \tag{2.110}
$$

The bulk-to-bulk Green's function $\mathbb{G}$ is given by:

$$
\begin{aligned}
\mathbb{G}(\zeta|\zeta_0,\omega,\ell) &= \frac{1}{W_{LR}(\zeta_0,\omega,\ell)} g_R(\zeta_\succ,\omega,\ell) g_L(\zeta_\prec,\omega,\ell) \\
&\equiv \frac{1}{W_{LR}(\zeta_0,\omega,\ell)} \begin{cases} g_R(\zeta,\omega,\ell) g_L(\zeta_0,\omega,\ell) & \text{if } \zeta \succ \zeta_0 \\ g_L(\zeta,\omega,\ell) g_R(\zeta_0,\omega,\ell) & \text{if } \zeta \prec \zeta_0 \end{cases}\ .
\end{aligned} \tag{2.111}
$$

where $\succ$ and $\prec$ respectively mean '*succeeds*' and '*precedes*' on the dS-SK contour. In (2.110), one can use integration by parts to rewrite it in the more conventional definition of the bulk-to-bulk Green function:

$$
\varphi_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m}) = -\oint dr_0\ r_0^{\mathcal{N}} D_-^0\, \mathbb{G}(\zeta|\zeta_0,\omega,\ell)\varrho_{\mathcal{N}}(\zeta_0,\omega,\ell,\vec{m})\ . \tag{2.112}
$$

[8]The corresponding bulk-to-bulk two-point functions in the case of black holes can be found in [102].

If we now repackage this new Green function into a 'Neumann' Green function defined by:

$$\varphi_{\mathcal{N}}(\zeta,\omega,\mathbb{L}) = \oint dr_0 \ \widetilde{\mathbb{G}}(\zeta|\zeta_0,\omega,\ell)\varrho_{\mathcal{N}}(\zeta_0,\omega,\mathbb{L}) \ , \tag{2.113}$$

we find the expression for the Neumann Green function in terms of bulk to boundary propagators as:

$$\begin{aligned} \widetilde{\mathbb{G}}(\zeta|\zeta_0,\omega,\ell) &= -r_0^{\mathcal{N}} D_-^0 \left[\frac{1}{W_{LR}(\zeta_0,\omega,\ell)} g_R(\zeta_\succ,\omega,\ell) g_L(\zeta_\prec,\omega,\ell)\right] \\ &= \frac{1}{W_{LR}(\zeta_0,\omega,\ell)} \begin{cases} g_R(\zeta,\omega,\ell)\pi_L(\zeta_0,\omega,\ell) & \text{if } \zeta\succ\zeta_0 \\ g_L(\zeta,\omega,\ell)\pi_R(\zeta_0,\omega,\ell) & \text{if } \zeta\prec\zeta_0 \end{cases} . \end{aligned} \tag{2.114}$$

In going from the first line of the equation to the second, we have used the fact that for any function $\mathfrak{f}(r)$:

$$D_- \left[\frac{\mathfrak{f}(r)}{W_{LR}}\right] = \frac{D_+\mathfrak{f}(r)}{W_{LR}}. \tag{2.115}$$

This Neumann Green function then solves the following differential equation:

$$\begin{aligned} \frac{1}{r^{\mathcal{N}}} D_+[r^{\mathcal{N}} D_+\widetilde{\mathbb{G}}] + \omega^2\widetilde{\mathbb{G}} + \frac{1-r^2}{4r^2}\Big\{(\mathcal{N}-1)^2 - 4\nu^2 + [4\mu^2-(\mathcal{N}+1)^2]r^2\Big\}\widetilde{\mathbb{G}} \\ + \frac{1}{r^{\mathcal{N}}} D_+ \left[r^{\mathcal{N}}(1-r^2)\delta_c(r-r_0)\right] = 0 \ . \end{aligned} \tag{2.116}$$

Spherical shell influence phase with Neumann boundary conditions

In this section, we will derive the influence phase of an extended source for scalar fields that obey Neumann boundary conditions. We will discretize the extended source into a set of spherical shells, centered at the origin, on both left and right static patches. The discontinuity in the field is given by the surface density σ_i^R for the sphere placed at $\zeta = 1+\zeta_i$ on the right patch and by the surface density σ_i^L for the sphere placed at $\zeta = \zeta_i$ on the left patch. One can think of such a source in terms of the bulk source ρ defined in the previous section as being given by:

$$r^{\mathcal{N}}\varrho_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m}) = \sum_i \sigma_i^R(\omega,\ell,\vec{m}) \ \delta_c(\zeta|1+\zeta_i) - \sum_i \sigma_i^L(\omega,\ell,\vec{m}) \ \delta_c(\zeta|\zeta_i) \ . \tag{2.117}$$

The crucial difference from the Dirichlet case is that when this source enters the inhomogeneous differential equation for φ, it is acted upon by a D_+ operator. The solution is given by:

$$\varphi_{\mathcal{N}} = \sum_i \frac{1}{W_{LR}(\zeta_i,\omega,\ell)} \begin{cases} e^{2\pi\omega} g_L(\zeta,\omega,\ell) \quad \left[\pi_L(1+\zeta_i,\omega,\ell)\ \sigma_i^L - \pi_R(1+\zeta_i,\omega,\ell)\ \sigma_i^R\right] \\ \qquad\qquad\qquad\qquad \text{if } \zeta \prec 1+\zeta_i\ , \\ \pi_R(\zeta_i,\omega,\ell) \quad \left[g_L(\zeta,\omega,\ell)\ \sigma_i^L - g_R(\zeta,\omega,\ell)\ \sigma_i^R\right] \\ \qquad\qquad\qquad\qquad \text{if } 1+\zeta_i \prec \zeta \prec \zeta_i\ , \\ g_R(\zeta,\omega,\ell) \quad \left[\pi_L(\zeta_i,\omega,\ell)\ \sigma_i^L - \pi_R(\zeta_i,\omega,\ell)\ \sigma_i^R\right] \\ \qquad\qquad\qquad\qquad \text{if } \zeta \succ \zeta_i\ . \end{cases} \tag{2.118}$$

We can substitute this solution into the action to obtain the effective action in terms of the surface charge densities of the shells. This yields the following:

$$\begin{aligned} S|_{\textbf{On-shell}} &= \frac{1}{2}\sum_{\ell\vec{m}} \int \frac{d\omega}{2\pi} \oint r^{\mathcal{N}} dr\ \varrho_{\mathcal{N}}^* D_+ \varphi_{\mathcal{N}}|_{\textbf{On-shell}} \\ &= \frac{1}{2}\sum_{i,j,\ell,\vec{m}} \int \frac{d\omega}{2\pi} \frac{\pi_R(\zeta_i,\omega,\ell,\vec{m})}{W_{LR}(\zeta_i,\omega,\ell,\vec{m})} \\ &\quad \times \Big\{ \sigma_j^{R*} \left[\pi_R(1+\zeta_j,\omega,\ell,\vec{m})\sigma_i^R - \pi_L(1+\zeta_j,\omega,\ell,\vec{m})\sigma_i^L\right] \\ &\quad -\sigma_j^{L*} \left[\pi_R(\zeta_j,\omega,\ell,\vec{m})\sigma_i^R - \pi_L(\zeta_j,\omega,\ell,\vec{m})\sigma_i^L\right]\Big\}\ . \end{aligned} \tag{2.119}$$

We can now use explicit expressions for the bulk to boundary propagators and rewrite the action in a convenient form. We make use of the following equations:

$$
\begin{aligned}
W_{LR}(\zeta_i,\omega,\ell) &= -(1+n_\omega)\left(\frac{1-r_i}{1+r_i}\right)^{i\omega}\left[K_{\text{Out}}-K_{\text{In}}\right]\,, \\
\frac{\pi_R(\zeta_i,\omega,\ell)}{W_{LR}(\zeta_i,\omega,\ell)} &= \left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}}(1-r_i^2)\partial_{r_i}\Xi_n(r_i,\omega,\ell)\,, \\
\pi_L(\zeta_i,\omega,\ell) &= -\left(\frac{1-r_i}{1+r_i}\right)^{\frac{i\omega}{2}}(1-r_i^2) \\
&\quad\times\partial_{r_i}\left\{\Xi_{nn}(r_i,\omega,\ell)+\left[n_\omega K_{\text{Out}}-(1+n_\omega)K_{\text{In}}\right]\Xi_n(r_i,\omega,\ell)\right\}\,, \\
\pi_L(1+\zeta_i,\omega,\ell) &= -n_\omega\left(\frac{1-r_i}{1+r_i}\right)^{\frac{i\omega}{2}}\left[K_{\text{Out}}-K_{\text{In}}\right](1-r_i^2)\partial_{r_i}\Xi_n(r_i,\omega,\ell)\,, \\
\pi_R(\zeta_i,\omega,\ell) &= -(1+n_\omega)\left(\frac{1-r_i}{1+r_i}\right)^{\frac{i\omega}{2}}\left[K_{\text{Out}}-K_{\text{In}}\right](1-r_i^2)\partial_{r_i}\Xi_n(r_i,\omega,\ell)\,, \\
\pi_R(1+\zeta_i,\omega,\ell) &= \left(\frac{1-r_i}{1+r_i}\right)^{\frac{i\omega}{2}}(1-r_i^2) \\
&\quad\times\partial_{r_i}\left\{\Xi_{nn}(r_i,\omega,\ell)-\left[(1+n_\omega)K_{\text{Out}}-n_\omega K_{\text{In}}\right]\Xi_n(r_i,\omega,\ell)\right\}\,.
\end{aligned}
\tag{2.120}
$$

Substituting these expressions in the above action yields the following:

$$
\begin{aligned}
S|_{\textbf{On-shell}} &= \frac{1}{2}\sum_{\ell\vec{m}}\int\frac{d\omega}{2\pi}\oint r^{\mathcal{N}}dr\ \varrho^*_{\mathcal{N}}D_+\varphi_{\mathcal{N}}|_{\textbf{On-shell}} \\
&= \frac{1}{2}\sum_{i,j,\ell,\vec{m}}\int\frac{d\omega}{2\pi}\left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}}\left(\frac{1-r_j}{1+r_j}\right)^{\frac{i\omega}{2}}(1-r_i^2)\partial_{r_i}\Xi_n(r_i,\omega,\ell) \\
&\Big\{\sigma_j^{R*}\Big[(1-r_j^2)\partial_{r_j}\left\{\Xi_{nn}(r_j,\omega,\ell)-\left[(1+n_\omega)K_{\text{Out}}-n_\omega K_{\text{In}}\right]\Xi_n(r_j,\omega,\ell)\right\}\sigma_i^R \\
&+n_\omega\left[K_{\text{Out}}-K_{\text{In}}\right](1-r_j^2)\partial_{r_j}\Xi_n(r_j,\omega,\ell)\sigma_i^L\Big] \\
&-\sigma_j^{L*}\Big[-(1+n_\omega)\left[K_{\text{Out}}-K_{\text{In}}\right](1-r_j^2)\partial_{r_j}\Xi_n(r_j,\omega,\ell)\sigma_i^R \\
&+(1-r_j^2)\partial_{r_j}\left\{\Xi_{nn}(r_j,\omega,\ell)+\left[n_\omega K_{\text{Out}}-(1+n_\omega)K_{\text{In}}\right]\Xi_n(r_j,\omega,\ell)\right\}\sigma_i^L\Big]\Big\}\,.
\end{aligned}
\tag{2.121}
$$

Simplifying the above equation further leads to:

$$
\begin{aligned}
S|_{\textbf{On-shell}} = &\frac{1}{2}\sum_{ij\ell}\int\frac{d\omega}{2\pi}\left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}}\left(\frac{1-r_j}{1+r_j}\right)^{\frac{i\omega}{2}} \\
&\times\Big\{(1-r_i^2)\partial_{r_i}\Xi_n(r_i,\omega,\ell)(1-r_j^2)\partial_{r_j}\Xi_{nn}(r_j,\omega,\ell)\ [\sigma_j^{R*}\sigma_i^R-\sigma_j^{L*}\sigma_i^L] \\
&\quad-(1-r_i^2)\partial_{r_i}\Xi_n(r_i,\omega,\ell)(1-r_j^2)\partial_{r_j}\Xi_n(r_j,\omega,\ell) \\
&\qquad\times K_{\text{Out}}(\sigma_j^R-\sigma_j^L)^*[(1+n_\omega)\sigma_i^R-n_\omega\sigma_i^L] \\
&\quad-(1-r_i^2)\partial_{r_i}\Xi_n(r_i,\omega,\ell)(1-r_j^2)\partial_{r_j}\Xi_n(r_j,\omega,\ell) \\
&\qquad\times K_{\text{In}}(\sigma_i^R-\sigma_i^L)[(1+n_{-\omega})\sigma_j^{R*}-n_{-\omega}\sigma_j^{L*}]\Big\}\ .
\end{aligned}
\tag{2.122}
$$

The last two lines of the above expression are related by relabelling $\omega\to-\omega$. These two terms can be interpreted physically by defining the radiative multipole moments:

$$
\begin{aligned}
\mathcal{J}_R(\omega,\ell,\vec{m}) &\equiv \sum_i\left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}}(1-r_i^2)\partial_{r_i}\Xi_n(r_i,\omega,\ell)\ \sigma_i^R \\
&\equiv \int_R dr\ r^{\mathcal{N}}(1-r^2)\partial_r\Xi_n(r,\omega,\ell)\ \left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}}\ \varrho_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m})\ , \\
\mathcal{J}_L(\omega,\ell,\vec{m}) &\equiv \sum_i\left(\frac{1-r_i}{1+r_i}\right)^{-\frac{i\omega}{2}}(1-r_i^2)\partial_{r_i}\Xi_n(r_i,\omega,\ell)\ \sigma_i^L \\
&\equiv -\int_L dr\ r^{\mathcal{N}}(1-r^2)\partial_r\Xi_n(r,\omega,\ell)\ \left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}}\ \varrho_{\mathcal{N}}(\zeta,\omega,\ell,\vec{m})\ .
\end{aligned}
\tag{2.123}
$$

This now allows us to recast the last two lines of the action into the cosmological influence phase we obtained for the point source:

$$
\begin{aligned}
S_{\text{CIP}}^{\text{Pt}} &\equiv -\sum_{\ell\vec{m}}\int\frac{d\omega}{2\pi}K_{\text{Out}}(\mathcal{J}_R-\mathcal{J}_L)^*[(1+n_\omega)\mathcal{J}_R-n_\omega\mathcal{J}_L] \\
&= -\sum_{\ell\vec{m}}\int\frac{d\omega}{2\pi}K_{\text{Out}}\ \mathcal{J}_D^*\ \left[\mathcal{J}_A+\left(n_\omega+\frac{1}{2}\right)\mathcal{J}_D\right]\ .
\end{aligned}
\tag{2.124}
$$

where we have defined the average and difference multipole moments as

$$
\mathcal{J}_A(\omega,\ell,\vec{m})\equiv\frac{1}{2}[\mathcal{J}_R(\omega,\ell,\vec{m})+\mathcal{J}_L(\omega,\ell,\vec{m})]\ ,
$$

and

$$
\mathcal{J}_D(\omega,\ell,\vec{m})\equiv\mathcal{J}_R(\omega,\ell,\vec{m})-\mathcal{J}_L(\omega,\ell,\vec{m})\ .
$$

2.3 Radiation reaction due to light scalar fields

In this section, we will evaluate the radiation reaction force on a dS point particle coupled to a scalar field. We will do this in a small curvature approximation, i.e., we begin with the leading order result in flat spacetime [85, 86] and then systematically correct it for curvature effects. In dS, Hubble constant H parametrises the deviation from flat spacetime, so the small curvature expansion is an expansion in H. We will also work within a non-relativistic expansion and a multipole expansion, and then eventually covariantise the final answer for the radiation reaction(RR).

To this end, consider a point-like source moving along a time-like worldline $x(\tau)$ in dS, where τ denotes the proper time of the source. We will assume that the particle trajectory is close to the south pole ($rH \ll 1$) and the radiation wavelength is taken to be much larger than the length scale of the particle trajectory ($\omega r \ll 1$), but much smaller than the curvature length scale ($\omega \gg H$). Further, we work in a non-relativistic limit ($v \ll 1$). Thus, we consider the following hierarchy of scales (See Fig.2.3):

$$H \ll \omega \ll 1/r \ . \tag{2.125}$$

In analogy with flat spacetime, we will refer to this expansion as the post-newtonian(PN) expansion in dS.

The source density for a moving source in dS is given by

$$\widetilde{\rho}(x') = \int \delta^{d+1}(x-x')d\tau = \sqrt{1 - H^2r^2 - \frac{\dot{r}^2}{1-H^2r^2} + \dot{r}^2 - v^2}\ \delta^d(\vec{x}-\vec{x}')\ , \tag{2.126}$$

where we have defined $v = \sqrt{\sum_{i=1}^{d} \dot{x}_i^2}$. Here the dots denote the derivative with respect to the standard time t. The time-dilation/length-contraction factor for the particle worldline can then be expanded as follows:

$$\begin{aligned}&\sqrt{1 - H^2r^2 - \frac{\dot{r}^2}{1-H^2r^2} + \dot{r}^2 - v^2}\\ &= -\sum_{n,s,k} \binom{n}{k} \frac{(2s+2n+2k-5)!!(2s+2k-5)!!}{2^{n+s-1}(2s+4k-5)!!\ n!(s-1)!} (Hr)^{2n} \dot{r}^{2k} v^{2s-2}\ .\end{aligned} \tag{2.127}$$

Here the sum is over all non-negative integers, and the binomial coefficient vanishes for all values of k outside the range $0 \leq k \leq n$. This expansion describes the red-shift

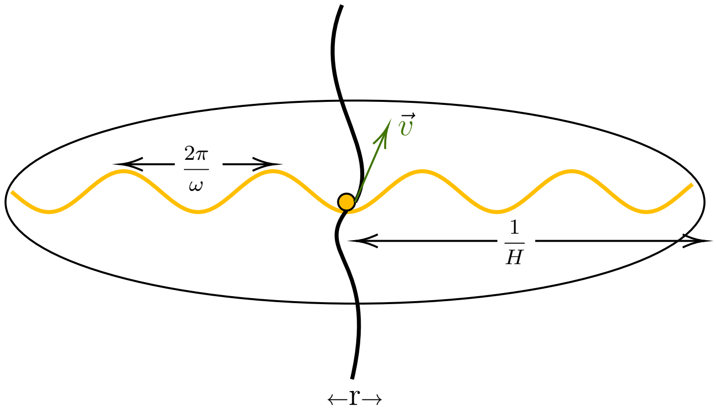

Figure 2.3: The RR is given in a post-newtonian expansion: the velocity is taken to be small ($v \ll 1$) and the trajectory is centred about the south pole ($\omega r \ll 1$) while the near-flat expansion requires that the curvature effects are small(i.e. $H \ll \omega$ and $rH \ll 1$).

of the particle within dS spacetime in a slow-motion approximation, assuming that both the cosmological redshift Hr as well as the Doppler red-shifts due to peculiar motions (proportional to $\dot{r}$ and v) are small. Our strategy below will be to use the above expansion to compute the symmetric trace-free(STF) multipole moments of the source, which can then be fed into the cosmological influence phase to compute the RR force.

We caution the reader that the source form given in Eq.(2.126) is specific to the KG scalar case with $\mathcal{N} = d-1$. This is *not* the correct form of the source for the scalar/vector/tensor sectors of EM field and gravity. For such cases, the explicit form of sources involves extra velocity/time-dilation factors, e.g., EM vector sector source is the electric current carried by the particle which has additional velocity dependence not captured by Eq.(2.126). Another related comment on EM/gravity sources is that the RR force coming from just one of the sectors is not expected to be covariant [86]: one should add in the contributions from all sectors to derive a covariant force expression. To do this for EM/linearised gravity, we need a theory of vector/tensor STF expansions (i.e., a formalism analogous to the one described in appendix A). We will derive such a formalism in the next chapter on electromagnetism: in this chapter, we will limit our RR force analysis to KG scalars.

In the dS-SK geometry, the above source will be doubled to a $\widetilde{\rho}_L$ and a $\widetilde{\rho}_R$ coming from left/right trajectories $x_L(\tau_L)$ and $x_R(\tau_R)$. The degrees of freedom of our open system are thus two copies of the position of the particle and its time derivatives: $\{x_L, x_R, \dot{x}_L, \dot{x}_R, \ddot{x}_L, \ddot{x}_R, \ldots\}$. The scalar ALD force and its post-newtonian corrections only require expressions linear in $x_D, \dot{x}_D, \ddot{x}_D, \ldots$ which are the difference in the positions and their derivatives. We will also keep terms up to cubic powers of x_D. In this approximation, the average and difference functions of the sources can be written in a simple way. Consider for illustration, the average and difference functions of just the position:

$$\frac{1}{2}\left[\mathfrak{f}\left(x_A+\frac{x_D}{2}\right)+\mathfrak{f}\left(x_A-\frac{x_D}{2}\right)\right]=\mathfrak{f}(x_A)+\frac{x_D^2}{4}\frac{\partial^2\mathfrak{f}}{\partial x_A^2}+O(x_D^4) \tag{2.128}$$

$$\mathfrak{f}\left(x_A+\frac{x_D}{2}\right)-\mathfrak{f}\left(x_A-\frac{x_D}{2}\right)=x_D\frac{\partial\,\mathfrak{f}}{\partial x_A}+\frac{x_D^3}{24}\frac{\partial^3\,\mathfrak{f}}{\partial x_A^3}+O(x_D^4) \tag{2.129}$$

In general, the sources will be functions not only of the positions but also their time derivatives: in such cases, the above formula should be interpreted as a multi-variable Madhava-Taylor expansion.

We will now substitute the particle source Eq.(2.126) into the multipole moments defined in Eq.(2.103) and obtain the lagrangian for RR force in PN expansion. We begin by expressing the influence phase in terms of STF moments of the particle density: we proceed similarly to how we rewrote the RR influence phase in flat spacetime (Eq.(A.74)) in terms of STF multipole moments (Eq.(A.78)). Using the STF addition theorem in Eq.(A.50), we can rewrite the dissipative part of Eq.(2.104) in time-domain as:

$$S_{RR}^{\text{Odd }d}=\sum_{\ell}\int\frac{d\omega}{2\pi}\frac{K_{\text{Out}}}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\frac{1}{\ell!}\mathcal{Q}_{D,STF}^{*<i_1i_2\ldots i_\ell>}\mathcal{Q}^{A,STF}_{<i_1i_2\ldots i_\ell>}\ . \tag{2.130}$$

where we have defined the time-domain STF multipole moments in dS as

$$\begin{aligned}\mathcal{Q}_{A,STF}^{i_1\ldots i_\ell}(t)&\equiv\Pi^{<i_1i_2\ldots i_\ell>}_{<j_1j_2\ldots j_\ell>}\int r^{d-1}dr\ \hat{r}^{j_1}\hat{r}^{j_2}\ldots\hat{r}^{j_\ell}\ \Xi_n(i\partial_t,r)\widetilde{\rho}_A(t,\vec{r})\ ,\\ \mathcal{Q}_{D,STF}^{i_1\ldots i_\ell}(t)&\equiv\Pi^{<i_1i_2\ldots i_\ell>}_{<j_1j_2\ldots j_\ell>}\int r^{d-1}dr\ \hat{r}^{j_1}\hat{r}^{j_2}\ldots\hat{r}^{j_\ell}\ \Xi_n(i\partial_t,r)\widetilde{\rho}_D(t,\vec{r})\ .\end{aligned} \tag{2.131}$$

We will now use the PN expansion for $\widetilde{\rho}_D$ in Eq.(2.127), the expansion for K_{Out} from Eq.(2.160) and the expansion for Ξ_n from Eq.(2.143) respectively. Keeping all terms in the action up to quartic order in amplitudes (i.e. in position x) and quartic order in

Hubble constant H, we get an effective lagrangian of the form

$$
\begin{aligned}
|\mathbb{S}^{d-1}|(d-2)!! \times (-1)^{\frac{d-1}{2}} L = & -[x_i]_D \mathbb{D}_1 [x^i]_A + \frac{1}{2}\left[x_i x_j - \frac{x^2}{d}\delta_{ij}\right]_D \mathbb{D}_2 \left[x^i x^j - \frac{x^2}{d}\delta_{ij}\right]_A \\
& - \left\{\frac{1}{2}(x_i)_D \mathbb{D}_1^X [x^i x^2]_A + \frac{1}{2}(x^2 x_i)_D \mathbb{D}_1^X [x^i]_A\right\} \\
& + \left\{\frac{1}{2}(x_i)_D \mathbb{D}_1^V [x^i v^2]_A + \frac{1}{2}(v^2 x_i)_D \mathbb{D}_1^V [x^i]_A\right\} \\
& + \frac{1}{2d}[x^2]_D \mathbb{D}_0^{XX} [x^2]_A + \frac{1}{4}[v^2]_D \mathbb{D}_0^{VV} [v^2]_A \\
& - \frac{1}{4}[x^2]_D \mathbb{D}_0^{XV} [v^2]_A - \frac{1}{4}[v^2]_D \mathbb{D}_0^{XV} [x^2]_A \ .
\end{aligned}
\tag{2.132}
$$

Here, we have the seven differential operators, each built out of a finite number of time-derivatives with constant coefficients, and labelled by $\{\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_1^X, \mathbb{D}_1^V, \mathbb{D}_0^{XX}, \mathbb{D}_0^{VV}, \mathbb{D}_0^{XV}\}$. We use the subscripts of these operators to denote the multipole number, whereas the superscripts are used to distinguish between different structures occurring at the same multipole number. The explicit form of these operators is tabulated in table 2.12. We note that terms beyond the quadrupole moment do not contribute to the quartic influence phase.

We can expand out the average and the difference multipole moments in terms of the average/difference in the particle position by using the following identities:

$$
\begin{aligned}
[Z]_d[Y^3]_a &= Z_d Y_a^3 + \frac{1}{4} Z_d \ (3Y_d^2 Y_a) \\
[Z^2]_d[Y^2]_a &= (2Z_d Z_a) Y_a^2 + \frac{1}{4}(2Z_d Z_a) Y_d^2 \\
[Z^3]_d[Y]_a &= (3Z_a^2 Z_d) Y_a + \frac{1}{4} Z_d^3 Y_a \ .
\end{aligned}
\tag{2.133}
$$

After integration by parts, the above lagrangian can be cast into the form:

$$
L = \frac{(-1)^{\frac{d-1}{2}}}{|\mathbb{S}^{d-1}|(d-2)!!}\left[F_i(x_A) x_D^i + \frac{1}{4} N_i(x_D) x_A^i\right]
\tag{2.134}
$$

where F^i are the Euler-Lagrange derivatives of the terms linear in x_D with respect to x_D^i. Similarly, N^i are the Euler-Lagrange derivatives of the terms linear in x_A with respect to x_A^i.

The terms in the lagrangian which are cubic in x_D give rise to noise terms N^i. These terms are different from pure noise terms, i.e. those quartic in x_D. The N^i contribute to the radiation reaction with a x_A^i dependent term. It should be noted that this noise is not thermal in origin. Rather, the origin of this noise can be understood as follows: Despite the scalar field coupling linearly to the multipole moments, the moments themselves are non-linear functions of the positions. Hence, the open system described in terms of position has extra noise terms.

Both the F^i and the N^i can be written in terms of the operators given in table 2.12 as:

$$\begin{aligned} F^i =& - \mathbb{D}_1[x^i] + x_j \mathbb{D}_2[x^i x^j] - \frac{x^i}{d}\mathbb{D}_2[x^2] \\ & - \left\{ \frac{1}{2}\mathbb{D}_1^X[x^i x^2] + \frac{1}{2}x^2 \mathbb{D}_1^X[x^i] + x^i x^j \mathbb{D}_1^X[x_j] \right\} \\ & + \left\{ \frac{1}{2}\mathbb{D}_1^V[x^i v^2] + \frac{1}{2}v^2 \mathbb{D}_1^V[x^i] - \partial_t \left(v^i x^j \mathbb{D}_1^V[x_j] \right) \right\} \\ & + \frac{x^i}{d}\mathbb{D}_0^{XX}[x^2] - \frac{1}{2}\partial_t \left(v^i \mathbb{D}_0^{VV}[v^2] \right) + \left\{ \frac{1}{2}\partial_t \left(v^i \mathbb{D}_0^{XV}[x^2] \right) - \frac{1}{2}x^i \mathbb{D}_0^{XV}[v^2] \right\} \ , \end{aligned} \tag{2.135}$$

as well as

$$\begin{aligned} N^i(x) =& x_j \mathbb{D}_2[x^i x^j] - \frac{x^i}{d}\mathbb{D}_2[x^2] \\ & + \left\{ \frac{1}{2}\mathbb{D}_1^X[x^i x^2] + \frac{1}{2}x^2 \mathbb{D}_1^X[x^i] + x^i x^j \mathbb{D}_1^X[x_j] \right\} \\ & - \left\{ \frac{1}{2}\mathbb{D}_1^V[x^i v^2] + \frac{1}{2}v^2 \mathbb{D}_1^V[x^i] - \partial_t \left(v^i x^j \mathbb{D}_1^V[x_j] \right) \right\} \\ & + \frac{x^i}{2d}\mathbb{D}_0^{XX}[x^2] - \frac{1}{2}\partial_t \left(v^i \mathbb{D}_0^{VV}[v^2] \right) + \left\{ \frac{1}{2}\partial_t \left(v^i \mathbb{D}_0^{XV}[x^2] \right) - \frac{1}{2}x^i \mathbb{D}_0^{XV}[v^2] \right\} \end{aligned} \tag{2.136}$$

Given the near flat expansion of the action, we can, in a controlled fashion, calculate the curvature corrections to the RR force. This force is given by the variation of the lagrangian with respect to x_D. The leading term in the PN expansion of the flat space RR is a scalar version of the Abraham-Lorentz-Dirac force and stems from the dipole moment of the particle. This term, in arbitrary d, with the first Hubble correction, we find to be:

$$F^i_{\text{ALD}} = \frac{(-1)^{\frac{d+1}{2}}}{|\mathbb{S}^{d-1}| d!!(d-2)!!} \left\{ \partial_t^d x^i - H^2 \frac{d}{6}\left(d^2-1\right) \partial_t^{d-2} x^i \right\} \ . \tag{2.137}$$

This expression gives an equation of motion that is third-order in derivative for dS_4.

Even higher time derivatives show up if we include higher-order post-newtonian corrections. This is a known effect in flat space calculations.

The overall sign of the leading term in the force agrees with the fact that this force is dissipative rather than anti-dissipative. To understand this, consider a 1d oscillator with an RR force of the form

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \lambda(-1)^{\frac{d+1}{2}} \frac{d^d x}{dt^d} \ , \tag{2.138}$$

where we will assume that d is odd and $\lambda > 0$. We would now like to argue that the RR force is dissipative. To see this, we note that the above equation is equivalent to a dispersion relation $\omega^2 = \omega_0^2 - i\lambda\omega^d$, which can be solved approximately to give $\omega \approx \omega_0 - i\lambda\omega_0^d$. Since the imaginary part of ω is negative, we conclude that the above force is indeed dissipative.

As noted in [86], the terms in the flat space PN expansion of the RR force add up to give a Poincare covariant expression. This is a non-trivial check for the accuracy of the result as both the structure of the multipole PN expansion and the requirement that it sums up to a covariant result leave little room for error. We similarly find that the curvature corrections obtained along with the flat space results are also tightly constrained: the contributions from our influence phase non-trivially sum up to expressions covariant under dS metric.

The final RR force then takes the form

$$F^{\mu}_{\rm RR} \equiv \frac{(-)^{\frac{d-1}{2}}}{|\mathbb{S}^{d-1}|(d-2)!!} f^{\mu}$$

where f^{μ} has an expansion of the form

$$\begin{aligned} f^{\mu}_{d} &= {}^{0}f^{\mu}_{d} - \frac{H^2}{4\times 3!} c_h \, {}^{0}f^{\mu}_{d-2} \\ &\quad + \frac{H^4}{8\times 6!}[5c_h^2 - 40(d+2)c_h + 32(d+2)(d^2-1)] \, {}^{0}f^{\mu}_{d-4} + O(H^6) \ . \end{aligned} \tag{2.139}$$

Here $c_h \equiv 12\mu^2 + d^2 - 4$ contains the information about the mass of the scalar, and the combinations ${}^0f^\mu_d$'s for odd values of d are :

$$
\begin{aligned}
{}^0f^\mu_1 &\equiv -v^\mu \ , \\
{}^0f^\mu_3 &\equiv \frac{P^{\mu\nu}}{3!!}\left\{-a^{(1)}_\nu\right\} - \frac{H^2}{3!!}\left\{v^\mu\right\} \ , \\
{}^0f^\mu_5 &\equiv \frac{P^{\mu\nu}}{5!!}\left\{-a^{(3)}_\nu + 5\,(a\cdot a)\,a^{(1)}_\nu + 10\,(a\cdot a^{(1)})\,a_\nu\right\} - H^2\frac{P^{\mu\nu}}{5!!}\left\{a^{(1)}_\nu\right\} + \frac{H^4}{5!!}\left\{-v^\mu\right\} \ , \\
{}^0f^\mu_7 &\equiv \frac{P^{\mu\nu}}{7!!}\Big\{-a^{(5)}_\nu + 14\,(a\cdot a)\,a^{(3)}_\nu + 70\,(a\cdot a^{(1)})\,a^{(2)}_\nu + 84\,(a\cdot a^{(2)})\,a^{(1)}_\nu + 42\,(a\cdot a^{(3)})\,a_\nu \\
&\qquad + \frac{224}{3}\,(a^{(1)}\cdot a^{(1)})\,a^{(1)}_\nu + 105\,(a^{(1)}\cdot a^{(2)})\,a_\nu + O(a^5)\Big\} \\
&\quad - H^2\frac{P^{\mu\nu}}{7!!}\left\{a^{(3)}_\nu + 15\,(a\cdot a)\,a^{(1)}_\nu + 37\,(a\cdot a^{(1)})\,a_\nu\right\} + H^4\frac{P^{\mu\nu}}{7!!}\left\{-a^{(1)}_\nu\right\} - \frac{H^6}{7!!}\left\{v^\mu\right\} \ .
\end{aligned}
\tag{2.140}
$$

Here $v^\mu = \frac{dx^\mu}{d\tau}$ is the proper velocity of the particle computed using dS metric, $a^\mu \equiv \frac{D^2x^\mu}{D\tau^2}$ is its proper acceleration, and $P^{\mu\nu} \equiv g^{\mu\nu} + v^\mu v^\nu$ is the transverse projector to the worldline. We use $a^{(k)}_\mu \equiv \frac{D^k a^\mu}{D\tau^k}$ to denote the proper-time derivatives of the acceleration. All the spacetime dot products are computed using the dS metric.

One remarkable feature of the above formula for radiation reaction is the recursive nature of the Hubble corrections. One can see that the $O(H^{2k})$ correction to the force in d dimensions is related to the RR force in $d-2k$ dimensions. It would be interesting to see whether there are specific quantum mechanical models which can reproduce such a recursive structure.

One of the consequences of this recurrence is that the H^{d-1} terms in dS_{d+1} resembles the RR effects in $d=1$ flat space. The flat space $d=1$ massless scalar RR was explored in [99]. However, as noted there, it is inconsistent to assume a constant coupling for a particle coupled to a massless scalar in 2D flat space. Similar issues emerge at $O(H^2)$ in $d=3$ dS [97] and in general in any d at $O(H^{d-1})$ due to the aforementioned recurrence relation. This is, in turn, related to the breakdown of the small ω expansion of the K_{Out} noted in footnote 5: an issue that can be cured by turning on a small mass for the scalar.

We have checked that the flat limit of the RR force coincides with the covariant expressions derived in [86]. However, there are sign mismatches with expressions of [87][9]. The expressions at order H^2 do not match the general curved space force in [87] restricted to dS. Since our methods differ significantly from [87], we are unable to comment further

[9]This sign mismatch was noted by [86] as well.

on the specific source of disagreement.

We have not yet succeeded in finding similar covariant expressions for the N^i.

2.3.1 Near flat expansions for odd d

In this subsection, we will describe how normalisable modes of the generalised scalar equation in dS can be thought of as perturbations of the corresponding Bessel J modes in the flat spacetime. In the context of radiation reaction problems, these modes are essential in defining the radiative multipole moments: their role is to appropriately smear the sources to take into account time-delay effects. Once such an expansion is obtained, it is easy to find the flat space expansion of the non-normalisable mode for even-dimensional dS just by analytical continuation.

The solution of the generalised scalar wave equation, regular at $r = 0$, is given by

$$\begin{aligned}\Xi_n \equiv & \frac{1}{2\nu} r^{\nu-\frac{d}{2}+1+\frac{1}{2}(d-1-\mathcal{N})}(1-H^2r^2)^{-\frac{i\omega}{2H}} \\ & \times {}_2F_1\left[\frac{1}{2}\left(1+\mu+\nu-\frac{i\omega}{H}\right), \frac{1}{2}\left(1-\mu+\nu-\frac{i\omega}{H}\right), 1+\nu, H^2r^2\right] .\end{aligned} \tag{2.141}$$

Here we have made all H factors explicit so that the $H \to 0$ limit can readily be taken. In such a limit, the above expression reduces to a Bessel J function. More explicitly, we will find it convenient to define a sequence of scaled Bessel J functions of the form

$$\mathfrak{B}_k \equiv \frac{r^{\nu-\frac{d}{2}+1+\frac{1}{2}(d-1-\mathcal{N})+2k}}{2\nu(\nu+1)\ldots(\nu+k)}\ {}_0F_1\left[1+k+\nu, -\frac{\omega^2r^2}{4}\right] = \frac{\Gamma(\nu)\ r^{\frac{1}{2}(1-\mathcal{N})+k}}{2(\omega/2)^{k+\nu}} J_{k+\nu}(\omega r) \tag{2.142}$$

in flat spacetime. In terms of these functions, the dS wavefunction Ξ_n has a small H expansion

$$\Xi_n = \sum_{k=0}^{\infty} \mathfrak{p}_k(\nu, H^2, \omega^2)\mathfrak{B}_k\ , \tag{2.143}$$

with $\mathfrak{p}_k(H^2, \omega^2)$ being a homogeneous polynomial of degree k in the variables H^2 and ω^2.

An explicit expression is given by

$$\mathfrak{p}_k \equiv \frac{H^{2k}}{k!}\sum_{m=0}^{k}(-)^m\binom{k}{m}\sum_{n=0}^{m}(-)^n\binom{m}{n}\sigma^{2k-2m}\frac{\Gamma(\alpha+m)\Gamma(1+\nu+m)}{\Gamma(\alpha+m-n)\Gamma(1+\nu+m-n)} \\ \times\frac{\Gamma(\alpha+i\sigma+m-n)\Gamma(\alpha-i\sigma+m-n)}{\Gamma(\alpha+i\sigma)\Gamma(\alpha-i\sigma)}\ , \tag{2.144}$$

where we have defined the variables

$$\alpha \equiv \frac{1}{2}(1+\nu-\mu)\ , \quad \sigma = \frac{\omega}{2H}\ . \tag{2.145}$$

A useful fact about these polynomials is the leading H scaling at small H of these polynomials, given by

$$\mathfrak{p}_{3n-2}\ ,\ \mathfrak{p}_{3n-1}\ ,\ \mathfrak{p}_{3n}\ \propto H^{2n}\ . \tag{2.146}$$

Thus, to obtain an answer accurate up to H^{2n}, we only need the polynomials till $\mathfrak{p}_{3n}$.

We will now outline a derivation for the above expansion as follows: first, we use the Euler transformation on the hypergeometric functions to write

$$\Xi_n = \frac{1}{2\nu}r^{\nu-\frac{d}{2}+\frac{1}{2}(d-1-\mathcal{N})}(1-H^2r^2)^{-\alpha}\ {}_2F_1\left[\alpha+i\sigma,\alpha-i\sigma,1+\nu,-\frac{H^2r^2}{1-H^2r^2}\right]\ , \tag{2.147}$$

where the variables α and σ are as defined above. In the next step, we employ the Mellin-Barnes representation of the hypergeometric function, viz. [103],

$${}_2F_1\left[a,b,c,x\right] = \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}(-x)^z\Gamma(-z)\frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+z)}\ . \tag{2.148}$$

and expand the resultant integrand using

$$(1-H^2r^2)^{-\alpha-z}\left(H^2r^2\right)^z = \sum_{k=0}^{\infty}\frac{(H^2r^2)^{k+z}}{k!}\frac{\Gamma\left[k+z+\frac{1}{2}(1+\nu-\mu)\right]}{\Gamma\left[\alpha+z+\frac{1}{2}(1+\nu-\mu)\right]}\ . \tag{2.149}$$

Shifting the Mellin-Barnes integration variable, we get the following Mellin-integral representation for Ξ_n:

$$\Xi_n = \frac{1}{2\nu}r^{\nu-\frac{d}{2}+\frac{1}{2}(d-1-\mathcal{N})}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\left(\frac{\omega r}{2}\right)^{2z}\Gamma(-z)\frac{\Gamma(c)}{\Gamma(c+z)}\widetilde{\Xi}(z)\ , \tag{2.150}$$

where the Mellin-transform

$$\begin{aligned}\widetilde{\Xi}(z) &\equiv \left(\frac{2H}{\omega}\right)^{2z}\sum_{k=0}^{\infty}(-)^k\binom{z}{k}\frac{\Gamma(a+z-k)\Gamma(b+z-k)\Gamma(c+z)}{\Gamma(a)\Gamma(b)\Gamma(c+z-k)}\frac{\Gamma(z+\alpha)}{\Gamma(z-k+\alpha)} \\ &= \left(\frac{2H}{\omega}\right)^{2z}\frac{\Gamma(a+z)\Gamma(b+z)}{\Gamma(a)\Gamma(b)}{}_3F_2\left[\begin{matrix}1-c-z, & -z, & 1-\alpha-z \\ 1-a-z & , & 1-b-z\end{matrix};1\right] .\end{aligned} \tag{2.151}$$

Here, $\alpha \equiv \frac{1}{2}(1+\nu-\mu)$ and a, b, c denote the parameters of the hypergeometric function appearing in Eq.(2.147). The Mellin-transform $\widetilde{\Xi}(z)$ evaluated at integer z is, in fact, a polynomial of degree z in the variable $(H/\omega)^2$: this can be gleaned from the fact that the series above truncates in this case with polynomial coefficients.

To determine the polynomials $\mathfrak{p}_n$, we should compare the polynomials $\widetilde{\Xi}(n)$ against the Mellin-transform of $\sum_k \mathfrak{p}_k \mathfrak{B}_k$. This can be done using the Mellin-Barnes representation of ${}_0F_1$, viz. [103],

$${}_0F_1\left[c, x\right] = \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}(-x)^z\Gamma(-z)\frac{\Gamma(c)}{\Gamma(c+z)} . \tag{2.152}$$

This, in turn, yields an expression of the form

$$\widetilde{\Xi}(z) = \sum_{k=0}^{\infty}\left(\frac{2}{\omega}\right)^{2k}\mathfrak{p}_k(\nu, H^2, \omega^2)\,\frac{\Gamma(k-z)}{\Gamma(-z)} . \tag{2.153}$$

This series also truncates for integer z, and the above relation can then be inverted to give

$$\mathfrak{p}_k = \frac{1}{k!}\left(\frac{\omega}{2}\right)^{2k}\sum_{m=0}^{k}(-)^m\binom{k}{m}\,\widetilde{\Xi}(m) . \tag{2.154}$$

The explicit expression quoted before follows from this equation. The first few polynomials are given by

$$
\begin{aligned}
\mathfrak{p}_0 &= 1 \ , \\
\mathfrak{p}_1 &= \frac{H^2}{2^2}(1+\nu+\mu)(1+\nu-\mu) \ , \\
\mathfrak{p}_2 &= \frac{H^2}{2^3}\left\{-\omega^2(2\nu+3) + \frac{H^2}{2^2}(1+\nu+\mu)(1+\nu-\mu)(3+\nu+\mu)(3+\nu-\mu)\right\} \ , \\
\mathfrak{p}_3 &= \frac{H^2}{2^2\times 3!}\left\{\omega^4 + \frac{H^2\omega^2}{2^2}[3\mu^2(2\nu+5)-(103+132\nu)-3(17\nu^2+2\nu^3)] + H^4(\ldots)\right\} \ ,
\end{aligned}
\tag{2.155}
$$

The polynomials $\mathfrak{p}_4$ and higher are proportional to H^4 and, hence the above expressions are sufficient to obtain an answer accurate up to order H^2 terms. To get terms accurate up to order H^4, we also need the leading terms of the next three polynomials:

$$
\begin{aligned}
\mathfrak{p}_4 &= \frac{H^4}{2^5\times 4!}\left\{\omega^4[-4\mu^2+8\nu(2\nu+13)+157] + H^2(\ldots)\right\} \ , \\
\mathfrak{p}_5 &= \frac{H^4}{2^2\times 5!}\left\{\omega^6(5\nu+18) + H^2(\ldots)\right\} \ , \\
\mathfrak{p}_6 &= \frac{H^4}{2^7\times 3^2}\left\{\omega^8 + H^2(\ldots)\right\} \ .
\end{aligned}
\tag{2.156}
$$

The polynomials $\mathfrak{p}_7$ and higher are proportional to H^6, and hence can be ignored at this order.

For odd values of d, the function Ξ_{nn} is related to Ξ_n simply by the transformation: $\nu \to -\nu$. This allows us to also obtain the flat space expansion for Ξ_{nn} in odd d:

$$
\Xi_{nn}|_{\text{Odd } d} = \sum_{k=0}^{\infty} \mathfrak{p}_k(-\nu, H^2, \omega^2)\mathfrak{G}_k \ , \tag{2.157}
$$

where the functions $\mathfrak{G}_k$ are related to the $\mathfrak{B}_k$ by $\nu \to -\nu$:

$$
\mathfrak{G}_k \equiv \frac{r^{-\nu-\frac{d}{2}+\frac{1}{2}(d-1-\mathcal{N})+2k}}{(-\nu+1)\ldots(-\nu+k)}\ {}_0F_1\left[1+k-\nu, -\frac{\omega^2 r^2}{4}\right] = -2\nu\frac{\Gamma(-\nu)\ r^{\frac{1}{2}(1-\mathcal{N})+k}}{2(\omega/2)^{k-\nu}} J_{k-\nu}(\omega r) \ . \tag{2.158}
$$

We will conclude by giving the near-flat/high-frequency expansion of K_{Out} in odd d.

This can be achieved using Stirling approximation, i.e.,

$$\Gamma(z) \sim \exp\left\{ \left(z - \tfrac{1}{2}\right) \ln z - z + \tfrac{1}{2} \ln(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} \right\} , \tag{2.159}$$

an approximation valid as long as $z \to \infty$ away from the negative real axis. We then obtain the following expansion for $K_{\rm Out}$ in odd d:

$$\begin{aligned} K_{\rm Out}|_{\rm Odd\ \it d} = \frac{2\pi i}{\Gamma(\nu)^2} \left(\frac{\omega}{2}\right)^{2\nu} \Bigg[& 1 + (\nu^2 + 3\mu^2 - 1) \frac{\nu}{3!!} \frac{H^2}{\omega^2} \\ & + \frac{5\nu^4 - 4\nu^3 + (30\mu^2 - 14)\nu^2 - (60\mu^2 - 16)\nu + (45\mu^2 - 90\mu^2 + 21)}{2 \times 3} \\ & \times \frac{\nu(\nu-1)}{5!!} \frac{H^4}{\omega^4} + O\left(\frac{H^6}{\omega^6}\right) \Bigg] . \end{aligned} \tag{2.160}$$

This expression describes how the radiation reaction kernel gets corrected due to the non-zero cosmological constant.

Symbol	${}^{0}f^{\mu}_{d}$	${}^{0}f^{\mu}_{d-2}$	${}^{0}f^{\mu}_{d-4}$
$\mathbb{D}_1$	$\frac{\partial_t^d}{d!!}$	$\frac{\partial_t^{d-2}}{(d-2)!!}$	$\frac{\partial_t^{d-4}}{(d-4)!!}$
$\mathbb{D}_2$	$\frac{\partial_t^{d+2}}{(d+2)!!} - \frac{H^2}{3!}(d+1)\frac{\partial_t^d}{d!!} + \frac{H^4}{5!}\frac{7}{3}(d^2-1)\frac{\partial_t^{d-2}}{(d-2)!!}$	$\frac{\partial_t^d}{d!!} - \frac{H^2}{3!}(d-1)\frac{\partial_t^{d-2}}{(d-2)!!} + \frac{H^4}{5!}\frac{7}{3}(d-1)(d-3)\frac{\partial_t^{d-4}}{(d-4)!!}$	$\frac{\partial_t^{d-2}}{(d-2)!!}$
$\mathbb{D}_1^X$	$\frac{\partial_t^{d+2}}{(d+2)!!} + \frac{H^2}{3}(d-2)\frac{\partial_t^d}{d!!} - \frac{H^4}{45}(d^2-1)\frac{\partial_t^{d-2}}{(d-2)!!}$	$\frac{\partial_t^d}{d!!} + \frac{H^2}{3}(d-4)\frac{\partial_t^{d-2}}{(d-2)!!} - \frac{H^4}{45}(d-1)(d-3)\frac{\partial_t^{d-4}}{(d-4)!!}$	$\frac{\partial_t^{d-2}}{(d-2)!!}$
$\mathbb{D}_1^V$	$\frac{\partial_t^d}{d!!}$	$\frac{\partial_t^{d-2}}{(d-2)!!}$	$\frac{\partial_t^{d-4}}{(d-4)!!}$
$\mathbb{D}_0^{XX}$	$\frac{d+2}{2}\frac{\partial_t^{d+2}}{(d+2)!!} + \frac{H^2}{4!}2(5d^2-15d-2)\frac{\partial_t^d}{d!!}$ $+\frac{H^4}{6!}(67d^3-526d^2+833d-14)$	$\frac{d+2}{2}\frac{\partial_t^d}{d!!} + \frac{H^2}{4!}2(5d^2-25d+2)\frac{\partial_t^{d-2}}{(d-2)!!}$ $+\frac{H^4}{6!}(67d^3-794d^2+2125d+42)$	$\frac{d+2}{2}\frac{\partial_t^{d-2}}{(d-2)!!}$
$\mathbb{D}_0^{VV}$	$\frac{\partial_t^{d-2}}{(d-2)!!} + \frac{H^2}{3!}(d-1)\frac{\partial_t^{d-4}}{(d-4)!!} + \frac{H^4}{5!}(d-1)(d-3)\frac{\partial_t^{d-6}}{(d-6)!!}$	$\frac{\partial_t^{d-4}}{(d-4)!!} + \frac{H^2}{3!}(d-3)\frac{\partial_t^{d-6}}{(d-6)!!} + \frac{H^4}{5!}(d-3)(d-5)\frac{\partial_t^{d-6}}{(d-6)!!}$	$\frac{\partial_t^{d-6}}{(d-6)!!}$
$\mathbb{D}_0^{XV}$	$\frac{\partial_t^d}{d!!} + \frac{H^2}{2!}(d-3)\frac{\partial_t^{d-2}}{(d-2)!!} + \frac{H^4}{4!}(d-1)(d-7)\frac{\partial_t^{d-4}}{(d-4)!!}$	$\frac{\partial_t^{d-2}}{(d-2)!!} + \frac{H^2}{2!}(d-5)\frac{\partial_t^{d-4}}{(d-4)!!} + \frac{H^4}{4!}(d-3)(d-9)\frac{\partial_t^{d-6}}{(d-6)!!}$	$\frac{\partial_t^{d-4}}{(d-4)!!}$

Table 2.12: The differential operators that appear in dS radiation reaction (for d odd). We have divided up the sum into three columns where each column combines into an expression covariant under dS isometries. The entries in the second column must be multiplied by a relative factor of $-\frac{H^2}{4\times 3!}c_h$ with $c_h \equiv 12\mu^2 + d^2 - 4$ and then added to the first column. Similarly, the third column should be multiplied by a relative factor of $\frac{H^4}{8\times 6!}[5c_h^2 - 40(d+2)c_h + 32(d+2)(d^2-1)]$ and then added to the first two contributions. The sum of these contributions should be further multiplied by a factor of $\frac{(-1)^{\frac{d-1}{2}}}{|S^{d-1}|(d-2)!!}$.

2.4 Interactions

In this section, we will describe how the computation of the on-shell action can be extended beyond the free-field examples. In particular, we would like to check that our prescription in Eq.(1.4) equating the cosmological influence phase to on-shell action still works after we include interactions. We will check this in a simple example: $\varphi_{\mathcal{N}}^3$ theory in dS_4. However, as will be clear below, our arguments can be easily adapted to set up perturbative diagrammatics for arbitrary interactions.

For $\varphi_{\mathcal{N}}^3$ theory in dS_4, we should simply evaluate the bulk on-shell action with the cubic interaction term. At leading order in perturbation theory, the cubic contribution to S_{CIP} is obtained by substituting the free field solutions into interaction terms of the action.[10] The interaction term should then be integrated over the full dS-SK geometry: this is the dS version of a Witten diagram vertex.

We saw in Sec§1.1 that the S_{CIP} we derived satisfies constraints due to SK collapse and KMS conditions. That this should be true for interacting theories as well is not clear a priori, but we will now show that these constraints are still satisfied at least at the level of contact diagrams. This is most easily seen in terms of the $P-F$ basis multipole moments defined in Eq.(2.53). In terms of these multipole moments, SK collapse and KMS conditions are equivalent to showing that there are no terms in the action with only $\mathcal{J}_{\bar{F}}$ or only $\mathcal{J}_{\bar{P}}$ [104].

To check these conditions, we use Eq.(2.51) and write the vertex contribution to the on-shell action as

$$-\frac{\lambda_3}{3!}\int d^{3+1}x\, \varphi_{\mathcal{N}}^3 = \sum_{\ell_i,m_i} \text{Gaunt}(\ell_i,m_i)\int_{\omega_1,\omega_2,\omega_3}\delta(\omega_1+\omega_2+\omega_3)$$
$$\times\left[\mathfrak{I}_{FFF}(\omega_i,\ell_i,m_i)+\mathfrak{I}_{FFP}(\omega_i,\ell_i,m_i)+\mathfrak{I}_{FPP}(\omega_i,\ell_i,m_i)+\mathfrak{I}_{PPP}(\omega_i,\ell_i,m_i)\right]\ . \tag{2.161}$$

Here the index i runs over $\{1,2,3\}$ and $\text{Gaunt}(\ell_i,m_i)$ are the Gaunt coefficients coming from the integral of 3 spherical harmonics over the sphere (see equation 34.3.22 of [103]). Time-translation invariance implies that the three frequencies ω_1, ω_2 and ω_3 are constrained by an energy-conserving δ function. The contributions to the cubic influence

[10] The argument here is similar to the one given in appendix C of [50] for gr-SK geometry.

phase are given by radial contour integrals, viz.,

$$\begin{aligned}
\mathcal{I}_{FFF}(\omega_i,\ell_i,m_i) &\equiv \frac{\lambda_3}{3!}\oint_\zeta G^{\rm Out}_{\mathcal{N}}(\zeta,\omega_1,\ell_1)\ G^{\rm Out}_{\mathcal{N}}(\zeta,\omega_2,\ell_2)\ G^{\rm Out}_{\mathcal{N}}(\zeta,\omega_3,\ell_3) \\
&\qquad\qquad\qquad \times \mathcal{J}_{\bar{F}}(\omega_1,\ell_1)\mathcal{J}_{\bar{F}}(\omega_2,\ell_2)\mathcal{J}_{\bar{F}}(\omega_3,\ell_3)\ , \\
\mathcal{I}_{FFP}(\omega_i,\ell_i,m_i) &\equiv -\frac{\lambda_3}{2!}\oint_\zeta e^{2\pi\omega_3(1-\zeta)}\ G^{\rm Out}_{\mathcal{N}}(\zeta,\omega_1,\ell_1)\ G^{\rm Out*}_{\mathcal{N}}(\zeta,\omega_2,\ell_2)\ G^{\rm Out*}_{\mathcal{N}}(\zeta,\omega_3,\ell_3) \\
&\qquad\qquad\qquad \times \mathcal{J}_{\bar{F}}(\omega_1,\ell_1)\mathcal{J}_{\bar{P}}(\omega_2,\ell_2)\mathcal{J}_{\bar{P}}(\omega_3,\ell_3)\ , \\
\mathcal{I}_{FPP}(\omega_i,\ell_i,m_i) &\equiv \frac{\lambda_3}{2!}\oint_\zeta e^{2\pi(\omega_2+\omega_3)(1-\zeta)}\ G^{\rm Out}_{\mathcal{N}}(\zeta,\omega_1,\ell_1)\ G^{\rm Out*}_{\mathcal{N}}(\zeta,\omega_2,\ell_2)\ G^{\rm Out*}_{\mathcal{N}}(\zeta,\omega_3,\ell_3) \\
&\qquad\qquad\qquad \times \mathcal{J}_{\bar{F}}(\omega_1,\ell_1)\mathcal{J}_{\bar{P}}(\omega_2,\ell_2)\mathcal{J}_{\bar{P}}(\omega_3,\ell_3)\ , \\
\mathcal{I}_{PPP}(\omega_i,\ell_i,m_i) &\equiv -\frac{\lambda_3}{3!}\oint_\zeta e^{2\pi(\omega_1+\omega_2+\omega_3)(1-\zeta)} \\
&\qquad\qquad \times G^{\rm Out*}_{\mathcal{N}}(\zeta,\omega_1,\ell_1)\ G^{\rm Out*}_{\mathcal{N}}(\zeta,\omega_2,\ell_2)\ G^{\rm Out*}_{\mathcal{N}}(\zeta,\omega_3,\ell_3) \\
&\qquad\qquad\qquad \times \mathcal{J}_{\bar{P}}(\omega_1,\ell_1)\mathcal{J}_{\bar{P}}(\omega_2,\ell_2)\mathcal{J}_{\bar{P}}(\omega_3,\ell_3)\ .
\end{aligned} \tag{2.162}$$

We now note that, since $G^{\rm Out}_{\mathcal{N}}$ is analytic, the integrands in $\mathcal{I}_{FFF}$ and $\mathcal{I}_{PPP}$ are analytic (to see the latter, we use energy conservation). This, in turn, implies that these integrals evaluate to zero by Cauchy's theorem. It is now evident that this argument generalises to all contact diagrams of $\varphi^n_{\mathcal{N}}$ type, thus demonstrating our claim about SK collapse and KMS conditions. A similar argument in the AdS blackhole case has been checked also for exchange diagrams [54, 100] and it would be interesting to check whether a similar claim holds here. Further, it would also be interesting to study the correction to the radiation reaction due to such non-linear interactions [105].

This concludes our technical treatment of the influence phase of an observer coupled to scalar fields. We will discuss some implications and applications of our calculations in this section further in section 4. In the next section, we will move on to understanding the electromagnetic influence phase, and we will connect the analysis to that of our designer scalar for particular values of $\mathcal{N}$ and μ from this section.

Chapter 3

Electromagnetic Observers

This chapter extends the analysis of the previous chapter to understand the electromagnetic influence phase of an observer modelled as a localised current density. The path followed is quite similar to the previous chapter, but interesting features coming from the vector nature of the fields, as well as from the gauge invariance properties, will become manifest as we proceed through it. Much of the notation used here has been developed for flat spacetime and compared with textbook treatments of flat space electromagnetism in the appendix C for the reader's convenience.

3.1 EM point source in dS

In this section, we will describe the EM fields of a comoving point source in dS_{d+1}. The discussion here is straightforward and mainly provides information to establish our conventions about multipole moments in dS. We will also frame our discussion in a way that the similarities to holographic renormalisation [33,72,73] are obvious. Our discussion here also closely parallels the discussion of EM fields around planar AdS black holes [52] but with crucial differences. Some other authors have also studied electromagnetic fields in de Sitter in various settings, such as [93, 106–111]

Consider then a classical point particle placed at the south pole of global de Sitter, at the centre of the southern static patch. We will characterise this point particle by its electric and magnetic multipole moments, which we take to be time-dependent. We will give a precise definition of these moments below for particles in dS, using how EM fields behave as we approach the particle. Our goal would be to characterise the radiative loss suffered by the particle in terms of these moments.

Our definition of near-zone multipole moments is strongly guided by the principle that they should simply extend the corresponding flat spacetime definitions. This fact should be contrasted with *far-zone* multipole moments defined, say, using asymptotic behaviour at future time-like infinity. As we shall see, given the different asymptotics of dS, such far-zone moments do not have any simple relation to their flat-space counterparts [112].

Let $\mathcal{C}_{\mu\nu}$ denote the EM field-strength due to the particle in the frequency domain corresponding to outgoing time u. In the outgoing coordinates of the static patch, we will expand each component of this field strength into appropriate scalar/vector spherical harmonics as follows:

$$\begin{aligned}
\mathcal{C}_{ru}(r,\omega,\Omega) &\equiv \sum_{\ell\vec{m}} \mathcal{E}_r(r,\omega,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\Omega) = \mathcal{C}^{ur}(r,\omega,\Omega)\ , \\
\mathcal{C}_{rI}(r,\omega,\Omega) &\equiv \sum_{\ell\vec{m}} \mathcal{B}_{rs}(r,\omega,\ell,\vec{m})\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}(\Omega) + \sum_{\alpha\ell\vec{m}} \mathcal{B}_{rv}(r,\omega,\alpha,\ell,\vec{m})\mathbb{V}_I^{\alpha\ell\vec{m}}(\Omega) \\
&= r^2\gamma_{IJ}\mathcal{C}^{Ju}(r,\omega,\Omega)\ , \\
\mathcal{C}_{Iu}(r,\omega,\Omega) &= \sum_{\ell\vec{m}} \mathcal{E}_s(r,\omega,\ell,\vec{m})\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}(\Omega) + \sum_{\alpha\ell\vec{m}} \mathcal{E}_v(r,\omega,\alpha,\ell,\vec{m})\mathbb{V}_I^{\alpha\ell\vec{m}}(\Omega)\ , \\
\mathcal{C}_{IJ}(r,\omega,\Omega) &\equiv \sum_{\alpha\ell\vec{m}} \mathcal{H}_{vv}(r,\omega,\alpha,\ell,\vec{m})\left[\mathscr{D}_I\mathbb{V}_J^{\alpha\ell\vec{m}}(\Omega) - \mathscr{D}_J\mathbb{V}_I^{\alpha\ell\vec{m}}(\Omega)\right] \\
&= r^4\gamma_{IK}\gamma_{JL}\mathcal{C}^{KL}(r,\omega,\alpha,\ell,\vec{m})\ .
\end{aligned} \tag{3.1}$$

Here Ω denotes the angular co-ordinates on the sphere $\mathbb{S}^{d-1}$, indices I, J, K denote vector indices on the sphere, γ_{IJ} denotes the unit-sphere metric and $\mathscr{D}_I$ is its associated covariant derivative. We have also indicated the relation to the tensor $\mathcal{C}^{\mu\nu}$ with raised indices for future convenience. Another useful combination is

$$\begin{aligned}
r^2\gamma_{IJ}\mathcal{C}^{rJ}(r,\omega,\Omega) &= (1-r^2)\mathcal{C}_{rI}(r,\omega,\Omega) + \mathcal{C}_{Iu}(r,\omega,\Omega) \\
&\equiv \sum_{\ell\vec{m}} \mathcal{H}_s(r,\omega,\ell,\vec{m})\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}(\Omega) + \sum_{\alpha\ell\vec{m}} \mathcal{H}_v(r,\omega,\alpha,\ell,\vec{m})\mathbb{V}_I^{\alpha\ell\vec{m}}(\Omega)\ ,
\end{aligned} \tag{3.2}$$

where we have defined

$$\mathcal{H}_s \equiv (1-r^2)\mathcal{B}_{rs} + \mathcal{E}_s\ , \quad \mathcal{H}_v \equiv (1-r^2)\mathcal{B}_{rv} + \mathcal{E}_v\ . \tag{3.3}$$

Our definitions here closely parallel the spherical harmonic expansion in flat space described in the appendix C. We use calligraphic letters here to encode the fact that the above quantities are defined in the frequency domain conjugate to u, which differ, in the flat limit, from frequency domain expressions conjugate to t by an additional factor of $e^{i\omega r}$. We use subscripts s and v to denote sphere vector indices in the scalar/vector sector, respectively. The time-reversal covariant combinations are $\{\mathcal{E}_s, \mathcal{E}_v, \mathcal{E}_r\}$ and $\{\mathcal{H}_s, \mathcal{H}_v, \mathcal{H}_{vv}\}$, with the intrinsic time-reversal parity being even and odd for the two sets, respectively.

Given the spherical harmonic decomposition given above, we can recast the Maxwell equations in terms of the above components. The source-free Maxwell equations (or equivalently the Bianchi identity of electromagnetism $\partial_{[\mu}\overline{\mathcal{C}}_{\nu\lambda]} = 0$) take the following form:

$$\mathcal{E}_v = i\omega\ \mathcal{H}_{vv}\ , \quad \mathcal{B}_{rv} = \partial_r \mathcal{H}_{vv}\ , \quad \mathcal{E}_r = \partial_r \mathcal{E}_s - i\omega\ \mathcal{B}_{rs}\ , \tag{3.4}$$

or an equivalent time-reversal covariant form

$$\mathcal{E}_v = i\omega\ \mathcal{H}_{vv}\ , \quad \mathcal{H}_v = D_+ \mathcal{H}_{vv}\ , \quad (1-r^2)\mathcal{E}_r = D_+ \mathcal{E}_s - i\omega\ \mathcal{H}_s\ , \tag{3.5}$$

where $D_+ \equiv (1-r^2)\partial_r + i\omega$. As usual, these equations can be solved by the introduction of a vector potential in some gauge. While we will indeed chose a gauge eventually, it is more illuminating to first analyse this system via gauge invariant observables (i.e., EM fields).

The sourced Maxwell equations outside the particle take the form $\partial_\mu \left[\sqrt{-g}\ \overline{\mathcal{C}}^{\mu\nu}\right] = 0$, which when written out in terms of the components become

$$\begin{aligned}
-i\omega\ \mathcal{E}_r + \frac{\ell(\ell+d-2)}{r^2}\mathcal{H}_s &= 0 \qquad \text{(r-Eqn)}\ , \\
\frac{1}{r^{d-1}}\partial_r\left[r^{d-1}\mathcal{E}_r\right] + \frac{\ell(\ell+d-2)}{r^2}\mathcal{B}_{rs} &= 0 \qquad \text{(u-Eqn)}\ , \\
\frac{1}{r^{d-3}}\partial_r\left[r^{d-3}\mathcal{H}_s\right] + i\omega\ \mathcal{B}_{rs} &= 0 \qquad \text{(I-Eqn scalar)}\ , \\
\frac{1}{r^{d-3}}D_+\left[r^{d-3}\mathcal{H}_v\right] - i\omega\ \mathcal{E}_v - \frac{(\ell+1)(\ell+d-3)}{r^2}\mathcal{H}_{vv} &= 0 \qquad \text{(I-Eqn vector)}\ .
\end{aligned} \tag{3.6}$$

Here, we have indicated the equation coming from each component. As is well-known, these equations are not all independent: the r-equation above is the Gauss constraint for radial evolution, which is preserved by the next two equations with radial derivatives.

Alternatively, the u-equation above is the Gauss constraint for u evolution, which is preserved by the first/third equations with time derivatives. Since our goal is to examine how the near zone data is given in terms of multipole moments of the particle, we will take the radial evolution perspective.

In terms of time reversal covariant quantities, we can recast the above equations in the form

$$\begin{aligned}
\frac{\ell(\ell+d-2)}{r^2}\mathcal{H}_s &= i\omega\ \mathcal{E}_r\ , \\
\frac{1}{r^{d-1}}D_+\left[r^{d-1}\mathcal{E}_r\right] &= \frac{\ell(\ell+d-2)}{r^2}\mathcal{E}_s\ , \\
\frac{1}{r^{d-3}}D_+\left[r^{d-3}\mathcal{H}_s\right] &= i\omega\ \mathcal{E}_s\ , \\
\frac{1}{r^{d-3}}D_+\left[r^{d-3}\mathcal{H}_v\right] &= i\omega\ \mathcal{E}_v + \frac{(\ell+1)(\ell+d-3)}{r^2}\mathcal{H}_{vv}\ .
\end{aligned} \tag{3.7}$$

To conclude, the set of seven equations in Eqs.(3.5) and (3.7) define the Maxwell system in dS_{d+1}.

Examining them as a set of coupled radial ODEs for the six quantities $\{\mathcal{E}_s, \mathcal{E}_v, \mathcal{E}_r, \mathcal{H}_s, \mathcal{H}_v, \mathcal{H}_{vv}\}$, we note the following structure: we can first solve the two equations with no radial derivatives to express $\{\mathcal{E}_v, \mathcal{H}_s\}$ in terms of rest four quantities. Once this is done, one of the radial equations involving $D_+\left[r^{d-3}\mathcal{H}_s\right]$ is also solved for 'free'. Among the seven equations, we are then left with four first-order radial evolution equations (one each for $\{\mathcal{E}_s, \mathcal{E}_r, \mathcal{H}_v, \mathcal{H}_{vv}\}$). Thus, given the near zone values of these four fields, one can radially evolve the above set of equations to get the EM fields far away from the particle.

Among all possible near-zone data, it is intuitively clear that roughly half would give rise to outgoing EM waves, whereas the other half would result in incoming EM waves.[1] More generally, if we wanted to also impose boundary conditions far away (or in the case of dS-SK on the near zone of the other branch), the correct thing to do is to constrain only two out of the four quantities. Motivated by the above heuristic argument, we will

[1]We say roughly since this argument ignores time-independent solutions, which are neither incoming nor outgoing. But such zero frequency solutions are a set of measure zero in the space of all solutions of the Maxwell equations and can hence be justifiably ignored in this counting.

seek EM fields that satisfy the following boundary conditions at $r = 0$:

$$\begin{aligned} \mathcal{J}^E(\omega,\ell,\vec{m}) &\equiv -\lim_{r\to 0}\ r^{\ell+d-2}\mathcal{E}_s(r,\omega,\ell,\vec{m})\ , \\ \mathcal{J}^B(\omega,\alpha,\ell,\vec{m}) &\equiv \lim_{r\to 0}\ r^{\ell+d-3}\mathcal{H}_{vv}(r,\omega,\alpha,\ell,\vec{m})\ . \end{aligned} \tag{3.8}$$

Here $\mathcal{J}^{E,B}$ are the electric/magnetic multipole moments of the particle, and we have chosen here the appropriate powers of r to agree with the flat space definitions in Eqs.(C.58) and (C.122). We then expect the near-zone behaviour of the form

$$\begin{aligned} &\mathcal{E}_r \sim (\ell+d-2)\frac{\mathcal{J}^E}{r^{\ell+d-1}}\ ,\ \mathcal{E}_s \sim -\frac{\mathcal{J}^E}{r^{\ell+d-2}}\ ,\ \mathcal{E}_v \sim \frac{i\omega\mathcal{J}^B}{r^{\ell+d-3}}\ , \\ &\mathcal{H}_s \sim \frac{i\omega\mathcal{J}^E}{\ell r^{\ell+d-3}}\ ,\ \mathcal{H}_v \sim -(\ell+d-3)\frac{\mathcal{J}^B}{r^{\ell+d-2}}\ ,\ \mathcal{H}_{vv} \sim \frac{\mathcal{J}^B}{r^{\ell+d-3}}\ . \end{aligned} \tag{3.9}$$

Here, the terms without $i\omega$ correspond exactly to the standard static multipole solutions in flat spacetime. The $i\omega$ terms represent the leading quasi-static correction in flat spacetime: the component $\mathcal{E}_v$ is the induced EMF due to changing magnetic moment, whereas the component $\mathcal{H}_s$ is the magnetic field due to Maxwell's displacement current. This fact gives a physical justification of why we treat the pair $\{\mathcal{E}_v, \mathcal{H}_s\}$ as being derived from the other four: at a given order in quasi-static expansion, these components can be obtained from the other four at one less order.

For a given multipole moment $\mathcal{J}^{E,B}$, we should then arrange the sub-leading near zone behaviour of the other two fields $\{\mathcal{E}_r, \mathcal{H}_v\}$ to get an outgoing wave solution: the powers that need to be controlled happen to be $1/r^{\ell-1}$ terms in $\mathcal{E}_r$ and $1/r^{\ell}$ terms in $\mathcal{H}_s$. In fact, as we shall show below, the near zone behaviour of these two other fields, appropriately renormalised, actually encodes the radiation reaction on the particle corresponding to these multipole moments. More precisely, we will show that

$$\begin{aligned} \lim_{r\to 0} r^{1-\ell}\left[\mathcal{E}_r + (\text{counter-term proportional to } \mathcal{E}_s)\right] &= \text{Radiation reaction on } \mathcal{J}^E\ , \\ \lim_{r\to 0} r^{-\ell}\left[\mathcal{H}_v + (\text{counter-term proportional to } \mathcal{H}_{vv})\right] &= \text{Radiation reaction on } \mathcal{J}^B\ . \end{aligned} \tag{3.10}$$

This statement relates the sub-dominant behaviour in the near zone to radiation reaction.

To make these statements precise, it is convenient to express the $\ell \neq 0$ EM field strengths in terms of electric/magnetic *Hertz-Debye scalars*, i.e., we write

$$\begin{gathered}
\mathcal{E}_r(r,\omega,\ell,\vec{m}) = \frac{\ell(\ell+d-2)}{r^{d-1}}\Phi_E(r,\omega,\ell,\vec{m}) \ , \\
\mathcal{E}_s(r,\omega,\ell,\vec{m}) = \frac{1}{r^{d-3}}D_+\Phi_E(r,\omega,\ell,\vec{m}) \ , \ \mathcal{E}_v(r,\omega,\alpha,\ell,\vec{m}) = i\omega\Phi_B(r,\omega,\alpha,\ell,\vec{m}) \ , \\
\mathcal{H}_s(r,\omega,\ell,\vec{m}) = \frac{i\omega}{r^{d-3}}\Phi_E(r,\omega,\ell,\vec{m}) \ , \ \mathcal{H}_v(r,\omega,\alpha,\ell,\vec{m}) = D_+\Phi_B(r,\omega,\alpha,\ell,\vec{m}) \ , \\
\mathcal{H}_{vv}(r,\omega,\alpha,\ell,\vec{m}) = \Phi_B(r,\omega,\alpha,\ell,\vec{m}) \ , \\
\mathcal{B}_{rs}(r,\omega,\ell,\vec{m}) = -\frac{1}{r^{d-3}}\partial_r\Phi_E(r,\omega,\ell,\vec{m}) \ , \ \mathcal{B}_{rv}(r,\omega,\alpha,\ell,\vec{m}) = \partial_r\Phi_B(r,\omega,\alpha,\ell,\vec{m}) \ .
\end{gathered} \tag{3.11}$$

The reader is encouraged to check that this form automatically satisfies the Maxwell equations in Eqs.(3.5) and (3.7), provided the Hertz-Debye scalar fields satisfy the following radial ODEs:

$$\begin{aligned}
&\frac{1}{r^{3-d}}D_+\left[r^{3-d}D_+\Phi_E\right] + \omega^2\Phi_E - (1-r^2)\frac{\ell(\ell+d-2)}{r^2}\Phi_E = 0 \ , \\
&\frac{1}{r^{d-3}}D_+\left[r^{d-3}D_+\Phi_B\right] + \omega^2\Phi_B - (1-r^2)\frac{(\ell+1)(\ell+d-3)}{r^2}\Phi_B = 0 \ .
\end{aligned} \tag{3.12}$$

In terms of the scalar fields, the Maxwell system reduces to a set of decoupled radial ODEs. An alternate route to get the same results is to write down the gauge potentials $\mathcal{V}_\mu$ in the *Hertz-Debye gauge*, i.e., we take

$$\begin{aligned}
\mathcal{V}_u(r,\omega,\Omega) &= r^{3-d}\sum_{\ell\vec{m}} D_+\Phi_E(r,\omega,\ell,\vec{m}) \ \mathscr{Y}_{\ell\vec{m}}(\Omega) \ , \\
\mathcal{V}_r(r,\omega,\Omega) &= r^{3-d}\sum_{\ell\vec{m}} \partial_r\Phi_E(r,\omega,\ell,\vec{m}) \ \mathscr{Y}_{\ell\vec{m}}(\Omega) \ , \\
\mathcal{V}_I(r,\omega,\Omega) &= \sum_{\alpha\ell\vec{m}} \Phi_B(r,\omega,\alpha,\ell,\vec{m}) \ \mathbb{V}_I^{\alpha\ell\vec{m}}(\Omega) \ .
\end{aligned} \tag{3.13}$$

In this gauge, we essentially set all the electric sector gauge fields to be normal to the sphere directions. The magnetic sector is gauge invariant due to the divergencelessness of the $\mathbb{V}^{\alpha\ell\vec{m}}$. It can then be checked that the field strengths derived from these potentials turn out to be the expressions in Eq.(3.11).

The radial ODEs for the Debye potentials can be solved with appropriate boundary conditions inherited from the corresponding boundary conditions on the field strengths defined in (3.9), which are:

$$
\begin{aligned}
-\lim_{r\to 0} r^{\ell+1} D_+ \Phi_E(r,\omega,\ell,\vec{m}) &= \mathcal{J}^E(\omega,\ell,\vec{m}) \ , \\
\lim_{r\to 0} r^{\ell+d-2} \Phi_B(r,\omega,\alpha,\ell,\vec{m}) &= \mathcal{J}^B(\omega,\alpha,\ell,\vec{m}) \ .
\end{aligned} \tag{3.14}
$$

We can now solve the fields in terms of bulk to boundary propagators for the two scalars. These propagators are special cases of the propagators we found for the generic class of scalars in chapter §2. We will review some of the properties of these propagators that we need in our analysis here, but we refer the reader to chapter §2 for a much more extensive discussion. One writes the Debye potentials as:

$$
\begin{aligned}
\Phi_E(r,\omega,\ell,\vec{m}) &= \frac{1}{\ell} G_E^{\text{Out}}(r,\omega,\ell) \mathcal{J}^E(\omega,\ell,\vec{m}) \ , \\
\Phi_B(r,\omega,\alpha,\ell,\vec{m}) &= G_B^{\text{Out}}(r,\omega,\ell) \mathcal{J}^B(\omega,\alpha,\ell,\vec{m}) \ .
\end{aligned} \tag{3.15}
$$

The extra factor of ℓ is a convenient normalisation for the Φ_E Debye scalar because the scalar satisfies a Neumann boundary condition at the origin.

The boundary to bulk propagators can be written explicitly in terms of Gauss hypergeometric functions, given as follows:

$$
\begin{aligned}
G_E^{\text{Out}}(r,\omega,\ell) &= \frac{r^{\ell+d-2}(1+r)^{-i\omega}}{\Gamma(1-i\omega)\Gamma\left(\ell+\frac{d}{2}-1\right)} \Gamma\left(\frac{\ell+2-i\omega}{2}\right) \Gamma\left(\frac{\ell+d-2-i\omega}{2}\right) \\
&\quad \times {}_2F_1\left[\frac{\ell+2-i\omega}{2}, \frac{\ell+d-2-i\omega}{2}; 1-i\omega; 1-r^2\right] \ , \\
G_B^{\text{Out}}(r,\omega,\ell) &= \frac{r^{\ell+1}(1+r)^{-i\omega}}{\Gamma(1-i\omega)\Gamma\left(\ell+\frac{d}{2}-1\right)} \Gamma\left(\frac{\ell+1-i\omega}{2}\right) \Gamma\left(\frac{\ell+d-1-i\omega}{2}\right) \\
&\quad \times {}_2F_1\left[\frac{\ell+1-i\omega}{2}, \frac{\ell+d-1-i\omega}{2}; 1-i\omega; 1-r^2\right] \ .
\end{aligned} \tag{3.16}
$$

The above hypergeometric functions, for odd values of d, are polynomials in r that generalise the reverse Bessel polynomials one obtains in the study of outgoing radiation in $3+1$ dimensional spacetime[2]. Later in this section, we show how the $G_{E/B}^{\text{Out}}$ gives Hubble corrections to the reverse Bessel polynomials in flat spacetimes with odd values of d. Much like the case in the corresponding flat spacetimes, for even values of d, the functions don't reduce to a polynomial form in r.

[2] In appendix C, we have generalised the reverse Bessel polynomials to arbitrary even-dimensional spacetimes.

The hypergeometric functions are defined by a series expansion about the points where the last argument goes to zero. This tells us that the above retarded boundary-to-bulk propagators have a nice expansion at the horizon $r = 1$. On the other hand, to obtain their behaviour at $r = 0$, the following equivalent form is instructive:

$$\begin{aligned}
G_E^{\text{Out}} &= r^{-\ell}(1+r)^{-i\omega} \\
&\times \Bigg\{ {}_2F_1\left[\frac{-\ell - i\omega}{2}, \frac{4-d-\ell-i\omega}{2}; 2 - \frac{d}{2} - \ell; r^2\right] \\
&\qquad -(1 + i\cot\nu\pi)\widehat{K}_E^{\text{Out}}(\omega,\nu)\frac{r^{2\nu}}{2\nu}\ {}_2F_1\left[\frac{2+\ell-i\omega}{2}, \frac{d+\ell-2-i\omega}{2}; \frac{d}{2} + \ell; r^2\right]\Bigg\} , \\
G_B^{\text{Out}} &= r^{3-d-\ell}(1+r)^{-i\omega} \\
&\times \Bigg\{ {}_2F_1\left[\frac{1-\ell - i\omega}{2}, \frac{3-d-\ell-i\omega}{2}; 2 - \frac{d}{2} - \ell; r^2\right] \\
&\qquad -(1 + i\cot\nu\pi)\widehat{K}_B^{\text{Out}}(\omega,\nu)\frac{r^{2\nu}}{2\nu}\ {}_2F_1\left[\frac{1+\ell-i\omega}{2}, \frac{d+\ell-1-i\omega}{2}; \frac{d}{2} + \ell; r^2\right]\Bigg\} .
\end{aligned} \tag{3.17}$$

Here, $\nu = \ell + \frac{d}{2} - 1$ as defined in the previous sections. The functions $\widehat{K}^{\text{Out}}$ are given by:

$$\begin{aligned}
\widehat{K}_E^{\text{Out}}(\omega,\nu) &= -e^{i\nu\pi}\frac{2\pi i}{\Gamma(\nu)^2}\frac{\Gamma\left(\frac{3-\frac{d}{2}+\nu-i\omega}{2}\right)\Gamma\left(\frac{-1+\frac{d}{2}+\nu-i\omega}{2}\right)}{\Gamma\left(\frac{3-\frac{d}{2}-\nu-i\omega}{2}\right)\Gamma\left(\frac{-1+\frac{d}{2}-\nu-i\omega}{2}\right)} , \\
\widehat{K}_B^{\text{Out}}(\omega,\nu) &= -e^{i\nu\pi}\frac{2\pi i}{\Gamma(\nu)^2}\frac{\Gamma\left(\frac{2-\frac{d}{2}+\nu-i\omega}{2}\right)\Gamma\left(\frac{\frac{d}{2}+\nu-i\omega}{2}\right)}{\Gamma\left(\frac{2-\frac{d}{2}-\nu-i\omega}{2}\right)\Gamma\left(\frac{\frac{d}{2}-\nu-i\omega}{2}\right)} .
\end{aligned} \tag{3.18}$$

When d is odd, ν takes half-integer values, and the above expression is well-defined. For even-dimensional spacetimes, the above expressions for the propagators should be treated as a limit as ν takes the desired integer value. This will play a role when we obtain renormalised boundary correlators on the worldline next.

Now that we have the fields satisfying the prescribed boundary conditions, we want to understand the *self-force* on the multipole moments due to the radiation, i.e. we would like to ask how the fields cause the multipole moments to dissipate energy. We claim that the radiation reaction is encoded in the boundary behaviour of *renormalised* components of the electric and magnetic fields $\mathcal{E}_r$ and $\mathcal{H}_v$. These field components exert a radial force on a spherical shell current density. One can think of the point source as the zero radius limit of such a shell. In this limit, the radial force naively diverges, but if one focuses on purely dissipative parts of the force, they are regular. One would like to remove the conservative pieces containing all the divergences by systematically subtracting them from the $\mathcal{E}_r$ and $\mathcal{H}_v$. This is accomplished by subtracting from the fields, terms proportional to $\mathcal{E}_s$ and $\mathcal{H}_{vv}$ as follows:

$$\begin{aligned} -\lim_{r\to 0} r^{-\ell}\left[r\mathcal{E}_r + \frac{\ell(\ell+d-2)}{\mathscr{C}_{3-d}}\mathcal{E}_s\right] &= \frac{\ell+d-2}{\ell}(1+i\cot\pi\nu)\widehat{K}_E^{\text{Out}}\ \mathcal{J}^E\ , \\ -\lim_{r\to 0} r^{-\ell}\left[\mathcal{H}_v + \frac{\mathscr{C}_{d-3}}{r}\mathcal{H}_{vv}\right] &= (1+i\cot\pi\nu)\widehat{K}_B^{\text{Out}}\ \mathcal{J}^B \end{aligned} \tag{3.19}$$

where the $\mathscr{C}$'s are :

$$\begin{aligned} \frac{\mathscr{C}_{d-3}}{1-r^2} &\equiv -r\frac{d}{dr}\ln\left\{r^{3-d-\ell}(1-r^2)^{-\frac{i\omega}{2}}{}_2F_1\left[\frac{1-\ell-i\omega}{2},\frac{3-d-\ell-i\omega}{2};2-\frac{d}{2}-\ell;r^2\right]\right\}\ , \\ \frac{\mathscr{C}_{3-d}}{1-r^2} &\equiv -r\frac{d}{dr}\ln\left\{r^{-\ell}(1-r^2)^{-\frac{i\omega}{2}}{}_2F_1\left[\frac{-\ell-i\omega}{2},\frac{4-d-\ell-i\omega}{2};2-\frac{d}{2}-\ell;r^2\right]\right\}\ . \end{aligned} \tag{3.20}$$

The $\mathscr{C}$'s are special cases of the counterterm $\mathscr{C}_{\mathbb{N}}$ that we obtained in chapter §2 for the generic class of scalars. The essential difference here is in the case of the electric sector, where the Debye scalar Φ_E satisfies a Neumann boundary condition. We review this counterterming procedure for the same generic class of scalars discussed in chapter §2, now satisfying Neumann boundary conditions, in section 2.2.3.

The radiation reaction kernel is encoded in the boundary values of the renormalised fields, which we will call $K_{E/B}^{\text{Out}}$. For odd values of d(half-integer values of ν), they are given by:

$$\begin{aligned} K_E^{\text{Out}}|_{\text{Odd d}} = (1+i\cot\pi\nu)\widehat{K}_E^{\text{Out}}|_{\text{Odd d}} &= -e^{i\nu\pi}\frac{2\pi i}{\Gamma(\nu)^2}\frac{\Gamma\left(\frac{3-\frac{d}{2}+\nu-i\omega}{2}\right)\Gamma\left(\frac{-1+\frac{d}{2}+\nu-i\omega}{2}\right)}{\Gamma\left(\frac{3-\frac{d}{2}-\nu-i\omega}{2}\right)\Gamma\left(\frac{-1+\frac{d}{2}-\nu-i\omega}{2}\right)}\ , \\ K_B^{\text{Out}}|_{\text{Odd d}} = (1+i\cot\pi\nu)\widehat{K}_B^{\text{Out}}|_{\text{Odd d}} &= -e^{i\nu\pi}\frac{2\pi i}{\Gamma(\nu)^2}\frac{\Gamma\left(\frac{2-\frac{d}{2}+\nu-i\omega}{2}\right)\Gamma\left(\frac{\frac{d}{2}+\nu-i\omega}{2}\right)}{\Gamma\left(\frac{2-\frac{d}{2}-\nu-i\omega}{2}\right)\Gamma\left(\frac{\frac{d}{2}-\nu-i\omega}{2}\right)}\ , \end{aligned} \tag{3.21}$$

The table 2.2 lists explicit expressions for this function in even-dimensional spacetimes up to quadrupole. The $K_{E/B}^{\text{Out}}$ for odd values of d are polynomials which signify the markovian nature of the electromagnetic radiation reaction. This is the same 'boundary two-point function' whose poles are used to analyse the quasinormal mode spectrum of the static patch [7, 94, 113].

Table 3.1: $\frac{K_{\text{Out}}}{-i\omega}$ for Electromagnetic radiation

Magnetic	$\ell=0$	$\ell=1$	$\ell=2$
$d=3$	1	ω^2+1	$\frac{\omega^4}{9}+\frac{5\omega^2}{9}+\frac{4}{9}$
$d=5$	ω^2+4	$\frac{\omega^4}{9}+\frac{10\omega^2}{9}+1$	$\frac{\omega^6}{225}+\frac{7\omega^4}{75}+\frac{28\omega^2}{75}+\frac{64}{225}$
$d=7$	$\frac{\omega^4}{9}+\frac{20\omega^2}{9}+\frac{64}{9}$	$\frac{\omega^6}{225}+\frac{7\omega^4}{45}+\frac{259\omega^2}{225}+1$	$\frac{\omega^8}{11025}+\frac{19\omega^6}{3675}+\frac{8\omega^4}{105}+\frac{3088\omega^2}{11025}+\frac{256}{1225}$
Electric	$\ell=0$	$\ell=1$	$\ell=2$
$d=3$	1	ω^2+1	$\frac{\omega^4}{9}+\frac{5\omega^2}{9}+\frac{4}{9}$
$d=5$	ω^2+1	$\frac{\omega^4}{9}+\frac{5\omega^2}{9}+\frac{4}{9}$	$\frac{\omega^6}{225}+\frac{14\omega^4}{225}+\frac{49\omega^2}{225}+\frac{4}{25}$
$d=7$	$\frac{\omega^4}{9}+\frac{10\omega^2}{9}+1$	$\frac{\omega^6}{225}+\frac{7\omega^4}{75}+\frac{28\omega^2}{75}+\frac{64}{225}$	$\frac{\omega^8}{11025}+\frac{13\omega^6}{3675}+\frac{19\omega^4}{525}+\frac{1261\omega^2}{11025}+\frac{4}{49}$

On the other hand, for even values of d, the $\cot \pi\nu$ diverges and one needs additional counterterms to obtain the correct radiation reaction kernel. These can be obtained by adding the following counterterm action to our electromagnetic action:

$$\begin{aligned} S_{ct,\text{Even}} = \sum_{\ell\vec{m}} \frac{1}{\nu-n} \int \frac{d\omega}{2\pi} \Bigg[& r^{d-4+2n} \Delta\left(n, \frac{d}{2}-1, \omega\right) \Phi_B^* \Phi_B|_{r_c} \\ & + \ell(\ell+d-2) r^{2-d+2n} \Delta\left(n, \frac{d}{2}-2, \omega\right) |D_+\Phi_E|^2|_{r_c} \Bigg] , \end{aligned} \tag{3.22}$$

where $n = \ell + \frac{d}{2} - 1$ and,

$$\begin{aligned} \Delta(n,\mu,\omega) \equiv \frac{(-)^n}{\Gamma(n)^2} \frac{\Gamma\left(\frac{1+n-\mu-i\omega}{2}\right)\Gamma\left(\frac{1+n+\mu-i\omega}{2}\right)}{\Gamma\left(\frac{1-n+\mu-i\omega}{2}\right)\Gamma\left(\frac{1-n-\mu-i\omega}{2}\right)} = \frac{1}{\Gamma(n)^2} \prod_{k=1}^{n} \left[\frac{\omega^2}{4} + \frac{1}{4}(\mu-n+2k-1)^2\right] \\ = \Delta^*(n,\mu,\omega) \ . \end{aligned} \tag{3.23}$$

With this counterterm, we obtain the following form of the radiation reaction kernel for even-dimensional spacetimes:

$$
\begin{aligned}
K_E^{\text{Out}}|_{\text{Even } d} = \Delta_{\mathcal{N}}\left(\nu, \frac{d}{2}-2, \omega\right) \Bigg[& \psi^{(0)}\left(\frac{3-\frac{d}{2}+\nu-i\omega}{2}\right) + \psi^{(0)}\left(\frac{-1+\frac{d}{2}\nu-i\omega}{2}\right) \\
& +\psi^{(0)}\left(\frac{3-\frac{d}{2}-\nu-i\omega}{2}\right) + \psi^{(0)}\left(\frac{-1+\frac{d}{2}-\nu-i\omega}{2}\right) - 4\psi^{(0)}(\nu)\Bigg] \ , \\
K_B^{\text{Out}}|_{\text{Even } d} = \Delta_{\mathcal{N}}\left(\nu, \frac{d}{2}-1, \omega\right) \Bigg[& \psi^{(0)}\left(\frac{2-\frac{d}{2}+\nu-i\omega}{2}\right) + \psi^{(0)}\left(\frac{\frac{d}{2}\nu-i\omega}{2}\right) \\
& +\psi^{(0)}\left(\frac{2-\frac{d}{2}-\nu-i\omega}{2}\right) + \psi^{(0)}\left(\frac{\frac{d}{2}-\nu-i\omega}{2}\right) - 4\psi^{(0)}(\nu)\Bigg]
\end{aligned}
\tag{3.24}
$$

Unlike the case for odd values of d, we see that these functions don't reduce to polynomials in ω, which is expected from the non-markovian nature of radiation reaction in odd spacetime dimensions [84].

In the limit where the Hubble constant is small, i.e. we are looking at sources moving much more rapidly compared to the cosmological time scales, the radiation reaction kernel reduces to its flat space analogues, which have the same behaviour for both magnetic and electric sectors [85,86]:

$$
K_{E/B}^{\text{Out}} \approx \begin{cases} \frac{2\pi i}{\Gamma(\nu)^2}\left(\frac{\omega}{2}\right)^{2\nu} & \text{for } d \text{ odd} \ , \\ \frac{1}{\Gamma(\nu)^2}\left(\frac{\omega}{2}\right)^{2\nu} \ln\left(\frac{\omega^4}{H^4}\right) & \text{for } d \text{ even} \ . \end{cases} \tag{3.25}
$$

The procedure we just illustrated provides us with a dS version of the famous Son-Starinets prescription [71] in AdS/CFT: in this prescription, the field's value ($\overline{\mathcal{V}}_\mu$ in our case) is fixed at the boundary and an outgoing boundary condition is imposed at the horizon. This corresponds to imposing a Dirichlet boundary condition on the Φ_B, whereas the Φ_E satisfies a Neumann boundary condition at $r = 0$. Then one takes the boundary limit of the conjugate field $\mathcal{C}^{r\mu}$ and renormalises it to obtain the radiation reaction kernel K^{Out}.

Given this procedure of obtaining the radiation reaction, we will now justify it as an on-shell action computation on the dS-SK geometry described in detail in 1.1. This is the de Sitter version of the real-time GKPW prescription: We specify boundary data as the observer's multipole moments, and the dS-SK saddle computes the effective action of the observer's dynamics.

We will begin by reviewing some useful geometric details required in our analysis here. We take the de Sitter static patch and complexify the radial coordinate. Then, we consider a hypersurface defined by a contour in the complex r plane. The following *mock tortoise coordinate* is useful to make this notion precise:

$$\zeta(r) = \frac{1}{i\pi}\int_{r}^{0-i\epsilon} \frac{dr'}{1-r'^2} = \frac{1}{2\pi i}\ln\left(\frac{1-r}{1+r}\right) \tag{3.26}$$

This integral has logarithmic branch points at $r = \pm 1$, and we pick the branch cut to run from $r = -1$ to $r = 1$. If we start from $r = 1 + i\epsilon$ and go around the branch cut to $r = 1 - i\epsilon$, the ζ coordinate is normalised to go from 1 to 0 in its real part. We define $\zeta = 0$ as the left boundary and $\zeta = 1$ as the right boundary. Given this geometry, we will now turn to the question of obtaining the electromagnetic fields on it.

We need ingoing counterparts of the outgoing propagators to define the correct boundary to bulk propagator on the dS-SK geometry. We can obtain the ingoing propagator simply by time-reversing the outgoing one:

$$G^{\text{In}}_{E/B}(r,\omega,\ell) = e^{-2\pi\omega\zeta} G^{\text{Out}*}_{E/B}(r,\omega,\ell) \tag{3.27}$$

The boundary-to-bulk propagators on the dS-SK geometry then turn out to be:

$$\begin{aligned} g_L^{E/B}(r,\omega,\ell) &\equiv n_\omega \Big[G^{\text{Out}}_{E/B}(r,\omega,\ell) - e^{2\pi\omega(1-\zeta)} G^{\text{Out}*}_{E/B}(r,\omega,\ell)\Big] \ , \\ g_R^{E/B}(r,\omega,\ell) &\equiv (1+n_\omega) \Big[G^{\text{Out}}_{E/B}(r,\omega,\ell) - e^{-2\pi\omega\zeta} G^{\text{Out}*}_{E/B}(r,\omega,\ell)\Big] \ . \end{aligned} \tag{3.28}$$

where $n_\omega = \frac{1}{e^{2\pi\omega}-1}$, is the Bose-Einstein factor. Essentially, the g_L connects the sources on the left boundary to the fields on the dS-SK geometry, whereas g_R does the same for the sources on the right boundary. These boundary-to-bulk propagators are hence defined so as to satisfy the following boundary conditions:

$$
\begin{aligned}
\lim_{\zeta\to 0} r^{\ell} g_L^E &= -1 \ , \quad \lim_{\zeta\to 0} r^{\ell} g_R^E = 0 \ , \\
\lim_{\zeta\to 1} r^{\ell} g_L^E &= 0 \ , \quad \lim_{\zeta\to 1} r^{\ell} g_R^E = 1, \\
\lim_{\zeta\to 0} r^{d-3+\ell} g_L^B &= -1 \ , \quad \lim_{\zeta\to 0} r^{d-3+\ell} g_R^B = 0 \ , \\
\lim_{\zeta\to 1} r^{d-3+\ell} g_L^B &= 0 \ , \quad \lim_{\zeta\to 1} r^{d-3+\ell} g_R^B = 1.
\end{aligned}
\tag{3.29}
$$

Hence, the $g_{L/R}$ is defined to be regular on the right/left boundary and to have a source singularity on the other boundary[3].

Using these boundary-to-bulk propagators, we can write down the solutions for Φ_E and Φ_B as:

$$
\begin{aligned}
\Phi_E(\zeta,\omega,\ell,\vec{m}) &= \frac{1}{\ell}\left\{ g_R^E(\zeta,\omega,\ell)\mathfrak{J}_R^E(\omega,\ell,\vec{m}) - g_L^E(\zeta,\omega,\ell)\mathfrak{J}_L^E(\omega,\ell,\vec{m})\right\} , \\
\Phi_B(\zeta,\omega,\alpha,\ell,\vec{m}) &= g_R^B(\zeta,\omega,\ell)\mathfrak{J}_R^B(\omega,\alpha,\ell,\vec{m}) - g_L^B(\zeta,\omega,\ell)\mathfrak{J}_L^B(\omega,\alpha,\ell,\vec{m}) \ ;
\end{aligned}
\tag{3.30}
$$

where $\mathfrak{J}_L^{E/B}$ and $\mathfrak{J}_R^{E/B}$ are the electric/magnetic multipole moments on the left and the right boundary, respectively. The Φ_E satisfies Neumann boundary conditions, whereas Φ_B satisfies Dirichlet boundary conditions at both the R and L boundaries of the dS-SK geometry. Correspondingly, we can also write the conjugate fields to $\pi_{E/B}$ in the following manner:

$$
\begin{aligned}
\pi_E(\zeta,\omega,\ell,\vec{m}) &= \frac{1}{\ell}\left\{ \pi_L^E(\zeta,\omega,\ell)\mathfrak{J}_L^E(\omega,\ell,\vec{m}) - \pi_R^E(\zeta,\omega,\ell)\mathfrak{J}_R^E(\omega,\ell,\vec{m})\right\} , \\
\pi_B(\zeta,\omega,\alpha,\ell,\vec{m}) &= \pi_L^B(\zeta,\omega,\ell)\mathfrak{J}_L^B(\omega,\alpha,\ell,\vec{m}) - \pi_R^B(r,\omega,\ell)\mathfrak{J}_R^B(\omega,\alpha,\ell,\vec{m})
\end{aligned}
\tag{3.31}
$$

where we have defined $\pi_{L/R}^{E/B} = D_+ g_{L/R}^{E/B}$. These fields are fixed such that they satisfy the following boundary conditions:

$$
\begin{aligned}
\lim_{\zeta\to 0} r^{\ell+d-2}\pi_E &= \mathfrak{J}_L^E \ , \qquad \lim_{\zeta\to 1} r^{\ell+d-2}\pi_E = \mathfrak{J}_R^E \ , \\
\lim_{\zeta\to 0} r^{\ell+d-3}\Phi_B &= \mathfrak{J}_L^B \ , \qquad \lim_{\zeta\to 1} r^{\ell+d-3}\Phi_B = \mathfrak{J}_R^B
\end{aligned}
\tag{3.32}
$$

Given the solutions with appropriate boundary conditions, one can now substitute

[3]In chapter §2, we give a more detailed analysis of these boundary-to-bulk propagators, along with explicit expressions, which the reader can refer to for further details.

them into the action to obtain the boundary Schwinger-Keldysh action:

$$
\begin{aligned}
S &= -\frac{1}{4}\int_{dSSK} d^{d+1}x\sqrt{-g}\overline{\mathfrak{C}}^{\mu\nu}\overline{\mathfrak{C}}_{\mu\nu} + S_{ct} \\
&= \frac{1}{2}\int_{dSSK} d^{d+1}x \left[\overline{\mathcal{V}}_\nu \partial_\mu\left(\sqrt{-g}\overline{\mathfrak{C}}^{\mu\nu}\right) - \partial_\mu\left(\sqrt{-g}\overline{\mathfrak{C}}^{\mu\nu}\overline{\mathcal{V}}_\nu\right)\right] + S_{ct} \\
&\underset{\text{(On-Shell)}}{=} \left[-\frac{1}{2}\int r^{d-1}dt\ d\Omega_{d-1}\overline{\mathfrak{C}}^{r\mu}\overline{\mathcal{V}}_\mu\right]_{r_c+i\epsilon}^{r_c-i\epsilon} + S_{ct}
\end{aligned}
\tag{3.33}
$$

Here, the first term on the second line can be set to zero using the equations of motion, and the second one evaluates to a boundary term. We can compute this boundary term using the dS-SK solutions:

$$
\begin{aligned}
\lim_{\zeta\to 1} r^{d-1}\mathfrak{C}^{ru}_{\text{ren}}\mathcal{V}^*_u &= \lim_{\zeta\to 1}\frac{1}{r^{\ell-1}}\mathfrak{C}^{ru}_{\text{ren}} \times \lim_{\zeta\to 1} r^{\ell+d-2}\mathcal{V}^*_u \\
&= -\frac{(\ell+d-2)}{\ell}\mathcal{J}^{E*}_R\left\{K^E_{LR}\mathcal{J}^E_R - K^E_{LL}\mathcal{J}^E_L\right\}\ , \\
\lim_{\zeta\to 0} r^{d-1}\mathfrak{C}^{ru}_{\text{ren}}\mathcal{V}^*_u &= \lim_{\zeta\to 0}\frac{1}{r^{\ell-1}}\mathfrak{C}^{ru}_{\text{ren}} \times \lim_{\zeta\to 0} r^{\ell+d-2}\mathcal{V}^*_u \\
&= -\frac{(\ell+d-2)}{\ell}\mathcal{J}^{E*}_L\left\{K^E_{RR}\mathcal{J}^E_R - K^E_{RL}\mathcal{J}^E_L\right\}\ , \\
\lim_{\zeta\to 1} r^{d-1}\mathfrak{C}^{rI}_{\text{ren}}\mathcal{V}^*_I &= \lim_{\zeta\to 1}\frac{1}{r^{\ell-2}}\mathfrak{C}^{rI}_{\text{ren}} \times \lim_{\zeta\to 1} r^{\ell+d-3}\mathcal{V}^*_I \\
&= -\mathcal{J}^{E*}_R\left\{K^B_{LR}\mathcal{J}^B_R - K^B_{LL}\mathcal{J}^B_L\right\}\ , \\
\lim_{\zeta\to 0} r^{d-1}\mathfrak{C}^{ru}_{\text{ren}}\mathcal{V}^*_u &= \lim_{\zeta\to 0}\frac{1}{r^{\ell-2}}\mathfrak{C}^{ru}_{\text{ren}} \times \lim_{\zeta\to 0} r^{\ell+d-3}\mathcal{V}^*_u \\
&= -\mathcal{J}^{B*}_L\left\{K^B_{RR}\mathcal{J}^B_R - K^B_{RL}\mathcal{J}^B_L\right\}\ .
\end{aligned}
\tag{3.34}
$$

We have defined these combinations of the wordline two-point functions:

$$
\begin{aligned}
K^{E/B}_{LL} &\equiv n_\omega K^{\text{Out}}_{E/B} - (1+n_\omega)K^{\text{Out}*}_{E/B}\ , \quad K^{E/B}_{LR} \equiv (1+n_\omega)\Big(K^{\text{Out}}_{E/B} - K^{\text{Out}*}_{E/B}\Big)\ , \\
K^{E/B}_{RL} &\equiv n_\omega\Big(K^{\text{Out}}_{E/B} - K^{\text{Out}*}_{E/B}\Big)\ , \quad K^{E/B}_{RR} \equiv (1+n_\omega)K^{\text{Out}}_{E/B} - n_\omega K^{\text{Out}*}_{E/B}\ .
\end{aligned}
\tag{3.35}
$$

Given the above expressions on the boundaries of dS-SK, we can write down the on-shell action as follows:

$$
\begin{aligned}
S_{\text{CIP}} = &-\sum_{\alpha\ell\vec{m}}\int\frac{d\omega}{2\pi}K^{\text{Out}}_B(\omega,\ell)\ \mathcal{J}^{B*}_D\ \left[\mathcal{J}^B_A + \left(n_\omega+\frac{1}{2}\right)\mathcal{J}^B_D\right] \\
&-\sum_{\ell\vec{m}}\frac{(\ell+d-2)}{\ell}\int\frac{d\omega}{2\pi}K^{\text{Out}}_E(\omega,\ell)\ \mathcal{J}^{E*}_D\ \left[\mathcal{J}^E_A + \left(n_\omega+\frac{1}{2}\right)\mathcal{J}^E_D\right]\ ,
\end{aligned}
\tag{3.36}
$$

where we have defined the average and difference combinations of the source multipole moments:

$$\mathcal{J}_A^{E/B} \equiv \frac{1}{2}\mathcal{J}_R^{E/B} + \frac{1}{2}\mathcal{J}_L^{E/B} \,, \qquad \mathcal{J}_D^{E/B} \equiv \mathcal{J}_R^{E/B} - \mathcal{J}_L^{E/B} \,. \tag{3.37}$$

This is otherwise known as the Keldysh basis, which is convenient for extracting the physics from these expressions. The average-difference terms capture the dissipative piece: they encode the physics, as we will see in the next section, of the Abraham-Lorentz-Dirac force in dS. The difference-difference term encodes the Hawking fluctuations. This can be seen through a Hubbard-Stratonovich transformation of the difference moments, which will induce a noise field whose fluctuations are controlled by the Hawking temperature[4]. The thermality of the correlators is encoded in the fact that the fluctuations are proportional to the dissipation, as can be seen from our action.

Lastly, we will show how the S_{ct} can be written gauge-invariantly. The counterterm lagrangian can be written in a gauge-invariant manner in the following way:

$$\begin{aligned}
S_{ct} = &\Bigg[\frac{1}{2}\int r^{d-1}du\; d\Omega_{d-1}\frac{r}{\mathscr{C}_{\mathcal{N}=3-d}}\left\{(\mathscr{D}^I\overline{\mathcal{C}}_{uI})\overline{\mathcal{V}}_u\right\} \\
&-\frac{1}{2}\int r^{d-1}du\; d\Omega_{d-1}\left\{\frac{\mathscr{C}_{\mathcal{N}=d-3}-\frac{r^2\partial_u^2}{\mathscr{C}_{\mathcal{N}=3-d}}}{(\ell+1)(\ell+d-3)}r\mathscr{D}_J\overline{\mathcal{C}}^{IJ}+\frac{r}{\mathscr{C}_{\mathcal{N}=3-d}}\partial_u\overline{\mathcal{C}}_u^{\ I}\right\}\overline{\mathcal{V}}_I\Bigg]_{r_c+i\epsilon}^{r_c-i\epsilon} \\
= &\Bigg[\frac{1}{2}\int r^{d-1}du\; d\Omega_{d-1}\frac{r}{\mathscr{C}_{\mathcal{N}=3-d}}\Big\{(\mathscr{D}^I\overline{\mathcal{C}}_{uI})\overline{\mathcal{V}}_u-(\partial_u\overline{\mathcal{C}}_u^{\ I})\overline{\mathcal{V}}_I \\
&\qquad\qquad +\frac{r^2}{(\ell+1)(\ell+d-3)}(\partial_u^2\mathscr{D}_J\overline{\mathcal{C}}^{IJ})\overline{\mathcal{V}}_I\Big\} \\
&-\frac{1}{2}\int r^{d-1}du\; d\Omega_{d-1}\frac{r\mathscr{C}_{\mathcal{N}=d-3}}{(\ell+1)(\ell+d-3)}(\mathscr{D}_J\overline{\mathcal{C}}^{IJ})\overline{\mathcal{V}}_I\Bigg]_{r_c+i\epsilon}^{r_c-i\epsilon} \\
= &\Bigg[\frac{1}{2}\int r^{d-1}du\; d\Omega_{d-1}\frac{r}{\mathscr{C}_{\mathcal{N}=3-d}}\left\{\overline{\mathcal{C}}_{uI}\overline{\mathcal{C}}_u^{\ I}-\frac{1}{2}\frac{r^2}{(\ell+1)(\ell+d-3)}(\partial_u\overline{\mathcal{C}}^{IJ})(\partial_u\overline{\mathcal{C}}_{IJ})\right\} \\
&-\frac{1}{4}\int r^{d-1}du\; d\Omega_{d-1}\frac{r\mathscr{C}_{\mathcal{N}=d-3}}{(\ell+1)(\ell+d-3)}\overline{\mathcal{C}}^{IJ}\overline{\mathcal{C}}_{IJ}\Bigg]_{r_c+i\epsilon}^{r_c-i\epsilon}
\end{aligned} \tag{3.38}$$

As we can see, the counterterm action is local in time and gauge invariant. This concludes our analysis of constructing a regularised effective action for a point source observer.

[4]See chapter §2 for a proper derivation of the fluctuating field in the long time limit.

3.1.1 Energy flux through the horizon

The Electromagnetic stress tensor, in our notation, is given by:

$$\overline{T}^{\mu\nu}_{\text{EM}} = \overline{\mathcal{C}}^{\mu\alpha}\overline{\mathcal{C}}^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}\overline{\mathcal{C}}_{\alpha\beta}\overline{\mathcal{C}}^{\alpha\beta} \tag{3.39}$$

We want to calculate the electromagnetic energy flux that exits through the horizon. This is encoded in the T^r_u component of the stress tensor, which we integrate over the sphere to obtain the total flux:

$$\begin{aligned}\int_{\mathbb{S}^{d-1}_r} r^{d-1}(\overline{T}_{\text{EM}})_u^{\ r} &= -\sum_{\ell\vec{m}}\int_\omega r^{d-3}\left[\sum_\alpha \mathcal{E}^*_v\mathcal{H}_v + \ell(\ell+d-2)\mathcal{H}^*_s\mathcal{E}_s\right] \\ &= \sum_{\ell\vec{m}}\int_\omega i\omega\left[\sum_\alpha r^{d-3}\ \Phi^*_B D_+\Phi_B + r^{3-d}\ \ell(\ell+d-2)\Phi^*_E D_+\Phi_E\right]\end{aligned} \tag{3.40}$$

Here, we have expressed the stress tensor in terms of the Debye scalars. In [112], the authors compute using covariant phase space formalism, the flux through $\mathscr{I}^+$ given by:

$$F = \int_{\mathscr{I}^+} dV_{\mathscr{I}^+}\ g^{\mu\nu}\left[\overline{\mathcal{C}}_{u\mu}\overline{\mathcal{C}}_{u\nu} - (1-r^2)\overline{\mathcal{C}}_{u\mu}\overline{\mathcal{C}}_{r\nu}\right] \tag{3.41}$$

Which matches our expression for the flux through a constant r slice.

The boundary to bulk retarded Green's functions in the de Sitter static patch for the Debye scalars are:

$$\begin{aligned}G_E^{\text{Out}}(r,\omega,\ell) &= H^{2\nu}r^{\ell+d-2}(1+Hr)^{-\frac{i\omega}{H}}\frac{\Gamma\left(\frac{\ell+2-\frac{i\omega}{H}}{2}\right)\Gamma\left(\frac{\ell+d-2-\frac{i\omega}{H}}{2}\right)}{\Gamma(1-\frac{i\omega}{H})\Gamma\left(\ell+\frac{d}{2}-1\right)} \\ &\quad\times\ {}_2F_1\left[\frac{\ell+2-\frac{i\omega}{H}}{2},\frac{\ell+d-2-\frac{i\omega}{H}}{2};1-\frac{i\omega}{H};1-r^2H^2\right]\ , \\ G_B^{\text{Out}}(r,\omega,\ell) &= H^{2\nu}r^{\ell+1}(1+Hr)^{-\frac{i\omega}{H}}\frac{\Gamma\left(\frac{\ell+1-\frac{i\omega}{H}}{2}\right)\Gamma\left(\frac{\ell+d-1-\frac{i\omega}{H}}{2}\right)}{\Gamma(1-\frac{i\omega}{H})\Gamma\left(\ell+\frac{d}{2}-1\right)} \\ &\quad\times\ {}_2F_1\left[\frac{\ell+1-\frac{i\omega}{H}}{2},\frac{\ell+d-1-\frac{i\omega}{H}}{2};1-\frac{i\omega}{H};1-H^2r^2\right]\ .\end{aligned} \tag{3.42}$$

This form of writing the propagators, allows us to easily read off the horizon behaviour as the hypergeometric function goes to 1. We have:

$$\begin{aligned}
\lim_{r\to H^{-1}} G_E^{\rm Out} &= 2^{-\frac{i\omega}{H}} H^{\ell} \frac{\Gamma\left(\frac{\ell+2-\frac{i\omega}{H}}{2}\right)\Gamma\left(\frac{\ell+d-2-\frac{i\omega}{H}}{2}\right)}{\Gamma(1-\frac{i\omega}{H})\Gamma\left(\ell+\frac{d}{2}-1\right)} \equiv H^{3-d}\mathfrak{f}_E(\omega) \ , \\
\lim_{r\to H^{-1}} G_B^{\rm Out} &= 2^{-\frac{i\omega}{H}} H^{\ell+d-3} \frac{\Gamma\left(\frac{\ell+1-\frac{i\omega}{H}}{2}\right)\Gamma\left(\frac{\ell+d-1-\frac{i\omega}{H}}{2}\right)}{\Gamma(1-\frac{i\omega}{H})\Gamma\left(\ell+\frac{d}{2}-1\right)} \equiv \mathfrak{f}_B(\omega) \ .
\end{aligned} \tag{3.43}$$

where we have defined the $\mathfrak{f}_E$ to have the same mass dimension as $\mathfrak{f}_B$. Similarly, we can also obtain the following:

$$\begin{aligned}
\lim_{r\to H^{-1}} D_+ G_E^{\rm Out} &= i\omega H^{3-d}\ \mathfrak{f}_E \ , \\
\lim_{r\to H^{-1}} D_+ G_B^{\rm Out} &= i\omega\ \mathfrak{f}_B \ .
\end{aligned} \tag{3.44}$$

To find the flux of outgoing radiation, we simply substitute the outgoing solutions evaluated at the horizon. Substituting this relation back into (3.40) evaluated at the horizon, we have:

$$\lim_{r\to H^{-1}} \int_{\mathbb{S}_r^{d-1}} r^{d-1} (\overline{T}_{\rm EM})_u^{\ r} = -\sum_{\ell\vec{m}} \int_\omega i\omega \left[\sum_\alpha K_B^{\rm Out} |\mathcal{J}^B|^2 + \frac{\ell+d-2}{\ell} K_E^{\rm Out} |\mathcal{J}^E|^2\right] \tag{3.45}$$

From the above expressions, we can obtain the behaviour of the electromagnetic fields at the horizon:

$$\begin{aligned}
&\lim_{r\to H^{-1}} \mathcal{E}_r = (\ell+d-2)H^2\ \mathfrak{f}_E\ \mathcal{J}^E \ , \quad \lim_{r\to H^{-1}} \mathcal{E}_s = i\omega\ \mathfrak{f}_E \frac{\mathcal{J}^E}{\ell} \ , \quad \lim_{r\to H^{-1}} \mathcal{H}_s = i\omega\ \mathfrak{f}_E \frac{\mathcal{J}^E}{\ell} \ , \\
&\lim_{r\to H^{-1}} \mathcal{H}_{vv} = \mathfrak{f}_B\ \mathcal{J}^B \ , \quad \lim_{r\to H^{-1}} \mathcal{E}_v = i\omega\ \mathfrak{f}_B\ \mathcal{J}^B \ , \quad \lim_{r\to H^{-1}} \mathcal{H}_v = i\omega\ \mathfrak{f}_B\ \mathcal{J}^B \ .
\end{aligned} \tag{3.46}$$

We can see that these expressions reproduce the flat space expressions as $H \to 0$ with r set to $\frac{1}{H}$. To see this, we give the flat space limits of the $\mathfrak{f}_{E/B}$(for odd values of d):

$$\lim_{H\to 0} H^{\frac{3-d}{2}} \mathfrak{f}_E = \lim_{H\to 0} H^{\frac{3-d}{2}} \mathfrak{f}_B = \frac{(-i\omega)^{\nu-\frac{1}{2}}}{(2\nu-2)!!} \ . \tag{3.47}$$

For generic values of d we have:

$$\lim_{H\to 0} H^{\frac{3-d}{2}} \mathfrak{f}_E = \lim_{H\to 0} H^{\frac{3-d}{2}} \mathfrak{f}_B = \frac{\sqrt{\pi}}{\Gamma(\nu)} \left(-\frac{i\omega}{2}\right)^{\nu-\frac{1}{2}} \ . \tag{3.48}$$

3.2 Extended EM sources in dS and radiation reaction

In this section, we want to describe in detail the results about extended EM sources in de Sitter that were alluded to in the main text. Our goal here is twofold: first, we want to describe radiative multipole expansion in de Sitter with correct normalisations for multipole moments, etc., which in the $H \to 0$ limit reproduces the flat space analysis. The second goal is to compute the analogue of the ALD force in de Sitter (they also have a $H \to 0$ limit). One main difference to our flat spacetime analysis is the following: the analysis in this section uses retarded time u instead of the Schwarzschild time t.

Magnetic Multipole Radiation

We will begin by describing the magnetic multipole radiation due to toroidal currents in dS_{d+1}. These currents are identically conserved; hence, conservation equations play no role in this sector, making the analysis conceptually simpler. One begins with decomposing the currents in terms of vector spherical harmonics on the sphere.

$$\bar{J}^u(r,u,\hat{r}) = \bar{J}^r(r,u,\hat{r}) = 0 \ , \quad \bar{J}^I(r,u,\hat{r}) = \sum_{\alpha\ell\vec{m}} \int_\omega J_V^{\alpha\ell\vec{m}}(r,\omega)\mathbb{V}^I_{\alpha\ell\vec{m}}(\hat{r}) \ . \tag{3.49}$$

As in the flat space as well as the point source analysis, we will use a convenient parameterisation of the gauge field in terms of the magnetic Debye scalar:

$$\overline{\mathcal{V}}_u(r,u,\hat{r}) = \overline{\mathcal{V}}_r(r,u,\hat{r}) = 0 \ , \quad \overline{\mathcal{V}}_I(r,u,\hat{r}) = \sum_{\alpha\ell\vec{m}} \int_\omega e^{-i\omega u}\Phi_B(r,\omega,\alpha,\ell,\vec{m}) \ \mathbb{V}_I^{\alpha\ell\vec{m}}(\hat{r}) \ . \tag{3.50}$$

We will remind the reader that due to the orthogonality of the VSH with $\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}$, the electromagnetic fields due to toroidal currents completely decouple from those due to charge densities and poloidal currents. Hence, we can proceed with independently analysing the effects of toroidal current distributions.

Given the above parametrisation of the sources and the gauge fields, the electromagnetic field equations reduce to the following inhomogeneous equation for the magnetic

Debye potential:

$$\frac{1}{r^{d-3}}D_+\left[r^{d-3}D_+\Phi_B\right]+\omega^2\Phi_B-\frac{(\ell+1)(\ell+d-3)(1-r^2)}{r^2}\Phi_B+r^2(1-r^2)J_V^{\alpha\ell\vec{m}}=0\ . \tag{3.51}$$

We will construct a Green function for solving the above inhomogeneous differential equation, such that it satisfies the following equation:

$$\begin{split}\frac{1}{r^{d-3}}D_+\left[r^{d-3}D_+\mathbb{G}_B(r,r_0;\ell)\right]+\omega^2\mathbb{G}_B(r,r_0;\ell)-\frac{(\ell+1)(\ell+d-3)(1-r^2)}{r^2}\mathbb{G}_B(r,r_0;\ell)\\+(1-r^2)\frac{\delta(r-r_0)}{r^{d-3}}=0\ .\end{split} \tag{3.52}$$

One interprets this green function as the field generated by a single shell of unit toroidal current placed at $r=r_0$. The appropriate boundary conditions for this problem are that the field should be outgoing at the horizon, i.e. at $r=1$ and that it should be finite at $r=0$. The field is also required to satisfy the correct jump condition at the sphere obtained by integrating the above equation about $r=r_0$:

$$r^{d-3}D_+\mathbb{G}_B\Big|_{r_0+}^{r_0-}=1\ . \tag{3.53}$$

We have already analysed the homogeneous solution to this equation that satisfies the outgoing boundary condition: G_B^{Out}, which dictates the field outside the sphere. We also need a homogeneous solution normalisable at the origin to obtain the appropriate Green function for the inhomogeneous solution. This normalisable solution is given by:

$$\begin{split}\left(\frac{1-r}{1+r}\right)^{\frac{i\omega}{2}}\Xi_n^B(r,\omega,\ell)\equiv\frac{1}{2\ell+d-2}r^{\ell+1}(1+r)^{-i\omega}\\\times\,{}_2F_1\left[\frac{\ell+1-i\omega}{2},\frac{\ell+d-1-i\omega}{2};\ell+\frac{d}{2};r^2\right]\ .\end{split} \tag{3.54}$$

The function $\Xi_n^B(r,\omega,\ell)$ is the corresponding normalisable solution in the Schwarzschild time t [7]. $\Xi_n^B(r,\omega,\ell)$ reproduces the flat space normalisable solution (the Bessel function) (C.34) in the $H\to 0$ limit(see section 3.2.1). Given these solutions to the homogeneous equation satisfying appropriate boundary conditions, we can construct the Green function for the above inhomogeneous equation such that the solution takes the form:

$$\Phi_B(r,\omega,\alpha,\ell,\vec{m}) = \int dr_0 r_0^{d-1} \mathbb{G}_B(r,r_0;\omega,\ell) J_V^{\alpha\ell\vec{m}}(r_0,\omega) \tag{3.55}$$

Imposing these boundary conditions and the appropriate jump condition at $r = r_0$, we can write the form of the Green function as follows:

$$\mathbb{G}_B(r,r_0;\omega,\ell) = \frac{1}{W_B(r_0,\omega,\ell)} \left(\frac{1-r_<}{1+r_<}\right)^{\frac{i\omega}{2}} \Xi_n^B(r_<,\omega,\ell) G_B^{\text{Out}}(r_>,\omega,\ell)\ , \tag{3.56}$$

where the Wronskian $W_B(r_0,\omega,\ell)$ is given by:

$$W_B(r_0,\omega,\ell) = \left(\frac{1-r_0}{1+r_0}\right)^{i\omega} \tag{3.57}$$

and

$$r_> \equiv \text{Max}(r,r_0)\ , \quad r_< \equiv \text{Min}(r,r_0)\ . \tag{3.58}$$

We have now solved for the Debye potential Φ_B given a poloidal current distribution. Given the potential, we can now use (3.11) to obtain the solutions for the fields:

$$\begin{aligned}
\mathcal{E}_v = i\omega\ \Phi_B &= \int_{\vec{r}_0} i\omega\ \mathbb{G}_B(r,r_0;\omega,\ell)\mathbb{V}_I^{\alpha\ell\vec{m}}(\hat{r}_0) J^I(\vec{r}_0,\omega)\ , \\
\mathcal{H}_v = D_+\Phi_B &= \int_{\vec{r}_0} D_+\mathbb{G}_B(r,r_0;\omega,\ell)\mathbb{V}_I^{\alpha\ell\vec{m}}(\hat{r}_0) J^I(\vec{r}_0,\omega)\ , \\
\mathcal{H}_{vv} = \Phi_B &= \int_{\vec{r}_0} \mathbb{G}_B(r,r_0;\omega,\ell)\mathbb{V}_I^{\alpha\ell\vec{m}}(\hat{r}_0) J^I(\vec{r}_0,\omega)\ .
\end{aligned} \tag{3.59}$$

These equations are analogous to the equations (C.57) in flat space. As pointed out in the flat space analysis, the time-dependent toroidal currents give rise not only to a magnetic field but also to an induced electric field. Given the expressions of the fields, we can now describe the fields outside the sources. We identify the magnetic multipole moment to be:

$$\begin{aligned}\mathcal{J}^B(\omega,\alpha,\ell,\vec{m}) &= \int_{\vec{r}_0}\left(\frac{1-r_0}{1+r_0}\right)^{-\frac{i\omega}{2}}\Xi^B_n(r_0,\omega,\ell)\mathbb{V}^{\alpha\ell\vec{m}}_I(\hat{r}_0)J^I(\vec{r}_0,\omega)\\ &= \frac{1}{2\ell+d-2}\int_{\vec{r}_0} r_0^{\ell+1}(1-r_0)^{-i\omega}\\ &\times {}_2F_1\left[\frac{\ell+1-i\omega}{2},\frac{\ell+d-1-i\omega}{2};\ell+\frac{d}{2};r_0^2\right]\mathbb{V}^{\alpha\ell\vec{m}}_I(\hat{r}_0)J^I(\vec{r}_0,\omega)\ .\end{aligned} \tag{3.60}$$

We can rewrite the multipole moment in Schwarzschild time to obtain a formula that can be compared directly with the corresponding flat space expression(C.58). For this, we need to convert our expressions in the Fourier transform of the outgoing time to expressions in terms of Schwarzschild time t. This can be achieved by first recognising that:

$$J^I(\vec{r},\omega) = \int du\ e^{i\omega u}\bar{J}^I(\vec{r},t) = \int dt\ e^{i\omega t}\left(\frac{1-r}{1+r}\right)^{\frac{i\omega}{2}}\bar{J}^I(\vec{r},t)\ . \tag{3.61}$$

This allows us to rewrite our expression for $\mathcal{J}_B$ as:

$$\mathcal{J}^B(\omega,\alpha,\ell,\vec{m}) = \int dt\ e^{i\omega t}\int_{\vec{r}_0}\Xi^B_n(r_0,i\partial_t,\ell)\mathbb{V}^{\alpha\ell\vec{m}}_I(\hat{r}_0)J^I(\vec{r}_0,t) \tag{3.62}$$

Electric Multipole Radiation

Charge density and poloidal currents source the 'electric' multipole radiation. Unlike toroidal currents, which are conserved identically and hence carry no constraint coming from conservation equations, poloidal currents are interlinked with the temporal change of charge density. One can rewrite the current density in a form where the conservation equation is manifest:

$$\begin{aligned}\bar{J}^u(r,u,\hat{r}) &= \sum_{\ell\vec{m}}\int_\omega\left[\frac{\ell(\ell+d-2)}{r^2}J_2(r,\omega,\ell,\vec{m}) - \frac{1}{r^{d-1}}\partial_r\left\{r^{d-1}J_1(r,\omega,\ell,\vec{m})\right\}\right]\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ ,\\ \bar{J}^r(r,u,\hat{r}) &= -\sum_{\ell\vec{m}}\int_\omega i\omega J_1(r,\omega,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ ,\\ \bar{J}^I(r,u,\hat{r}) &= -\sum_{\ell\vec{m}}\int_\omega\frac{i\omega}{r^2}J_2(r,\omega,\ell,\vec{m})\mathscr{D}^I\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ .\end{aligned} \tag{3.63}$$

The electric parity multipole radiation has a gauge redundancy in its description tied to the above conservation equation. Due to such a gauge redundancy, one cannot impose the same gauge conditions as we did for free electromagnetic fields in the presence of extended sources. In particular, the gauge field parametrisation, in terms of the Debye scalar, requires modification with additional source-local terms to satisfy the sourced Maxwell equations. The gauge field parametrisation then becomes:

$$\begin{aligned}
\overline{\mathcal{V}}_u(r,u,\hat{r}) &= \sum_{\ell\vec{m}} \int_\omega e^{-i\omega u} \left[r^{3-d} D_+ \Phi_E(r,\omega,\ell,\vec{m}) - (1-r^2) J_2 \right] \mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\
\overline{\mathcal{V}}_r(r,u,\hat{r}) &= \sum_{\ell\vec{m}} \int_\omega e^{-i\omega u} \left[r^{3-d} \partial_r \Phi_E(r,\omega,\ell,\vec{m}) - J_2 \right] \mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\
\overline{\mathcal{V}}_I(r,u,\hat{r}) &= \sum_{\alpha\ell\vec{m}} \int_\omega e^{-i\omega u} \Phi_B(r,\omega,\ell,\vec{m}) \ \mathbb{V}_I^{\alpha\ell\vec{m}}(\hat{r}) \ .
\end{aligned} \tag{3.64}$$

One can check the consistency of this gauge choice by plugging it into the Maxwell equations, which are satisfied contingent on the fact that the Debye scalar is a solution to second-order inhomogeneous differential equations:

$$\begin{aligned}
\frac{1}{r^{3-d}} D_+ \left[r^{3-d} D_+ \Phi_E \right] + \omega^2 \Phi_E - &\frac{\ell(\ell+d-2)(1-r^2)}{r^2} \Phi_E \\
&- r^{d-3} \left[D_+ \left[(1-r^2) J_2 \right] - (1-r^2) J_1 \right] = 0 \ .
\end{aligned} \tag{3.65}$$

The particular combination of the source parameters J_1 and J_2 that appears on the RHS of the EOM of Φ_E can be considered the 'radiative' source combination. This combination is solely responsible for the radiative energy loss.

Given the above parametrisation of the gauge field, the electromagnetic field strengths in the presence of sources can be computed to give:

$$\begin{aligned}
\mathcal{E}_r &= \frac{\ell(\ell+d-2)}{r^{d-1}} \Phi_E - J_1 \ , \\
\mathcal{E}_s &= \frac{1}{r^{d-3}} D_+ \Phi_E - (1-r^2) J_2 \ , \\
\mathcal{H}_s &= \frac{i\omega}{r^{d-3}} \Phi_E \ .
\end{aligned} \tag{3.66}$$

One can think of the above formulae alternatively as being obtained from the gauge field parametrisation in terms of the electromagnetic fields:

$$\begin{aligned}
\overline{\mathcal{V}}_u(r,u,\hat{r}) &= \sum_{\ell\vec{m}}\int_\omega e^{-i\omega u}\mathcal{E}_s(r,\omega,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\
\overline{\mathcal{V}}_r(r,u,\hat{r}) &= \sum_{\ell\vec{m}}\int_\omega \frac{e^{-i\omega u}}{1-r^2}\left[\mathcal{E}_s(r,\omega,\ell,\vec{m}) - \mathcal{H}_s(r,\omega,\ell,\vec{m})\right]\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\
\overline{\mathcal{V}}_I(r,u,\hat{r}) &= 0.
\end{aligned} \tag{3.67}$$

Given the inhomogeneous equations of motion for Φ_E, one can solve for the electromagnetic fields in de Sitter by finding the corresponding Green's functions. The natural boundary condition to impose in de Sitter is the outgoing boundary condition at the horizon. The outgoing Green's functions can be written as:

$$\mathbb{G}_E(r,r_0;\omega,\ell) = \frac{1}{W(r_0,\omega,\ell)}\left(\frac{1-r_<}{1+r_<}\right)^{\frac{i\omega}{2}}\Xi_n^E(r_<,\omega,\ell)G_E^{\text{Out}}(r_>,\omega,\ell) \ . \tag{3.68}$$

where the G_E^{Out} is the boundary-to-bulk outgoing propagator defined in the previous section and Ξ_n^E is the *normalisable* mode:

$$\begin{aligned}
\Xi_n^E(r,\omega,\ell) \equiv{}& \frac{1}{2\ell+d-2}r^{\ell+d-2}(1-r^2)^{-\frac{i\omega}{2}} \\
&\times {}_2F_1\left[\frac{\ell+2-i\omega}{2},\frac{\ell+d-2-i\omega}{2};\ell+\frac{d}{2};r^2\right] \ .
\end{aligned} \tag{3.69}$$

For odd values of d, the above expressions are well-defined, but for even values of d, one should evaluate the expressions as a limiting case. With these Green's functions, we can write the Debye scalars as:

$$\begin{aligned}
\Phi_E(r,\omega,\ell,\vec{m}) = \int & dr_0\mathbb{G}_E(r,r_0;\omega,\ell) \\
&\times\left[J_1(r_0,\omega,\ell,\vec{m}) - \frac{1}{1-r^2}D_+^0\left\{(1-r^2)J_2(r_0,\omega,\ell,\vec{m})\right\}\right] \ .
\end{aligned} \tag{3.70}$$

The above solutions in terms of the source parameters J_1 and J_2 can be rewritten in terms of the source currents to give:

$$
\begin{aligned}
\Phi_E =& -\frac{1}{i\omega}\int dr_0 \int d\Omega_{d-1} \\
&\left[\mathscr{Y}^*_{\ell\vec{m}} J^r \mathbb{G}_E(r,r_0;\omega,\ell) - \frac{1}{\ell(\ell+d-2)} r_0^2 J^I \mathscr{D}_I \mathscr{Y}^*_{\ell\vec{m}} D^0_- \mathbb{G}_E(r,r_0;\omega,\ell)\right] \\
\mathcal{H}_s =& -\frac{1}{r^{d-3}}\int dr_0 \int d\Omega_{d-1} \\
&\left[\mathscr{Y}^*_{\ell\vec{m}} J^r \mathbb{G}_E(r,r_0;\omega,\ell) - \frac{1}{\ell(\ell+d-2)} r_0^2 J^I \mathscr{D}_I \mathscr{Y}^*_{\ell\vec{m}} D^0_- \mathbb{G}_E(r,r_0;\omega,\ell)\right] \\
\mathcal{E}_r =& \frac{1}{r^{d-1}}\int dr_0 \int d\Omega_{d-1} \mathscr{Y}^*_{\ell\vec{m}} r_0^2 \left[J^t D^0_- \mathbb{G}_E(r,r_0;\omega,\ell) + i\omega J^r \mathbb{G}_E(r,r_0;\omega,\ell)\right]
\end{aligned}
\tag{3.71}
$$

The expression for $\mathcal{E}_s$ can be obtained from the above by integrating the first Bianchi as we did in flat space. The first Bianchi in EF coordinates in dS is:

$$
D_+\mathcal{E}_s = (1-r^2)\mathcal{E}_r + i\omega\mathcal{H}_s \tag{3.72}
$$

To integrate this equation, we will rewrite it as:

$$
\begin{aligned}
\partial_r\left(e^{i\omega r_*}\mathcal{E}_s\right) &= e^{i\omega r_*}\left[\mathcal{E}_r + \frac{i\omega}{(1-r^2)}\mathcal{H}_s\right] \\
&= e^{i\omega r_*}\int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} \mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\ \mathbb{G}_E(r,r_0;\omega,\ell)\Big\{i\omega\frac{r_0^2-\frac{r^2}{1-r^2}}{r^{d-1}} J^r(\vec{r}_0,\omega) \\
&\quad -\frac{1}{1-r_0^2}D^0_+\left[\frac{(1-r_0^2)r_0^2}{r^{d-1}}J^t(\vec{r}_0,\omega) + i\omega\frac{(1-r_0^2)r_0^2}{r^{d-3}}\frac{\mathscr{D}_I J^I(\vec{r}_0,\omega)}{\ell(\ell+d-2)}\right]\Big\}
\end{aligned}
\tag{3.73}
$$

where r_* is the tortoise coordinate. We can now integrate this equation to write an expression for $\mathcal{E}_s$ as follows:

$$
\begin{aligned}
\mathcal{E}_s &= e^{-i\omega r_*}\int dr_1 e^{i\omega r_{1*}}\left[\mathcal{E}_r(r_1) + \frac{i\omega}{(1-r_1^2)}\mathcal{H}_s(r_1)\right] \\
&= e^{-i\omega r_*}\int dr_1 e^{i\omega r_{1*}}\int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} \mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\ \mathbb{G}_E(r_1,r_0;\omega,\ell)\Big\{i\omega\frac{r_0^2-\frac{r_1^2}{1-r_1^2}}{r_1^{d-1}} J^r(\vec{r}_0,\omega) \\
&\quad -\frac{1}{1-r_0^2}D^0_+\left[\frac{(1-r_0^2)r_0^2}{r_1^{d-1}}J^t(\vec{r}_0,\omega) + i\omega\frac{(1-r_0^2)r_0^2}{r_1^{d-3}}\frac{\mathscr{D}_I J^I(\vec{r}_0,\omega)}{\ell(\ell+d-2)}\right]\Big\}
\end{aligned}
\tag{3.74}
$$

Action reduction

We must evaluate the on-shell action on the dS-SK geometry to obtain the effective action describing the extended observer. Although the explicit derivation specific to the dS-SK geometry is treated in the next section, we want to simplify the sourced Maxwell action in this section after imposing the equations of motion to bring it to a simpler form. We begin with the Maxwell action:

$$S_{EM} = -\int d^{d+1}x \left[\frac{1}{4}\bar{\mathcal{C}}_{\mu\nu}\bar{\mathcal{C}}^{\mu\nu} - \bar{\mathcal{V}}_{\mu}J^{\mu}\right] \tag{3.75}$$

We perform, as before, an expansion in the spherical harmonics and a Fourier transform on the outgoing time to obtain, in terms of the field strengths:

$$\begin{aligned} S_{EM} = -\frac{1}{2}\sum_{\ell\vec{m}}\int_{\omega}\int dr \frac{r^{d-1}}{1-r^2}\Bigg[&(1-r^2)|\mathcal{E}_r|^2 - \frac{\ell(\ell+d-2)}{r^2}\left\{|\mathcal{H}_s|^2 - |\mathcal{E}_s|^2\right\} + 2i\omega\mathcal{H}_s J_1^* \\ &+ \mathcal{E}_s\left\{\frac{1}{r^{d-1}}D_-(r^{d-1}J_1^*) - \frac{\ell(\ell+d-2)}{r^2}(1-r^2)J_2^*\right\}\Bigg] \end{aligned} \tag{3.76}$$

Using integration by parts and the Φ_E EOM:

$$\begin{aligned} S_{\text{on-shell}} = \frac{1}{2}\sum_{\ell\vec{m}}\int\frac{d\omega}{2\pi}\Bigg[&\partial_r\left\{-r^2(D_+\Phi_E)^*J_1 + r^2(1-r^2)\ell(\ell+d-2)\Phi_E^*J_2 + r^{d+1}J_1J_2^*\right\} \\ &+\ell(\ell+d-2)\Phi_E^*\left\{\frac{1}{1-r^2}D_+\left[(1-r^2)J_2\right] + J_1\right\} \\ &-r^{d-1}|J_1|^2 - r^{d+1}\ell(\ell+d-2)(1-r^2)|J_2|^2\Bigg] \end{aligned} \tag{3.77}$$

To obtain the parameters J_1 and J_2 given a current density, we invert the equations in (3.63) to obtain:

$$\begin{aligned}J_1(r,\omega,\ell,\vec{m}) &= \int d\Omega_{d-1}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r})\frac{J^r(r,\omega,\hat{r})}{-i\omega}\ ,\\ J_2(r,\omega,\ell,\vec{m}) &= -\int d\Omega_{d-1}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r})\frac{r^2}{-i\omega\ell(\ell+d-2)}\mathscr{D}_I J^I(r,\omega,\hat{r})\\ &= \int d\Omega_{d-1}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r})\frac{1}{\ell(\ell+d-2)}\frac{1}{1-r^2}\left[\frac{1}{-i\omega r^{d-3}}D_+\left(r^{d-1}J^r\right)-r^2 J_u\right]\ .\end{aligned} \tag{3.78}$$

Here we have rewritten the $\mathscr{D}_I J^I$ term using the conservation equation:

$$\mathscr{D}_I J^I = -\frac{1}{1-r^2}\left[i\omega J_u+\frac{1}{r^{d-1}}D_+\left(r^{d-1}J^r\right)\right]\ . \tag{3.79}$$

3.2.1 Radiative multipole moments on dS-SK

We will use the result of the previous section and identify the average/difference electromagnetic radiative multipole moments on the dS-SK geometry. The equation of motion for Φ_E shows that the J_2 acts like a 'Neumann' source, i.e., the Φ_E is sourced by a derivative operator acting on J_2. On the other hand, J_1 acts like a 'Dirichlet' source. For the 'Neumann' case, we identify ρ with $\frac{J_2}{r^{3-d}}$. Consider the contribution to the 'R' electric multipole moment from the J_2 source takes the form:

$$\begin{aligned}&-\ell\int_R dr\ \left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}}(1-r^2)\partial_r\Xi^E_n(r,\omega,\ell)\ J_2(r,\omega,\ell,\vec{m})\\ &\qquad=\frac{1}{\ell+d-2}\int_R dr\ \left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}}\partial_r\Xi^E_n(r,\omega,\ell)\\ &\qquad\qquad\int d\Omega_{d-1}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r})\left[r^2 J_u+\frac{1}{i\omega r^{d-3}}D_+\left(r^{d-1}J^r\right)\right]\ .\end{aligned} \tag{3.80}$$

We can use integration by parts on the J^r term along with the following identity for Ξ_n:

$$\begin{aligned}&D_-\left[\left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}}r^{3-d}(1-r^2)\partial_r\Xi^E_n\right]\\ &\qquad=-\left\{\omega^2-\frac{\ell(\ell+d-2)}{r^2}(1-r^2)\right\}\left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}}r^{3-d}\Xi^E_n\ .\end{aligned} \tag{3.81}$$

The full radiative multipole moment then becomes:

$$\begin{aligned}
\mathfrak{J}^E_R(r,\omega,\ell,\vec{m}) &= \frac{1}{2\nu(\ell+d-2)}\int_R d^dx\ \mathscr{Y}^*_{\ell\vec{m}}(\hat{r})\frac{1}{r^{d-3}}\left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}} \\
&\qquad \times\left[\partial_r\Xi^E_n(r,\omega,\ell)J_u(r,\omega,\hat{r})+i\omega\ \Xi^E_n(r,\omega,\ell)J^r(r,\omega,\hat{r})\right] \\
\mathfrak{J}^E_L(r,\omega,\ell,\vec{m}) &= \frac{1}{2\nu(\ell+d-2)}\int_R d^dx\ \mathscr{Y}^*_{\ell\vec{m}}(\hat{r})\frac{1}{r^{d-3}}\left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}} \\
&\qquad \times\left[\partial_r\Xi^E_n(r,\omega,\ell)J_u(r,\omega,\hat{r})+i\omega\ \Xi^E_n(r,\omega,\ell)J^r(r,\omega,\hat{r})\right]\ .
\end{aligned} \tag{3.82}$$

The computation of the magnetic multipole moments on the dS-SK geometry, on the other hand, is straightforward, as there is no ambiguity due to conservation equations. We find:

$$\begin{aligned}
\mathfrak{J}^B_R(\omega,\alpha,\ell,\vec{m}) &= \int_R d^dr\left(\frac{1-r_0}{1+r_0}\right)^{-\frac{i\omega}{2}}\Xi^B_n(r_0,\omega,\ell)\mathbb{V}^{\alpha\ell\vec{m}}_I(\hat{r}_0)J^I(\vec{r}_0,\omega)\ , \\
\mathfrak{J}^B_L(\omega,\alpha,\ell,\vec{m}) &= \int_L d^dr\left(\frac{1-r_0}{1+r_0}\right)^{-\frac{i\omega}{2}}\Xi^B_n(r_0,\omega,\ell)\mathbb{V}^{\alpha\ell\vec{m}}_I(\hat{r}_0)J^I(\vec{r}_0,\omega)\ .
\end{aligned} \tag{3.83}$$

Corresponding to these multipole moments, we can write down the STF moments, which are better suited for a post-newtonian expansion:

$$\begin{aligned}
{}^E\mathcal{Q}^{i_1\ldots i_\ell}_{A,STF}(\omega) &\equiv \frac{1}{(\ell+d-2)}(\Pi^S)^{<i_1i_2\ldots i_\ell>}_{<j_1j_2\ldots j_\ell>}\int d^dr\ \hat{r}^{j_1}\hat{r}^{j_2}\ldots\hat{r}^{j_\ell} \\
&\qquad\times\frac{2\nu}{r^{d-3}}\left[\partial_r\Xi^E_n(r,\omega,\ell)J_u(r,\omega,\hat{r})+i\omega\ \Xi^E_n(r,\omega,\ell)J^r(r,\omega,\hat{r})\right]\ , \\
{}^E\mathcal{Q}^{i_1\ldots i_\ell}_{D,STF}(\omega) &\equiv \frac{1}{(\ell+d-2)}(\Pi^S)^{<i_1i_2\ldots i_\ell>}_{<j_1j_2\ldots j_\ell>}\int d^dr\ \hat{r}^{j_1}\hat{r}^{j_2}\ldots\hat{r}^{j_\ell} \\
&\qquad\times\frac{2\nu}{r^{d-3}}\left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}}\Big[\partial_r\Xi^E_n(r,\omega,\ell)J_u(r,\omega,\hat{r}) \\
&\qquad\qquad+i\omega\Xi^E_n(r,\omega,\ell)J^r(r,\omega,\hat{r})\Big]\ .
\end{aligned} \tag{3.84}$$

Similarly, the magnetic multipole moments on dS-SK can be written as:

$$
\begin{aligned}
{}^B\mathcal{Q}_A^{i<i_1\ldots i_\ell>} &\equiv (\Pi^V)^{i<i_1 i_2\ldots i_\ell>}_{j<j_1 j_2\ldots j_\ell>}\int d^d x\ x^{j_1}x^{j_2}\ldots x^{j_\ell}\ \frac{2\nu}{r^{\ell+1}}\left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}}\ \Xi_n^B(r,\omega,\ell)J_A^j \\
{}^B\mathcal{Q}_D^{i<i_1\ldots i_\ell>} &\equiv (\Pi^V)^{i<i_1 i_2\ldots i_\ell>}_{j<j_1 j_2\ldots j_\ell>}\int d^d x\ x^{j_1}x^{j_2}\ldots x^{j_\ell}\ \frac{2\nu}{r^{\ell+1}}\left(\frac{1-r}{1+r}\right)^{-\frac{i\omega}{2}}\ \Xi_n^B(r,\omega,\ell)J_D^j
\end{aligned}
\tag{3.85}
$$

The dissipative part of the on-shell action in terms of these STF multipoles can then be written as:

$$
\begin{aligned}
S_{RR}^{\text{Odd } d} = -\sum_\ell \int \frac{d\omega}{2\pi}\frac{1}{4\nu^2\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\frac{1}{\ell!}\Bigg[&K_E^{\text{Out}}\frac{\ell+d-2}{\ell}\,{}^E\mathcal{Q}^{*<i_1 i_2\ldots i_\ell>}_{D,STF}\,{}^E\mathcal{Q}^{A,STF}_{<i_1 i_2\ldots i_\ell>} \\
&+K_B^{\text{Out}}\ {}^B\mathcal{Q}^{*i<i_1 i_2\ldots i_\ell>}_{D,STF}\,{}^B\mathcal{Q}^{A,STF}_{i<i_1 i_2\ldots i_\ell>}\Bigg]\,.
\end{aligned}
\tag{3.86}
$$

Near Flat Expansion

In the next section, we will calculate the radiation reaction of a point particle moving along an arbitrary trajectory. This result is obtained in a Hubble expansion about flat spacetime up to order H^4 terms. To facilitate this calculation, we quote the Hubble expansions of some useful quantities in this section[5]. Since the radiation reaction force is well defined only in dS_{d+1} for odd d, we will restrict our analysis to that particular case.

The radiation reaction kernels $K^{\text{Out}}_{E/B}$ have the following expansions in the small H limit:

$$
\begin{aligned}
K_B^{\text{Out}}|_{\text{Odd } d} = \frac{2\pi i}{\Gamma(\nu)^2}\left(\frac{\omega}{2}\right)^{2\nu}\Bigg[&1+\left\{\nu^2+\frac{3}{4}d(d-4)+2\right\}\frac{\nu}{3!!}\frac{H^2}{\omega^2} \\
&+c_B\frac{\nu(\nu-1)}{5!!}\frac{H^4}{\omega^4}+O\left(\frac{H^6}{\omega^6}\right)\Bigg]\,.
\end{aligned}
\tag{3.87}
$$

$$
\begin{aligned}
K_E^{\text{Out}}|_{\text{Odd } d} = \frac{2\pi i}{\Gamma(\nu)^2}\left(\frac{\omega}{2}\right)^{2\nu}\Bigg[&1+\left\{\nu^2+\frac{3}{4}d(d-8)+11\right\}\frac{\nu}{3!!}\frac{H^2}{\omega^2} \\
&+c_E\frac{\nu(\nu-1)}{5!!}\frac{H^4}{\omega^4}+O\left(\frac{H^6}{\omega^6}\right)\Bigg]\,.
\end{aligned}
\tag{3.88}
$$

[5] The expressions quoted here are special cases of those derived in 2.3.1. There, one can find a detailed derivation of these formulae.

Here, we have defined the coefficients

$$\begin{aligned}
c_B &\equiv \frac{1}{2\times 3}\Bigg[5\nu^4-4\nu^3+\left\{\frac{15d(d-4)+32}{2}\right\}\nu^2 \\
&\qquad -\{15d(d-4)+44\}\,\nu+\frac{45}{16}d^2(d-4)^2-24\Bigg]\ , \\
c_E &\equiv \frac{1}{2\times 3}\Bigg[5\nu^4-4\nu^3+\left\{\frac{15d(d-8)+212}{2}\right\}\nu^2 \\
&\qquad -\{15d(d-8)+224\}\,\nu+\frac{45}{16}d(d-8)\{d(d-8)+24\}+381\Bigg]\ .
\end{aligned} \tag{3.89}$$

These expressions can be obtained using the Stirling approximation. In the flat limit, the combination appearing in the influence phase evaluates to

$$\frac{K^{\rm Out}|_{\text{Odd } d}}{4\nu^2\mathcal{N}_{d,\ell}}=\frac{\omega^{2\ell+d-2}}{(d-2)!!(2\ell+d-2)!!}\ , \tag{3.90}$$

so that the above expressions can be used to give an explicit expression for the influence phase in a small H expansion.

The smearing functions for the multipole moments $\Xi_n^{E/B}$ can also be expanded about small H in the following manner:

$$\Xi_n^{E/B}=\sum_{k=0}^{\infty}\mathfrak{p}_k^{E/B}(\nu,H^2,\omega^2)\mathfrak{B}_k^{E/B}\ , \tag{3.91}$$

where,

$$\begin{aligned}
\mathfrak{B}_k^E &\equiv \frac{r^{\nu+\frac{d}{2}-1+2k}}{2\nu(\nu+1)\dots(\nu+k)}\ {}_0F_1\left[1+k+\nu,-\frac{\omega^2r^2}{4}\right]=\frac{\Gamma(\nu)\ r^{\frac{d-2}{2}+k}}{2(\omega/2)^{k+\nu}}J_{k+\nu}(\omega r)\ , \\
\mathfrak{B}_k^B &\equiv \frac{r^{\nu-\frac{d}{2}+2+2k}}{2\nu(\nu+1)\dots(\nu+k)}\ {}_0F_1\left[1+k+\nu,-\frac{\omega^2r^2}{4}\right]=\frac{\Gamma(\nu)\ r^{2-\frac{d}{2}+k}}{2(\omega/2)^{k+\nu}}J_{k+\nu}(\omega r)\ ,
\end{aligned} \tag{3.92}$$

and

$$\begin{aligned}
\mathfrak{p}_k^{E/B} \equiv \frac{H^{2k}}{k!}\sum_{m=0}^{k}(-)^m\binom{k}{m}\sum_{n=0}^{m}(-)^n\binom{m}{n}\sigma^{2k-2m}\frac{\Gamma(\alpha_{E/B}+m)\Gamma(1+\nu+m)}{\Gamma(\alpha_{E/B}+m-n)\Gamma(1+\nu+m-n)} \\
\times\frac{\Gamma(\alpha_{E/B}+i\sigma+m-n)\Gamma(\alpha_{E/B}-i\sigma+m-n)}{\Gamma(\alpha_{E/B}+i\sigma)\Gamma(\alpha_{E/B}-i\sigma)}\ .
\end{aligned} \tag{3.93}$$

The electric vs magnetic parity smearing function only differs in the α parameter in the above formula:

$$\alpha_E \equiv \frac{1}{2}(3 - \frac{d}{2} + \nu) \ , \quad \alpha_B \equiv \frac{1}{2}(2 - \frac{d}{2} + \nu) \ , \quad \sigma = \frac{\omega}{2H} \ . \tag{3.94}$$

This expansion was derived for the generic case of the designer scalar in chapter §2, which we have used for the specific cases of $\{\mathcal{N} = 3 - d, \mu = \frac{d}{2} - 2\}$ for the electric and $\{\mathcal{N} = d - 3, \mu = \frac{d}{2} - 1\}$ for the magnetic smearing function.

3.2.2 Non-relativistic expansion

We will now derive the Abraham-Lorentz-Dirac force in arbitrary dimensions and find curvature corrections in dS spacetime. We will also find terms, up to cubic order in amplitude, contributing to the full radiation reaction(RR) force. We will follow the technique used in the previous chapter with some crucial differences for a charged particle interacting with electromagnetic fields.

Let us start with a point source travelling along a worldline $x(\tau)$. We will evaluate the RR force in a non-relativistic approximation. We will also take the particle to move close to the south pole, i.e. $rH \ll 1$. The wavelength of the radiation is much larger than the 'amplitude' of the trajectory about the south pole($\omega r \ll 1$) but much smaller than the Hubble constant ($\omega \gg H$). In [101], we referred to these approximations as the post-newtonian(PN) approximations adapted to dS.

The 4-current density associated with a charged particle in dS is given by,

$$\bar{J}^\mu(x') = \int \frac{dx^\mu}{d\tau}\delta^{d+1}(x(\tau) - x')d\tau = \frac{dx^\mu}{dt}\delta(\vec{x} - \vec{x}') \tag{3.95}$$

In the doubled dS-SK geometry, this source will also be doubled, i.e. given by two worldlines $x_L(\tau)$ and $x_R(\tau)$. Correspondingly, they will source the electromagnetic fields by current densities $\bar{J}_L$ and $\bar{J}_R$. The particle degrees of freedom, on which the effective action of radiation reaction is defined, are the positions of the two particles on either side of the geometry, as well as their time derivatives i.e. $\{x_L, x_R, \dot{x}_L, \dot{x}_R, \ddot{x}_L, \ddot{x}_R, \dots\}$. In the RR lagrangian, we will retain only up to quartic terms in the x's. The RR force is determined by the terms linear in the difference of their positions as well as time derivatives, i.e. $\{x_D, \dot{x}_D, \ddot{x}_D, \dots\}$. The terms cubic in x_D give rise to non-thermal fluctuations, which will be discussed later. The fact that there are only linear and cubic x_D terms in the lagrangian follows from the fact that the action is odd under $R - L$ exchange.

Such an amplitude expansion of the lagrangian allows us to evaluate the forces and fluctuations in a straightforward way by Taylor expanding the lagrangian about the point where x_D and its time derivatives are zero. As an illustration, consider a given function $\mathfrak{f}$ of position. Its corresponding average and difference are obtained as:

$$\frac{1}{2}\left[\mathfrak{f}\left(x_A + \frac{x_D}{2}\right) + \mathfrak{f}\left(x_A - \frac{x_D}{2}\right)\right] = \mathfrak{f}(x_A) + \frac{x_D^2}{4}\frac{\partial^2 \mathfrak{f}}{\partial x_A^2} + O(x_D^4) \tag{3.96}$$

$$\mathfrak{f}\left(x_A + \frac{x_D}{2}\right) - \mathfrak{f}\left(x_A - \frac{x_D}{2}\right) = x_D\frac{\partial \mathfrak{f}}{\partial x_A} + \frac{x_D^3}{24}\frac{\partial^3 \mathfrak{f}}{\partial x_A^3} + O(x_D^4) \tag{3.97}$$

In general, we will need to expand functions which are not just functions of positions but also depend on the time derivatives of the positions, in which case one uses a multi-variable Madhava-Taylor expansion.

The electric sector on-shell action gives the following RR lagrangian:

$$\begin{aligned}
&|\mathbb{S}^{d-1}|(d-2)!! \times (-1)^{\frac{d+1}{2}} L_E \\
=&(d-1)[x_i]_D \mathbb{D}_1 [x^i]_A - \frac{d}{4}\left[x_i x_j - \frac{x^2}{d}\delta_{ij}\right]_D \mathbb{D}_2 \left[x^i x^j - \frac{x^2}{d}\delta_{ij}\right]_A \\
&+ \left\{\frac{1}{2}(x_i)_D \mathbb{D}_1^X [x^i x^2]_A + \frac{1}{2}(x^2 x_i)_D \mathbb{D}_1^X [x^i]_A\right\} \\
&- \left\{(x_i)_D \mathbb{D}_1^V \partial_t [(\vec{x}\cdot\vec{v})x^i]_A + ((\vec{x}\cdot\vec{v})x^i)_D \mathbb{D}_1^V \partial_t [x^i]_A \right\} .
\end{aligned} \tag{3.98}$$

Only the dipole and quadrupole terms contribute to this order. The differential operators used in the above expressions $\{\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_1^X, \mathbb{D}_1^V\}$ are given explicitly in table 3.2. The number on the differential operator indicates which multipole contributes to that particular term, whereas the subscripts signify the structure on which this operator acts. Similar to the scalar case, the magnetic sector action gives the following RR lagrangian:

$$|\mathbb{S}^{d-1}|(d-2)!! \times (-1)^{\frac{d+1}{2}} L_B = \frac{1}{4}(x_i v_j - x_j v_i)_D \mathbb{D}_{\mathbb{V}} [x^i v^j - x^j v^i]_A . \tag{3.99}$$

In this case, only the dipole contributes to the quartic lagrangian, and we only have one differential operator. The full lagrangian is just a sum of these two contributions.

Given the lagrangian, we can use integration by parts to rewrite it in a way that one can read off the RR force:

$$L = \frac{(-1)^{\frac{d-1}{2}}}{|\mathbb{S}^{d-1}|(d-2)!!}\left[f_i(x_A)x_D^i + \frac{1}{4}N_i(x_D)x_A^i\right] \ . \tag{3.100}$$

Here, f^i are the Euler-Lagrange derivatives of the terms linear in x_D with respect to x_D^i. Similarly, N^i are the Euler-Lagrange derivatives of the terms linear in x_A with respect to x_A^i. The f^i's can be written as:

$$\begin{aligned} f^i =& -(d-1)\mathbb{D}_1[x^i] + \frac{d}{2}x_j\mathbb{D}_2[x^ix^j] - \frac{x^i}{2}\mathbb{D}_2[x^2] \\ &- \left\{\frac{1}{2}\mathbb{D}_1^X[x^ix^2] + \frac{1}{2}x^2\mathbb{D}_1^X[x^i] + x^ix^j\mathbb{D}_1^X[x_j]\right\} \\ &+ \left\{\mathbb{D}_1^V\partial_t\left(x^i(x_jv^j)\right) + (x_jv^j)\mathbb{D}_1^V[v^i] + (x^jv^i)\mathbb{D}_1^V[v_j] - \partial_t\left(x^ix^j\mathbb{D}_1^V[v_j]\ \right)\right\} \\ &+ v_j\mathbb{D}_1^{\mathbb{V}}[x^jv^i - x^iv^j] + \frac{1}{2}x_j\mathbb{D}_1^{\mathbb{V}}[x^ja^i - x^ia^j] \ , \end{aligned} \tag{3.101}$$

Symbol	f_d^μ
$\mathbb{D}_1$	$\frac{\partial_t^d}{d!!} - \frac{H^2}{3!}(d^2-6d+11)\frac{\partial_t^{d-2}}{(d-2)!!} + \frac{H^4}{5!}\frac{(d-1)(d-3)}{3}(5d^2-48d+127)\frac{\partial_t^{d-4}}{(d-4)!!}$
$\mathbb{D}_2$	$\frac{\partial_t^{d+2}}{(d+2)!!} - \frac{H^2}{3!}(d^2-5d+12)\frac{\partial_t^d}{d!!} + \frac{H^4}{5!}\frac{(d-1)(d-2)}{3}(5d^2-43d+132)\frac{\partial_t^{d-2}}{(d-2)!!}$
$\mathbb{D}_1^X$	$(d+1)\frac{\partial_t^{d+2}}{(d+2)!!} - \frac{H^2}{3!}(d-3)(d^2-4d+1)\frac{\partial_t^d}{d!!} + \frac{H^4}{5!}\frac{(d-1)}{3}(5d^4-78d^3+420d^2-946d+711)\frac{\partial_t^{d-2}}{(d-2)!!}$
$\mathbb{D}_1^V$	$\frac{\partial_t^d}{d!!} - \frac{H^2}{3!}(d^2-6d+11)\frac{\partial_t^{d-2}}{(d-2)!!} + \frac{H^4}{5!}\frac{(d-1)(d-3)}{3}(5d^2-48d+127)\frac{\partial_t^{d-4}}{(d-4)!!}$
$\mathbb{D}_1^{\mathbb{V}}$	$\frac{\partial_t^d}{d!!} - \frac{H^2}{3!}(d-1)(d-2)\frac{\partial_t^{d-2}}{(d-2)!!} + \frac{H^4}{5!}\frac{(d-1)(d-3)(d-4)(5d+2)}{3}\frac{\partial_t^{d-4}}{(d-4)!!}$

Table 3.2: The differential operators that appear in de Sitter electromagnetic radiation reaction (for d odd).

3.2.3 dS covariantisation

The f^i's can be obtained from the following de Sitter covariant vectors:

$$
\begin{aligned}
f_3^\mu &\equiv \frac{P^{\mu\nu}}{3!!}\Big\{ -2a_\nu^{(1)}\Big\}, \\
f_5^\mu &\equiv \frac{P^{\mu\nu}}{5!!}\Big\{ -4a_\nu^{(3)} + 10(a\cdot a)a_\nu^{(1)} + 30(a\cdot a^{(1)})a_\nu\Big\} - H^2\frac{P^{\mu\nu}}{5!!}\Big\{16a_\nu^{(1)}\Big\}, \\
f_7^\mu &\equiv \frac{P^{\mu\nu}}{7!!}\Big\{ -6a_\nu^{(5)} + 42(a\cdot a)a_\nu^{(3)} + 210(a\cdot a^{(1)})a_\nu^{(2)} + 224(a\cdot a^{(2)})a_\nu^{(1)} \\
&\qquad + \frac{574}{3}(a^{(1)}\cdot a^{(1)})a_\nu^{(1)} + 126(a\cdot a^{(3)})a_\nu + 280(a^{(1)}\cdot a^{(2)})a_\nu + O(a^5)\Big\} \\
&\qquad + H^2\frac{P^{\mu\nu}}{7!!}\Big\{120a_\nu^{(3)} - 342(a\cdot a)a_\nu^{(1)} - 978(a\cdot a^{(1)})a_\nu\Big\} \\
&\qquad - H^4\frac{P^{\mu\nu}}{7!!}\Big\{384a_\nu^{(1)}\Big\}.
\end{aligned}
\tag{3.102}
$$

Here $v^\mu = \frac{dx^\mu}{d\tau}$ is the proper velocity of the particle computed using dS metric, $a^\mu \equiv \frac{D^2x^\mu}{D\tau^2}$ is its proper acceleration and $P^{\mu\nu} \equiv g^{\mu\nu} + v^\mu v^\nu$ is the transverse projector to the worldline. We use $a_\mu^{(k)} \equiv \frac{D^k a^\mu}{D\tau^k}$ to denote the proper-time derivatives of the acceleration. All the spacetime dot products are computed using the dS metric.

The problem of flat space electromagnetic radiation reaction in 3+1 dimensions has been discussed in textbooks of classical electrodynamics (see for reference [114, 115]). The corresponding RR force in higher dimensions has been treated in many works [85–87, 89–91]. In [86], the authors compute the electromagnetic radiation reaction action in arbitrary dimensions. The flat space limit of our action matches the one they obtained. We match their post-newtonian expansion of the RR force with ours for the cases $d = 3$ and $d = 5$ given in their paper. We also match the flat limit of our covariant expressions to those given in previous works. Our expressions match the flat space results from [86] and [87] for $d = 3$ and $d = 5$. Galakhov [88] gives covariant expressions up to $d = 7$, which match ours up to signs of certain terms. We disagree with the curved space results of [87] at the H^2 order and higher, even though we match the flat space result. The source of this disagreement is unclear due to the very different nature of our derivations.

Chapter 4

Summary and Discussion

In this thesis, we have proposed a de Sitter-Schwinger Keldysh(dS-SK) geometry formed by two copies of the static patch stitched together at their future horizons. We then showed how the influence phase of a dS observer could be obtained by evaluating the on-shell action on this geometry. Our proposal yields results that pass a variety of checks: first, from a broad structural point of view, it satisfies the constraints imposed on it from bulk unitarity (SK collapse) and the dS version of Kubo-Martin-Schwinger (KMS) conditions.

Another check is the flat space limit, where we showed that, for point-like sources, the dissipative part of the action correctly produces the flat space radiation reaction. This also allows us to calculate Hubble corrections to the radiation reaction in odd spatial dimensions, and show that they combine into generally covariant expressions on the dS background, which serves another non-trivial check on our computation. As a technical aside, we have also shown how we can counter-term the influence phase for localised sources with multipole moments by using a Dirac-Deitweiler-Whiting type decomposition of the dS Green functions.

Here, we have focused on the scalar and the electromagnetic self-force. But we hope that many of these ideas directly generalise to the gravitational case. In particular, we hope that the method of covariant counterterms introduced for the gauge field analysis can be extended to linearised gravitational perturbations. This might be a useful alternative to existing methods to regularise the self-field [84, 96, 116–118]. We intend to explore some of this in our upcoming work [119].

The more challenging analysis is considering gravitational non-linearities. We hope it will be possible to have a well-defined perturbation theory analogous to the multipolar-post-minkowskian (MPM) analysis in flat spacetime. A much simpler analysis is to consider scalar interactions in this setup, which might provide insights into the more complicated problem of gravity. Our analysis can readily be extended to interactions, following techniques invented in the AdS context [50, 54, 102, 120, 121], as we sketched in section §2.4. This aspect will be explored in detail elsewhere [105]. It would also be interesting to explore whether the familiar tools of conformal invariance, e.g., conformal block decomposition, can shed more light on the structure of radiation reaction at a non-linear level. On the face of it, the presence of the observer breaks the dS isometries to just rotations/time-translations around the observer's worldline. But the re-emergence of the full dS isometry in the effective action that we described above suggests that conformal techniques could be fruitfully exploited to understand the structure of Hubble corrections to the radiation reaction.

In this work, we have advocated a point of view that real-world cosmology is fruitfully framed in terms of a cosmological influence phase $S_{\rm CIP}$ for an observer's worldline. It is interesting to ask whether realistic FLRW cosmology from Λ-CDM and the CMB phenomenology can indeed be rewritten in these terms. To this end, it would be interesting to extend our analysis to time-varying cosmological spacetimes: perhaps, one should begin by extending our framework to simpler time-dependent extensions involving sudden/adiabatic approximations.

More broadly, we can enquire about the role played by radiation reaction in cosmology. Given the enhanced electromagnetic dissipation any given radiative source experiences due to the presence of the cosmological constant, one could ask if there are any astrophysical/cosmological phenomena where such a dissipation would be relevant. The time(length) scales associated with such phenomena would be of the order of billions of (light-)years. This may rule out many sub-galactic-cluster scales but would contribute to intergalactic and large-scale structure dynamics. Understanding the gravitational radiation reaction at the galactic/extra-galactic scales might be crucial to predicting the stochastic gravitational wave background [122–124].

Even though our analysis focuses on the dissipative terms obtained in the effective action for the extended source, the conservative effects often dominate the dynamics of astrophysically/cosmologically relevant phenomena; e.g., the presence of a cosmological constant plays an important role in the relative dynamics of the local group with respect to the Virgo cluster [125]. This motivates the study of the orbital dynamics of two interacting bodies in dS, which can be solved using the conservative pieces in our effective action for extended sources. The actual problem of galactic dynamics requires the gravitational effective action, but the scalar/electromagnetic counterpart serves as a simpler toy model to understand the binary problem. We hope to explore this avenue in upcoming work[1].

Real world cosmology is tied to the standard model(SM) of particle physics through several observationally relevant questions: CMB, nucleosynthesis, neutrino masses, baryogenesis, etc. Indeed, any proposed holographic dual needs to be rich enough to encode the SM fields in appropriate physical regimes. At a formal level, this leads one to think about how the solipsistic observer interacts with such fields and if a similar langevin description emerges when these fields are integrated out. At the quadratic order, one would expect the non-abelian gauge theory to have similar features to the electromagnetic influence phase we have obtained in this thesis. On the other hand, one expects the fermionic fields to have qualitative differences: the Hawking fluctuations will be subject to Pauli exclusion and would follow a Fermi-Dirac distribution (see [127, 128] for the corresponding analyses of fermions in the context of AdS blackholes). But such expectations need to be verified through proper calculations.

Much of what we say about dS radiation reaction can readily be adapted to the AdS case, with a change of signs. This statement is expected to be true at short times, where the cosmological constant can be treated perturbatively, and its sign does not result in any qualitatively new features. Thus, at short times, we expect generally covariant expressions for the radiation reaction felt by an AdS observer, very similar to the ones we derive in this work. However, we expect qualitative differences at long time scales due to reflection at AdS asymptotia, resulting in long-time tails in radiation reaction. Further, we do not expect an analogue of dS Hawking radiation in AdS. It might be worthwhile to make these intuitions more precise and understand the dual CFT interpretation of these statements. This would be a good test of the existing proposals describing bulk observers within AdS/CFT [29–31].

We began this thesis by motivating our work in the context of solipsistic holography. We see the results here as a first step towards constructing an open system whose details

[1]A similar question has been addressed in the case of pure AdS in the newtonian limit [126] where it can have interesting reflections upon the dual CFT.

can be compared against proposed dual quantum mechanical models. Following the examples like BFSS matrix model [9][2], it is natural to expect some sort of a large N matrix quantum mechanics to give rise to the same influence phase as what we derive here.[3] To check this, it would be good to construct a formalism for computing the influence phase of slow macroscopic observables in a large N matrix model: our computations suggest that a clean separation of slow/fast modes is possible at least when there is a dual gravity description. These slow observables describing the dS observer should not be entirely gauge-invariant but rather have the structure of partially gauge-fixed probes [141–143]. Whether this is so is yet to be seen.

[2] See [10, 11] for a review and [129, 130] for matrix model proposals in dS.

[3] An especially interesting avenue is to cleverly use known AdS/CFT to derive dS duals: this can be done either by embedding a dS bubble within AdS [131–135] or by $T\bar{T}$-like deformation of the dual CFT [136–140]. It would be interesting to see how the radiation reaction viewpoint we advocate here fits within such proposals. We thank the referee for bringing some of these works to our attention.

Appendix A

STF tensors and multipole expansion

We will begin by reviewing the notion of symmetric trace-free tensors, which are the appropriate tools to discuss multipole expansion. The $d = 3$ version of this story is discussed in a variety of places.[1] The generalisation to arbitrary dimensions is straightforward, if somewhat involved. In the course of this work, we had to use a variety of identities involving STF tensors in arbitrary dimensions scattered across these references. The goal of this section is to review this theory for the reader's benefit.

We will begin with a more traditional account of electrostatic multipole expansion in $\mathbb{R}^d$ via orthonormal spherical harmonics on $\mathbb{S}^{d-1}$. This is the generalisation of familiar multipole expansion in $d = 3$, and we will use it to set the stage for a more modern account of multipole expansion using symmetric, trace-free (STF) tensors in the later subsections. We conclude this appendix with a discussion of radiation reaction in flat spacetime using these tools.

A.1 Orthonormal Spherical harmonics on $\mathbb{S}^{d-1}$

Let us begin by considering the problem of electrostatics in $\mathbb{R}^d$. Our goal in this subsection would be to describe the multipole expansion in this case, given an orthonormal basis of spherical harmonics on $\mathbb{S}^{d-1}$. Later in this subsection, we will give an explicit construction of such an orthonormal basis, which can, in principle, be used in explicit computations.

Given a charge distribution $\rho(\vec{r})$, the electric potential produced by such a distribution

[1]See [144] for a textbook discussion. We will refer the reader to [85, 86, 145–150] for a discussion of STF tensors in general dimensions.

is given in terms of the Newton-Coulomb integral

$$\phi(\vec{r}) = \int d^d r_0 \frac{\rho(\vec{r}_0)}{(d-2)|\mathbb{S}^{d-1}||\vec{r}-\vec{r}_0|^{d-2}} \ . \tag{A.1}$$

Here, we have denoted the volume of a unit sphere $\mathbb{S}^{d-1}$ via

$$|\mathbb{S}^{d-1}| \equiv \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \ , \tag{A.2}$$

and have fixed our normalisations such that the Poisson equation takes the form $\nabla^2\phi = -\rho$. While the above integral is indeed correct, a more useful answer is obtained by performing a multipole expansion of the Newton-Coulomb potential in terms of Legendre polynomials. In $d = 3$, this is a well-known statement from undergraduate physics courses, and we will now describe a quick way to generalise this statement to arbitrary dimensions.

To this end, consider a simple problem where the answer due to multipole expansion is straightforward: we imagine a spherical shell of radius R in $\mathbb{R}^d$ carrying a surface charge density $\sigma_{\ell\vec{m}}(\hat{r})$ proportional to a spherical harmonic $\mathscr{Y}_{\ell\vec{m}}(\hat{r})$, i.e., a spherical harmonic which under the sphere laplacian has an eigenvalue $-\ell(\ell+d-2)$ and we use $\vec{m}$ to denote the additional labels required to furnish an orthonormal basis within this eigenspace. The above eigenvalue follows from demanding that $r^\ell \mathscr{Y}_{\ell\vec{m}}(\hat{r})$ be a harmonic function annihilated by the Laplacian operator

$$\nabla^2_{\mathbb{R}^d} \equiv \frac{1}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\nabla^2_{\mathbb{S}^{d-1}} \ . \tag{A.3}$$

By symmetry, the potential due to such a problem should also be proportional to the same spherical harmonic as the charge distribution. The potential should be a harmonic function for $r \neq R$, regular at the origin, vanishing at infinity, be continuous at $r = R$, but have a derivative discontinuity at the shell equal to the charge density. These requirements uniquely determine the solution to be

$$\frac{R\sigma_{\ell\vec{m}}(\hat{r})}{(2\ell+d-2)}\left\{\Theta(r<R)\frac{r^\ell}{R^\ell} + \Theta(r>R)\frac{R^{\ell+d-2}}{r^{\ell+d-2}}\right\} \ . \tag{A.4}$$

This answer can be generalised to an arbitrary charge distribution, once it is realised that any distribution can be built shell by shell and ℓ by ℓ. Using an orthonormal basis of spherical harmonics to do the projection to every ℓ, we can then write the potential for an arbitrary charge distribution as

$$\int d^d r_0 \ \rho(\vec{r}_0) \ \sum_{\ell\vec{m}} \frac{\mathscr{Y}_{\ell\vec{m}}(\hat{r})\mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)}{(2\ell+d-2)r_0^{d-2}} \left\{ \Theta(r<r_0)\frac{r^\ell}{r_0^\ell} + \Theta(r>R)\frac{r_0^{\ell+d-2}}{r^{\ell+d-2}} \right\} . \tag{A.5}$$

Comparing this against the Newton-Coulomb integral, we obtain the multipole expansion formula in $\mathbb{R}^d$:

$$\begin{aligned} &\frac{1}{(d-2)|\mathbb{S}^{d-1}||\vec{r}-\vec{r}_0|^{d-2}} \\ &\quad = \sum_{\ell\vec{m}} \frac{\mathscr{Y}_{\ell\vec{m}}(\hat{r})\mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)}{(2\ell+d-2)r_0^{d-2}} \left\{ \Theta(r<r_0)\frac{r^\ell}{r_0^\ell} + \Theta(r>R)\frac{r_0^{\ell+d-2}}{r^{\ell+d-2}} \right\} . \end{aligned} \tag{A.6}$$

If we define the spherical multipole moments of the charge distribution $\rho(\vec{r})$ by

$$q_{\ell\vec{m}} \equiv \frac{1}{2\ell+d-2} \int d^d r_0 \ \rho(\vec{r}_0) \ r_0^\ell \mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0) \ , \tag{A.7}$$

we can write the potential far outside the charge distribution as

$$\sum_{\ell\vec{m}} \frac{1}{r^{\ell+d-2}} \ q_{\ell\vec{m}} \mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ . \tag{A.8}$$

This is the basic content of multipole expansion in electrostatics. However, to actually compute these multipole moments for a give charge distribution $\rho(\vec{r})$, we will need the explicit form of the spherical harmonics $\mathscr{Y}_{\ell\vec{m}}(\hat{r})$ on $\mathbb{S}^{d-1}$: we will now proceed to address this in the rest the subsection.

The first step in constructing the spherical harmonics is to derive the most symmetric among them: the Legendre polynomials. We will do this by recasting the above expansion in terms of the Legendre polynomial. In the formula above, the sum over orthonormal spherical harmonics of a given ℓ can be performed through a higher-dimensional generalisation of the addition theorem for spherical harmonics, viz.,

$$\sum_{\vec{m}} \mathscr{Y}_{\ell\vec{m}}(\hat{r})\mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0) = \frac{N_{HH}(d,\ell)}{|\mathbb{S}^{d-1}|} P_\ell(d, \hat{r}\cdot\hat{r}_0) \ . \tag{A.9}$$

Here $N_{HH}(d,\ell)$ is the number of orthonormal spherical harmonics of degree ℓ, with the notation here inspired by the fact that it is also the number of linearly independent, homogeneous, harmonic polynomials (HHPs) of degree ℓ in $\mathbb{R}^d$. We will elaborate on this and get an explicit expression for $N_{HH}(d,\ell)$ below. For now, we move on to note that $P_\ell(d,x)$ is the generalisation of the Legendre polynomial to $\mathbb{R}^d$: it is the unique spherical harmonic invariant under $SO(d-1)$ rotations which keep two poles of $\mathbb{S}^{d-1}$ fixed and is normalised to unity at the north pole, i.e., $P_\ell(d, x=1) \equiv 1$.

With the above definitions, we can argue for the above addition theorem as follows: first of all, the sum over orthonormal spherical harmonics of a given ℓ should be a spherical harmonic which only depends on the relative orientation of $\hat{r}$ and $\hat{r}_0$ and hence, the above sum should be proportional to $P_\ell(d,\hat{r}\cdot\hat{r}_0)$. The constant of proportionality can then be fixed by setting $\hat{r}=\hat{r}_0$ and integrating over the sphere $\mathbb{S}^{d-1}$ using orthonormality.

As a corollary of the above addition theorem, we note the following formula for the inner product between Legendre harmonics of two different orientations:

$$\int_{\mathbb{S}^{d-1}} P_\ell(d,\hat{r}\cdot\hat{r}_0)P_{\ell'}(d,\hat{r}\cdot\hat{r}_0') = \delta_{\ell\ell'}\frac{|\mathbb{S}^{d-1}|}{N_{HH}(d,\ell)}P_\ell(d,\hat{r}_0\cdot\hat{r}_0')\ . \tag{A.10}$$

This statement follows directly by the use of addition theorem followed by the fact that $\mathscr{Y}_{\ell\vec{m}}(\hat{r})$ are assumed to be orthonormal.For $\hat{r}_0=\hat{r}_0'$, we get the Legendre orthogonality relation

$$\int_0^\pi d\vartheta\ \sin^{d-2}\vartheta\ P_\ell(d,\cos\vartheta)P_{\ell'}(d,\cos\vartheta) = \delta_{\ell\ell'}\frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^{d-2}|N_{HH}(d,\ell)}\ . \tag{A.11}$$

With the addition theorem, we can recast the multipole expansion in terms of the Legendre polynomial as[2]

$$\begin{aligned}&\frac{1}{(d-2)|\vec{r}-\vec{r}_0|^{d-2}}\\ &\quad=\sum_\ell \frac{N_{HH}(d,\ell)P_\ell(d,\hat{r}\cdot\hat{r}_0)}{(2\ell+d-2)r_0^{d-2}}\left\{\Theta(r<r_0)\frac{r^\ell}{r_0^\ell}+\Theta(r>R)\frac{r_0^{\ell+d-2}}{r^{\ell+d-2}}\right\}\ .\end{aligned} \tag{A.13}$$

[2]This expansion is often used to define Gegenbauer polynomials $C_\ell^\mu(z)$, which differ from the generalised Legendre polynomials introduced here merely by an overall normalisation. These polynomials are also proportional to the associated Legendre functions. The explicit relations are given by

$$C_\ell^{\frac{d}{2}-1}(z) \equiv (d-2)\frac{N_{HH}(d,\ell)}{2\ell+d-2}P_\ell(d,z)\ ,\quad P_\lambda^{-\mu}(z)\equiv\frac{\left(\sqrt{1-z^2}\right)^\mu}{2^\mu\mu!}P_{\lambda-\mu}(2\mu+3;z)\ . \tag{A.12}$$

As is well-known in $d = 3$ case, this series expansion can be used to derive an explicit expression for $P_\ell(d,x)$.

The steps involved are as follows: we take the case $r_0 < r$, set $t = \frac{r_0}{r} < 1$ and $x = \hat{r}\cdot\hat{r}_0$ to write

$$N_{HH}(d,\ell)P_\ell(d,x) = (2\ell+d-2) \times \text{Coefficient of } t^\ell \text{ in } \frac{1}{(d-2)(1-2xt+t^2)^{\frac{d}{2}-1}} \ . \tag{A.14}$$

To extract the t^ℓ coefficient, we use

$$\frac{(2\ell+d-2)}{(d-2)(1-2xt+t^2)^{\frac{d}{2}-1}} = \frac{\ell+\frac{d}{2}-1}{\Gamma(\frac{d}{2})}\int_0^\infty ds\ s^{\frac{d}{2}-2}\ e^{-s+2xst-st^2} \ , \tag{A.15}$$

expand the exponentials involving t and integrate to obtain

$$\begin{aligned} N_{HH}(d,\ell)P_\ell(d,x) &= \frac{\ell+\frac{d}{2}-1}{\Gamma(\frac{d}{2})}\sum_k \int_0^\infty ds\ s^{\frac{d}{2}-2}\ e^{-s}\frac{(2xs)^{\ell-2k}}{(\ell-2k)!}\frac{(-s)^k}{k!} \\ &= \frac{2^\ell\Gamma\left(\ell+\frac{d}{2}\right)}{\Gamma(\frac{d}{2})}\sum_k \frac{\Gamma\left(\ell+\frac{d}{2}-1-k\right)}{\Gamma(\ell+\frac{d}{2}-1)}\frac{(-)^k}{2^{2k}k!}\frac{x^{\ell-2k}}{(\ell-2k)!} \ . \end{aligned} \tag{A.16}$$

Here, the sum over k runs from $k = 0$ until the combination $\ell - 2k$ is non-negative. Defining the normalisation factor[3]

$$\mathcal{N}_{d,\ell} \equiv \frac{\Gamma(\frac{d}{2})}{2^\ell\Gamma\left(\ell+\frac{d}{2}\right)} \ , \qquad \nu \equiv \frac{d}{2}+\ell-1 \ , \tag{A.18}$$

we finally obtain an explicit expression for the generalised Legendre polynomial as

$$\mathcal{N}_{d,\ell}N_{HH}(d,\ell)P_\ell(d,x) = \sum_k \frac{\Gamma(\nu-k)}{2^{2k}k!\Gamma(\nu)}\frac{(-)^k x^{\ell-2k}}{(\ell-2k)!} \ . \tag{A.19}$$

Incidentally, the same expansion at $x = 1$ also gives the number of orthonormal spherical

[3]The interpretation of this ubiquitous normalisation factor will become clearer when we describe STF tensors in the next subsection. For now, we will note that $\mathcal{N}_{d,\ell}$ is an inverse integer which has the following alternate forms

$$\mathcal{N}_{d,\ell} \equiv \frac{|\mathbb{S}^{d+2\ell-1}|}{|\mathbb{S}^1|^\ell|\mathbb{S}^{d-1}|} = \frac{(d-2)!!}{(d+2\ell-2)!!} \ . \tag{A.17}$$

harmonics of degree ℓ as

$$\begin{aligned} N_{HH}(d,\ell) &= \frac{2\ell+d-2}{d-2} \times \text{Coefficient of } t^\ell \text{ in } \frac{1}{(1-t)^{d-2}} \\ &= \frac{2\ell+d-2}{d-2}\binom{\ell+d-3}{\ell} . \end{aligned} \tag{A.20}$$

Next, we will give a recursive construction of a complete orthonormal basis of spherical harmonics on $\mathbb{S}^{d-1}$, just using the Legendre polynomials constructed above. We begin with an explicit spherical coordinate system in $\mathbb{R}^d$ given by

$$\begin{aligned} x_1 &= r\ \sin\vartheta_{d-2}\ \sin\vartheta_{d-3}\ \ldots\ \sin\vartheta_2\ \sin\vartheta_1\ \cos\varphi\ , \\ x_2 &= r\ \sin\vartheta_{d-2}\ \sin\vartheta_{d-3}\ \ldots\ \sin\vartheta_2\ \sin\vartheta_1\ \sin\varphi\ , \\ x_3 &= r\ \sin\vartheta_{d-2}\ \sin\vartheta_{d-3}\ \ldots\ \sin\vartheta_2\ \cos\vartheta_1\ , \\ x_4 &= r\ \sin\vartheta_{d-2}\ \sin\vartheta_{d-3}\ \ldots\ \cos\vartheta_2\ , \\ &\ldots\ , \\ x_{d-2} &= r\ \sin\vartheta_{d-2}\ \sin\vartheta_{d-3}\ \cos\vartheta_{d-4}\ , \\ x_{d-1} &= r\ \sin\vartheta_{d-2}\ \cos\vartheta_{d-3}\ , \\ x_d &= r\ \cos\vartheta_{d-2}\ . \end{aligned} \tag{A.21}$$

Here the radius r varies from 0 to ∞ whereas the allowed values of angles is $\vartheta_i \in [0,\pi]$ and $\varphi \in [0,2\pi)$. In these coordinates, we can write the metric of $\mathbb{S}^{d-1}$ as

$$\begin{aligned} d\Omega_{d-1}^2 &= d\vartheta_{d-2}^2 + \sin^2\vartheta_{d-2} d\Omega_{d-2}^2 \\ &= d\vartheta_{d-2}^2 + \sin^2\vartheta_{d-2} d\vartheta_{d-3}^2 + \sin^2\vartheta_{d-2}\sin^2\vartheta_{d-3} d\vartheta_{d-4}^2 + \ldots \\ &\quad + \prod_{k=j+1}^{d-2} \sin^2\vartheta_k\ d\vartheta_j^2 + \ldots + \prod_{k=1}^{d-2}\sin^2\vartheta_k\ d\varphi^2. \end{aligned} \tag{A.22}$$

The volume form

$$\int_{\mathbb{S}^{d-1}} (\ldots) \equiv \int d\vartheta_1 \wedge d\vartheta_2 \ldots d\vartheta_{d-2} \wedge d\varphi\ \prod_{k=1}^{d-2} \sin^k\vartheta_k\ (\ldots)\ . \tag{A.23}$$

We are interested in constructing an orthonormal basis of spherical harmonics in these coordinates. As we described above, the simplest spherical harmonic is the Legendre harmonic $P_\ell(d, \cos\vartheta_{d-2})$ which depends only on ϑ_{d-2}. It obeys the second-order ODE

$$\left[\frac{1}{\sin^{d-2}\vartheta}\frac{d}{d\vartheta}\sin^{d-2}\vartheta\frac{d}{d\vartheta} + \ell(\ell+d-2)\right] P_\ell(d, \cos\vartheta) = 0 \ . \tag{A.24}$$

The function $P_\ell(d, \cos\vartheta)$ is the unique ℓ^{th} degree polynomial in $\cos\vartheta$ that solves the above ODE and is normalised to $P_\ell(d, \cos\vartheta = 1) = 1$. In general, spherical harmonics of degree ℓ obey the eigenvalue equation $\left[\nabla^2_{\mathbb{S}^{d-1}} + \ell(\ell+d-2)\right] \mathcal{S}_\ell(\Omega_{d-1}) = 0$, or in more detail

$$\left[\frac{1}{\sin^{d-2}\vartheta_{d-2}}\frac{\partial}{\partial\vartheta_{d-2}}\sin^{d-2}\vartheta_{d-2}\frac{\partial}{\partial\vartheta_{d-2}} + \frac{1}{\sin^2\vartheta_{d-2}}\nabla^2_{\mathbb{S}^{d-2}} + \ell(\ell+d-2)\right] \mathcal{S}_\ell(\Omega_{d-1}) = 0 \ . \tag{A.25}$$

This equation can be solved via a separation of variables ansatz

$$\mathcal{S}_\ell = (\sin\vartheta_{d-2})^m P_{\ell-m}(d+2m, \cos\vartheta_{d-2})\widehat{\mathcal{S}}_m(\Omega_{d-2}) \ , \tag{A.26}$$

for a non-negative integer $0 \leq m \leq \ell$. Substituting this ansatz into the equation above yields the eigenvalue equation $\left[\nabla^2_{\mathbb{S}^{d-2}} + m(m+d-3)\right] \widehat{\mathcal{S}}_m(\Omega_{d-2}) = 0$ in the lower dimensional sphere, i.e., the function $\widehat{\mathcal{S}}_m(\Omega_{d-2})$ is actually a spherical harmonic of degree m on $\mathbb{S}^{d-2}$. This gives rise to

$$\sum_{m=0}^{\ell} N_{HH}(d-1, m) = N_{HH}(d, \ell) \tag{A.27}$$

number of spherical harmonics of degree ℓ on $\mathbb{S}^{d-1}$ (to get the above equality, we have used Eq.(A.20)). Recursing this construction, we get a set of spherical harmonics of the form

$$\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \equiv \mathcal{C}_{\ell\vec{m}}\ e^{\pm i m_1 \varphi} \left[\prod_{k=1}^{d-2}(\sin\vartheta_k)^{m_k} P_{m_{k+1}-m_k}(k+2+2m_k, \cos\vartheta_k)\right]_{m_{d-1}=\ell} \ , \tag{A.28}$$

one for every non-decreasing sequence of non-negative integers

$$0 \leq m_1 \leq m_2 \ldots \leq m_{d-2} \leq m_{d-1} = \ell \ . \tag{A.29}$$

Here $\mathcal{C}_{\ell\vec{m}}$ is a normalisation constant which we shall determine below.

We will now argue that these spherical harmonics form an orthonormal set: any two harmonics with distinct $e^{i\varphi}$ factors are evidently orthogonal. Thus, we need to address only the case where $e^{i\varphi}$ factors are the same. Without loss of generality, let us assume that the dependences on ϑ_k for all $k < i$ are also the same between the two spherical harmonics for some $i < d-1$, and they differ first on their ϑ_i dependence, i.e., we consider two spherical harmonics in the above set with $m_k = m'_k$ for all $k \leq i$, but have $m_{i+1} \neq m'_{i+1}$. The inner product between these two spherical harmonics then has a factor

$$\int_0^\pi d\vartheta_i (\sin\vartheta_i)^{i+2m_i} P_{m_{i+1}-m_i}(i+2+2m_i, \cos\vartheta_i) P_{m'_{i+1}-m_i}(i+2+2m_i, \cos\vartheta_i) \ , \tag{A.30}$$

which then vanishes using Legendre orthogonality (see Eq.(A.11)) on $\mathbb{S}^{i+2m_i+1}$. The mutual orthogonality along with the counting in Eq.(A.27) proves then that we have indeed constructed a complete set of spherical harmonics of degree ℓ on $\mathbb{S}^{d-1}$.

To reiterate our construction so that its application to later construction of VSHs is clearer, the SSHs written in Eq.(A.28) are simultaneous eigenfunctions of the laplacian on lower spheres $\mathbb{S}^{d-1}, \mathbb{S}^{d-2}, \ldots, \mathbb{S}^1$ respectively. The lower spheres are obtained by successively dropping the angles $\vartheta_{d-2}, \vartheta_{d-3}, \ldots$. In fact, the set of m_i's in this construction are indeed related to the lower sphere laplacians, viz.,

$$-\mathscr{D}^2_{\mathbb{S}^{I+1}} \mathscr{Y}_{\ell\vec{m}}(\hat{r}) = -\gamma_{II} \sum_{J=0}^{I} \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial\theta_J} \left\{ \sqrt{\gamma}\ \gamma^{JJ} \frac{\partial}{\partial\theta_J} \mathscr{Y}_{\ell\vec{m}}(\hat{r}) \right\} = m_{I+1}(m_{I+1}+I)\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ . \tag{A.31}$$

We will conclude this discussion by normalising the spherical harmonics constructed above. The norm computation reduces to a product integral like the one above, which can then be evaluated using Eq.(A.11). Thus, the normalisation of the spherical harmonic given in Eq.(A.28) is given by

$$\begin{aligned} |\mathcal{C}_{\ell\vec{m}}|^{-2} &\equiv 2\pi \prod_{i=1}^{d-2} \int_0^\pi d\vartheta_i (\sin\vartheta_i)^{i+2m_i} P^2_{m_{i+1}-m_i}(i+2+2m_i, \cos\vartheta_i) \\ &= 2\pi \prod_{i=1}^{d-2} \frac{|\mathbb{S}^{i+2m_i+1}|}{|\mathbb{S}^{i+2m_i}| N_{HH}(i+2m_i+2, m_{i+1}-m_i)} \ , \end{aligned} \tag{A.32}$$

With this, we have a concrete realisation of the orthonormal spherical harmonics $\mathscr{Y}_{\ell\vec{m}}(\hat{r})$, using which multipole moments could be computed for a given charge distribution.

We will conclude this section with a comment on the counting of SSHs. We will can obtain a formula for $\mathcal{N}_{HH}(d,\ell)$ by an explicit counting of the number of $\vec{m}$ satisfying the condition given in Eq.(A.29), i.e.,

$$\mathcal{N}_{HH}(d,m_{d-1}) = \sum_{m_{d-2}=0}^{m_{d-1}} \sum_{m_{d-3}=0}^{m_{d-2}} \cdots \sum_{m_2=0}^{m_3} \left\{ 1 + \sum_{m_1=1}^{m_2} 2 \right\} . \tag{A.33}$$

As a check, setting $d=3$ yields the well-known result that there are $2m_2+1$ SSHs on $\mathbb{S}^2$ corresponding to the eigenvalue $-m_2(m_2+1)$. We can perform this sum as follows: first, we note that the above identity implies a recursion relation of the form

$$\mathcal{N}_{HH}(d,\ell) = \sum_{m=0}^{\ell} \mathcal{N}_{HH}(d-1,m) . \tag{A.34}$$

Using this relation, we can get the number of SSHs on $\mathbb{S}^{d-1}$ by starting from the count in $\mathbb{S}^2$ and then recursively summing the answer to obtain (A.20).

A.2 STF tensors in $\mathbb{R}^d$ and cartesian multipole moments

Till now, we have described the multipole expansion in terms of an orthonormal basis of spherical harmonics $\mathscr{Y}_{\ell\vec{m}}(\hat{r})$ and the corresponding spherical multipole moments $q_{\ell\vec{m}}$. We will now describe an alternate formalism based on a more symmetric, but over-complete basis of spherical harmonics made of Legendre polynomials about arbitrary directions (we will call this basis an STF basis). A general spherical harmonic in STF basis is naturally described by symmetric trace-free (STF) tensors with constant cartesian components.

For definiteness, we consider spherical harmonics of the form

$$\begin{aligned} \mathcal{N}_{d,\ell} N_{HH}(d,\ell) P_\ell(d,\hat{\kappa}\cdot\hat{r}) &= \sum_k \frac{\Gamma(\nu-k)}{2^{2k}k!\Gamma(\nu)} \frac{(-)^k(\hat{\kappa}\cdot\hat{r})^{\ell-2k}}{(\ell-2k)!} \\ &= \frac{1}{\ell!}\hat{\kappa}_{i_1}\hat{\kappa}_{i_2}\ldots\hat{\kappa}_{i_\ell}\hat{r}^{<i_1}\hat{r}^{i_2}\ldots\hat{r}^{i_\ell>} \\ &= \frac{1}{\ell!}\hat{\kappa}_{i_1}\hat{\kappa}_{i_2}\ldots\hat{\kappa}_{i_\ell}\hat{r}^{j_1}\hat{r}^{j_2}\ldots\hat{r}^{j_\ell}\Pi^{<i_1i_2\ldots i_\ell>}_{<j_1j_2\ldots j_\ell>} , \end{aligned} \tag{A.35}$$

where $\hat{\kappa}$ is an arbitrary unit vector, and in the last line we have written the spherical harmonic as a projected contraction of two tensors. The angular bracket here denotes the symmetric trace-free (STF) projection and Π is the STF-projector. An explicit expression that follows from the above definition is

$$\hat{r}^{<i_1}\hat{r}^{i_2}\dots\hat{r}^{i_\ell>} = \sum_k \frac{(-)^k\Gamma(\nu-k)}{2^k\Gamma(\nu)} \times \left\{\hat{r}^{i_1}\hat{r}^{i_2}\dots\hat{r}^{i_{\ell-2k}}\delta^{i_{\ell+1-2k}i_{\ell+2-2k}}\dots\delta^{i_{\ell-1}i_\ell} + \text{distinct index permutations}\right\}. \tag{A.36}$$

Here the sum within the curly braces sums over all index permutations of the set $\{i_1,\dots,i_\ell\}$ which give distinct answers. The number of such distinct permutations can be counted as follows: there are $\binom{\ell}{2k}$ ways of choosing the subset of indices that go into Kronecker deltas, and $\frac{(2k)!}{2^k k!} = (2k-1)!!$ distinct ways of pairing a given subset.[4] Thus, the total number of distinct permutations is $(2k-1)!!\binom{\ell}{2k} = \frac{\ell!}{2^k k!(\ell-2k)!}$. With this counting of distinct permutations, it is then easy to check that contracting $\hat{r}^{<i_1}\hat{r}^{i_2}\dots\hat{r}^{i_\ell>}$ with $\frac{1}{\ell!}\hat{\kappa}_{i_1}\hat{\kappa}_{i_2}\dots\hat{\kappa}_{i_\ell}$ does give $\mathcal{N}_{d,\ell}\mathcal{N}_{HH}(d,\ell)P_\ell(d,\hat{\kappa}\cdot\hat{r})$. The STF projector can also be given a closed-form expression as:

$$(\Pi^S_{d,\ell})^{<i_1 i_2\dots i_\ell>}_{<j_1 j_2\dots j_\ell>} = \sum_k \frac{(-)^k\Gamma(\nu-k)}{2^k k!(\ell-2k)!\Gamma(\nu)} \times \delta^{(i_1}_{(j_1}\delta^{i_2}_{j_2}\dots\delta^{i_{\ell-2k}}_{j_{\ell-2k}}\delta^{i_{\ell+1-2k}i_{\ell+2-2k}}\dots\delta^{i_{\ell-1}i_\ell)}\delta_{j_{\ell+1-2k}j_{\ell+2-2k}}\dots\delta_{j_{\ell-1}j_\ell)}, \tag{A.37}$$

where the $(i_1\dots i_\ell)$ denotes a symmetric projection. To elucidate the arguments above, we will now write down the explicit expressions of $\hat{r}^{<i_1}\hat{r}^{i_2}\dots\hat{r}^{i_\ell>}$ for $\ell \leq 5$. We have

$$\begin{aligned} &\hat{r}^{<i_1>} \equiv \hat{r}^{<i_1>}\ ,\ \hat{r}^{<i_1}\hat{r}^{i_2>} \equiv \hat{r}^{i_1}\hat{r}^{i_2} - \frac{1}{d}\delta^{i_1 i_2} \\ &\hat{r}^{<i_1}\hat{r}^{i_2}\hat{r}^{i_3>} \equiv \hat{r}^{i_1}\hat{r}^{i_2}\hat{r}^{i_3} - \frac{1}{d+2}\left(\hat{r}^{i_1}\delta^{i_2 i_3} + \hat{r}^{i_2}\delta^{i_1 i_3} + \hat{r}^{i_3}\delta^{i_1 i_2}\right), \end{aligned} \tag{A.38}$$

[4]The number of pairings can be counted as follows: the $(2k)!$ ways to permute the subset of indices on Kronecker deltas. Exchanging an index within a pair, as well as permuting the pair as a whole does not change the final resultant pairings, i.e., there is a $(\mathbb{Z}_2)^k \times \mathbb{S}_k$ automorphism group which acts freely and transitively on the equivalence class of permutations which result in a given pairing. We hence obtain the number of distinct pairings by dividing out the cardinality of the automorphism group.

for $\ell \leq 3$. For $\ell = 4$, we have

$$
\begin{aligned}
&\hat{r}^{<i_1}\hat{r}^{i_2}\hat{r}^{i_3}\hat{r}^{i_4>} \equiv \hat{r}^{i_1}\hat{r}^{i_2}\hat{r}^{i_3}\hat{r}^{i_4} \\
&\quad - \frac{1}{d+4}\left(\hat{r}^{i_1}\hat{r}^{i_2}\delta^{i_3i_4} + \hat{r}^{i_1}\hat{r}^{i_3}\delta^{i_2i_4} + \hat{r}^{i_1}\hat{r}^{i_4}\delta^{i_2i_3} + \hat{r}^{i_2}\hat{r}^{i_3}\delta^{i_1i_4} + \hat{r}^{i_2}\hat{r}^{i_4}\delta^{i_1i_3} + \hat{r}^{i_3}\hat{r}^{i_4}\delta^{i_1i_2}\right) \\
&\quad + \frac{1}{(d+4)(d+2)}\left(\delta^{i_1i_2}\delta^{i_3i_4} + \delta^{i_1i_3}\delta^{i_2i_4} + \delta^{i_1i_4}\delta^{i_2i_3}\right) \ ,
\end{aligned}
\tag{A.39}
$$

and for $\ell = 5$, we get

$$
\begin{aligned}
&\hat{r}^{<i_1}\hat{r}^{i_2}\hat{r}^{i_3}\hat{r}^{i_4}\hat{r}^{i_5>} \equiv \hat{r}^{i_1}\hat{r}^{i_2}\hat{r}^{i_3}\hat{r}^{i_4}\hat{r}^{i_5} \\
&\quad - \frac{1}{d+6}\Big(\hat{r}^{i_1}\hat{r}^{i_2}\hat{r}^{i_3}\delta^{i_4i_5} + \hat{r}^{i_1}\hat{r}^{i_2}\hat{r}^{i_4}\delta^{i_3i_5} + \hat{r}^{i_1}\hat{r}^{i_3}\hat{r}^{i_4}\delta^{i_2i_5} + \hat{r}^{i_2}\hat{r}^{i_3}\hat{r}^{i_4}\delta^{i_1i_5} + \hat{r}^{i_2}\hat{r}^{i_4}\hat{r}^{i_5}\delta^{i_1i_3} \\
&\quad +\hat{r}^{i_5}\hat{r}^{i_1}\hat{r}^{i_2}\delta^{i_3i_4} + \hat{r}^{i_5}\hat{r}^{i_1}\hat{r}^{i_3}\delta^{i_2i_4} + \hat{r}^{i_5}\hat{r}^{i_1}\hat{r}^{i_4}\delta^{i_2i_3} + \hat{r}^{i_5}\hat{r}^{i_2}\hat{r}^{i_3}\delta^{i_1i_4} + \hat{r}^{i_5}\hat{r}^{i_3}\hat{r}^{i_4}\delta^{i_1i_2}\Big) \\
&\quad + \frac{1}{(d+6)(d+4)}\Big(\hat{r}^{i_1}\delta^{i_2i_3}\delta^{i_4i_5} + \hat{r}^{i_1}\delta^{i_2i_4}\delta^{i_3i_5} + \hat{r}^{i_1}\delta^{i_2i_5}\delta^{i_3i_4} \\
&\quad + \hat{r}^{i_2}\delta^{i_1i_3}\delta^{i_4i_5} + \hat{r}^{i_2}\delta^{i_1i_4}\delta^{i_3i_5} + \hat{r}^{i_2}\delta^{i_1i_5}\delta^{i_3i_4} + \hat{r}^{i_3}\delta^{i_1i_2}\delta^{i_4i_5} + \hat{r}^{i_3}\delta^{i_1i_4}\delta^{i_2i_5} + \hat{r}^{i_3}\delta^{i_1i_5}\delta^{i_2i_4} \\
&\quad +\hat{r}^{i_4}\delta^{i_1i_2}\delta^{i_3i_5} + \hat{r}^{i_4}\delta^{i_1i_3}\delta^{i_2i_5} + \hat{r}^{i_4}\delta^{i_1i_5}\delta^{i_2i_3} + \hat{r}^{i_5}\delta^{i_1i_2}\delta^{i_3i_4} + \hat{r}^{i_5}\delta^{i_1i_3}\delta^{i_2i_4} + \hat{r}^{i_5}\delta^{i_1i_4}\delta^{i_2i_3}\Big) \ .
\end{aligned}
\tag{A.40}
$$

The reader can check that the expressions in the RHS are completely symmetric under permutations of indices, and vanish if we take a trace over any two indices. Further, our counting of distinct permutations can also be checked for every term written above.

A more succinct way to summarise the permutations/symmetrisations described above is to work instead with the homogeneous harmonic polynomials (HHPs) in cartesian coordinates

$$
\begin{aligned}
x^{<i_1}x^{i_2}\dots x^{i_\ell>} &\equiv r^\ell\ \hat{r}^{<i_1}\hat{r}^{i_2}\dots\hat{r}^{i_\ell>} \\
&= \left[\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\Gamma(\nu-k)}{k!\ \Gamma(\nu)}\left(\frac{r}{2}\right)^{2k}(-\nabla^2)^k\right]_{\nu=\frac{d}{2}+\ell-1} x^{i_1}x^{i_2}\dots x^{i_\ell}\ .
\end{aligned}
\tag{A.41}
$$

The relation to generalised Legendre polynomials then follows from

$$\begin{aligned}
\frac{1}{\ell!}\kappa_{i_1}\kappa_{i_2}\ldots\kappa_{i_\ell}x^{<i_1}x^{i_2}\ldots x^{i_\ell>} &\equiv \left[\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\Gamma(\nu-k)}{k!\,\Gamma(\nu)}\left(\frac{r}{2}\right)^{2k}(-\nabla^2)^k\right]_{\nu=\frac{d}{2}+\ell-1}\frac{(\vec{\kappa}\cdot\vec{r})^\ell}{\ell!} \\
&= \left[\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\Gamma(\nu-k)}{k!\,\Gamma(\nu)}\left(-\frac{\kappa^2r^2}{4}\right)^{k}\frac{(\vec{\kappa}\cdot\vec{r})^{\ell-2k}}{(\ell-2k)!}\right]_{\nu=\frac{d}{2}+\ell-1} \\
&= \mathcal{N}_{d,\ell}N_{HH}(d,\ell)(\kappa r)^\ell P_\ell\left(d,\hat{\kappa}\cdot\hat{r}\right)\ ,
\end{aligned} \tag{A.42}$$

where, in the last step, we have used Eq.(A.19). The STF basis for multipole expansion in flat spacetime is often introduced in terms of these cartesian HHPs (See e.g. [144]). In dS spacetime (and more generally in cosmology), the absence of global cartesian coordinates limits their scope. The STF basis for spherical harmonics is, however, a useful tool for multipole expansion in such spacetimes, since isotropy is still a true symmetry.

We will now describe how the STF basis relates to the description of spherical harmonics given before. For any given ℓ, we can form

$$N_H(d,\ell) \equiv \binom{\ell+d-1}{\ell} \tag{A.43}$$

number of STF harmonics of the form $\hat{r}^{<i_1}\hat{r}^{i_2}\ldots\hat{r}^{i_\ell>}$. The above binomial coefficient counts the number of ways d directions can be filled into ℓ indices. The combinatorics here is identical to the Bose counting problem familiar from elementary statistical mechanics, where one counts the ways in which d bosons could be filled into ℓ degenerate energy levels. All the STF harmonics are not, however, linearly independent; they obey $N_H(d,\ell-2)$ number of conditions of the form

$$\delta_{i_1i_2}\hat{r}^{<i_1}\hat{r}^{i_2}\ldots\hat{r}^{i_\ell>} = 0\ . \tag{A.44}$$

They hence span a vector space of spherical harmonics of dimension

$$N_H(d,\ell) - N_H(d,\ell-2) = N_{HH}(d,\ell)\ , \tag{A.45}$$

where the equality folows by using the explicit forms in Eqs.(A.20) and (A.43). This shows that the harmonics $\hat{r}^{<i_1}\hat{r}^{i_2}\ldots\hat{r}^{i_\ell>}$ indeed form an overcomplete basis of spherical harmonics of degree ℓ.

The completeness means the following: say we are given a spherical harmonic $\mathscr{Y}_\ell(\hat{r})$ of degree ℓ on $\mathbb{S}^{d-1}$. We can then define a symmetric trace-free (STF) tensor $\mathscr{Y}_{i_1 i_2 \ldots i_\ell}$ of rank ℓ in $\mathbb{R}^d$ such that

$$\mathscr{Y}_\ell(\hat{r}) = \frac{1}{\ell!}\mathscr{Y}_{i_1 i_2 \ldots i_\ell} \hat{r}^{<i_1}\hat{r}^{i_2}\ldots\hat{r}^{i_\ell>} \ . \tag{A.46}$$

The orthonormal basis of spherical harmonics constructed in the previous subsection then defines an orthonormal set of STF tensors

$$\mathscr{Y}_{\ell\vec{m}}(\hat{r}) = \frac{1}{\ell!}\mathscr{Y}^{(\ell\vec{m})}_{i_1 i_2 \ldots i_\ell} \hat{r}^{<i_1}\hat{r}^{i_2}\ldots\hat{r}^{i_\ell>} \ . \tag{A.47}$$

Further, the inner product on the space of STF tensors is induced from the standard inner product on the space of functions on $\mathbb{S}^{d-1}$. To get an explicit expression, consider the following integral

$$\begin{aligned}
&\int_{\hat{r}\in\mathbb{S}^{d-1}} \frac{1}{\ell!}\kappa_{i_1}\kappa_{i_2}\ldots\kappa_{i_\ell}\hat{r}^{<i_1}\hat{r}^{i_2}\ldots\hat{r}^{i_\ell>} \times \frac{1}{\ell!}\bar{\kappa}_{j_1}\bar{\kappa}_{j_2}\ldots\bar{\kappa}_{j_\ell}\hat{r}^{<j_1}\hat{r}^{j_2}\ldots\hat{r}^{j_\ell>} \\
&\qquad = [\mathcal{N}_{d,\ell} N_{HH}(d,\ell)]^2(\kappa\bar{\kappa})^\ell \int_{\hat{r}\in\mathbb{S}^{d-1}} P_\ell(d,\hat{\kappa}\cdot\hat{r})P_\ell(d,\hat{\bar{\kappa}}\cdot\hat{r}) \\
&\qquad = \mathcal{N}^2_{d,\ell} N_{HH}(d,\ell)|\mathbb{S}^{d-1}|(\kappa\bar{\kappa})^\ell P_\ell(d,\hat{\kappa}\cdot\hat{\bar{\kappa}}) \\
&\qquad = \mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|\ \frac{1}{\ell!}\kappa_{<i_1}\kappa_{i_2}\ldots\kappa_{i_\ell>}\bar{\kappa}^{<i_1}\bar{\kappa}^{i_2}\ldots\bar{\kappa}^{i_\ell>} \ .
\end{aligned} \tag{A.48}$$

For example, the STF tensors corresponding to the orthonormal spherical harmonics have an inner product given by

$$\frac{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}{\ell!}\mathscr{Y}^{*<i_1 i_2\ldots i_\ell>}_{(\ell\vec{m}')}\mathscr{Y}^{(\ell\vec{m})}_{<i_1 i_2\ldots i_\ell>} = \delta^{\vec{m}}_{\vec{m}'} \ . \tag{A.49}$$

We recognise $\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|$ here as the conversion factor between the STF tensor inner product and the standard functional inner product between the spherical harmonics. The same factor also appears in the statement of spherical harmonic addition theorem, stated in terms of STF tensors:

$$\frac{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}{\ell!}\sum_{\vec{m}}\mathscr{Y}^{*<i_1 i_2\ldots i_\ell>}_{(\ell\vec{m})}\mathscr{Y}^{(\ell\vec{m})}_{<j_1 j_2\ldots j_\ell>} = (\Pi^S_{d,\ell})^{<i_1 i_2\ldots i_\ell>}_{<j_1 j_2\ldots j_\ell>} \ . \tag{A.50}$$

This important relation can be proved in many ways: one way is to use Eq.(A.35) to convert the standard addition theorem into STF tensors. Another ab initio derivation is to first argue that LHS should be proportional to RHS for symmetry reasons and then fix the normalisation by using the orthonormality relation Eq.(A.49).

Given any two vectors $\vec{r}$ and $\vec{\kappa}$, we define the following projected contraction

$$\begin{aligned}\Pi^S_{d,\ell}(\vec{r}|\vec{\kappa}) \equiv \Pi^S_{d,\ell}(\vec{\kappa}|\vec{r}) &\equiv \frac{1}{\ell!}\kappa^{i_1}\dots\kappa^{i_\ell}(\Pi^S_{d,\ell})^{<j_1\dots j_\ell>}_{<i_1\dots i_\ell>}\, r_{j_1}\dots r_{j_\ell} \\ &= \mathcal{N}_{d,\ell}\mathcal{N}_{HH}(d,\ell)(\kappa r)^\ell P_\ell\left(d,\hat{\kappa}\cdot\hat{r}\right) \\ &= \left[\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\Gamma(\nu-k)}{k!\,\Gamma(\nu)}\left(-\frac{\kappa^2r^2}{4}\right)^k\frac{(\vec{\kappa}\cdot\vec{r})^{\ell-2k}}{(\ell-2k)!}\right]_{\nu=\frac{d}{2}+\ell-1}\,.\end{aligned} \tag{A.51}$$

This is an ℓ^{th} degree homogeneous polynomial in both $\vec{r}$ and $\vec{\kappa}$, and it is harmonic in both these variables, viz.,

$$\nabla^2\Pi^S_{d,\ell}(\vec{r}|\vec{r}_0) = \nabla^2_0\Pi^S_{d,\ell}(\vec{r}|\vec{r}_0) = 0\ . \tag{A.52}$$

It is, in fact, the unique polynomial which satisfies these properties up to an overall normalisation. The STF projector itself can then be obtained by differentiating this polynomial to strip off the x_i and κ_i factors.

Another representation of the STF projector, derived from the standard addition theorem for orthonormal SSHs, is

$$(\Pi^S_{d,\ell})^{<i_1i_2\dots i_\ell>}_{<j_1j_2\dots j_\ell>} = \frac{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}{\ell!}\sum_{\vec{m}}\mathscr{Y}^{*<i_1i_2\dots i_\ell>}_{\ell\vec{m}}\mathscr{Y}^{\ell\vec{m}}_{<j_1j_2\dots j_\ell>}\ , \tag{A.53}$$

where $\mathscr{Y}^{(\ell\vec{m})}_{<i_1i_2\dots i_\ell>}$ are the STF tensors which convert between the orthonormal basis and the STF basis. Equivalently, by contracting the STF indices with arbitrary vectors, we can write

$$\Pi^S_{d,\ell}(\hat{r}_0|\hat{r}) = \mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|\sum_{\vec{m}}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\mathscr{Y}^{\ell\vec{m}}(\hat{r})\ . \tag{A.54}$$

The above expression relates the STF projector to the standard inner product on SSHs: one gets an extra factor of $\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|$ relative to an orthonormal basis because of the overcompleteness of the STF basis. The same factor appears in the inner product computed in the STF basis:

$$\int_{\mathbb{S}^{d-1}} \left[\frac{1}{\ell!}\mathscr{Y}_{<i_1 i_2 \ldots i_\ell>} \hat{r}^{<i_1} \ldots \hat{r}^{i_\ell>}\right] \left[\frac{1}{\ell!}\overline{\mathscr{Y}}_{<j_1 j_2 \ldots j_\ell>} \hat{r}^{<j_1} \ldots \hat{r}^{j_\ell>}\right] = \frac{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}{\ell!} \mathscr{Y}^{<i_1 i_2 \ldots i_\ell>} \overline{\mathscr{Y}}_{<i_1 i_2 \ldots i_\ell>} \ . \tag{A.55}$$

This is, in fact, necessary for the sum in Eq.(A.53) to be a projector, i.e., for the idempotent property

$$(\Pi^S_{d,\ell})^{<i_1 i_2 \ldots i_\ell>}_{<k_1 k_2 \ldots k_\ell>} (\Pi^S_{d,\ell})^{<k_1 k_2 \ldots k_\ell>}_{<j_1 j_2 \ldots j_\ell>} = (\Pi^S_{d,\ell})^{<i_1 i_2 \ldots i_\ell>}_{<j_1 j_2 \ldots j_\ell>} \tag{A.56}$$

to hold. This concludes our brief overview of SSHs in the language of STF tensors. We will refer the reader to Appendix (A.2) of [101] for a more detailed exposition with explicit expressions and derivations. We will generalise these ideas to VSHs on $\mathbb{S}^{d-1}$ in appendix B.

A.3 Green functions in Minkowski spacetime

We will begin by briefly reviewing the Green functions of the wave operator (i.e., the massless scalar operator) in $\mathbb{R}^{d,1}$. This theory is standard, although the notations and normalisations for Green functions in $d \neq 2, 3$ are non-standard. Thus, this subsection mainly serves to establish our notation. We will state our results with an eye towards their generalisation to dS Green functions.

We begin with the unique spherically symmetric eigenfunction of the Laplacian in $\mathbb{R}^d$ with eigenvalue $-\omega^2$:

$$J_0(d, \omega r) \equiv {}_0F_1\left[\frac{d}{2}, -\frac{\omega^2 r^2}{4}\right] \ . \tag{A.57}$$

We can construct a whole tower of descendants from this eigenfunction by taking an STF derivative

$$\mathscr{Y}_\ell(-\vec{\nabla}) J_0(d, \omega r) \equiv \omega^\ell \ J_\ell(d, \omega r) \ \mathscr{Y}_\ell(\vec{n}) = \omega^{\nu - \frac{d}{2} + 1} \ J_\ell(d, \omega r) \ \mathscr{Y}_\ell(\vec{n}) \ , \tag{A.58}$$

where we have defined (we remind the reader that $\nu \equiv \ell + \frac{d}{2} - 1$)

$$J_\ell(d, \omega r) \equiv \Gamma\left(\frac{d}{2}\right)\left(\frac{\omega r}{2}\right)^{1-\frac{d}{2}} J_\nu(\omega r) \equiv \frac{\Gamma(d/2)}{\Gamma(1+\nu)}\left(\frac{\omega r}{2}\right)^{\nu-\frac{d}{2}+1} {}_0F_1\left[1+\nu, -\frac{\omega^2 r^2}{4}\right] . \tag{A.59}$$

The notation is motivated by the fact that the functions that appear here generalise the Bessel J functions in the $d = 2$ version of the above problem. We can also define the functions analogous to Neumann and Hankel functions. We will define the *Neumann Green function* via

$$\begin{aligned} N_\ell(d, \omega r) &\equiv -\frac{1}{4}\frac{Y_\nu(\omega r)}{(2\pi\omega r)^{\frac{d}{2}-1}} \\ &= \frac{\Gamma(\nu)}{(4\pi)^{d/2}}\left(\frac{\omega r}{2}\right)^{-\nu-\frac{d}{2}+1} \\ &\quad \times \left\{ {}_0F_1\left[1-\nu, -\frac{\omega^2 r^2}{4}\right] - \frac{\pi \cot \nu\pi}{\Gamma(\nu)\Gamma(1+\nu)}\left(\frac{\omega r}{2}\right)^{2\nu} {}_0F_1\left[1+\nu, -\frac{\omega^2 r^2}{4}\right] \right\} . \end{aligned} \tag{A.60}$$

In the above definition, for half-integer ν (i.e., for d odd), we can set $\cot \nu\pi = 0$, whereas for integer ν, the divergence in the $\cot \nu\pi$ cancels the divergence in the first term and this formula should be interpreted as a limit. Green functions are normalised such that

$$-(\vec{\nabla}^2 + \omega^2)[\omega^{\nu+\frac{d}{2}-1} N_\ell(d, \omega r) \mathscr{Y}_\ell(\vec{n})] = \mathscr{Y}_\ell(-\vec{\nabla})\delta^d(\vec{r}) . \tag{A.61}$$

Since the RHS here is a multipole source, the combination $\omega^{\nu+\frac{d}{2}-1} N_\ell(d, \omega r)\mathscr{Y}_\ell(\vec{n})$ should then be interpreted as the amplitude of standing wave sourced by such a multipole source. We term this a *standing wave* since it is an even function of frequency. In contrast, the outgoing/ingoing waves are denoted by $\omega^{\nu+\frac{d}{2}-1} H_\ell^\pm(d, \omega r)$ respectively. We will refer to them as *Hankel Green functions.* Given a spherical harmonic $\mathscr{Y}_\ell(\vec{n})$ of degree ℓ on $\mathbb{S}^{d-1}$, both these Green functions satisfy

$$-(\vec{\nabla}^2 + \omega^2)[\omega^{\nu+\frac{d}{2}-1} H_\ell^\pm(d, \omega r) \mathscr{Y}_\ell(\vec{n})] = \mathscr{Y}_\ell(-\vec{\nabla})\delta^d(\vec{r}) . \tag{A.62}$$

The outgoing/ingoing conditions are imposed by taking $H_\ell^\pm(d, \omega r)$ to be analytic in the upper/lower half plane of complex frequency, respectively. The notation here is again motivated by the fact that these functions generalise the Hankel functions in $d = 2$ (up to normalisations). Their explicit forms are given by

$$\begin{aligned}
H^{\pm}_{\ell}(d,\omega r) &\equiv \frac{\pm i}{4}\frac{H^{1,2}_{\nu}(\omega r)}{(2\pi\omega r)^{\frac{d}{2}-1}} \equiv N_\ell(d,\omega r) \pm \frac{i\pi}{\Gamma(d/2)(4\pi)^{d/2}} J_\ell(d,\omega r) \\
&= \frac{\Gamma(\nu)}{(4\pi)^{d/2}}\left(\frac{\omega r}{2}\right)^{-\nu-\frac{d}{2}+1}\left\{ {}_0F_1\left[1-\nu, -\frac{\omega^2 r^2}{4}\right]\right. \\
&\qquad \left. \pm (1 \pm i\cot\nu\pi)\frac{2\pi i}{\Gamma(\nu)^2}\frac{1}{2\nu}\left(\frac{\omega r}{2}\right)^{2\nu} {}_0F_1\left[1+\nu, -\frac{\omega^2 r^2}{4}\right]\right\} .
\end{aligned} \tag{A.63}$$

As in the case of Neumann Green functions, for half-integer ν (i.e., for d odd), we can set $\cot\nu\pi = 0$, whereas for integer ν the above expression is indeterminate and should be interpreted as a limit.

As in the case of Bessel J functions, these Green functions could also be obtained by STF-differentiating their corresponding primary eigenfunction at $\ell = 0$, viz.,

$$\begin{aligned}
\mathscr{Y}_\ell(-\vec{\nabla})N_0(d,\omega r) &= \omega^\ell\ N_\ell(d,\omega r)\ \mathscr{Y}_\ell(\vec{n}) = \omega^{\nu-\frac{d}{2}+1}\ N_\ell(d,\omega r)\ \mathscr{Y}_\ell(\vec{n})\ , \\
\mathscr{Y}_\ell(-\vec{\nabla})H^{\pm}_0(d,\omega r) &= \omega^\ell\ H^{\pm}_\ell(d,\omega r)\ \mathscr{Y}_\ell(\vec{n}) = \omega^{\nu-\frac{d}{2}+1}\ H^{\pm}_\ell(d,\omega r)\ \mathscr{Y}_\ell(\vec{n})\ .
\end{aligned} \tag{A.64}$$

A related statement is the *multipole-expansion* of these Green functions, which, in our normalisations, takes the following form:

$$\begin{aligned}
J_0(d,\omega|\vec{r}-\vec{r}_0|) &= \sum_{\ell m} |\mathbb{S}^{d-1}|\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\mathscr{Y}_{\ell\vec{m}}(\hat{r}_0)^* J_\ell(d,\omega r)J_\ell(d,\omega r_0)\ , \\
N_0(d,\omega|\vec{r}-\vec{r}_0|) &= \sum_{\ell m} |\mathbb{S}^{d-1}|\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\mathscr{Y}_{\ell\vec{m}}(\hat{r}_0)^* \\
&\quad \times \Big\{\Theta(r<r_0)J_\ell(d,\omega r)N_\ell(d,\omega r_0) + \Theta(r>r_0)J_\ell(d,\omega r_0)N_\ell(d,\omega r)\Big\}\ , \\
H^{\pm}_0(d,\omega|\vec{r}-\vec{r}_0|) &= \sum_{\ell\vec{m}} |\mathbb{S}^{d-1}|\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\mathscr{Y}_{\ell\vec{m}}(\hat{r}_0)^* \\
&\quad \times \Big\{\Theta(r<r_0)J_\ell(d,\omega r)H^{\pm}_\ell(d,\omega r_0) + \Theta(r>r_0)J_\ell(d,\omega r_0)H^{\pm}_\ell(d,\omega r)\Big\}\ .
\end{aligned} \tag{A.65}$$

Here, the set of functions $\mathscr{Y}_{\ell\vec{m}}(\hat{r})$ for different $\vec{m}$ denote an orthonormal basis of $\mathbb{S}^{d-1}$ spherical harmonics of degree ℓ. Further, in the equation above, the symbol

$$|\mathbb{S}^{d-1}| \equiv \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \tag{A.66}$$

denotes the volume of the unit sphere. The argument for the above expansion is well-known within the theory of Green functions: we first expand the LHS in terms of eigenfunctions and then fix the coefficients by demanding continuity and a unit jump in the radial derivative. The jump can be readily evaluated using the Wronskian formulae[5]

$$\mathcal{W}[N_\ell(d,z), J_\ell(d,z)] = \mathcal{W}[H_\ell^\pm(d,z), J_\ell(d,z)] = \frac{1}{|\mathbb{S}^{d-1}|z^{d-1}} \ . \tag{A.67}$$

We will be interested here in the multipole expansion of the retarded/outgoing Green function $\omega^{d-2}H_0^+(d,\omega|\vec{r}-\vec{r}_0|)$, which, using the relations quoted earlier, we can rewrite entirely in terms of ${}_0F_1$ functions:

$$\begin{aligned}
&\omega^{d-2}H_0^+(d,\omega|\vec{r}-\vec{r}_0|) = \omega^{d-2}H_0^+(d,\omega|\vec{r}_0-\vec{r}|) \\
&= \frac{i\pi}{2}\sum_{\ell\vec{m}} \frac{(rr_0)^{\nu-\frac{d}{2}+1}}{\Gamma(1+\nu)^2}\left(\frac{\omega}{2}\right)^{2\nu} \mathscr{Y}_{\ell\vec{m}}(\hat{r})\mathscr{Y}_{\ell\vec{m}}(\hat{r}_0)^* \ {}_0F_1\left[1+\nu,-\frac{\omega^2r^2}{4}\right] {}_0F_1\left[1+\nu,-\frac{\omega^2r_0^2}{4}\right] \\
&\quad + \sum_{\ell\vec{m}} \frac{1}{2\nu}\frac{r_<^{\nu-\frac{d}{2}+1}}{r_>^{\nu+\frac{d}{2}-1}} \mathscr{Y}_{\ell\vec{m}}(\hat{r})\mathscr{Y}_{\ell\vec{m}}(\hat{r}_0)^* \ {}_0F_1\left[1+\nu,-\frac{\omega^2r_<^2}{4}\right] \\
&\qquad \times\left\{ {}_0F_1\left[1-\nu,-\frac{\omega^2r_>^2}{4}\right] - \frac{\pi\cot\nu\pi}{\Gamma(\nu)\Gamma(1+\nu)}\left(\frac{\omega r_>}{2}\right)^{2\nu} {}_0F_1\left[1+\nu,-\frac{\omega^2r_>^2}{4}\right]\right\} .
\end{aligned} \tag{A.68}$$

Here we have used a commonly used notation in such expansions, viz.,

$$r_> \equiv \mathrm{Max}(r,r_0) \ , \quad r_< \equiv \mathrm{Min}(r,r_0) \ . \tag{A.69}$$

Further, we have also separated out the real and the imaginary parts of the radial functions.

Consider double integrals of the form

$$\int d^dr \ \rho_1(\vec{r},\omega)\int d^dr_0 \ \rho_2(\vec{r}_0,\omega) \ f(d,\omega|\vec{r}-\vec{r}_0|) \tag{A.70}$$

where f could be any one of the functions discussed above. Using the multipole expansion, such a double integral can be decomposed into an infinite sum of factorised integrals, one for every spherical harmonic. For the practical computation of radiation reaction, it is then convenient to convert the spherical harmonic sum into an STF expression using

[5]Our Wronskian convention is $\mathcal{W}[f(z),g(z)] \equiv f\partial_z g - g\partial_z f$.

eq.(A.50).

Let us illustrate the above remarks by computing the flat spacetime scalar radiation reaction, which, in a slightly different notation, is explained in detail in references [85,86]. Say we have an extended scalar source whose emissive part at frequency ω is the average source $\rho_A(\omega,\vec{r})$, and whose absorptive part is the difference source $\rho_D(\omega,\vec{r})$. Ignoring all fluctuation effects, the flat spacetime influence phase for this source, after integrating out the massless scalar field about the vacuum, can be written down as

$$S_{RR}^{\text{bare}} = \int \frac{d\omega}{2\pi} \int d^d r_0 \int d^d r \; [\rho_D(\vec{r}_0,\omega)]^* \; \rho_A(\vec{r},\omega) \; \omega^{d-2} H_0^+(d,\omega|\vec{r}_0-\vec{r}|) \; . \tag{A.71}$$

Here $\omega^{d-2}H_0^+(d,\omega|\vec{r}_0-\vec{r}|)$ is the outgoing Green function for the scalar field, and the superscript 'bare' indicates that this expression is divergent and has to be counter-termed before it makes sense. We do not give here a derivation of the above influence phase except for the heuristic that the above action describes the causal propagation of a free scalar about the Minkowski vacuum. The above influence phase is also natural if one applies the original Feynman-Vernon argument in [56] for harmonic oscillators to each Minkowski mode of the scalar field and sums the result. A more proper derivation should involve a careful discussion of the fall-offs near space-like, time-like, and null asymptotia. We do not attempt such a discussion here because, as we shall see later, dS-SK geometry naturally incorporates such boundary conditions. Our dS answer in an appropriate limit will reduce to the above result.

We substitute the multipole expansion Eq.(A.68) into the influence phase S_{RR}^{bare}. For simplicity, we will take the number of spatial dimensions (i.e., d) to be odd, so that $\nu \equiv \ell + \frac{d}{2} - 1$ is a half-integer (and $\cot \nu\pi = 0$). The above action then has two sets of terms: the first set of terms, odd under time reversal, are

$$\begin{aligned} &\sum_{\ell\vec{m}} \int \frac{d\omega}{2\pi} \frac{i\pi}{2} \left(\frac{\omega}{2}\right)^{2\nu} \frac{1}{\Gamma(1+\nu)^2} \\ &\times \int d^d r_0 \left\{ \rho_D(\vec{r}_0,\omega) r_0^{\nu-\frac{d}{2}+1} \mathscr{Y}_{\ell\vec{m}}(\hat{r}_0) \; {}_0F_1\left[1+\nu, -\frac{\omega^2 r_0^2}{4}\right] \right\}^* \\ &\qquad \times \int d^d r \; \left\{ \rho_A(\vec{r},\omega) r^{\nu-\frac{d}{2}+1} \mathscr{Y}_{\ell\vec{m}}(\hat{r}) \; {}_0F_1\left[1+\nu, -\frac{\omega^2 r^2}{4}\right] \right\} \; . \end{aligned} \tag{A.72}$$

The combinations appearing in the second and the third line are the Bessel-smeared radiative multipole moments[6] of the sources, i.e.,

$$\begin{aligned}\mathcal{J}_A(\omega,\ell,\vec{m}) &\equiv \frac{1}{2\nu}\int d^d r\ \rho_A(\vec{r},\omega)\ r^{\nu-\frac{d}{2}+1}\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ {}_0F_1\left[1+\nu,-\frac{\omega^2 r^2}{4}\right]\ ,\\ \mathcal{J}_D(\omega,\ell,\vec{m}) &\equiv \frac{1}{2\nu}\int d^d r\ \rho_D(\vec{r},\omega)\ r^{\nu-\frac{d}{2}+1}\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ {}_0F_1\left[1+\nu,-\frac{\omega^2 r^2}{4}\right]\ .\end{aligned} \tag{A.73}$$

We can then write the time reversal of odd terms in the form

$$\sum_{\ell\vec{m}}\int \frac{d\omega}{2\pi}\frac{2\pi i}{\Gamma(\nu)^2}\left(\frac{\omega}{2}\right)^{2\nu}\mathcal{J}_D^*(\omega,\ell,\vec{m})\mathcal{J}_A(\omega,\ell,\vec{m})\ . \tag{A.74}$$

Given that ${}_0F_1$ functions are completely regular when their first argument is positive (i.e., when $1+\nu=\ell+\frac{d}{2}>0$), we conclude that these multipole moments are finite, even for point-like sources. Hence, each term in Eq.(A.72) is finite. The reader should contrast this with the second set of terms, even under time reversal:

$$\begin{aligned}&\sum_{\ell\vec{m}}\int\frac{d\omega}{2\pi}\int d^d r\int d^d r_0\ [\rho_D(\vec{r},\omega)]^*\ \rho_A(\vec{r},\omega)\\ &\times\frac{1}{2\nu}\frac{r_<^{\nu-\frac{d}{2}+1}}{r_>^{\nu+\frac{d}{2}-1}}\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\mathscr{Y}_{\ell\vec{m}}(\hat{r}_0)^*\ {}_0F_1\left[1+\nu,-\frac{\omega^2 r_<^2}{4}\right]{}_0F_1\left[1-\nu,-\frac{\omega^2 r_>^2}{4}\right]\ .\end{aligned} \tag{A.75}$$

which are divergent due to the Green functions ${}_0F_1(1-\nu,\ldots)$. Fortunately, since these are all even under time reversal, one can counter-term away these terms. In other words, these terms in the influence phase serve to renormalise the non-dissipative terms already present in the action of the source.

Let us return to the terms in Eq.(A.72): they are odd in ω, and hence *cannot* be countertermed or absorbed into the non-dissipative action. We can simplify these remaining terms by substituting the STF definition of spherical harmonics (see Eq.(A.47)) and invoking the STF addition theorem in Eq.(A.50). We then get the radiation-reaction

[6]The reader should compare this definition against electrostatic multipole moments defined in Eq.(A.7), remembering $\nu\equiv\ell+\frac{d}{2}-1$.

influence phase as

$$
\begin{aligned}
S_{RR}^{\text{Odd } d} = \sum_{\ell} \int \frac{d\omega}{2\pi} \frac{i\pi}{2\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|} \left(\frac{\omega}{2}\right)^{2\nu} \frac{1}{\Gamma(1+\nu)^2} \frac{1}{\ell!} (\Pi^{S}_{d,\ell})^{<i_1 i_2 \ldots i_\ell>}_{<j_1 j_2 \ldots j_\ell>} \\
\times \int d^d r_0 \left\{ \rho_D(\vec{r}_0, \omega) x_0^{j_1} x_0^{j_2} \ldots x_0^{j_\ell} \; {}_0F_1 \left[1+\nu, -\frac{\omega^2 r_0^2}{4} \right] \right\}^* \\
\times \int d^d r \; \left\{ \rho_A(\vec{r}, \omega) x_{i_1} x_{i_2} \ldots x_{i_\ell} \; {}_0F_1 \left[1+\nu, -\frac{\omega^2 r^2}{4} \right] \right\} \; .
\end{aligned}
\tag{A.76}
$$

Here, we recognise the STF multipole moments of the sources' absorptive and emissive parts. We will find it convenient to define our STF multipole moments as

$$
\begin{aligned}
\mathcal{Q}^{i_1 \ldots i_\ell}_{A,STF}(\omega) \equiv \frac{1}{2\nu} \Pi^{<i_1 i_2 \ldots i_\ell>}_{<j_1 j_2 \ldots j_\ell>} \int d^d r \; \rho_A(\vec{r}, \omega) x^{j_1} x^{j_2} \ldots x^{j_\ell} \; {}_0F_1 \left[1+\nu, -\frac{\omega^2 r^2}{4} \right] \; , \\
\mathcal{Q}^{i_1 \ldots i_\ell}_{D,STF}(\omega) \equiv \frac{1}{2\nu} \Pi^{<i_1 i_2 \ldots i_\ell>}_{<j_1 j_2 \ldots j_\ell>} \int d^d r \; \rho_D(\vec{r}, \omega) x^{j_1} x^{j_2} \ldots x^{j_\ell} \; {}_0F_1 \left[1+\nu, -\frac{\omega^2 r^2}{4} \right] \; .
\end{aligned}
\tag{A.77}
$$

In terms of these STF multipole moments, the action for radiation reaction takes the form

$$
\begin{aligned}
S_{RR}^{\text{Odd } d} &= \sum_{\ell \vec{m}} \int \frac{d\omega}{2\pi} \frac{2\pi i}{\Gamma(\nu)^2} \left(\frac{\omega}{2}\right)^{2\nu} \mathcal{J}^*_D(\omega, \ell, \vec{m}) \mathcal{J}_A(\omega, \ell, \vec{m}) \\
&= \sum_{\ell} \int \frac{d\omega}{2\pi} \frac{2\pi i}{\Gamma(\nu)^2} \left(\frac{\omega}{2}\right)^{2\nu} \frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|} \frac{1}{\ell!} \mathcal{Q}^{*<i_1 i_2 \ldots i_\ell>}_{D,STF} \mathcal{Q}^{A,STF}_{<i_1 i_2 \ldots i_\ell>} \; .
\end{aligned}
\tag{A.78}
$$

In the first line, we have quoted the answer in terms of the spherical multipole moments for comparison. The multipole action above could also be derived entirely by using Cartesian STF harmonics from the very beginning (See [151] for a detailed derivation). Given the absence of Cartesian coordinates valid everywhere on the static patch, we will employ a judicious mix of spherical harmonic and STF harmonic expansions to compute the influence phase. The flat spacetime derivation we have given here closely mimics the strategy we will eventually use for dS.

Let us conclude this flat spacetime discussion by commenting on the case where d is even and $\nu \in \mathbb{Z}$. We will tackle this case by a dimensional regularisation via analytic continuation in ν. From our discussion of multipole expansion, it is clear that the time reversal of even terms in Eq.(A.75) is the same for any ν and can be counter-termed away similarly.

The terms in Eq.(A.72), on the other hand, get multiplied by a factor of $(1 + i\cot\pi\nu)$

for a general ν: this can be seen, e.g., in Eq.(A.68). The $\cot\pi\nu$ factor leads to novel divergences as ν approaches an integer, necessitating further counter-terms.

To compute the counter-terms as $\nu \to n \in \mathbb{Z}$, we need the following expansion:

$$(1 + i\cot\pi\nu)\frac{2\pi i}{\Gamma(\nu)^2}\left(\frac{\omega}{2H}\right)^{2\nu} = \frac{1}{\Gamma(n)^2}\left(\frac{\omega}{2H}\right)^{2n} \left\{\frac{2}{\nu - n} - 4\psi^{(0)}(n) + \ln\left(\frac{\omega}{2H}\right)^4 + O(\nu - n)\right\} . \tag{A.79}$$

Here H is the characteristic scale for dimensional regularisation and $\psi^{(0)}(x) \equiv \frac{d}{dx}\ln\Gamma(x)$ is the di-gamma function. Using a version of modified minimal subtraction, we counter-term away the first two terms inside the bracket of RHS. Thus, the influence phase due to radiation reaction for even spatial dimensions is

$$S_{RR}^{\text{Even } d} = \sum_\ell \int \frac{d\omega}{2\pi}\frac{1}{\Gamma(\nu)^2}\left(\frac{\omega}{2}\right)^{2\nu}\ln\left(\frac{\omega^4}{H^4}\right)\frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\frac{1}{\ell!}\mathcal{Q}^{*<i_1 i_2 \ldots i_\ell>}_{D,STF}\mathcal{Q}^{A,STF}_{<i_1 i_2 \ldots i_\ell>} , \tag{A.80}$$

where we have reset n again everywhere to the variable ν. What we have here is a classical renormalisation group running of the multipole couplings present in the world line action, i.e., an RGE induced by the classical radiation reaction. Such classical RGE is, in fact, common in many radiation reaction problems (See e.g. discussions in [39,40,86,152]). We will see later how this non-local influence phase gets further modified in dS spacetime.

Appendix B

Theory of vector spherical harmonics

In this chapter, we will give explicit expressions for the vector spherical harmonics(VSH) on $\mathbb{S}^{d-1}$. We will construct higher-dimensional analogues of the standard and well-known expressions for $\mathbb{S}^2$ [153–155], which we refer to as orthonormal VSH. Past discussions of vector spherical harmonics on higher dimensional spheres appear in [145,146,156–158]. In particular, Higuchi [145] gives a recursive construction for an arbitrary tensor harmonic on $\mathbb{S}^{d-1}$ in terms of tensor harmonics on $\mathbb{S}^{d-2}$. In what follows, we will write explicit forms for the VSHs on $\mathbb{S}^{d-1}$, which agree with his recursion.

Along with the usual orthonormal VSH expressed in spherical polar coordinates, one can also construct them in terms of symmetric trace-free(STF) tensors on the ambient $\mathbb{R}^d$. Their construction in cartesian coordinates helps in the post-newtonian expansions of multipole moments, and hence, they find their natural home in the literature on gravitational waves [159, 160]. Their higher-dimensional analogues can be found in [161, 162]. We will construct these cartesian STF VSH in all dimensions and show how they connect to previous constructions on $\mathbb{S}^2$ as well as for higher-dimensional spheres.

We begin with an explicit spherical coordinate system in $\mathbb{R}^d$ given by

$$
\begin{aligned}
x_1 &= r\ \sin\vartheta_{d-2}\ \sin\vartheta_{d-3}\ \dots\ \sin\vartheta_2\ \sin\vartheta_1\ \cos\varphi\ ,\\
x_2 &= r\ \sin\vartheta_{d-2}\ \sin\vartheta_{d-3}\ \dots\ \sin\vartheta_2\ \sin\vartheta_1\ \sin\varphi\ ,\\
x_3 &= r\ \sin\vartheta_{d-2}\ \sin\vartheta_{d-3}\ \dots\ \sin\vartheta_2\ \cos\vartheta_1\ ,\\
x_4 &= r\ \sin\vartheta_{d-2}\ \sin\vartheta_{d-3}\ \dots\ \cos\vartheta_2\ ,\\
&\dots\ ,\\
x_{d-2} &= r\ \sin\vartheta_{d-2}\ \sin\vartheta_{d-3}\ \cos\vartheta_{d-4}\ ,\\
x_{d-1} &= r\ \sin\vartheta_{d-2}\ \cos\vartheta_{d-3}\ ,\\
x_d &= r\ \cos\vartheta_{d-2}\ .
\end{aligned}
\tag{B.1}
$$

Here the radius r varies from 0 to ∞ whereas the allowed values of angles is $\vartheta_i \in [0,\pi]$ and $\varphi \in [0, 2\pi)$. We will set $\vartheta_0 \equiv \varphi$ and denote the coordinates on $\mathbb{S}^{d-1}$ as ϑ_I with $I = 0, 1, \dots, d-2$. The sphere metric in these coordinates takes the form

$$
\begin{aligned}
d\Omega^2_{d-1} &\equiv \gamma_{IJ} d\vartheta_I d\vartheta_J = d\vartheta^2_{d-2} + \sin^2\vartheta_{d-2} d\Omega^2_{d-2} = \dots\\
&= d\vartheta^2_{d-2} + \sin^2\vartheta_{d-2} d\vartheta^2_{d-3} + \dots + \prod_{K=J+1}^{d-2} \sin^2\vartheta_K\ d\vartheta_J^2 + \dots + \prod_{K=1}^{d-2} \sin^2\vartheta_K\ d\varphi^2.
\end{aligned}
\tag{B.2}
$$

In other words, the explicit metric coefficients are given by

$$
\gamma_{IJ} = \begin{cases} \prod_{K=I+1}^{d-2} \sin^2\vartheta_K & \text{when } I = J\ ,\\ 0 & \text{otherwise.} \end{cases}
\tag{B.3}
$$

Since the metric is diagonal, its inverse is given by inverting the diagonal entries, i.e., $\gamma^{II} = \gamma_{II}^{-1}$. Another result we will need is the volume measure on the sphere $\sqrt{\gamma} = \prod_{J=1}^{d-2} \sin^J\vartheta_J$. By integrating this measure, we obtain the volume of $\mathbb{S}^{d-1}$ as

$$
|\mathbb{S}^{d-1}| \equiv \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\ ,
\tag{B.4}
$$

We will denote the covariant derivative associated with the unit sphere metric as $\mathscr{D}_I$. For some of the conversions between partial derivatives in the spherical coordinates to cartesian coordinates, the following formula is useful:

$$\begin{aligned}\frac{\partial}{\partial\vartheta_I} &= -r\prod_{j=I}^{d-2}\sin\vartheta_j\ \frac{\partial}{\partial x_{I+2}} + \frac{\cos\vartheta_I}{\sin\vartheta_I}\sum_{j=1}^{I+1} x_j\frac{\partial}{\partial x_j} \\ &= \frac{1}{\prod_{j=I}^{d-2}\sin\vartheta_j}\sum_{j=1}^{I+1} x_j\left\{x_{I+2}\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_{I+2}}\right\}\ .\end{aligned} \tag{B.5}$$

This is the push-forward of the coordinate basis vector fields on $\mathbb{S}^{d-1}$ to $\mathbb{R}^d$.

B.1 Toroidal derivatives and VSHs

We will now move on to the subject of vector spherical harmonics (VSHs), i.e., divergence-free vector fields on $\mathbb{S}^{d-1}$, which are also eigenvectors of the sphere laplacian. From the cartesian viewpoint, these correspond to homogeneous, harmonic, divergence-free, polynomial vector fields on $\mathbb{R}^d$ that have no radial component.

Harmonic vector fields in $\mathbb{R}^d$

We will now construct such harmonic vector fields by applying an appropriate derivative operator on homogeneous harmonic polynomials $x^{<i_1}x^{i_2}\ldots x^{i_\ell>}$. Such a construction is well-known in $d=3$ where the toroidal operator $\vec{r}\times\vec{\nabla}$ will do the job. Given that there is no cross-product for $d>3$, this statement does not generalise as stated: *there is, in fact, no one derivative operator that constructs all Vector polynomials from scalar polynomials in* $d>3$. However, we will now show that if we allow for two derivatives, we can indeed construct a full set of toroidal derivative operators for $d>3$. The standard $d=3$ construction will then be recovered as a degenerate special case. As far as we are aware, such a construction of toroidal operators for general dimensions has not appeared elsewhere and is entirely new.

Let $\mathbb{L}_{ij}\equiv x_i\partial_j - x_j\partial_i$ be the rotation Killing vectors of $\mathbb{R}^d$ obeying $SO(d)$ Lie-algebra

$$[\mathbb{L}_{ij},\mathbb{L}_{kl}] = \delta_{ik}\mathbb{L}_{lj} - \delta_{jk}\mathbb{L}_{li} - \delta_{il}\mathbb{L}_{kj} + \delta_{jl}\mathbb{L}_{ki}\ . \tag{B.6}$$

These operators obey relations of the form

$$x_k \mathbb{L}_{ij} + x_i \mathbb{L}_{jk} + x_j \mathbb{L}_{ki} = 0 = \partial_k \mathbb{L}_{ij} + \partial_i \mathbb{L}_{jk} + \partial_j \mathbb{L}_{ki} \ . \tag{B.7}$$

A useful corollary of the above relations is a sum of the form

$$\sum_{ik} x_i \mathbb{L}_{jk} \mathbb{L}_{ik} = \frac{1}{2} x_j \sum_{ik} \mathbb{L}_{ik} \mathbb{L}_{ik} \ , \quad \sum_{ik} \partial_i \mathbb{L}_{jk} \mathbb{L}_{ik} = \frac{1}{2} \partial_j \sum_{ik} \mathbb{L}_{ik} \mathbb{L}_{ik} \ , \tag{B.8}$$

where the sums over i and k are performed over the same subset of indices. These properties motivate the following definition of the toroidal operators

$$\mathbb{A}^{(\alpha)}_{i,\alpha+2} f \equiv \begin{cases} \sum_{k=1}^{\alpha+1} \mathbb{L}_{k,\alpha+2} \mathbb{L}_{ki} f & \text{for } 1 \leq i \leq \alpha+1 \ , \\ -\frac{1}{2} \sum_{j,k=1}^{\alpha+1} \mathbb{L}_{jk} \mathbb{L}_{jk} f & \text{for } i = \alpha+2 \ , \\ 0 & \text{for } i > \alpha+2 \ , \end{cases} \tag{B.9}$$

acting on an arbitrary function f on $\mathbb{R}^d$. Here, α takes on values $\alpha = 1, 2, \ldots, (d-2)$, and the reason for our notation will become clear shortly. Equations Eq.(B.8) imply that the vector field $\mathbb{A}^{(\alpha)}_{i,\alpha+2} f$ is tangential to the sphere and is divergence-free for any f, viz.,

$$\partial_i \mathbb{A}^{(\alpha)}_{i,\alpha+2} f = 0 \ , \ x^i \mathbb{A}^{(\alpha)}_{i,\alpha+2} f = 0 \ . \tag{B.10}$$

Further, since $\mathbb{L}_{ij}$s commute with the laplacian in $\mathbb{R}^d$, $\mathbb{A}^{(\alpha)}_{i,\alpha+2} f$ is a harmonic vector field if f is harmonic. We can then take f to be any homogeneous harmonic polynomial in $\mathbb{R}^d$ to get a homogeneous harmonic vector field. Thus, an overcomplete basis of homogeneous harmonic vector fields of degree ℓ can be constructed by taking

$$\mathbb{A}^{(\alpha)}_{i,\alpha+2} [x^{<i_1} x^{i_2} \ldots x^{i_\ell>}] \frac{\partial}{\partial x_i} \tag{B.11}$$

for $\alpha = 1, 2, \ldots, (d-2)$. Such vector fields, when restricted to $\mathbb{S}^{d-1}$, yield a vector spherical harmonic (VSH). In the next subsection, we will construct an orthonormal basis for such VSHs.

The above set of toroidal operators can be generalised as follows. Say we are given a subspace $\mathbb{R}^{\alpha+2} \subseteq \mathbb{R}^d$. We can then define a toroidal operator corresponding to this subspace and a direction j within that subspace via

$$\mathbb{\Delta}^{(\alpha)}_{ij} f \equiv \Big\{ \sum_{k\in\mathbb{R}^{\alpha+2}} \mathbb{L}_{kj}\mathbb{L}_{ki} - \frac{1}{2}\delta_{ij} \sum_{k,l\in\mathbb{R}^{\alpha+2}} \mathbb{L}_{kl}\mathbb{L}_{kl} \Big\}_{i,j\in\mathbb{R}^{\alpha+2}} f \ . \tag{B.12}$$

This formula should be interpreted as follows: first of all, we get a non-zero answer only if i,j directions are tangent to the subspace $\mathbb{R}^{\alpha+2}$ under question. Further sums indicated inside the bracket are over directions within $\mathbb{R}^{\alpha+2}$. If we take the subspace $\mathbb{R}^{\alpha+2}$ spanned by the cartesian directions $\{x_1, x_2, \ldots, x_{\alpha+2}\}$ and choose j to be equal along $x^{\alpha+2}$, and using

$$\begin{gathered}\sum_{k=1}^{\alpha+2} \mathbb{L}_{k,\alpha+2}\mathbb{L}_{ki} = \sum_{k=1}^{\alpha+1} \mathbb{L}_{k,\alpha+2}\mathbb{L}_{ki} \text{ for } 1 \le i \le \alpha+1 \ , \\ \sum_{k=1}^{\alpha+2} \mathbb{L}_{k,\alpha+2}\mathbb{L}_{k,\alpha+2} - \frac{1}{2}\sum_{j,k=1}^{\alpha+2} \mathbb{L}_{jk}\mathbb{L}_{jk} = -\frac{1}{2}\sum_{j,k=1}^{\alpha+1} \mathbb{L}_{jk}\mathbb{L}_{jk} \ ,\end{gathered} \tag{B.13}$$

we get back the toroidal operators defined before in Eq.(B.9).

A couple of remarks about the above form: first, if the function f is invariant under the $SO(\alpha+1)$ that rotates $\{x_1, x_2, \ldots \ , x_{\alpha+1}\}$, all the $\mathbb{L}_{ij}$s in Eq.(B.9) annihilates f, and we get a vector field that is identically zero. Thus, *the toroidal operators we have defined above have non-trivial kernels which become smaller as α increases.* Relatedly, at a given α, the harmonic vector field should necessarily break $SO(\alpha+1)$. A second remark is that, for $\alpha = 1$, the Eq.(B.9) reduces to

$$\mathbb{\Delta}^{(\alpha=1)}_{i,3} f \equiv \begin{cases} -\mathbb{L}_{23}\mathbb{L}_{12} f & \text{for } i=1 \ , \\ -\mathbb{L}_{31}\mathbb{L}_{12} f & \text{for } i=2 \ , \\ -\mathbb{L}_{12}^2 f & \text{for } i=3 \ , \\ 0 & \text{for } i>3 \ . \end{cases} \tag{B.14}$$

We recognise in RHS the familiar $3d$ toroidal operator $-\vec{r}\times\vec{\nabla}$ acting on $\mathbb{L}_{12} f$. As remarked above, the kernel of the above operator is the largest among all the toroidal operators: it is the set of $SO(2)$ invariant functions, where the $SO(2)$ rotates the 12 plane. But, in this special case of $\alpha = 1$ (and only in this case), we can improve our toroidal operator by dropping an $\mathbb{L}_{12}$ and defining

$$\mathbb{A}^{(\alpha=1)}_{i,3}|_{\text{New}} f \equiv \begin{cases} -\mathbb{L}_{23} f & \text{for } i = 1 \ , \\ -\mathbb{L}_{31} f & \text{for } i = 2 \ , \\ -\mathbb{L}_{12} f & \text{for } i = 3 \ , \\ 0 & \text{for } i > 3 \ . \end{cases} \tag{B.15}$$

The kernel of this 'improved' toroidal operator is smaller and is the set of $SO(3)$ invariant functions, where the $SO(3)$ rotates the x_1, x_2 and x_3, i.e., the kernel is of the same size as the $\alpha = 2$ toroidal operator. The usual 3d toroidal operator is improved in this sense.

Vector spherical harmonics on $\mathbb{S}^{d-1}$

We now turn to a description in spherical coordinates. The toroidal double-derivative operators on $\mathbb{S}^{d-1}$ take the following form:

$$\mathbb{A}^{\alpha}_{I} f \equiv \sqrt{\gamma_{\alpha\alpha}} \times \begin{cases} \frac{1}{\sin^{\alpha-2}\vartheta_\alpha} \frac{\partial}{\partial\vartheta_\alpha} \frac{\partial}{\partial\vartheta_I} \left[\sin^{\alpha-1}\vartheta_\alpha \ f\right] & \text{for } 0 \le I \le \alpha - 1 \ , \\ -\frac{1}{\sin\vartheta_\alpha} \mathscr{D}^2_{\mathbb{S}^\alpha} f & \text{for } I = \alpha \ , \\ 0 & \text{for } I > \alpha \ . \end{cases} \tag{B.16}$$

Here, f is an arbitrary function on the sphere, the index α takes values from 1 to $(d-2)$, thus defining $(d-2)$ different derivative operators. The index $I = 0, 1, \ldots, (d-2)$ denotes the vector directions on $\mathbb{S}^{d-1}$, $\mathscr{D}^2_{\mathbb{S}^{I+1}}$ is the lower sphere laplacian defined in Eq.(A.31), and γ_{IJ} are the sphere metric coefficients defined in Eq.(B.3). The above derivative operators exhibit the following useful properties, as can be established via direct computation:

- For an arbitrary function f on $\mathbb{S}^{d-1}$, the corresponding vector field $\mathbb{A}^{\alpha}_{I} f$ is divergenceless.

- The derivative operators $\mathbb{A}^{\alpha}_{I}$ obey the following commutation relation with the sphere laplacian:

$$[\mathscr{D}^2, \mathbb{A}^{\alpha}_{I}] f = \mathbb{A}^{\alpha}_{I} f \ . \tag{B.17}$$

If we distinguish between the scalar and the vector laplacians on the sphere by subscripts S and V respectively, the relation above can also be stated as $\mathscr{D}^2_V \mathbb{A}^{\alpha}_{I} f =$

$\mathbb{\Delta}_I^\alpha(\mathscr{D}_S^2+1)f$.

- These vector fields are mutually orthogonal in the following sense: for any two functions f and g on $\mathbb{S}^{d-1}$, we have

$$\int_{\mathbb{S}^{d-1}} \gamma^{IJ}\ (\mathbb{\Delta}_I^\alpha f)\ (\mathbb{\Delta}_J^{\alpha'} g) = 0 \quad \text{for } \alpha \neq \alpha'. \tag{B.18}$$

  When $\alpha = \alpha'$, the same inner product evaluates to

$$\int_{\mathbb{S}^{d-1}} \gamma^{IJ}\ (\mathbb{\Delta}_I^\alpha f)\ (\mathbb{\Delta}_J^\alpha g) = \int_{\mathbb{S}^{d-1}} (\mathscr{D}^2_{\mathbb{S}^\alpha} f)\Big(\mathscr{D}^2_{\mathbb{S}^{\alpha+1}} g + (1-\alpha) g\Big)\ . \tag{B.19}$$

Once these statements are established, the $\mathbb{\Delta}_I^\alpha$ operators can be used to give an explicit form for the vector spherical harmonics (VSHs).

To this end, consider the vector fields defined[1] by acting $\mathbb{\Delta}_I^\alpha$'s on the orthonormal SSHs of Eq.(A.28):

$$\begin{aligned}\mathbb{V}_I^{\alpha\ell\vec{m}} &\equiv C^V_{\alpha\ell\vec{m}} \mathbb{\Delta}_I^\alpha \mathscr{Y}_{\ell\vec{m}}(\hat{r}) \\ &= C^V_{\alpha\ell\vec{m}}\sqrt{\gamma_{\alpha\alpha}} \times \begin{cases} \frac{1}{\sin^{\alpha-2}\vartheta_\alpha}\frac{\partial}{\partial\vartheta_\alpha}\frac{\partial}{\partial\vartheta_I}\left[\sin^{\alpha-1}\vartheta_\alpha\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\right] & \text{for } 0 \leq I \leq \alpha-1\ , \\ \frac{1}{\sin\vartheta_\alpha} m_\alpha(m_\alpha+\alpha-1)\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I = \alpha\ , \\ 0 & \text{for } I > \alpha\ . \end{cases}\end{aligned} \tag{B.20}$$

Here, we have simplified the $I=\alpha$ component by using Eq.(A.31), and $C^V_{\alpha\ell\vec{m}}$ is a convenient normalization factor to be determined shortly. Using the first two properties $\mathbb{\Delta}_I^\alpha$ enumerated above, we conclude that $\mathbb{V}^I_{\alpha\ell\vec{m}}$ is a divergence-free vector field satisfying

$$\left[\mathscr{D}^2 + \ell(\ell+d-2) - 1\right] \mathbb{V}^I_{\alpha\ell\vec{m}} = 0\ . \tag{B.21}$$

Eq.(B.18) then ensures the orthogonality of $\mathbb{V}^I_{\alpha\ell\vec{m}}$ and $\mathbb{V}^I_{\alpha'\ell'\vec{m}'}$ for $\alpha \neq \alpha'$. For $\alpha = \alpha'$, we

[1] Our definitions here are consistent with the recursive construction by Higuchi [145]. See also appendix A.2 of [14] where Higuchi's construction is reviewed).

use Eq.(B.19) and Eq.(A.31) to get

$$\int_{\mathbb{S}^{d-1}} \gamma^{IJ} \mathbb{V}_I^{\alpha\ell\vec{m}} \mathbb{V}_J^{\alpha\ell'\vec{m}'*}$$
$$= C^V_{\alpha\ell\vec{m}} C^{V*}_{\alpha\ell'\vec{m}'} m_\alpha(m_\alpha+\alpha-1)(m'_{\alpha+1}+1)(m'_{\alpha+1}+\alpha-1) \int_{\mathbb{S}^{d-1}} \mathscr{Y}_{\ell\vec{m}}(\hat{r}) \mathscr{Y}^*_{\ell'\vec{m}'}(\hat{r}) \ . \tag{B.22}$$

From this, we conclude that $\mathbb{V}^I_{\alpha\ell\vec{m}}$ and $\mathbb{V}^I_{\alpha'\ell'\vec{m}'}$ are orthogonal unless $\alpha = \alpha', \ell = \ell'$ and $\vec{m} = \vec{m}'$. The above computation also determines the normalisation factor $C^V_{\alpha\ell\vec{m}}$ via

$$|C^V_{\alpha\ell\vec{m}}|^{-2} \equiv (m_{\alpha+1}+\alpha-1)(m_{\alpha+1}+1)m_\alpha(m_\alpha+\alpha-1) \ . \tag{B.23}$$

We note that this normalization factor diverges if $m_\alpha = 0$, or if $\alpha = 1$ and $m_{\alpha+1} = m_2 = 0$. This means that, unless the function multiplying this normalisation factor in Eq.(B.20) goes to zero in these cases, we have to discard the result for being non-normalisable. If instead, in any of these cases, *all* the components in Eq.(B.20) vanish, it is possible to get a finite result by taking a limit. We have to decide which case corresponds to which possibility. A careful analysis leads to the following conclusions:

- For $\alpha > 1$, the above expression yields a normalised VSH only if $m_\alpha > 0$. Thus, in this case, we should exclude the possibility that $m_\alpha = 0$.
- For $\alpha = 1$, $m_\alpha = m_1$ can be taken to be zero and Eq.(B.20) gives a finite answer when understood as a limit, provided $m_{\alpha+1} = m_2 > 0$.

As an example to illustrate the second point, consider the following $\alpha = 1$ VSH on $\mathbb{S}^4$:

$$\mathbb{V}_I^{1\ell\vec{m}}|_{\mathbb{S}^4} \equiv \frac{\sin\vartheta_3 \sin\vartheta_2}{m_1\sqrt{m_2(m_2+1)}} \begin{cases} \sin\vartheta_1 \frac{\partial}{\partial\vartheta_1}\frac{\partial}{\partial\varphi}\mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=0 \ , \\ \frac{1}{\sin\vartheta_1} m_1^2 \ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=1 \ , \\ 0 & \text{for } I=2,3 \ . \end{cases} \tag{B.24}$$

From Eq.(A.28), we can write $\frac{\partial}{\partial\varphi}\mathscr{Y}_{\ell\vec{m}}(\hat{r}) = \pm i m_1 \ \mathscr{Y}_{\ell\vec{m}}(\hat{r})$. It is then clear that we get a finite result in the expression above as we take $m_1 \to 0$ (provided $m_2 > 0$):

$$\lim_{m_1\to 0}(\mp i)\mathbb{V}_I^{1\ell\vec{m}}|_{\mathbb{S}^4} \equiv \frac{\sin\vartheta_3\sin\vartheta_2\sin\vartheta_1}{\sqrt{m_2(m_2+1)}} \begin{cases} \frac{\partial}{\partial\vartheta_1}\mathscr{Y}_{\ell\vec{m}}(\hat{r})|_{m_1=0} & \text{for } I=0 \ , \\ 0 & \text{for } I=1,2,3 \ . \end{cases} \tag{B.25}$$

Alternately, we can avoid this subtlety for $\alpha = 1$ altogether, by redefining the derivative operator $\mathbb{\Delta}_I^{\alpha=1}$ by stripping off a $\frac{\partial}{\partial\varphi}$ from it, i.e., we define

$$\mathbb{\Delta}_I^{\alpha=1} f|_{\text{new}} \equiv \sqrt{\gamma_{11}} \times \begin{cases} -\sin\vartheta_1 \frac{\partial f}{\partial\vartheta_1} & \text{for } I = 0 \ , \\ \frac{1}{\sin\vartheta_1}\frac{\partial f}{\partial\varphi} & \text{for } I = 1 \ , \\ 0 & \text{for } I > 1 \ . \end{cases} \tag{B.26}$$

This corresponds exactly to the 'improvement' of $\alpha = 1$ toroidal operator described before in the cartesian language. Since $\frac{\partial}{\partial\varphi}$ is an isometry, this redefinition does not change any of the properties of $\mathbb{\Delta}_I^{\alpha=1}$ except for its overall normalisation (a factor of m_1 has to be dropped). The orthogonality Eq.(B.18) still holds, whereas Eq.(B.19) becomes

$$\int_{\mathbb{S}^{d-1}} \gamma^{IJ} \ (\mathbb{\Delta}_I^{\alpha=1} f)_{\text{New}} \ (\mathbb{\Delta}_J^{\alpha=1} g)_{\text{New}} = -\int_{\mathbb{S}^{d-1}} f \ \mathscr{D}^2_{\mathbb{S}^2} g \ . \tag{B.27}$$

With this norm, the VSH quoted above then becomes

$$\mathbb{V}_I^{1\ell\vec{m}}|_{\mathbb{S}^4,\text{New}} \equiv \frac{\sin\vartheta_3 \sin\vartheta_2}{\sqrt{m_2(m_2+1)}} \begin{cases} -\sin\vartheta_1 \frac{\partial}{\partial\vartheta_1} \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I = 0 \ , \\ \frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi} \ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I = 1 \ , \\ 0 & \text{for } I = 2, 3 \ . \end{cases} \tag{B.28}$$

The expression appearing here is, in fact, the standard VSH on $\mathbb{S}^2$ constructed via the toroidal operator $\vec{r} \times \vec{\nabla}$: rewritten in this form, no subtle limiting procedure is necessary to deal with the $m_1 = 0$ case. Adopting this new definition, we give in table B.1 the explicit form of VSHs in $\mathbb{S}^2, \mathbb{S}^3, \mathbb{S}^4$ and $\mathbb{S}^5$.

Before we conclude, it is often convenient to have a simple VSH for any given ℓ written down explicitly, on which computations can be done with ease. We will end this subsection by providing two such examples. The first example is the VSH corresponding to

$$\alpha = 1 \ , \ m_1 = 0 \ , \ m_2 = m_3 = \ldots = m_{d-2} = 1 \leq m_{d-1} \equiv \ell \ . \tag{B.29}$$

VSHs on $\mathbb{S}^2$
$\mathbb{V}_I^{1\ell\vec{m}}\vert_{\rm New} \equiv \frac{1}{\sqrt{\ell(\ell+1)}} \begin{cases} -\sin\vartheta_1 \frac{\partial}{\partial\vartheta_1}\mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=0 \ , \\ \frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi}\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=1 \ . \end{cases}$
VSHs on $\mathbb{S}^3$
$\mathbb{V}_I^{2\ell\vec{m}} \equiv \frac{1}{(\ell+1)\sqrt{m_2(m_2+1)}} \begin{cases} \frac{\partial}{\partial\vartheta_2}\frac{\partial}{\partial\vartheta_I}\left[\sin\vartheta_2\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\right] & \text{for } I=0,1 \ , \\ \frac{1}{\sin\vartheta_2} m_2(m_2+1)\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=2 \ . \end{cases}$ $\mathbb{V}_I^{1\ell\vec{m}}\vert_{\rm New} \equiv \frac{\sin\vartheta_2}{\sqrt{m_2(m_2+1)}} \begin{cases} -\sin\vartheta_1 \frac{\partial}{\partial\vartheta_1}\mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=0 \ , \\ \frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi}\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=1 \ , \\ 0 & \text{for } I=2 \ . \end{cases}$
VSHs on $\mathbb{S}^4$
$\mathbb{V}_I^{3\ell\vec{m}} \equiv \frac{1}{\sqrt{(\ell+1)(\ell+2)m_3(m_3+2)}} \begin{cases} \frac{1}{\sin\vartheta_3}\frac{\partial}{\partial\vartheta_3}\frac{\partial}{\partial\vartheta_I}\left[\sin^2\vartheta_3\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\right] & \text{for } I=0,1,2 \ , \\ \frac{1}{\sin\vartheta_3} m_3(m_3+2)\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=3 \ . \end{cases}$ $\mathbb{V}_I^{2\ell\vec{m}} \equiv \frac{\sin\vartheta_3}{(m_3+1)\sqrt{m_2(m_2+1)}} \begin{cases} \frac{\partial}{\partial\vartheta_2}\frac{\partial}{\partial\vartheta_I}\left[\sin\vartheta_2\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\right] & \text{for } I=0,1 \ , \\ \frac{1}{\sin\vartheta_2} m_2(m_2+1)\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=2 \ , \\ 0 & \text{for } I=3 \ . \end{cases}$ $\mathbb{V}_I^{1\ell\vec{m}}\vert_{\rm New} \equiv \frac{\sin\vartheta_3\sin\vartheta_2}{\sqrt{m_2(m_2+1)}} \begin{cases} -\sin\vartheta_1 \frac{\partial}{\partial\vartheta_1}\mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=0 \ , \\ \frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi}\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=1 \ , \\ 0 & \text{for } I=2,3 \ . \end{cases}$
VSHs on $\mathbb{S}^5$
$\mathbb{V}_I^{4\ell\vec{m}} \equiv \frac{1}{\sqrt{(\ell+1)(\ell+3)m_4(m_4+3)}} \begin{cases} \frac{1}{\sin^2\vartheta_4}\frac{\partial}{\partial\vartheta_4}\frac{\partial}{\partial\vartheta_I}\left[\sin^3\vartheta_4\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\right] & \text{for } I=0,1,2,3 \ , \\ \frac{1}{\sin\vartheta_4} m_4(m_4+3)\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=4 \ . \end{cases}$ $\mathbb{V}_I^{3\ell\vec{m}} \equiv \frac{\sin\vartheta_4}{\sqrt{(m_4+1)(m_4+2)m_3(m_3+2)}} \begin{cases} \frac{1}{\sin\vartheta_3}\frac{\partial}{\partial\vartheta_3}\frac{\partial}{\partial\vartheta_I}\left[\sin^2\vartheta_3\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\right] & \text{for } I=0,1,2 \ , \\ \frac{1}{\sin\vartheta_3} m_3(m_3+2)\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=3 \ , \\ 0 & \text{for } I=4 \ . \end{cases}$ $\mathbb{V}_I^{2\ell\vec{m}} \equiv \frac{\sin\vartheta_4\sin\vartheta_3}{(m_3+1)\sqrt{m_2(m_2+1)}} \begin{cases} \frac{\partial}{\partial\vartheta_2}\frac{\partial}{\partial\vartheta_I}\left[\sin\vartheta_2\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\right] & \text{for } I=0,1 \ , \\ \frac{1}{\sin\vartheta_2} m_2(m_2+1)\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=2 \ , \\ 0 & \text{for } I=3,4 \ . \end{cases}$ $\mathbb{V}_I^{1\ell\vec{m}}\vert_{\rm New} \equiv \frac{\sin\vartheta_4\sin\vartheta_3\sin\vartheta_2}{\sqrt{m_2(m_2+1)}} \begin{cases} -\sin\vartheta_1 \frac{\partial}{\partial\vartheta_1}\mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=0 \ , \\ \frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\varphi}\ \mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=1 \ , \\ 0 & \text{for } I=2,3,4 \ . \end{cases}$

Table B.1: Explicit expressions for vector spherical harmonics (VSHs).

The normalised SSH for this $\vec{m}$ is given by Eq.(A.28) as

$$\mathscr{Y}_{\ell\vec{m}}(\hat{r}) = \sqrt{2\pi\frac{\mathcal{N}_{HH}(d+2,\ell-1)}{|\mathbb{S}^{d+1}|}}P_{\ell-1}(d+2,\cos\vartheta_{d-2})\cos\vartheta_1\prod_{J=2}^{d-2}\sin\vartheta_J \tag{B.30}$$

The corresponding VSH is given by

$$\begin{aligned}\mathbb{V}_I &= \frac{\prod_{J=2}^{d-2}\sin\vartheta_J}{\sqrt{m_2(m_2+1)}|_{m_2=1}}\begin{cases} -\sin\vartheta_1\frac{\partial}{\partial\vartheta_1}\mathscr{Y}_{\ell\vec{m}}(\hat{r}) & \text{for } I=0\ , \\ 0 & \text{for } I=1,2,3,\ldots,d-2\ . \end{cases}\\ &= \sqrt{\pi\frac{\mathcal{N}_{HH}(d+2,\ell-1)}{|\mathbb{S}^{d+1}|}}P_{\ell-1}(d+2,\cos\vartheta_{d-2})\begin{cases} \prod_{J=1}^{d-2}\sin^2\vartheta_J & \text{for } I=0\ , \\ 0 & \text{for } I=1,\ldots,d-2\ . \end{cases}\end{aligned} \tag{B.31}$$

We can also present this as a vector field on $\mathbb{S}^{d-1}$ by raising the sphere index, viz.,

$$\mathbb{V}^I\frac{\partial}{\partial\vartheta_I} = \sqrt{\pi\frac{\mathcal{N}_{HH}(d+2,\ell-1)}{|\mathbb{S}^{d+1}|}}P_{\ell-1}(d+2,\cos\vartheta_{d-2})\frac{\partial}{\partial\varphi}\ . \tag{B.32}$$

This vector-field can be pushed-forward to $\mathbb{R}^d$ using Eq.(B.5): we then get a vector field which varies as

$$P_{\ell-1}\left(d+2,\frac{x_d}{r}\right)\left\{x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_2}\right\}\ . \tag{B.33}$$

It is then evident that this vector field is invariant under the $SO(2)$ rotations of $x_1 - x_2$ plane as well as $SO(d-3)$ rotations of $x_3, x_4, \ldots, x_{d-1}$. This is, hence, a simple VSH with a large group of symmetries, and we found it to be a convenient example to check our computations.

The second example we discuss is a VSH with an even bigger symmetry of $SO(d-2)$ that rotates $x_1, x_2, \ldots, x_{d-2}$. This is the most symmetric of all VSHs (for $d > 4$) and will play an important role when we discuss the VSH addition theorem. The $SO(d-2)$-invariant VSH is obtained by taking

$$\alpha = d-2\ ,\ m_1 = m_2 = m_3 = \ldots = m_{d-3} = 0\ ,\ m_{d-2} = 1 \leq m_{d-1} \equiv \ell\ . \tag{B.34}$$

The normalised SSH in this case is

$$\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \;= (d-1)\sqrt{\frac{1}{2\pi}\frac{\mathcal{N}_{HH}(d+2,\ell-1)}{|\mathbb{S}^{d+1}|}}\sin\vartheta_{d-2}P_{\ell-1}(d+2,\cos\vartheta_{d-2})\cos\vartheta_{d-3}\ . \tag{B.35}$$

The corresponding VSH has the following form

$$\begin{aligned}\mathbb{V}^I\frac{\partial}{\partial\vartheta_I} &= \frac{(d-1)}{\sqrt{(d-2)(\ell+1)(\ell+d-3)}}\sqrt{\frac{1}{2\pi}\frac{\mathcal{N}_{HH}(d+2,\ell-1)}{|\mathbb{S}^{d+1}|}} \\ &\quad\times\Big\{(d-2)\cos\vartheta_{d-3}P_{\ell-1}(d+2,\cos\vartheta_{d-2})\frac{\partial}{\partial\vartheta_{d-2}} \\ &\qquad -\frac{\sin\vartheta_{d-3}}{\sin^{d-2}\vartheta_{d-2}}\frac{d}{d\vartheta_{d-3}}[\sin^{d-2}\vartheta_{d-2}P_{\ell-1}(d+2,\cos\vartheta_{d-2})]\ \frac{\partial}{\partial\vartheta_{d-3}}\Big\}\ .\end{aligned} \tag{B.36}$$

Stripping of the normalization pre-factor, its push-forward to $\mathbb{R}^d$ at radius r (computed via Eq.(B.5)) yields

$$\begin{aligned}&\frac{1}{d-1}r^{\ell-1}P_{\ell-1}\left(d+2,\frac{x_d}{r}\right)\left\{x_d\frac{\partial}{\partial x_{d-1}}-x_{d-1}\frac{\partial}{\partial x_d}\right\} \\ &\quad+\frac{(\ell-1)(\ell+d-1)}{(d+1)(d-1)(d-2)}r^{\ell-2}P_{\ell-2}\left(d+4,\frac{x_d}{r}\right)\sum_{j=1}^{d-2}x_j\left\{x_{d-1}\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_{d-1}}\right\}\ .\end{aligned} \tag{B.37}$$

Here, we have used the identity

$$\frac{d}{dz}P_\ell(d,z) = \frac{\ell(\ell+d-2)}{d-1}P_{\ell-1}(d+2,z) \tag{B.38}$$

to compute the derivative of the generalised Legendre polynomials.

Addendum: Counting of VSHs

Our explicit construction can be used to count the total number of VSHs for a given $\ell = m_{d-1}$: we will denote this by $\mathcal{N}^V_{HH}(d,\ell) = \mathcal{N}^V_{HH}(d,m_{d-1})$. To begin with, if there were no constraints on $\vec{m}$, all VSHs are obtained by acting $(d-2)$ derivative operators on $\mathcal{N}_{HH}(d,m_{d-1})$ number of SSHs and $\mathcal{N}^V_{HH}(d,m_{d-1})$ should just be $(d-2)\mathcal{N}_{HH}(d,m_{d-1})$. But given the constraints on $\vec{m}$ described above, this is an overcounting, and a more careful counting is needed.

For a given $\alpha > 1$, the number of VSHs we obtain is given by

$$\begin{aligned}\mathcal{N}_{HH}^{V,\alpha}(d,m_{d-1}) &= \sum_{m_{d-2}=1}^{m_{d-1}} \sum_{m_{d-3}=1}^{m_{d-2}} \cdots \sum_{m_\alpha=1}^{m_{\alpha+1}} \sum_{m_{\alpha-1}=0}^{m_\alpha} \sum_{m_{\alpha-2}=0}^{m_{\alpha-1}} \cdots \left\{1+\sum_{m_1=1}^{m_2} 2\right\} \\ &= \sum_{m_{d-2}=1}^{m_{d-1}} \cdots \sum_{m_\alpha=1}^{m_{\alpha+1}} \mathcal{N}_{HH}(\alpha+1,m_\alpha)\ .\end{aligned} \tag{B.39}$$

Here, we have imposed the constraint that $m_{d-1} \geq m_{d-2} \geq \ldots m_\alpha \geq 1$ and have used Eq.(A.33) in the second line. In the next steps, we should systematically subtract out the forbidden m_i's, e.g., the next few steps are given by

$$\begin{aligned}\mathcal{N}_{HH}^{V,\alpha}(d,m_{d-1}) &= \sum_{m_{d-2}=1}^{m_{d-1}} \cdots \sum_{m_{\alpha+1}=1}^{m_{\alpha+2}} \{\mathcal{N}_{HH}(\alpha+2,m_{\alpha+1})-1\} \\ &= \sum_{m_{d-2}=1}^{m_{d-1}} \cdots \sum_{m_{\alpha+2}=1}^{m_{\alpha+3}} \{\mathcal{N}_{HH}(\alpha+3,m_{\alpha+2})-1-m_{\alpha+2}\} \\ &= \sum_{m_{d-2}=1}^{m_{d-1}} \cdots \sum_{m_{\alpha+3}=1}^{m_{\alpha+4}} \Big\{\mathcal{N}_{HH}(\alpha+4,m_{\alpha+3})-1 \\ &\qquad -\binom{m_{\alpha+3}}{1}-\binom{m_{\alpha+3}+1}{2}\Big\}\ .\end{aligned} \tag{B.40}$$

We can show that, after k steps, the above expression generalises to

$$\mathcal{N}_{HH}^{V,\alpha}(d,m_{d-1}) = \sum_{m_{d-2}=1}^{m_{d-1}} \cdots \sum_{m_{\alpha+k}=1}^{m_{\alpha+k+1}} \left\{\mathcal{N}_{HH}(\alpha+k+1,m_{\alpha+k}) - \sum_{j=0}^{k-1}\binom{m_{\alpha+k}+j-1}{j}\right\}\ . \tag{B.41}$$

This follows from the recursive use of the binomial identity

$$\sum_{m=1}^{j}\binom{m+k-1}{k} = \binom{j+k}{k+1}\ . \tag{B.42}$$

The recursion terminates when $k = d-1-\alpha$ and we get

$$\mathcal{N}_{HH}^{V,\alpha>1}(d,m_{d-1}) = \mathcal{N}_{HH}(d,m_{d-1}) - \sum_{j=0}^{d-2-\alpha}\binom{m_{d-1}+j-1}{j}. \tag{B.43}$$

For $\alpha = 1$, we have to take $m_{d-1} \geq m_{d-2} \geq \ldots m_2 \geq 1$, but m_1 is allowed to vanish. This is identical to the $\alpha = 2$ case of the counting above. Thus, we get

$$\mathcal{N}^{V,\alpha=1}_{HH}(d, m_{d-1}) = \mathcal{N}^{V,\alpha=2}_{HH}(d, m_{d-1}) = \mathcal{N}_{HH}(d, m_{d-1}) - \sum_{j=0}^{d-4} \binom{m_{d-1}+j-1}{j} . \quad \text{(B.44)}$$

We can now sum over α to obtain the the total number of VSHs on $\mathbb{S}^{d-1}$ as

$$\mathcal{N}^{V}_{HH}(d, m_{d-1}) = \sum_{\alpha=1}^{d-2} \mathcal{N}^{V,\alpha}_{HH}(d, m_{d-1}) = (d-2)\mathcal{N}_{HH}(d, m_{d-1}) - \mathcal{N}_{HH}(d-2, m_{d-1}+1) \quad \text{(B.45)}$$

As we shall describe in more detail later, $\mathcal{N}^{V}_{HH}(d, \ell)$ counts the number of transverse, divergence-free, homogeneous harmonic polynomial vector fields of degree ℓ in $\mathbb{R}^d$. We have the following explicit expression [146, 147]

$$\mathcal{N}^{V}_{HH}(d, \ell) = \frac{\ell(\ell+d-2)}{(\ell+1)(\ell+d-3)}(d-2)\mathcal{N}_{HH}(d, \ell) = (2\ell+d-2)\frac{\ell}{\ell+d-3}\binom{\ell+d-2}{\ell+1} . \quad \text{(B.46)}$$

The first expression shows that $(d-2)\mathcal{N}_{HH}(d, \ell)$ is actually a good approximation to $\mathcal{N}^{V}_{HH}(d, \ell)$ at large ℓ. We also see that in $d = 3$, the number of VSHs is identical to the number of SSHs.

There is also a VSH analogue of the recursion relation Eq.(A.34) given by

$$\mathcal{N}^{V}_{HH}(d, \ell) = \sum_{m=1}^{\ell} [\mathcal{N}^{V}_{HH}(d-1, m) + \mathcal{N}_{HH}(d-1, m)] , \quad \text{(B.47)}$$

which can be shown using the above formula. Such a recursion relation is automatic in Higuchi's recursive construction of VSHs [145], whereby VSHs in $\mathbb{S}^{d-1}$ are constructed from VSHs and SSHs in $\mathbb{S}^{d-2}$.

B.2 VSH projector and addition theorem

Till now, we have described vector spherical harmonics in terms of an orthonormal basis. While such a basis is ideal for defining a set of linearly independent multipole moments associated with an extended source distribution, it is often convenient to shift to a different basis based on symmetric trace-free (STF) tensors. While the set of harmonics defined this way is overcomplete, it is often easier to compute the multipole moments in this basis. This is especially so for moving particle sources, where convoluting the orthonormal VSHs against the world line would be a tedious exercise, resulting in unwieldy expressions. We will also prove in this subsection the addition theorem for vector spherical harmonics, a crucial tool in going back and forth between the spherical vs the cartesian description.[2]

We will begin by recasting our results on orthonormal VSHs in terms of STF tensors. We will proceed in analogy with our description of SSHs. The orthonormal VSHs defined in Eq.(B.20) can be pushed-forward into $\mathbb{R}^d$ as follows:

$$\frac{1}{r}\left(\frac{\partial x_i}{\partial \vartheta_J}\right)\mathbb{V}^J_{\alpha\ell\vec{m}}(\hat{r}) \equiv \frac{1}{\ell!}\mathbb{V}^{\alpha\ell\vec{m}}_{i<i_1 i_2\ldots i_\ell>}\hat{r}^{<i_1}\hat{r}^{i_2}\ldots\hat{r}^{i_\ell>} \ . \tag{B.48}$$

Note that we have already indicated here the STF structure of the cartesian tensor in RHS. This can be justified as follows: we begin with the observation that the vector field $r^\ell\mathbb{V}^I_{\alpha\ell\vec{m}}(\hat{r})$ is a divergence-free harmonic vector field in $\mathbb{R}^d$. This means that each cartesian component should be harmonic separately: in fact, they should all be homogeneous harmonic polynomials of degree ℓ. This means that the vector field

$$\vec{\mathbb{V}}^{\alpha\ell\vec{m}}(\vec{r}) \equiv \frac{1}{\ell!}\hat{e}_i\mathbb{V}^{\alpha\ell\vec{m}}_{i<i_1 i_2\ldots i_\ell>}x^{<i_1}x^{i_2}\ldots x^{i_\ell>} \ , \tag{B.49}$$

defined using the tensor above should be harmonic in every cartesian component. This is possible if and only if the collection of indices $i_1 i_2 \ldots i_\ell$ is symmetric and trace-free as indicated.

Since this vector field $\vec{\mathbb{V}}^{\alpha\ell\vec{m}}$ is obtained by a push-forward of a divergence-free vector field on $\mathbb{S}^{d-1}$, we conclude that $\vec{\mathbb{V}}^{\alpha\ell\vec{m}}$ is a divergence-free vector field in $\mathbb{R}^d$, transverse to the radial direction, i.e.,

$$\vec{\nabla}\cdot\vec{\mathbb{V}}^{\alpha\ell\vec{m}} = 0 \ , \quad \vec{r}\cdot\vec{\mathbb{V}}^{\alpha\ell\vec{m}} = 0 \ . \tag{B.50}$$

The first condition implies the vanishing of the following contraction

$$\delta^{ii_1}\mathbb{V}^{\alpha\ell\vec{m}}_{i<i_1 i_2\ldots i_\ell>} = 0 \ , \tag{B.51}$$

[2] VSH addition theorem for $d = 3$ are discussed in [163–165]. This should not be confused with the conceptually completely different 'translational' addition theorems [166–169].

whereas the second one indicates the vanishing of the full symmetrisation:

$$\mathbb{V}^{\alpha\ell\vec{m}}_{i<i_1 i_2 \ldots i_\ell>} x^i x^{i_1} x^{i_2} \ldots x^{i_\ell} = 0 \ . \tag{B.52}$$

These properties imply that $\mathbb{V}^{\alpha\ell\vec{m}}_{i<i_1 i_2 \ldots i_\ell>}$ is a irreducible cartesian tensor of $SO(d)$ corresponding to the Young tableaux

$$\begin{array}{|c|c|c|c|c|c|}\hline i_1 & i_2 & i_3 & \ldots & i_{\ell-1} & i_\ell \\ \hline i & & & & & \\ \hline \end{array} \ .$$

The theory of VSHs then becomes equivalent to the study of cartesian tensors with such symmetry.

Many of the results of SSHs directly generalise. For example, the inner product formula in Eq.(A.55) implies a similar formula for VSHs:

$$\begin{split}\int_{\mathbb{S}^{d-1}} \left[\frac{1}{\ell!}\mathbb{V}_{i<i_1 i_2 \ldots i_\ell>} \hat{r}^{<i_1} \ldots \hat{r}^{i_\ell>}\right] \left[\frac{1}{\ell!}\overline{\mathbb{V}}_{i<j_1 j_2 \ldots j_\ell>} \hat{r}^{<j_1} \ldots \hat{r}^{j_\ell>}\right] \\ = \frac{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}{\ell!} \mathbb{V}^{i<i_1 i_2 \ldots i_\ell>} \overline{\mathbb{V}}_{i<i_1 i_2 \ldots i_\ell>} \ ,\end{split} \tag{B.53}$$

true for arbitrary $\mathbb{V}$ and $\overline{\mathbb{V}}$ with constant cartesian components. It then follows that the orthonormality of $\mathbb{V}^I_{\alpha\ell\vec{m}}$ can then be cast in terms of STF tensors as

$$\frac{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}{\ell!} \mathbb{V}^{*i<i_1 i_2 \ldots i_\ell>}_{\alpha\ell\vec{m}'} \mathbb{V}^{\beta\ell\vec{m}}_{i<i_1 i_2 \ldots i_\ell>} = \delta^\alpha_\beta \delta^{\vec{m}}_{\vec{m}'} \ . \tag{B.54}$$

Note the same extra factor of $\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|$ which appears in this inner product, as in the STF inner product for SSHs. With this extra factor, we can then define a vector STF projector analogous to the scalar STF projector defined in Eq.(A.53):

$$(\Pi^V_{ij})^{<i_1 i_2 \ldots i_\ell>}_{<j_1 j_2 \ldots j_\ell>} \equiv \frac{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}{\ell!} \sum_{\alpha\vec{m}} \mathbb{V}^{*i<i_1 i_2 \ldots i_\ell>}_{\alpha\ell\vec{m}} \mathbb{V}^{\alpha\ell\vec{m}}_{j<j_1 j_2 \ldots j_\ell>} \ . \tag{B.55}$$

Given the above definition, the orthonormality relation then guarantees the idempotence of Π^V, i.e., we have

$$(\Pi^V_{ik})^{<i_1 i_2 \ldots i_\ell>}_{<k_1 k_2 \ldots k_\ell>} (\Pi^V_{kj})^{<k_1 k_2 \ldots k_\ell>}_{<j_1 j_2 \ldots j_\ell>} = (\Pi^V_{ij})^{<i_1 i_2 \ldots i_\ell>}_{<j_1 j_2 \ldots j_\ell>} \ . \tag{B.56}$$

As we shall see later, this vector STF projector plays a crucial role in the vector multipole expansion for moving particle sources in dS. For this reason, in the rest of this subsection, we will provide a detailed treatment of this projector. Specifically, we seek explicit expressions to allow quick computations, as well as a catalogue of useful properties.

As we did for the scalar STF projector, it is convenient to define a projected contraction

$$\Pi^V_{ij}(\vec{r}|\vec{\kappa}) \equiv \Pi^V_{ji}(\vec{\kappa}|\vec{r}) \equiv \frac{1}{\ell!}\kappa^{i_1}\dots\kappa^{i_\ell}(\Pi^V_{ij})^{<j_1\dots j_\ell>}_{<i_1\dots i_\ell>}\, r_{j_1}\dots r_{j_\ell} \quad . \tag{B.57}$$

Using Eq.(B.55), this projected contraction can equivalently be defined via an *addition theorem* for VSHs

$$\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|\sum_{\alpha\vec{m}}\mathbb{V}^{*\alpha}_{i\ell\vec{m}}(\hat{r}_0)\mathbb{V}^{\alpha}_{j\ell\vec{m}}(\hat{r}) = \Pi^V_{ij}(\hat{r}_0|\hat{r})_{d,\ell} = \Pi^V_{ji}(\hat{r}|\hat{r}_0)_{d,\ell}\ . \tag{B.58}$$

This is equivalent to the following addition theorem for orthonormal VSHs:

$$\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|\sum_{\alpha\vec{m}}\mathbb{V}^{*\alpha}_{I\ell\vec{m}}(\hat{r}_0)\mathbb{V}^{\alpha}_{J\ell\vec{m}}(\hat{r}) = \Pi^V_{IJ}(\hat{r}_0|\hat{r})_{d,\ell} = \Pi^V_{JI}(\hat{r}|\hat{r}_0)_{d,\ell}\ . \tag{B.59}$$

An appropriate push-forward relates these two formulae:

$$\Pi^V_{ij}(\hat{r}_0|\hat{r})_{d,\ell} = \frac{1}{rr_0}\frac{\partial x_{0i}}{\partial\theta^I_0}\frac{\partial x_j}{\partial\theta^J}\Pi^{IJ}_V(\hat{r}_0|\hat{r})_{d,\ell}\ . \tag{B.60}$$

The vector STF projector satisfies the following equations:

$$\begin{aligned}
\nabla^2\overleftrightarrow{\Pi}^V(\vec{r}|\vec{r}_0) = \nabla_0^2\overleftrightarrow{\Pi}^V(\vec{r}|\vec{r}_0) = 0\ ,\\
\overleftarrow{\nabla}_0\cdot\overleftrightarrow{\Pi}^V(\vec{r}|\vec{r}_0) = \overrightarrow{\nabla}\cdot\overleftrightarrow{\Pi}^V(\vec{r}|\vec{r}_0) = 0\ ,\\
\vec{r}\cdot\overleftrightarrow{\Pi}^V(\vec{r}|\vec{r}_0) = \overleftrightarrow{\Pi}^V(\vec{r}|\vec{r}_0)\cdot\vec{r}_0 = 0\ ,
\end{aligned} \tag{B.61}$$

where the cartesian tensor $\overleftrightarrow{\Pi}^V(\vec{r}|\vec{r}_0)$ is defined via $\overleftrightarrow{\Pi}^V(\vec{r}|\vec{r}_0) \equiv \Pi^V_{ij}(\hat{r}|\hat{r}_0)\hat{e}_i\otimes\hat{e}_j$, and left arrow signifies vector dot product acting on the second index. Upto an overall normalisation, $\overleftrightarrow{\Pi}^V(\vec{r}|\vec{r}_0)$ is the unique tensor that is homogeneous of degree ℓ in both $\vec{r}$ and $\vec{r}_0$, as well as satisfying the above equations. We will now argue that the normalisation of

Π^V can be fixed via the following relation:

$$\delta^{ij}\Pi^V_{ij}(\vec{r}|\vec{r}_0) = \frac{\mathcal{N}^V_{HH}(d,\ell)}{\mathcal{N}_{HH}(d,\ell)}\Pi^S(\vec{r}|\vec{r}_0)_{d,\ell} = \frac{\ell(\ell+d-2)}{(\ell+1)(\ell+d-3)}(d-2)\Pi^S(\vec{r}|\vec{r}_0)_{d,\ell} \ . \tag{B.62}$$

First, for purely symmetry reasons, we should have $\delta^{ij}\Pi^V_{ij} \propto \Pi^S$. The reason is as follows: the combination $\delta^{ij}\Pi^V_{ij}$ is a ℓ^{th} degree harmonic polynomial in $\vec{r}$ and $\vec{r}_0$, invariant under simultaneous rotation of $\vec{r}$ and $\vec{r}_0$ and any such object should be proportional to Π^S. The constant of proportionality can then be fixed by comparing the integral

$$\frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}}\delta^{ij}\Pi^V_{ij}(\hat{r}|\hat{r}) = \sum_{\alpha\vec{m}}\int_{\mathbb{S}^{d-1}}\mathbb{V}^{*\alpha}_{i\ell\vec{m}}(\hat{r})\mathbb{V}^{i\alpha}_{\ell\vec{m}}(\hat{r}) = \mathcal{N}^V_{HH}(d,\ell) \ , \tag{B.63}$$

against the integral

$$\frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}}\Pi^S_{d,\ell}(\hat{r}|\hat{r}) = \sum_{\vec{m}}\int_{\mathbb{S}^{d-1}}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r})\mathscr{Y}_{\ell\vec{m}}(\hat{r}) = \mathcal{N}_{HH}(d,\ell) \ . \tag{B.64}$$

With this normalisation fixed, we have established Eqs.(B.61) and (B.62), which then serve to uniquely define Π^V. With this normalisation, we also have the following over-completeness relation for STF VSHs:

$$\begin{aligned}
&\frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\int_{\hat{r}\in\mathbb{S}^{d-1}}\Pi^V_{ij}(\hat{r}_1|\hat{r})\Pi^V_{jk}(\hat{r}|\hat{r}_2)\\
&= \mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|\sum_{\alpha_1\vec{m}_1}\sum_{\alpha_2\vec{m}_2}\mathbb{V}^{*i\alpha_1}_{\ell\vec{m}_1}(\hat{r}_1)\mathbb{V}^{i\alpha_2}_{\ell\vec{m}_2}(\hat{r}_2)\int_{\hat{r}\in\mathbb{S}^{d-1}}\mathbb{V}^{\alpha_1}_{i\ell\vec{m}_1}(\hat{r})\mathbb{V}^{*i\alpha_2}_{\ell\vec{m}_2}(\hat{r})\\
&= \mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|\sum_{\alpha\vec{m}}\mathbb{V}^{*i\alpha}_{\ell\vec{m}}(\hat{r}_1)\mathbb{V}^{i\alpha}_{\ell\vec{m}}(\hat{r}_2) = \Pi^V_{ik}(\hat{r}_1|\hat{r}_2) \ .
\end{aligned} \tag{B.65}$$

We will now use the above properties to explicitly construct Π^V. To this end, we remind the reader of our construction of the orthonormal VSHs via second-order differential operators on orthonormal SSHs. We will now employ a similar construction to derive the vector STF projector in terms of the scalar STF projector. We will start with the ansatz that $\Pi^V_{ij}(\vec{r}|\vec{r}_0) = \mathbb{A}_{ij}\Pi^S(\vec{r}|\vec{r}_0)_{d,\ell}$ with $\mathbb{A}_{ij}$ being a 2-derivative operator in $\vec{r}$. Since both $\Pi^S(\vec{r}|\vec{r}_0)_{d,\ell}$ as well as $\Pi^V_{ij}(\vec{r}|\vec{r}_0)$ have the same homogeneity (i.e., they are both of degree ℓ in $\vec{r}$), the derivative should not change the number of x's in each term. This leaves us with four possibilities:

$$\delta_{ij}\ ,\ x_j\partial_i\ ,\ x_i\partial_j\ ,\ x^2\partial_i\partial_j\ . \tag{B.66}$$

Here, we have used the fact that when acting on a homogeneous polynomial, the operator $x_i\partial_i$ reduces to a number.

Next, we impose the constraint that $x^i\Pi^V_{ij}(\vec{r}|\vec{r}_0) = 0$, which in turn implies that only the following two combinations can occur in $\mathbb{A}_{ij}$:

$$\ell\delta_{ij} - x_j\partial_i\ ,\ (\ell-1)x_i\partial_j - x^2\partial_i\partial_j\ . \tag{B.67}$$

Finally, imposing that $\Pi^V_{ij}(\vec{r}|\vec{\kappa})_{d,\ell}$ should be divergence-free in its first index, we conclude that only one combination is admissible, viz.,

$$\mathbb{A}_{ij} \propto \left\{\ell\ \delta_{ij} - x_j\partial_i - \frac{1}{\ell+d-3}[(\ell-1)\ x_i - x^2\partial_i]\partial_j\right\}\ . \tag{B.68}$$

As a consistency check of our ansatz, one may also check that this yields a harmonic tensor in $\vec{r}$. Fixing the normalisation via Eq.(B.62), we then obtain

$$\begin{aligned}\Pi^V_{ij}(\vec{r}|\vec{r}_0)_{d,\ell} &\equiv \mathbb{A}_{ij}\Pi^S(\vec{r}|\vec{r}_0)_{d,\ell}\\ &\equiv \frac{1}{\ell+1}\left\{\ell\ \delta_{ij} - x_j\partial_i - \frac{1}{\ell+d-3}[(\ell-1)\ x_i - x^2\partial_i]\partial_j\right\}\Pi^S_{d,\ell}(\vec{r}|\vec{r}_0)\ .\end{aligned} \tag{B.69}$$

The $\mathbb{A}_{ij}$ is also a toroidal operator similar to the ones described in §§B.1. Using the series form of Π^S, and performing the derivatives, we obtain the following form for the VSH projector:

$$\begin{aligned}\Pi^V_{ij}(\vec{r}|\vec{r}_0)_{d,\ell} \equiv &\frac{1}{(\ell+1)(\ell+d-3)}\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\Gamma(\nu-k)}{k!\ \Gamma(\nu)}\left(-\frac{r_0^2r^2}{4}\right)^k\\ &\times\left\{\delta_{ij}\left[\ell(\ell+d-2)-(\ell-2k)^2\right]\frac{(\vec{r}_0\cdot\vec{r})^{\ell-2k}}{(\ell-2k)!} - (d-2)\ x_{0i}x_j\frac{(\vec{r}_0\cdot\vec{r})^{\ell-2k-1}}{(\ell-2k-1)!}\right.\\ &\left.+\left[-(x_ix_{0j}+x_{0i}x_j)(\vec{r}_0\cdot\vec{r}) + \delta_{ij}(\vec{r}_0\cdot\vec{r})^2 + r_0^2x_ix_j + r^2x_{0i}x_{0j}\right]\frac{(\vec{r}_0\cdot\vec{r})^{\ell-2k-2}}{(\ell-2k-2)!}\right\}\ .\end{aligned} \tag{B.70}$$

A more useful form is obtained by grouping together the terms above to make transversality manifest:

$$
\begin{aligned}
\Pi^V_{ij}(\vec{r}|\vec{r}_0)_{d,\ell}V^j = &\frac{1}{\ell+1}\Bigg\{\frac{(\vec{r}_0\cdot\vec{r})^{\ell-1}}{(\ell-1)!}[x\cdot(x_0\wedge V)]_i \\
&-\frac{[r^2V_{\perp i}-x_i(\vec{r}\cdot\vec{V}_\perp)]}{\ell+d-3}\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor-1}\frac{\Gamma(\nu-k)}{k!\;\Gamma(\nu)}\left(-\frac{r_0^2r^2}{4}\right)^k\frac{(\vec{r}_0\cdot\vec{r})^{\ell-2k-2}}{(\ell-2k-2)!} \\
&-\frac{r^2}{4}\frac{[(\vec{r}_0\cdot\vec{r})V_{\perp i}-x_{0i}(\vec{r}\cdot\vec{V}_\perp)]}{(\ell+d-3)} \\
&\times\sum_{k=1}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\Gamma(\nu-k)}{k!\;\Gamma(\nu)}\left(-\frac{r_0^2r^2}{4}\right)^{k-1}\frac{(\vec{r}_0\cdot\vec{r})^{\ell-2k-1}}{(\ell-2k-1)!}(\ell+d-3-2k)\Bigg\}_{\nu=\ell+\frac{d}{2}-1}.
\end{aligned}
\tag{B.71}
$$

Here, we have used the notation

$$
(x_0\wedge V)_{ij}\equiv x_{0i}V_j-x_{0j}V_i\;,\qquad V_{\perp i}\equiv[x_0\cdot(x_0\wedge V)]_i=r_0^2V_i-(x_0\cdot V)x_{0i} \tag{B.72}
$$

to simplify our expressions. The formula above is manifestly transverse to $\vec{r}$ in the first index and $\vec{r}_0$ in the second index. The simplest case is where we take $\vec{r}_0=\hat{e}_d$ and $\vec{V}=\hat{e}_{d-1}$ where $\hat{e}_i$ are the cartesian basis vectors in $\mathbb{R}^d$. The above formula then simplifies to

$$
\begin{aligned}
\Pi^V_{i,d-1}(\vec{r}|\hat{e}_d)\frac{\partial}{\partial x_i} = &\,\mathcal{N}^V_{HH}(d,\ell)\;\mathcal{N}_{d,\ell} \\
&\times\frac{1}{d-1}\Bigg[r^{\ell-1}P_{\ell-1}\left(d+2,\frac{x_d}{r}\right)\left\{x_d\frac{\partial}{\partial x_{d-1}}-x_{d-1}\frac{\partial}{\partial x_d}\right\} \\
&+\frac{(\ell-1)(\ell+d-1)}{(d+1)(d-2)}r^{\ell-2}P_{\ell-2}\left(d+4,\frac{x_d}{r}\right)\sum_{j=1}^{d-2}x_j\left\{x_{d-1}\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_{d-1}}\right\}\Bigg]\;,
\end{aligned}
\tag{B.73}
$$

where $\mathcal{N}^V_{HH}(d,\ell)$ is the number of VSHs of degree ℓ on $\mathbb{S}^{d-1}$ (See Eq.(B.46)). We recognise here the appearance of the $SO(d-2)$ invariant harmonic vector field obtained by push-forward of the most symmetric VSH (see Eq.(B.37)). This is then the vector analogue of the statement that the scalar projector is proportional to the Legendre harmonic, i.e.,

$$
\Pi^S_{d,\ell}(\vec{r}|\hat{e}_d)=\mathcal{N}_{d,\ell}\mathcal{N}_{HH}(d,\ell)r^\ell P_\ell\left(d,\frac{x_d}{r}\right)\;. \tag{B.74}
$$

This follows directly from Eq.(A.51).

B.3 VSH projector using Young tableau methods

In the last section, we remarked that VSHs correspond to irreducible cartesian tensors of $SO(d)$ with the symmetry of their indices specified by the Young tableau

$$\begin{array}{|c|c|c|c|c|c|}\hline i_1 & i_2 & i_3 & \ldots & i_{\ell-1} & i_\ell \\ \hline i & \multicolumn{5}{l}{} \\ \cline{1-1}\end{array} \; .$$

For $SO(d)$, irreducible representations are obtained by Young tableaux with the number of rows less than $\frac{d}{2}$. This gives all irreducible tensor representations, except the $SO(2n)$ tensors with self-dual or anti-self-dual form indices (which correspond to reducible tableau with exactly $\frac{d}{2}$ rows).

We remind the reader how such a tableau should be interpreted in the context of $SO(d)$: the rows of the tableau indicate symmetrisation+trace-removal, and the columns indicate anti-symmetrisation. These two steps are done sequentially, then give an irreducible tensor with an appropriate symmetry. Given a Young tableau, standard theorems in $SO(d)$ representation theory give formulae for dimension, character, Clebsh-Gordon decomposition, etc., of the corresponding irreducible representation. As an example, if n_α is the number of boxes in the α^{th} row, the dimension of the corresponding $SO(d)$ irrep is [170](eqns. (3.1) and (3.2))

$$\prod_{\alpha>\beta|\alpha,\beta=1}^{\lfloor\frac{d}{2}\rfloor} \frac{(\alpha-n_\alpha)-(\beta-n_\beta)}{\alpha-\beta} \times \prod_{\alpha\geq\beta|\alpha,\beta=1}^{\lfloor\frac{d}{2}\rfloor} \frac{d-(\alpha-n_\alpha)-(\beta-n_\beta)}{d-\alpha-\beta} \tag{B.75}$$

for odd d. For even d, a similar formula holds provided we drop all the $\alpha=\beta$ terms in the second product. The structure of these products is such that only non-empty rows contribute to β: if β^{th} row is empty (i.e., if $n_\beta=0$) so is the α^{th} row (i.e., $n_\alpha=0$), and the contribution to the product is unity.

As an example, we can count the number of SSHs at a given d and ℓ by taking $n_1=\ell$ and $n_2=\ldots=n_{\lfloor\frac{d}{2}\rfloor}=0$. The two product factors become

$$\frac{2-(1-\ell)}{1}\frac{3-(1-\ell)}{2}\ldots\frac{\lfloor\frac{d}{2}\rfloor-(1-\ell)}{\lfloor\frac{d}{2}\rfloor-1} = \binom{\ell+\lfloor\frac{d}{2}\rfloor-1}{\ell}\,, \tag{B.76}$$

and

$$\frac{d-2(1-\ell)}{d-2}\frac{d-2-(1-\ell)}{d-2-1}\frac{d-3-(1-\ell)}{d-3-1}\cdots\frac{d-\lfloor\frac{d}{2}\rfloor-(1-\ell)}{d-\lfloor\frac{d}{2}\rfloor-1}$$
$$=\frac{2\ell+d-2}{d-2}\binom{\ell+d-3}{\ell}\binom{\ell+\lfloor\frac{d}{2}\rfloor-1}{\ell}^{-1} . \tag{B.77}$$

Here, we have indicated by red the $\alpha=\beta$ factor present only in odd d. The net product then matches with the explicit count of SSHs given in Eq.(A.20). In a similar vein, we can count spin-s spherical harmonics by taking

$$n_1=\ell,\ n_2=s,\ n_3=\ldots=n_{\lfloor\frac{d}{2}\rfloor}=0 . \tag{B.78}$$

A similar count as above, performed separately in odd vs even dimensions, gives the number of spin-s spherical harmonics as

$$\begin{aligned}\mathcal{N}^{(s)}_{HH}(d,\ell)&\equiv\frac{(2\ell+d-2)(2s+d-4)(\ell-s+1)(\ell+s+d-3)}{(d-2)(d-4)(\ell+1)(\ell+d-3)}\binom{\ell+d-3}{\ell}\binom{s+d-5}{s}\\&=\frac{(2s+d-4)(\ell-s+1)(\ell+s+d-3)}{(d-4)(\ell+1)(\ell+d-3)}\binom{s+d-5}{s}\mathcal{N}_{HH}(d,\ell) .\end{aligned} \tag{B.79}$$

As for VSHs, we can set $s=1$ in the above formula and recover Eq.(B.46).

The Young tableau methods can also be used to derive the VSH projector (up to an overall normalisation). In the rest of this subsection, we will review some of the Young tableau based methods existing in the literature. No new results are derived here, however, so an uninterested reader may safely skip this subsection.

Young-symmetriser

We will follow the recent works by Henry, Faye and Blanchet(HFB) [161] as well as Amalberti, Larrouturou and Yang(ALY) [162] to construct the projector corresponding to the above tableau. We begin with the following definition of the trace-free projector (see Eq.(A6) of ALY or 2nd line of Eq.(A4) of HFB):

$$
\begin{aligned}
\mathbb{TF}[V^a x^{<i_1} x^{i_2} \dots x^{i_\ell>}] \equiv\ & V^a x^{<i_1} x^{i_2} \dots x^{i_\ell>} \\
& - \frac{\ell(\nu-1)}{(\ell+d-3)\nu} \mathrm{STF}_{i_1 \dots i_\ell}[\delta^{a i_\ell} V_b x^{<i_1} x^{i_2} \dots x^{i_{\ell-1}} x^{b>}] \\
\equiv\ & V^a x^{<i_1} x^{i_2} \dots x^{i_\ell>} \\
& - \frac{\ell(\nu-1)}{(\ell+d-3)\nu} \mathrm{Sym}_{i_1 \dots i_\ell}[\delta^{a i_\ell} V_b x^{<i_1} x^{i_2} \dots x^{i_{\ell-1}} x^{b>}] \\
& + \frac{1}{2} \frac{\ell(\ell-1)}{(\ell+d-3)\nu} \mathrm{Sym}_{i_1 \dots i_\ell}[\delta^{i_\ell i_{\ell-1}} V_b x^{<i_1} x^{i_2} \dots x^{i_{\ell-2}} x^a x^{b>}] \ .
\end{aligned}
\tag{B.80}
$$

Here $\nu \equiv \ell + \frac{d}{2} - 1$ and STF/Sym denote the STF/symmetric projector onto its subscript indices, respectively. This trace-free projection is the first step in the construction of the VSH projector.

To understand the relation of this trace-free projector in our language, we define the following polynomial vector field

$$
\begin{aligned}
U^a \equiv\ & \frac{1}{\ell!} \kappa_{i_1} \dots \kappa_{i_\ell} \mathbb{TF}[V^a x^{<i_1} x^{i_2} \dots x^{i_\ell>}] \\
\equiv\ & V^a \Pi^S_{d,\ell}(\vec{\kappa}|\vec{r}) \\
& - \kappa^a \frac{\nu-1}{(\ell+d-3)\nu} \times \frac{1}{(\ell-1)!} \kappa_{i_1} \dots \kappa_{i_{\ell-1}} V_b x^{<i_1} x^{i_2} \dots x^{i_{\ell-1}} x^{b>} \\
& + \frac{\vec{\kappa}^2}{2} \frac{1}{(\ell+d-3)\nu} \times \frac{1}{(\ell-2)!} \kappa_{i_1} \dots \kappa_{i_{\ell-2}} V_b x^{<i_1} x^{i_2} \dots x^{i_{\ell-2}} x^a x^{b>} \\
=\ & V^b \left\{ \delta^a_b - \frac{\nu-1}{(\ell+d-3)\nu} \kappa^a \frac{\partial}{\partial \kappa_b} + \frac{\vec{\kappa}^2}{2} \frac{1}{(\ell+d-3)\nu} \frac{\partial^2}{\partial \kappa_a \partial \kappa_b} \right\} \Pi^S_{d,\ell}(\vec{\kappa}|\vec{r}) \ .
\end{aligned}
\tag{B.81}
$$

We will now show that $\mathbb{TF}$ is indeed the trace-free projector as claimed by ALY, i.e., we have

$$
\delta_{a i_\ell} \mathbb{TF}[V^a x^{<i_1} x^{i_2} \dots x^{i_\ell>}] = 0 \ . \tag{B.82}
$$

In our notation, this is equivalent to the assertion that $\frac{\partial U^a}{\partial \kappa_a} = 0$. From the

$$\begin{aligned}\frac{\partial U^a}{\partial \kappa_a} &= V^b \frac{\partial}{\partial \kappa_a}\left\{\delta^a_b - \frac{\nu-1}{(d+\ell-3)\nu}\kappa^a\frac{\partial}{\partial \kappa_b} + \frac{\vec{\kappa}^2}{2}\frac{1}{(d+\ell-3)\nu}\frac{\partial^2}{\partial \kappa_a \partial \kappa_b}\right\}\Pi^S_{d,\ell}(\vec{\kappa}|\vec{r}) \\ &= \left\{1 - \frac{\nu-1}{(d+\ell-3)\nu}\left(\kappa^a\frac{\partial}{\partial \kappa_a} + d\right) + \frac{1}{(d+\ell-3)\nu}\left(\kappa^a\frac{\partial}{\partial \kappa_a}\right)\right\} V^b\frac{\partial}{\partial \kappa_b}\Pi^S_{d,\ell}(\vec{\kappa}|\vec{r}) \ .\end{aligned} \tag{B.83}$$

We have used the harmonicity of $\Pi^S_{d,\ell}(\vec{\kappa}|\vec{r})$ to simplify the last term in the last line. Since $V^b\frac{\partial}{\partial \kappa_b}\Pi^S_{d,\ell}$ is a homogeneous polynomial in κ of degree $(\ell-1)$, we can use Euler's homogeneous function theorem to replace all $\kappa^a\frac{\partial}{\partial \kappa_a}$ above by $(\ell-1)$. With this replacement, the prefactor above vanishes, showing that $\mathbb{TF}$ is indeed the trace-free projector.

A similar computation yields

$$\begin{aligned}\nabla^2_\kappa U^a &= V^b\left\{-2\frac{\nu-1}{(d+\ell-3)\nu}\frac{\partial^2}{\partial \kappa^a \partial \kappa_b} + \frac{1}{(\ell+d-3)\nu}\left(2\kappa^c\frac{\partial}{\partial \kappa_c} + d\right)\frac{\partial^2}{\partial \kappa_a \partial \kappa_b}\right\}\Pi^S_{d,\ell}(\vec{\kappa}|\vec{r}) \\ &= 0 \ .\end{aligned} \tag{B.84}$$

Thus, the vector field U^a is a harmonic, divergence-free vector field in the κ_i variables. This, in turn, implies that U^a is a linear combination of the gradient of HHPs and toroidal vector fields. The gradient part has to be subtracted to get an irreducible tensor: as we shall see shortly, removing this gradient is equivalent to anti-symmetrisation imposed by the first column of the Young tableau.

The gradient part can be isolated by looking at the radial component

$$\kappa_a U^a = V^b\left\{\kappa_b - \frac{\vec{\kappa}^2}{2\nu}\frac{\partial}{\partial \kappa_b}\right\}\Pi^S_{d,\ell}(\vec{\kappa}|\vec{r}) \ . \tag{B.85}$$

Removing this gradient then gives a toroidal vector field

$$\begin{aligned}U^a_\perp &\equiv (\ell+1)U^a - \frac{\partial}{\partial \kappa_a}[\kappa_b U^b] \\ &= V^b\left\{\ell\delta^a_b - \kappa_b\frac{\partial}{\partial \kappa_a} - \frac{\ell-1}{\ell+d-3}\kappa^a\frac{\partial}{\partial \kappa_b} + \frac{\vec{\kappa}^2}{\ell+d-3}\frac{\partial^2}{\partial \kappa_a \partial \kappa_b}\right\}\Pi^S_{d,\ell}(\vec{\kappa}|\vec{r}) \\ &= (\ell+1)\Pi^V_{ab}(\vec{\kappa}|\vec{r})V^b \ ,\end{aligned} \tag{B.86}$$

where we have used our formula for Π^V derived in Eq.(B.69). From the definition of U^a, we also have

$$
\begin{aligned}
U_\perp^a &\equiv (\ell+1)U^a - \frac{\partial}{\partial\kappa_a}[\kappa_b U^b] = \ell U^a - \kappa_b\frac{\partial U^b}{\partial \kappa_a} = \kappa_b\left(\frac{\partial U^a}{\partial\kappa_b} - \frac{\partial U^b}{\partial\kappa_a}\right) \\
&= \frac{1}{(\ell-1)!}\kappa_{i_1}\ldots\kappa_{i_{\ell-1}}\kappa_b\left(\delta^{bi_\ell}\mathbb{TF}[V^a x^{<i_1}x^{i_2}\ldots x^{i_\ell>}] - \delta^{ai_\ell}\mathbb{TF}[V^b x^{<i_1}x^{i_2}\ldots x^{i_\ell>}]\right) \\
&= \frac{1}{(\ell-1)!}\kappa_{i_1}\ldots\kappa_{i_{\ell-1}}\kappa_b \text{ Anti}_{ab}\left[\mathbb{TF}[V^a x^{<i_1}x^{i_2}\ldots x^{i_{\ell-1}}x^{b>}]\right] \ .
\end{aligned}
\tag{B.87}
$$

Comparing, we get

$$
\frac{\ell+1}{\ell}\Pi^V_{ab}(\vec{\kappa}|\vec{r})V^b = \frac{1}{\ell!}\kappa_{i_1}\ldots\kappa_{i_\ell}\text{ Anti}_{ai_\ell}\left[\mathbb{TF}[V^a x^{<i_1}x^{i_2}\ldots x^{i_\ell>}]\right] \ . \tag{B.88}
$$

Stripping off the dummy κ factors, we obtain the following relation

$$
\begin{aligned}
(\Pi^V)^{i<i_1i_2\ldots i_\ell>}_{j<j_1j_2\ldots j_\ell>}V^j x^{j_1}x^{j_2}\ldots x^{j_\ell} = &\frac{1}{\ell+1}\text{Anti}_{ii_1}\left[\mathbb{TF}[V^i x^{<i_1}x^{i_2}\ldots x^{i_\ell>}]\right] \\
&+ \frac{1}{\ell+1}\text{Anti}_{ii_2}\left[\mathbb{TF}[V^i x^{<i_1}x^{i_2}\ldots x^{i_\ell>}]\right] \\
&+ \ldots + \frac{1}{\ell+1}\text{Anti}_{ii_\ell}\left[\mathbb{TF}[V^i x^{<i_1}x^{i_2}\ldots x^{i_\ell>}]\right] \ .
\end{aligned}
\tag{B.89}
$$

This is then the crucial relation we are after: it connects our vector projector to the sequential process of trace-removal followed by anti-symmetrisation. This relation can also be used to compare the results quoted in [161, 162] against our expressions.

VSH projector via weight shifting operators

We will next describe a slightly different route to constructing the projector using Young tableau- the method of *weight-shifting operators* [171, 172]. We will first state here the general algorithm behind this method without proof, and then apply it to the special case of vector projector.[3]

Say we need a projector for a representation corresponding to an arbitrary Young tableau with h rows. Then we proceed as follows

- We first introduce h number of cartesian positions: say we denote them by $x_{i,\alpha}$ corresponding to the α^{th} row. We will call $x_{i,\alpha}$ as the α^{th} row position.

[3]We would like to thank Arnab Rudra and Kushal Chakraborty for explaining this method to us and sharing with us their notes on this subject.

- Next, we construct a *seed* polynomial, which is a product of factors, one factor for each column. The factor for a column is the completely anti-symmetric polynomial of the row positions, made out of all the rows which contribute to that column (see below for how this works for toroidal harmonic vector fields). The seed polynomial of a tableau with n boxes is hence a polynomial of total homogeneity n. Further, the degree of homogeneity in each row position is the number of boxes in that row.

- The third step is to apply a *weight-shifting* differential operator on this seed polynomial. The weight-shifting differential operator is given by the matrix product of derivative operators, one each for every row. The $d \times d$ matrix of derivatives for α^{th} row is given by

$$\left(\delta^j_k - \frac{x_{j,\alpha}}{N_\alpha}\frac{\partial}{\partial x_{k,\alpha}}\right) \tag{B.90}$$

  where $N_\alpha \equiv d - 1 - h + n_h - \alpha + n_\alpha$ with n_α denoting the number of boxes in α^{th} row. The only exception is the last row, for which we take a $d \times 1$ column matrix of derivatives of the form

$$\left(\delta^j_p - \frac{x_{j,h}}{N_h - 1}\frac{\partial}{\partial x_{p,h}}\right)\frac{\partial}{\partial x_{p,h}} \ . \tag{B.91}$$

  The product of the square matrices with this column matrix then yields a $d \times 1$ column matrix of derivative operators.

- The fourth step is to apply the projector corresponding to a Young tableaux with one less box in the last row.

The claim is that the resultant polynomial gives a recursive construction for the required projector (up to some overall normalisation). We will refer the reader to [172] for a more detailed exposition of this algorithm with a variety of examples. Our interest is in applying this to the Young tableaux

$$\begin{array}{|c|c|c|c|c|c|}\hline i_1 & i_2 & i_3 & \ldots & i_{\ell-1} & i_\ell \\ \hline i & \multicolumn{5}{c}{} \\ \cline{1-1}\end{array}$$

corresponding to VSHs. For this case, we have $h = 2, n_h = 1$ so that $d-1-h+n_h = d-2$ and the numbers $N_1 = \ell+d-3$ and $N_2 = d-3$. Projector applied after the weight-shifting operator is determined by the tableau with the i box removed:

$$\begin{array}{|c|c|c|}\hline i_1 & i_2 & i_3 \\ \hline \end{array} \ \dots \ \begin{array}{|c|c|}\hline i_{\ell-1} & i_\ell \\ \hline \end{array}$$

This is the tableau for symmetric trace-free tensors, and the final projector needed is just the SSH projector Π^S.

Let us begin by assigning the row positions x_i and y_i corresponding to the two rows of the tableaux above. The seed polynomial is then given by

$$x_{[i_1} y_{i]} x_{i_2} \dots x_{i_\ell} = (x_{i_1} y_i - x_i y_{i_1}) x_{i_2} \dots x_{i_\ell}. \tag{B.92}$$

Here, the first anti-symmetric factor corresponds to the first column, whereas the rest of the monomials are contributions from the rest of the columns.

The weight shifting differential operator is a column of derivative operators given by

$$\left(\delta^j_k - \frac{x^j}{\ell+d-3}\frac{\partial}{\partial x^k}\right)\left(\delta^k_p - \frac{y^k}{d-4}\frac{\partial}{\partial y^p}\right)\frac{\partial}{\partial y^p}. \tag{B.93}$$

Since the seed polynomial in this case is linear in y, we can drop all the second derivatives in y to simplify this to

$$\left(\delta^j_k - \frac{x^j}{\ell+d-3}\frac{\partial}{\partial x^k}\right)\frac{\partial}{\partial y^k}. \tag{B.94}$$

The fourth step involves the projector of a smaller Young tableaux: in this case, this is just the scalar projector that can be realised via STF projecting the differential operator on x^i. To summarise, the weight-shifting algorithm gives the following expression for the VSH projector:

$$\begin{aligned}
&(\Pi^V)^{i<i_1\dots i_\ell>}_{j<j_1\dots j_\ell>} x^{j_1} x^{j_2} \dots x^{j_\ell} \\
&\quad \propto \left[\sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} \frac{\Gamma(\nu-k)}{k!\,\Gamma(\nu)}\left(\frac{r}{2}\right)^{2k}(-\nabla^2)^k\right]_{\nu=\frac{d}{2}+\ell-1} \\
&\qquad\qquad \left(\delta^j_p - \frac{x^j}{\ell+d-3}\frac{\partial}{\partial x^p}\right)\frac{\partial}{\partial y^p}[x^{[i_1} y^{i]} x^{i_2} \dots x^{i_\ell}].
\end{aligned} \tag{B.95}$$

We will now prove this relation and determine the proportionality constant. It is easier to work with the form obtained by contracting the free STF indices with a dummy variable and differentiating, viz.,

$$
\begin{aligned}
&\frac{1}{\ell!}\left[\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\Gamma(\nu-k)}{k!\,\Gamma(\nu)}\left(\frac{r}{2}\right)^{2k}(-\nabla^2)^k\right]_{\nu=\frac{d}{2}+\ell-1}\\
&\qquad\left(\delta_p^j-\frac{x^j}{\ell+d-3}\frac{\partial}{\partial x^p}\right)[\delta_p^i(\vec{\kappa}\cdot\vec{r})^\ell-x^i\kappa^p(\vec{\kappa}\cdot\vec{r})^{\ell-1}]\\
&\quad=\frac{1}{\ell!}\left[\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\Gamma(\nu-k)}{k!\,\Gamma(\nu)}\left(\frac{r}{2}\right)^{2k}(-\nabla^2)^k\right]_{\nu=\frac{d}{2}+\ell-1}\\
&\qquad\left(\delta_{ij}(\vec{\kappa}\cdot\vec{r})^\ell-x^i\kappa^j(\vec{\kappa}\cdot\vec{r})^{\ell-1}-\frac{(\ell-1)\kappa^i x^j}{\ell+d-3}(\vec{\kappa}\cdot\vec{r})^{\ell-1}+(\ell-1)\frac{\kappa^2x^ix^j}{\ell+d-3}(\vec{\kappa}\cdot\vec{r})^{\ell-2}\right)\ .
\end{aligned}
\tag{B.96}
$$

This form can be converted to a more familiar form by rewriting it in terms of κ derivatives as

$$
\begin{aligned}
&\left[\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\Gamma(\nu-k)}{k!\,\Gamma(\nu)}\left(\frac{r}{2}\right)^{2k}(-\nabla^2)^k\right]_{\nu=\frac{d}{2}+\ell-1}\\
&\quad\frac{1}{\ell}\left(\ell\delta_{ij}-\kappa^j\frac{\partial}{\partial\kappa^i}-\frac{(\ell-1)}{\ell+d-3}\kappa^i\frac{\partial}{\partial\kappa^j}+\frac{\kappa^2}{\ell+d-3}\frac{\partial^2}{\partial\kappa^i\partial\kappa^j}\right)\frac{(\vec{\kappa}\cdot\vec{r})^\ell}{\ell!}\\
&=\frac{1}{\ell}\left(\ell\delta_{ij}-\kappa^i\frac{\partial}{\partial\kappa^j}-\frac{(\ell-1)}{\ell+d-3}\kappa^j\frac{\partial}{\partial\kappa^i}+\frac{\kappa^2}{\ell+d-3}\frac{\partial^2}{\partial\kappa^i\partial\kappa^j}\right)\Pi^S(\vec{\kappa}|\vec{r})\ .
\end{aligned}
\tag{B.97}
$$

We recognise here the derivative operator mapping SSH projector to VSH projector (see Eq.(B.69)). We have thus proved that

$$
\begin{aligned}
\frac{\ell+1}{\ell}\Pi^V_{ij}(\vec{\kappa}|\vec{r})&=\frac{1}{\ell!}\kappa_{i_1}\kappa_{i_2}\dots\kappa_{i_\ell}\\
&\times\left[\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\Gamma(\nu-k)}{k!\,\Gamma(\nu)}\left(\frac{r}{2}\right)^{2k}(-\nabla^2)^k\right]_{\nu=\frac{d}{2}+\ell-1}\\
&\qquad\left(\delta_p^j-\frac{x^j}{\ell+d-3}\frac{\partial}{\partial x^p}\right)\frac{\partial}{\partial y^p}[x^{[i_1}y^{i]}x^{i_2}\dots x^{i_\ell}].
\end{aligned}
\tag{B.98}
$$

This is an especially succinct formula for the VSH projector. The above derivation also shows the crucial difference between the weight-shifting method vs the method of toroidal operator acting on Π^S: in the weight-shifting case, the SSH projection is done at the very end. Such an exercise can hopefully be generalised to give a closed-form expression for spin-s projectors.

Appendix C

Electromagnetic Radiation in Flat Spacetime

This appendix describes the multipole expansion for Maxwell's theory in $d+1$ dimensional flat space. Our primary motivation is to have a benchmark to compare the $H \to 0$ limit of our dS expressions. We will emphasise two ways to think about multipole expansion: first in terms of orthonormal spherical harmonics, and second in terms of symmetric trace-free (STF) cartesian tensors. Both these formalisms have their own advantage, and both of them are necessary to compute radiation reaction in de Sitter.

The $d = 3$ version of orthonormal multipoles is standard and is described in classic textbooks and articles [114, 153–155, 173–178]. The normalisations and conventions, however, differ from one reference to the other, and often even within the same textbook between statics and radiation. Discussion in most references is also incomplete in a variety of ways, e.g., they often do not describe what fields look like within sources.

The STF multipoles in EM are discussed mainly in gravitational wave literature [151, 160] where STF tensors are used widely. We did not find any reference systematically describing the relation between these two kinds of multipoles, especially at the level of detail needed for our work.[1] Our goal here is to clearly describe the connection between the two kinds of multipoles in flat space EM: corresponding objects in dS can then be understood as a generalisation. For the convenience of the reader, we provide a detailed

[1]The relation is straightforward in electrostatics, where only scalar spherical harmonics are involved: see Kip-Thorne's review [159], appendix A of [179], the textbook by Poisson-Will [144]), or the book by Soffel-Han [180].The references [181, 182] discuss applications to celestial mechanics, and [183] describe the effects of gravitational vector moments. What we did not find is a similar discussion of the conversion rules for magnetostatics and beyond.

comparison between $d = 3$ normalisations in §C.9.

The discussion of multipoles for general dimensional EM can be found in [85,149,162]. The main novelty in general d is the fact that the magnetic field B_{ij} is a 2-form and is no more a pseudo-vector field. All references cited above emphasise the STF viewpoint.[2] We will show here that the results of the previous appendix §B can be used to give a description of both STF and orthonormal multipole moments in general d.

C.1 Multipole expansion in statics I : toroidal currents

Magnetic fields due to toroidal currents

Let us begin with the simpler setting of magnetostatics and then generalise to time-dependent situations involving magnetic multipole radiation. Consider the following problem in magnetostatics: imagine a steady *toroidal* charge current, i.e., a constant, divergence-free current that flows everywhere tangentially to a thin spherical shell of radius R. Explicitly, we take a charge current density of the form

$$\bar{J}^r = 0 \ , \ \bar{J}^I(\vec{r}) = \delta(r-R)\ \mathcal{K}^I(\hat{r}) = \delta(r-R)\sum_{\alpha\ell\vec{m}} \mathcal{K}^I_{\alpha\ell\vec{m}}(\hat{r}) \ , \tag{C.1}$$

where the index I denotes the sphere directions and $\{\alpha, \ell, \vec{m}\}$ label an orthonormal basis of divergence-free vector fields on $\mathbb{S}^{d-1}$ denoted by $\mathbb{V}^I_{\alpha\ell\vec{m}}(\hat{r})$. We will find it convenient to take $\mathbb{V}^I_{\alpha\ell\vec{m}}(\hat{r})$ to be an orthonormal basis of *Vector Spherical Harmonics*(VSHs) on $\mathbb{S}^{d-1}$, i.e.,

$$\left[\mathscr{D}^2 + \ell(\ell+d-2) - 1\right]\mathbb{V}^I_{\alpha\ell\vec{m}} = 0 \ , \quad \mathscr{D}_I \mathbb{V}^I_{\alpha\ell\vec{m}} = 0 \ , \quad \int_{\mathbb{S}^{d-1}} \gamma_{IJ}\mathbb{V}^{I*}_{\alpha'\ell'\vec{m}'}\mathbb{V}^J_{\alpha\ell\vec{m}} = \delta_{\alpha'\alpha}\delta_{\ell'\ell}\delta_{\vec{m}'\vec{m}} \ . \tag{C.2}$$

Here γ_{IJ} is the standard metric on $\mathbb{S}^{d-1}$ and $\mathscr{D}_I$ is the corresponding covariant derivative. We will also define VSH with lower indices as $\mathbb{V}_I^{\alpha\ell\vec{m}} \equiv \gamma_{IJ}\mathbb{V}^J_{\alpha\ell\vec{m}}$, i.e., in our conventions, the VSH indices will always be lowered using the unit sphere metric rather than the spacetime metric. To avoid confusion, all other raising and lowering of sphere indices will be written out explicitly using the unit sphere metric. We will also need the *VSH*

[2] The reference [149] takes a hybrid viewpoint, but its treatment of VSHs is closer to the cartesian STF approach.

addition theorem (see the discussion around Eq.(B.59))

$$\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|\sum_{\alpha\vec{m}}\mathbb{V}_I^{*\alpha\ell\vec{m}}(\hat{r}_0)\ \mathbb{V}_J^{\alpha\ell\vec{m}}(\hat{r})=\Pi_{IJ}^V(\hat{r}_0|\hat{r})=\Pi_{JI}^V(\hat{r}|\hat{r}_0)\ , \tag{C.3}$$

where $|\mathbb{S}^{d-1}|$ is the volume of $\mathbb{S}^{d-1}$ and $\mathcal{N}_{d,\ell}$ is an inverse integer given by

$$|\mathbb{S}^{d-1}|\equiv\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\ ,\quad \mathcal{N}_{d,\ell}\equiv\frac{(d-2)!!}{(d+2\ell-2)!!}\ . \tag{C.4}$$

Further details about VSHs are explained in appendix B, but they do not matter for present purposes. It suffices to note that we can decompose the surface current $\mathcal{K}^I$ into terms proportional to VSHs, i.e., we assume $\mathcal{K}^I_{\alpha\ell\vec{m}}(\hat{r})\propto\mathbb{V}^I_{\alpha\ell\vec{m}}(\hat{r})$. The coefficients in these decompositions can be determined using the orthonormality of VSHs:

$$\mathcal{K}^I_{\alpha\ell\vec{m}}(\hat{r})\equiv\mathbb{V}^I_{\alpha\ell\vec{m}}(\hat{r})\int_{\hat{r}_0\in\mathbb{S}^{d-1}}[\mathbb{V}_J^{\alpha\ell\vec{m}}(\hat{r}_0)]^*\mathcal{K}^J(\hat{r}_0)\ . \tag{C.5}$$

The advantage of such a decomposition is that we can solve for the magnetic field due to each component quite easily. The final answer is then obtained by superposition. Since the symmetry properties of each VSH under $SO(d)$ rotation is different, the vector potential $\overline{\mathcal{V}}_\mu$ produced by $\mathcal{K}^I_{\alpha\ell\vec{m}}$ should be proportional to $\mathcal{K}^I_{\alpha\ell\vec{m}}$ upto an r dependent pre-factor.

Further, since the equation for this radial pre-factor can only depend on the eigenvalue of the spherical laplacian, the r-dependent factor can only depend on ℓ, i.e., we can take

$$\overline{\mathcal{V}}_r=0\ ,\quad \overline{\mathcal{V}}_I=\sum_\ell f_\ell(r)\sum_{\vec{m}\alpha}\gamma_{IJ}\overline{\mathcal{K}}^J_{\alpha\ell\vec{m}}\ . \tag{C.6}$$

This vector potential automatically satisfies the Coulomb gauge condition. Using the above ansatz, Maxwell equations can all be reduced to a single vector Poisson equation of the form[3]

$$-\frac{1}{r^{d-1}}\partial_r[r^{d-3}\partial_r\overline{\mathcal{V}}_I]+\frac{1}{r^4}(-\mathscr{D}^2+d-2)\overline{\mathcal{V}}_I=\gamma_{IJ}\bar{J}^J\ . \tag{C.7}$$

[3]We work in SI units and set the Maxwell coupling $g_{\text{EM}}=\sqrt{\mu_0}=1$. While it is not relevant to the magnetostatics discussion, we will also set $c=1$ when we later discuss radiation.

For $\overline{\mathcal{V}}_I$ varying as ℓ^{th} VSH, we can replace $-\mathscr{D}^2$ by $\ell(\ell+d-2)-1$. Since

$$\ell(\ell+d-2)-1+d-2=(\ell+1)(\ell+d-3)\ ,$$

we conclude that, away from the spherical shell, $f_\ell(r)$ should vary as $r^{\ell+1}$ or as $r^{-(\ell+d-3)}$. We should stitch together these two solutions continuously with an appropriate derivative discontinuity given by the current. We obtain the final answer

$$\overline{\mathcal{V}}_r=0\ ,\quad \overline{\mathcal{V}}_I=\sum_{\ell\vec{m}\alpha}\frac{R^3\gamma_{IJ}\overline{\mathcal{K}}^J_{\alpha\ell\vec{m}}}{2\ell+d-2}\left[\frac{r^{\ell+1}}{R^{\ell+1}}\Theta(r<R)+\frac{R^{\ell+d-3}}{r^{\ell+d-3}}\Theta(r>R)\right]\ . \tag{C.8}$$

It is instructive to rewrite this answer in terms of the original data, i.e., the currents before decomposition into VSHs. This can be done by using Eq.(C.5). We get

$$\begin{aligned}\overline{\mathcal{V}}_I&=\int_{\hat{r}_0\in\mathbb{S}^{d-1}}\sum_\ell\left[\frac{r^{\ell+1}}{R^{\ell+1}}\Theta(r<R)+\frac{R^{\ell+d-3}}{r^{\ell+d-3}}\Theta(r>R)\right]\frac{\sum_{\vec{m}\alpha}\mathbb{V}_I^{\alpha\ell\vec{m}}(\hat{r})\mathbb{V}_J^{\alpha\ell\vec{m}*}(\hat{r}_0)}{2\ell+d-2}R^3\overline{\mathcal{K}}^J(\hat{r}_0)\\&=\sum_\ell\left[\frac{r^{\ell+1}}{R^{\ell+1}}\Theta(r<R)+\frac{R^{\ell+d-3}}{r^{\ell+d-3}}\Theta(r>R)\right]\frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\int_{\hat{r}_0\in\mathbb{S}^{d-1}}\frac{\Pi^V_{IJ}(\hat{r}|\hat{r}_0)_{d,\ell}}{2\ell+d-2}R^3\overline{\mathcal{K}}^J(\hat{r}_0)\ .\end{aligned} \tag{C.9}$$

Here, we have used the VSH addition theorem at the second step.

Equivalently, we can rewrite this expression in terms of the full current density of the spherical shell, i.e.,

$$\begin{gathered}\overline{\mathcal{V}}_I=\int_{\vec{r}_0}\sum_\ell\frac{\mathbb{G}_B(r,r_0;\ell)}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\Pi^V_{IJ}(\hat{r}|\hat{r}_0)_{d,\ell}\bar{J}^J(\vec{r}_0)\quad\text{with}\\ \mathbb{G}_B(r,r_0;\ell)\equiv\frac{1}{2\ell+d-2}\left\{\frac{r^{\ell+1}}{r_0^{\ell+d-3}}\Theta(r<r_0)+\frac{r_0^{\ell+1}}{r^{\ell+d-3}}\Theta(r>r_0)\right\}\ .\end{gathered} \tag{C.10}$$

This expression is, in fact, applicable to a more general toroidal current distribution of the form

$$\bar{J}^r=0\ ,\ \ \bar{J}^I=\sum_{\alpha\ell\vec{m}}\bar{J}_V(r,\alpha,\ell,\vec{m})\mathbb{V}^I_{\alpha\ell\vec{m}}(\hat{r})\ . \tag{C.11}$$

As such, an arbitrary current distribution can be thought of as built from infinitely many thin spherical current sheets. The principle of superposition implies then that the resultant vector potential is still given by Eq.(C.10). The corresponding static magnetic fields are given by

$$\begin{aligned}\overline{\mathcal{C}}_{rI} &= \int_{\vec{r}_0} \sum_{\ell} \frac{\partial_r \mathbb{G}_B(r, r_0; \ell)}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|} \Pi^V_{IJ}(\hat{r}|\hat{r}_0)_{d,\ell} \bar{J}^J(\vec{r}_0) \ , \\ \overline{\mathcal{C}}_{IJ} &= \int_{\vec{r}_0} \sum_{\ell} \frac{\mathbb{G}_B(r, r_0; \ell)}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|} \mathscr{D}_{[I} \Pi^V_{J]K}(\hat{r}|\hat{r}_0)_{d,\ell} \bar{J}^K(\vec{r}_0) \ .\end{aligned} \tag{C.12}$$

Here we use the notation $A_{[I}B_{J]} \equiv A_I B_J - A_J B_I$. For future comparison, we will also write the decomposition into orthonormal VSHs:

$$\begin{aligned}&\overline{\mathcal{V}}_r = 0 \ , \quad \overline{\mathcal{V}}_I \equiv \sum_{\ell \vec{m} \alpha} \overline{\Phi}_B(r, \alpha, \ell, \vec{m}) \mathbb{V}^{\alpha \ell \vec{m}}_I(\hat{r}) \ , \\ &\overline{\mathcal{C}}_{rI} \equiv \sum_{\ell \vec{m} \alpha} \overline{H}_v(r, \alpha, \ell, \vec{m}) \mathbb{V}^{\alpha \ell \vec{m}}_I(\hat{r}) \ , \quad \overline{\mathcal{C}}_{IJ} \equiv \sum_{\ell \vec{m} \alpha} \overline{H}_{vv}(r, \alpha, \ell, \vec{m}) \mathscr{D}_{[I} \mathbb{V}^{\alpha \ell \vec{m}}_{J]}(\hat{r}) \ .\end{aligned} \tag{C.13}$$

Thus, knowing the scalar field $\overline{\Phi}_B$ for every spherical mode is sufficient to characterise the magnetic field. We will call $\overline{\Phi}_B$ as the *magnetic Debye field*. The field strength can then be determined via

$$\overline{H}_v = \partial_r \overline{\Phi}_B \ , \quad \overline{H}_{vv} = \overline{\Phi}_B \ . \tag{C.14}$$

The magnetic Debye field for the most general static current contribution can then be written as

$$\overline{\Phi}_B \equiv \int_{\vec{r}_0} \mathbb{G}_B(r, r_0; \ell) \mathbb{V}^{\alpha \ell \vec{m} *}_I(\hat{r}_0) \bar{J}^I(\vec{r}_0) = \int_0^\infty dr_0 \ r_0^{d-1} \mathbb{G}_B(r, r_0; \ell) \bar{J}_V(r_0) \ , \tag{C.15}$$

where we have defined $\bar{J}_V(r_0) \equiv \int_{\hat{r}_0 \in \mathbb{S}^{d-1}} \mathbb{V}^{\alpha \ell \vec{m} *}_I(\hat{r}_0) \bar{J}^I(\vec{r}_0)$.

Multipole expansion outside the sources

The expressions above simplify considerably if we focus on the fields outside the currents. Defining the spherical magnetic multipole moments via

$$\overline{\mathcal{J}}^B(\alpha, \ell, \vec{m}) \equiv \frac{1}{2\ell + d - 2} \int_{\vec{r}_0} r_0^{\ell+1} \mathbb{V}^{\alpha \ell \vec{m} *}_J(\hat{r}_0) \bar{J}^J(\vec{r}_0) \ , \tag{C.16}$$

we can then write the magnetic Debye potential outside the currents as

$$\overline{\Phi}^{\text{Out}}_{B}(r,\alpha,\ell,\vec{m}) = \frac{\overline{\mathscr{J}}^{B}(\alpha,\ell,\vec{m})}{r^{\ell+d-3}} \ , \tag{C.17}$$

and the corresponding magnetic field components are given by

$$\overline{H}^{\text{Out}}_{v}(r,\alpha,\ell,\vec{m}) = -(\ell+d-3)\frac{\overline{\mathscr{J}}^{B}(\alpha,\ell,\vec{m})}{r^{\ell+d-2}} \ , \qquad \overline{H}^{\text{Out}}_{vv}(r,\alpha,\ell,\vec{m}) = \frac{\overline{\mathscr{J}}^{B}(\alpha,\ell,\vec{m})}{r^{\ell+d-3}} \ . \tag{C.18}$$

These expressions constitute the magnetostatic multipole expansion in $\mathbb{R}^d$. We will see later that the dS multipoles reduce to these expressions in an appropriate limit.

The magnetostatic multipole expansion can also be recast into a cartesian STF form. To do this, we will sum over the $\vec{m}\alpha$ indices, multiply by appropriate powers of r and transform the spherical indices to get a harmonic vector field in $\mathbb{R}^d$. Each cartesian component of the vector field is then a ℓ^{th} degree homogeneous harmonic polynomial in cartesian coordinates. Such a vector field is of the form

$$\frac{1}{\ell!}\ {}^{B}\mathcal{Q}_{k<i_1 i_2 \ldots i_\ell>} x^{i_1}\ldots x^{i_\ell} \ , \tag{C.19}$$

where ${}^{B}\mathcal{Q}^{k}_{<i_1 i_2\ldots i_\ell>}$ is a constant irreducible tensor corresponding to the Young tableaux

$$\begin{array}{|c|c|c|c|c|c|}\hline i_1 & i_2 & i_3 & \ldots & i_{\ell-1} & i_\ell \\ \hline k \\ \cline{1-1}\end{array} \ .$$

The explicit relation between the magnetic multipole moment tensor and the spherical magnetic moments is

$$\frac{1}{\ell!}\ {}^{B}\overline{\mathcal{Q}}_{k<i_1 i_2\ldots i_\ell>} x^{i_1}\ldots x^{i_\ell} \equiv \mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|\left(\frac{\partial x_k}{\partial \vartheta_I}\right)\sum_{\vec{m}\alpha}\overline{\mathscr{J}}^{B}(\alpha,\ell,\vec{m})r^{\ell-1}\mathbb{V}^{I}_{\alpha\ell\vec{m}}(\hat{r}) \ . \tag{C.20}$$

We have followed here the steps outlined above and included an additional normalisation factor of $\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}| = (2\ell+d-2)\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|$ for convenience. We will now convert everything to cartesian basis using (B.60) and

$$r_0^2\gamma_{JK}(\theta_0)\overline{J}^{K}(\vec{r}_0) = \frac{\partial x_{0j}}{\partial \theta_0^J}\overline{J}^{j}(\vec{r}_0) \ . \tag{C.21}$$

Further, we sum over $(\alpha, \vec{m})$ by invoking addition theorem, and end up with

$$\frac{1}{\ell!}\ {}^B\overline{\mathcal{Q}}_{k<i_1 i_2 \ldots i_\ell>} x^{i_1} \ldots x^{i_\ell} \equiv \int_{\vec{r}_0} \Pi^V_{kj}(\vec{r}|\vec{r}_0) \bar{J}^j(\vec{r}_0)\ , \tag{C.22}$$

or equivalently,

$${}^B\overline{\mathcal{Q}}_{k<i_1 i_2 \ldots i_\ell>} \equiv (\Pi^V_{kj})^{<i_1 i_2 \ldots i_\ell>}_{<j_1 j_2 \ldots j_\ell>} \int_{\vec{r}_0} x_0^{j_1} \ldots x_0^{j_\ell} \bar{J}^j(\vec{r}_0)\ . \tag{C.23}$$

This equation gives a way to directly compute the cartesian moments from the current without going through orthonormal VSHs: we only need the vector STF projector constructed in Appendix§B.2. We will give explicit expressions for cartesian moments below.

The outside vector potential/magnetic field can be written in terms of the STF magnetic moment as[4]

$$\begin{aligned} \overline{\mathcal{V}}^{\text{Out}}_k &= \sum_\ell \frac{1}{\ell! \mathcal{N}_{d,\ell-1} |\mathbb{S}^{d-1}|} {}^B\overline{\mathcal{Q}}_{k<i_1 i_2 \ldots i_\ell>} \frac{x^{i_1} \ldots x^{i_\ell}}{r^{2\ell+d-2}}\ , \\ \overline{\mathcal{C}}^{\text{Out}}_{jk} &= \text{Anti}_{jk} \sum_\ell \frac{1}{\ell! \mathcal{N}_{d,\ell-1} |\mathbb{S}^{d-1}|} {}^B\overline{\mathcal{Q}}_{j<i_1 i_2 \ldots i_\ell>} [(2\ell+d-2) x^k x^{i_\ell} - r^2 \ell \delta^{k i_\ell}] \frac{x^{i_1} \ldots x^{i_{\ell-1}}}{r^{2\ell+d}}\ . \end{aligned} \tag{C.24}$$

These expressions give the cartesian multipole expansion for magnetostatics in $\mathbb{R}^d$. It is conventional in STF literature to rewrite the potential as a derivative of the Newton-Coulomb potential using

$$\partial^{i_1} \partial^{i_2} \ldots \partial^{i_\ell} \left\{ \frac{1}{(d-2)|\mathbb{S}^{d-1}| r^{d-2}} \right\} = \frac{(-)^\ell}{\mathcal{N}_{d,\ell-1} |\mathbb{S}^{d-1}|} \frac{x^{<i_1} \ldots x^{i_\ell>}}{r^{2\ell+d-2}}\ . \tag{C.25}$$

This formula can be established by direct differentiation and using the harmonicity of the Newton-Coulomb potential away from the origin. Using this, we can write the vector potential as a series of derivatives acting on the Newton-Coulomb potential:

$$\overline{\mathcal{V}}^{\text{Out}}_k = \sum_\ell \frac{(-)^\ell}{\ell!} \partial^{i_1} \partial^{i_2} \ldots \partial^{i_\ell} \left\{ \frac{{}^B\overline{\mathcal{Q}}_{k<i_1 i_2 \ldots i_\ell>}}{(d-2)|\mathbb{S}^{d-1}| r^{d-2}} \right\}\ . \tag{C.26}$$

Our derivation here follows closely the EM multipole expansion in $d = 3$ described in [86, 175] and the discussion in general d by [85]. The Cartesian multipole expansion in

[4] Here we use the notation $\text{Anti}_{jk}[T_{jk}] \equiv T_{jk} - T_{kj}$ for the anti-symmetrisation operator.

$d = 3$ can be derived directly without using orthonormal VSHs as shown in [151, 160](See [162] for a recent generalisation to arbitrary dimensions). Such a method, however, does not readily generalise to the de Sitter static patch, as there are no static Cartesian coordinates in dS.

We will conclude our discussion with an explicit formula for STF magnetic moments ${}^B\overline{\mathcal{Q}}_{k<i_1\ldots i_\ell>}$. Using the explicit form of the vector STF projector in Eq.(B.71), we can write

$$\begin{aligned}
\frac{1}{\ell!}{}^B\overline{\mathcal{Q}}_{j<i_1\ldots i_\ell>}\kappa^{i_1}\ldots\kappa^{i_\ell} &= \frac{1}{\ell+1}\int_{\vec{r}}\Bigg\{\frac{(\kappa\cdot r)^{\ell-1}}{(\ell-1)!}[\kappa\cdot(x\wedge\bar{J})]_j \\
&-\frac{[\kappa^2\bar{I}_j-\kappa_j(\kappa\cdot\bar{I})]}{\ell+d-3}\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor-1}\frac{\Gamma(\nu-k)}{k!\,\Gamma(\nu)}\left(-\frac{\kappa^2r^2}{4}\right)^k\frac{(\vec{\kappa}\cdot\vec{r})^{\ell-2k-2}}{(\ell-2k-2)!} \\
&-\frac{\kappa^2}{4}\frac{[(\kappa\cdot r)\bar{I}_j-(\kappa\cdot\bar{I})x_j]}{(\ell+d-3)} \\
&\times\sum_{k=1}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\Gamma(\nu-k)}{k!\,\Gamma(\nu)}\left(-\frac{\kappa^2r^2}{4}\right)^{k-1}\frac{(\vec{\kappa}\cdot\vec{r})^{\ell-2k-1}}{(\ell-2k-1)!}(\ell+d-3-2k)\Bigg\}_{\nu=\ell+\frac{d}{2}-1}.
\end{aligned} \tag{C.27}$$

Here κ^i is a dummy variable introduced to simplify our expressions, and we have used the notation

$$(x\wedge\bar{J})_{ij}\equiv x_i\bar{J}_j-x_j\bar{J}_i\ ,\qquad \bar{I}_i\equiv[x\cdot(x\wedge\bar{J})]_i=r^2\bar{J}_i-(x\cdot\bar{J})x_i \tag{C.28}$$

to simplify our expressions. For the first few ℓs, the above formula evaluates to

$$\begin{aligned}
{}^B\overline{\mathcal{Q}}_{ki_1}\kappa^{i_1} &= \frac{1}{2!}\int_{\vec{r}}[\kappa\cdot(x\wedge\bar{J})]^k\ , \\
\frac{1}{2!}{}^B\overline{\mathcal{Q}}_{k<i_1i_2>}\kappa^{i_1}\kappa^{i_2} &= \frac{2}{3!}\int_{\vec{r}}\left\{(\kappa\cdot r)[\kappa\cdot(x\wedge\bar{J})]^k-\frac{1}{d-1}[\kappa^2\bar{I}^k-\kappa^k(\kappa\cdot\bar{I})]\right\}, \\
\frac{1}{3!}{}^B\overline{\mathcal{Q}}_{k<i_1i_2i_3>}\kappa^{i_1}\kappa^{i_2}\kappa^{i_3} &= \frac{3}{4!}\int_{\vec{r}}\Bigg\{(\kappa\cdot r)^2[\kappa\cdot(x\wedge\bar{J})]^k-\frac{2}{d}(\kappa\cdot r)[\kappa^2\bar{I}^k-\kappa^k(\kappa\cdot\bar{I})] \\
&\qquad -\frac{d-2}{d(d+2)}\kappa^2[(\kappa\cdot r)\bar{I}^k-(\kappa\cdot\bar{I})x^k]\Bigg\}\ ,
\end{aligned} \tag{C.29}$$

$$
\begin{aligned}
\frac{1}{4!}{}^{B}\overline{\mathrm{Q}}_{k<i_1i_2i_3i_4>}\kappa^{i_1}\ldots\kappa^{i_4} = \frac{4}{5!}\int_{\vec{r}}\Bigg\{ & (\kappa\cdot r)^3[\kappa\cdot(x\wedge\bar{J})]^k \\
& -3\frac{(d+4)(\kappa\cdot r)^2-\kappa^2r^2}{(d+1)(d+4)}[\kappa^2\bar{I}^k-\kappa^k(\kappa\cdot\bar{I})] \\
& -3\frac{d-1}{(d+1)(d+4)}\kappa^2(\kappa\cdot r)[(\kappa\cdot r)\bar{I}^k-(\kappa\cdot\bar{I})x^k]\Bigg\}\ , \\
\frac{1}{5!}{}^{B}\overline{\mathrm{Q}}_{k<i_1i_2i_3i_4i_5>}\kappa^{i_1}\ldots\kappa^{i_5} = \frac{5}{6!}\int_{\vec{r}}\Bigg\{ & (\kappa\cdot r)^4[\kappa\cdot(x\wedge\bar{J})]^k \\
& -4\frac{(d+6)(\kappa\cdot r)^3-\kappa^2r^2(\kappa\cdot r)}{(d+2)(d+6)}[\kappa^2\bar{I}^k-\kappa^k(\kappa\cdot\bar{I})] \\
& -3\frac{2d(d+4)(\kappa\cdot r)^2-(d-2)\kappa^2r^2}{(d+2)(d+4)(d+6)}\kappa^2[(\kappa\cdot r)\bar{I}^k-(\kappa\cdot\bar{I})x^k]\Bigg\}\ .
\end{aligned}
\tag{C.30}
$$

Explicit tensor expressions can then be obtained by repeatedly differentiating the above formulae with respect to κ^i to yield explicitly symmetrised expressions. For example, the STF magnetic dipole/quadrupole tensors are

$$
\begin{aligned}
{}^{B}\overline{\mathrm{Q}}_{ki_1} &\equiv \frac{1}{2}\int_{\vec{r}}(x\wedge\bar{J})^{i_1k}\ , \\
{}^{B}\overline{\mathrm{Q}}_{k<i_1i_2>} &\equiv \frac{1}{3}\int_{\vec{r}}\Bigg\{(x\wedge\bar{J})^{i_1k}x^{i_2}+(x\wedge\bar{J})^{i_2k}x^{i_1}+\frac{1}{d-1}(\bar{I}^{i_1}\delta^{ki_2}+\bar{I}^{i_2}\delta^{ki_1}-2\bar{I}^k\delta^{i_1i_2})\Bigg\}\ ,
\end{aligned}
\tag{C.31}
$$

whereas the magnetic octopole tensor has the form

$$
\begin{aligned}
{}^{B}\overline{\mathrm{Q}}_{k<i_1i_2i_3>} \equiv \frac{1}{4}\int_{\vec{r}}\Bigg\{ & (x\wedge\bar{J})^{i_1k}x^{i_2}x^{i_3}+(x\wedge\bar{J})^{i_2k}x^{i_3}x^{i_1}+(x\wedge\bar{J})^{i_3k}x^{i_1}x^{i_2} \\
& -\frac{3d+2}{d(d+2)}\bar{I}^k(x^{i_1}\delta^{i_2i_3}+x^{i_2}\delta^{i_3i_1}+x^{i_3}\delta^{i_1i_2})+\frac{d-2}{d(d+2)}x^k\left(\bar{I}^{i_1}\delta^{i_2i_3}+\bar{I}^{i_2}\delta^{i_3i_1}+\bar{I}^{i_3}\delta^{i_1i_2}\right) \\
& +\frac{\delta^{ki_1}}{d}(\bar{I}^{i_2}x^{i_3}+\bar{I}^{i_3}x^{i_2})+\frac{\delta^{ki_2}}{d}(\bar{I}^{i_3}x^{i_1}+\bar{I}^{i_1}x^{i_3})+\frac{\delta^{ki_3}}{d}(\bar{I}^{i_1}x^{i_2}+\bar{I}^{i_2}x^{i_1})\Bigg\}\ .
\end{aligned}
\tag{C.32}
$$

C.2 Magnetic multipole radiation

The expressions we found in our study of magnetostatics can be readily generalised to time-dependent toroidal currents. This is best done in frequency domain. The vector Poisson equation in (C.7) generalises to the vector Helmholtz equation

$$-\frac{1}{r^{d-3}}\partial_r[r^{d-3}\partial_r\mathcal{V}_I]-\omega^2\mathcal{V}_I+\frac{1}{r^2}(-\mathscr{D}^2+d-2)\mathcal{V}_I=r^2\gamma_{IJ}J^J\ , \tag{C.33}$$

where $\mathcal{V}_I(\vec{r},\omega)$ and $J^I(\vec{r},\omega)$ are Fourier transforms of $\overline{\mathcal{V}}_I(\vec{r},t)$ and $\bar{J}^I(\vec{r},t)$ respectively. More generally, we use overline for functions of time and remove them to denote the Fourier transforms.

Homogenous spherical waves

Let us begin by finding the homogeneous solution for the above equation by separation of variables. We are interested in solutions whose angular dependence is given by the VSH $\mathbb{V}_I^{\alpha\ell\vec{m}}$. As we will briefly review, the radial dependencies are then controlled by Bessel-like functions.

For a given $(\alpha,\ell,\vec{m})$, there is a unique solution which is regular everywhere. It is given by:

$$r^{\nu-\frac{d}{2}+2}{}_0F_1\left[1+\nu,-\frac{\omega^2r^2}{4}\right]\mathbb{V}_I^{\alpha\ell\vec{m}}=\Gamma(1+\nu)\left(\frac{\omega}{2}\right)^{-\nu}r^{2-\frac{d}{2}}J_\nu(\omega r)\mathbb{V}_I^{\alpha\ell\vec{m}}\ , \tag{C.34}$$

where we have defined $\nu\equiv\ell+\frac{d}{2}-1$ to denote the rank of the Bessel function. Here, we have given two forms of the solution: one in terms of ${}_0F_1$ and another in terms of the Bessel function. Although the Bessel form is standard among textbooks, we find the ${}_0F_1$ notation to be the most convenient. As we shall see below, the ${}_0F_1$ function above becomes the *time-smearing function* used to define multipole moments for extended sources. In the gravitational wave literature, this ${}_0F_1$ function often appears in the following integral form [160, 161]

$${}_0F_1\left[1+\nu,-\frac{\omega^2r^2}{4}\right]=\int_{-1}^{1}dz\ \frac{\Gamma\left(1+\nu\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\nu\right)}(1-z^2)^{\nu-\frac{1}{2}}e^{i\omega zr}\equiv\int_{-1}^{1}dz\ \delta_{\nu-\frac{1}{2}}(z)\ e^{i\omega zr}\ . \tag{C.35}$$

The 'multipole delta function' $\delta_{\nu-\frac{1}{2}}(z)$ gives an even, positive, normalised measure on the interval $[-1,1]$. The above integral representation then interprets the ${}_0F_1$ function as a *weighed superposition of time-delays.*

There is another solution which is regular everywhere except the origin:

$$\begin{aligned}\frac{1}{r^{\nu+\frac{d}{2}-2}}\left\{{}_0F_1\left[1-\nu,-\frac{\omega^2r^2}{4}\right]-\frac{\pi\cot\nu\pi}{\Gamma(\nu)\Gamma(1+\nu)}\left(\frac{\omega r}{2}\right)^{2\nu}\ {}_0F_1\left[1+\nu,-\frac{\omega^2r^2}{4}\right]\right\}\mathbb{V}_I^{\alpha\ell\vec{m}}\\ =-\left(\frac{\omega}{2}\right)^{\nu}\frac{\pi\ r^{2-\frac{d}{2}}}{\Gamma(\nu)}Y_\nu(\omega r)\mathbb{V}_I^{\alpha\ell\vec{m}}\ .\end{aligned} \tag{C.36}$$

Here, Y_ν is the Neumann function: when $\nu\notin\mathbb{Z}$, this solution is obtained from the regular solution by the replacement $\nu\to-\nu$ and adding an appropriate amount of the regular solution. The expression in terms of ${}_0F_1$ should, however be carefully interpreted whenever $\nu\equiv\ell+\frac{d}{2}-1$ is an integer (i.e., whenever the number of space dimensions d is even). When ν is a positive integer, the hypergeometric series for ${}_0F_1[1-\nu,z]$ is divergent, and the $\cot\nu\pi$ factor is also divergent. However, these two divergences cancel each other in the above expression, so the limit $\nu\to$ Integer exists and converges to the Neumann function. These statements should be contrasted against the case when d is odd and ν is a half-integer: in this case, the hypergeometric series for ${}_0F_1[1-\nu,z]$ is convergent, and the factor $\cot\nu\pi$ evaluates to zero.

Alternately, one can characterise the solutions according to their behaviours at $r=\infty$, i.e. either as outgoing or ingoing solutions:

$$\begin{aligned}&\frac{1}{r^{\nu+\frac{d}{2}-2}}\left\{{}_0F_1\left[1-\nu,-\frac{\omega^2r^2}{4}\right]\right.\\ &\qquad\left.\pm(1\pm i\cot\nu\pi)\frac{2\pi i}{\Gamma(\nu)^2}\frac{1}{2\nu}\left(\frac{\omega r}{2}\right)^{2\nu}\ {}_0F_1\left[1+\nu,-\frac{\omega^2r^2}{4}\right]\right\}\mathbb{V}_I^{\alpha\ell\vec{m}}\\ &=\pm i\left(\frac{\omega}{2}\right)^{\nu}\frac{\pi\ r^{2-\frac{d}{2}}}{\Gamma(\nu)}H_\nu^{\pm}(\omega r)\mathbb{V}_I^{\alpha\ell\vec{m}}\end{aligned} \tag{C.37}$$

Here, the solution with H^+ denotes the outgoing waves, while the H^- denotes the incoming waves. Our comments regarding the case where d is even and ν is a positive integer still apply. Using the identity

$$\frac{(+i)(1+i\cot\nu\pi)}{(-i)(1-i\cot\nu\pi)}=[e^{-i\pi}]^{2\nu}\ , \tag{C.38}$$

it can be seen that the two solutions here are related by time-reversal, i.e., under $\omega \mapsto e^{i\pi}\omega$, the outgoing wave is mapped to the incoming wave, and under $\omega \mapsto e^{-i\pi}\omega$ the incoming wave is mapped to the outgoing wave. For radiation reaction, we are mainly interested in outgoing waves whose radial part is given by

$$\begin{aligned} G^{\text{Out}}_{B}(r,\omega,\ell) \equiv \frac{1}{r^{\nu+\frac{d}{2}-2}} \Bigg\{ {}_0F_1\left[1-\nu, -\frac{\omega^2 r^2}{4}\right] \\ +(1+i\cot\nu\pi)\frac{2\pi i}{\Gamma(\nu)^2}\frac{1}{2\nu}\left(\frac{\omega r}{2}\right)^{2\nu} {}_0F_1\left[1+\nu, -\frac{\omega^2 r^2}{4}\right]\Bigg\} \ . \end{aligned} \tag{C.39}$$

When ν is a half-integer, the function G^{Out}_{B} can be greatly simplified by the use of *reverse Bessel polynomials*. They are defined via

$$\begin{aligned} \theta_{\nu-\frac{1}{2}}(z) &\equiv \sqrt{\frac{\pi}{2}} e^z z^\nu K_\nu(z) \\ &= \frac{2^\nu \Gamma(\nu)}{\sqrt{2\pi}} e^z \left\{ {}_0F_1\left[1-\nu, \frac{z^2}{4}\right] + \frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{z}{2}\right)^{2\nu} {}_0F_1\left[1+\nu, \frac{z^2}{4}\right]\right\} \\ &= \sum_{n=0}^{\nu-\frac{1}{2}} \frac{z^{\nu-\frac{1}{2}-n}}{2^n n!} \frac{\left(\nu-\frac{1}{2}+n\right)!}{\left(\nu-\frac{1}{2}-n\right)!} \end{aligned} \tag{C.40}$$

Here $K_\nu(z)$ is the Macdonald function, and the second line shows that $\theta_{\nu-\frac{1}{2}}(z)$ is a polynomial of degree $\nu - \frac{1}{2}$ with positive integer coefficients, the coefficient of $z^{\nu-\frac{1}{2}}$ being normalised to unity. Explicit forms of the first few reverse Bessel polynomials are tabulated below:

Table C.1: $\theta_{\nu-\frac{1}{2}}(z)$ for various values of ν

ν	$\theta_{\nu-\frac{1}{2}}(z)$
$\frac{1}{2}$	1
$\frac{3}{2}$	$1+z$
$\frac{5}{2}$	$3+3z+z^2+r^2z$
$\frac{7}{2}$	$15+15z+6z^2+z^3$
$\frac{9}{2}$	$105+105z+45z^2+10z^3+z^4$
$\frac{11}{2}$	$945+945z+420z^2+105z^3+15z^4+z^5$
$\frac{13}{2}$	$10395+10395z+4725z^2+1260z^3+210z^4+21z^5+z^6$

Another useful property of the reverse Bessel polynomials evident from the above table is the value of the constant term in these polynomials

$$\theta_{\nu-\frac{1}{2}}(0) = (2\nu-2)!! = \frac{(d-2)!!}{\mathcal{N}_{d,\ell-1}} \ . \tag{C.41}$$

The radial part of the outgoing waves can then be written in the form

$$G_B^{\text{Out}}(r,\omega,\ell) = \frac{\theta_{\nu-\frac{1}{2}}(-i\omega r)}{\theta_{\nu-\frac{1}{2}}(0)} \frac{e^{i\omega r}}{r^{\nu+\frac{d}{2}-2}} = \mathcal{N}_{d,\ell-1} \frac{\theta_{\nu-\frac{1}{2}}(-i\omega r)}{(d-2)!!} \frac{e^{i\omega r}}{r^{\nu+\frac{d}{2}-2}} \tag{C.42}$$

for $\nu \in \mathbb{Z}+\frac{1}{2}$. The large r asymptotics of this outgoing spherical wave is given by taking the largest power in the reverse Bessel polynomial:

$$G_B^{\text{Out}}(r,\omega,\ell) \to \mathcal{N}_{d,\ell-1} \frac{(-i\omega)^{\nu-\frac{1}{2}}}{(d-2)!!} \frac{e^{i\omega r}}{r^{\frac{d-3}{2}}} \quad \text{as} \quad r \to \infty \ . \tag{C.43}$$

This asymptotic formula also holds for d even and $\nu \in \mathbb{Z}$, provided the double factorial for even integers is defined recursively with the convention that $0!! \equiv \sqrt{\frac{2}{\pi}}$. For general ν, we can justify this via the asymptotic expansion of Hankel functions (see https://dlmf.nist.gov/10.17):

$$\pm i \left(\frac{\omega}{2}\right)^{\nu} \frac{\pi\, r^{2-\frac{d}{2}}}{\Gamma(\nu)} H_\nu^{\pm}(\omega r) \to \left(\frac{\omega\ e^{\mp i\frac{\pi}{2}}}{2}\right)^{\nu-\frac{1}{2}} \frac{\sqrt{\pi}}{\Gamma(\nu)} \frac{e^{\pm i\omega r}}{r^{\frac{d-3}{2}}} \quad \text{as} \quad r \to \infty \ . \tag{C.44}$$

We will conclude this discussion with some useful identities: the raising and lowering relations for Hankel functions lead to

$$\begin{aligned}
-\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{G^{\text{Out}}_{B}(r,\omega,\ell)}{\mathcal{N}_{d,\ell-1}r^{\ell+1}}\right] &= \frac{G^{\text{Out}}_{B}(r,\omega,\ell+1)}{\mathcal{N}_{d,\ell}r^{\ell+2}} \ , \\
\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{G^{\text{Out}}_{B}(r,\omega,\ell)}{\mathcal{N}_{d,\ell-1}\ r^{3-d-\ell}}\right] &= \omega^2\frac{G^{\text{Out}}_{B}(r,\omega,\ell-1)}{\mathcal{N}_{d,\ell-2}\ r^{4-d-\ell}} \ .
\end{aligned} \tag{C.45}$$

These identities can be used to give the following formulae for the derivative of G^{Out}_{B}, i.e.,

$$\begin{aligned}
\partial_r G^{\text{Out}}_{B}(r,\omega,\ell) &= -\frac{(\ell+d-3)}{r}G^{\text{Out}}_{B}(r,\omega,\ell) \ + \frac{\omega^2}{2\ell+d-4}G^{\text{Out}}_{B}(r,\omega,\ell-1) \\
&= -\left(\ell+d-3\right)G^{\text{Out}}_{B}(r,\omega,\ell+1) + \frac{\ell+1}{2\ell+d-2}\frac{\omega^2}{2\ell+d-4}G^{\text{Out}}_{B}(r,\omega,\ell-1) \ .
\end{aligned} \tag{C.46}$$

Another useful identity arises from the cartesian version of Eq.(C.45):

$$-\partial_i\left[\frac{G^{\text{Out}}_{B}(r,\omega,\ell)}{\mathcal{N}_{d,\ell-1}r^{\ell+1}}\right] = x^i\frac{G^{\text{Out}}_{B}(r,\omega,\ell+1)}{\mathcal{N}_{d,\ell}r^{\ell+2}} \ , \quad \partial_i\left[\frac{G^{\text{Out}}_{B}(r,\omega,\ell)}{\mathcal{N}_{d,\ell-1}\ r^{3-d-\ell}}\right] = \omega^2 x^i\frac{G^{\text{Out}}_{B}(r,\omega,\ell-1)}{\mathcal{N}_{d,\ell-2}\ r^{4-d-\ell}} \ . \tag{C.47}$$

By repeated application of the first identity, we get the following relation, which expresses the radial part of ℓ^{th} spherical wave as a derivative of the $\ell = 0$ wave:

$$(-1)^\ell\partial^{<i_1}\partial^{i_2}\dots\partial^{i_\ell>}\left\{\frac{G^{\text{Out}}_{B}(r,\omega,\ell=0)}{(d-2)r}\right\} = \frac{G^{\text{Out}}_{B}(r,\omega,\ell)}{\mathcal{N}_{d,\ell-1}r^{\ell+1}}x^{<i_1}\dots x^{i_\ell>} \ . \tag{C.48}$$

Here we work with the convention that $\mathcal{N}_{d,-1} = (d-2)$ which is the correct analytic extension of $\mathcal{N}_{d,\ell}$ to negative ℓs. In the above relation, the STF projection on the indices ensures that the derivatives acting on x^is always give zero at every step.

Green function for magnetic radiation

Now that we understand the homogeneous solutions, we can solve the full inhomogeneous Helmholtz equation in Eq.(C.33) via Green functions. As we did in magnetostatics, we will begin with an ansatz for the vector potential:

$$\overline{\mathcal{V}}_t(\vec{r},t) = \overline{\mathcal{V}}_r(\vec{r},t) = 0 \ , \quad \overline{\mathcal{V}}_I(\vec{r},t) \equiv \sum_{\alpha\ell\vec{m}}\int_\omega e^{-i\omega t}\Phi_B(r,\omega,\alpha,\ell,\vec{m})\mathbb{V}^{\alpha\ell\vec{m}}_I(\hat{r}) \tag{C.49}$$

where $\Phi_{_B}$ denotes the frequency domain magnetic Debye field. Substituting this ansatz into Eq.(C.33), we get a sourced Helmholtz equation for $\Phi_{_B}$:

$$-\frac{1}{r^{d-3}}\partial_r[r^{d-3}\partial_r\Phi_{_B}]-\omega^2\Phi_{_B}+\frac{1}{r^2}(\ell+1)(\ell+d-3)\Phi_{_B}=r^2J_V(r,\omega)\ , \tag{C.50}$$

where the source appearing in the RHS is

$$J_V(r,\omega)\equiv\int_{\hat{r}\in\mathbb{S}^{d-1}}\mathbb{V}_I^{\alpha\ell\vec{m}*}(\hat{r})J^I(\vec{r},\omega)\ . \tag{C.51}$$

We will posit a solution of the form

$$\begin{aligned}\Phi_{_B}(r,\omega,\alpha,\ell,\vec{m})&\equiv\int_{\vec{r}_0}\mathbb{G}_B(r,r_0;\omega,\ell)\mathbb{V}_J^{\alpha\ell\vec{m}*}(\hat{r}_0)J^J(\vec{r}_0,\omega)\\&=\int_0^\infty dr_0\ r_0^{d-1}\mathbb{G}_B(r,r_0;\omega,\ell)J_V(r_0,\omega)\ ,\end{aligned} \tag{C.52}$$

which generalises the static expression in Eq.(C.15). The Green function $\mathbb{G}_B$ obeys

$$-\partial_r[r^{d-3}\partial_r\mathbb{G}_B]-\omega^2r^{d-3}\mathbb{G}_B+(\ell+1)(\ell+d-3)r^{d-5}\mathbb{G}_B=\delta(r-r_0)\ , \tag{C.53}$$

and is built by stitching together the homogeneous solutions of the vector Helmholtz equation. Our normalisations here are such that $\mathbb{G}_B$ generalises the static Green function defined in Eq.(C.10). The conditions on the Green function $\mathbb{G}_B$ are that it should be continuous, its derivative should have an appropriate discontinuity, and it should match onto an outgoing wave far away from currents. These determine

$$\begin{aligned}\mathbb{G}_B(r,r_0;\omega,\ell)&=\frac{1}{2\nu}\frac{r_<^{\nu-\frac{d}{2}+2}}{r_>^{\nu+\frac{d}{2}-2}}{}_0F_1\left[1+\nu,-\frac{\omega^2r_<^2}{4}\right]\\&\times\left\{{}_0F_1\left[1-\nu,-\frac{\omega^2r_>^2}{4}\right]\right.\\&\qquad\left.+(1+i\cot\nu\pi)\frac{2\pi i}{\Gamma(\nu)^2}\frac{1}{2\nu}\left(\frac{\omega r_>}{2}\right)^{2\nu}\ {}_0F_1\left[1+\nu,-\frac{\omega^2r_>^2}{4}\right]\right\}\\&=\frac{1}{2\nu}r_<^{\nu-\frac{d}{2}+2}{}_0F_1\left[1+\nu,-\frac{\omega^2r_<^2}{4}\right]G_B^{\rm Out}(r_>,\omega,\ell)\ ,\end{aligned} \tag{C.54}$$

where we have defined:

$$r_> \equiv \mathrm{Max}(r, r_0) \ , \quad r_< \equiv \mathrm{Min}(r, r_0) \ . \tag{C.55}$$

The reader can check that the above expression reduces to Eq.(C.10) in the static (i.e., $\omega \to 0$) limit. Once we have the vector potential, we can compute the the corresponding electric/magnetic field components. The VSH expansion of the field strengths takes the form

$$\begin{aligned}
\overline{\mathfrak{C}}_{rt}(\vec{r}, t) &= 0 \ , \\
\overline{\mathfrak{C}}_{It}(\vec{r}, t) &= \sum_{\alpha\ell\vec{m}} \int_\omega e^{-i\omega t} E_v(r, \omega, \alpha, \ell, \vec{m}) \ \mathbb{V}_I^{\alpha\ell\vec{m}}(\vec{r}) \ , \\
\overline{\mathfrak{C}}_{rI}(\vec{r}, t) &= \sum_{\alpha\ell\vec{m}} \int_\omega e^{-i\omega t} H_v(r, \omega, \alpha, \ell, \vec{m}) \ \mathbb{V}_I^{\alpha\ell\vec{m}}(\vec{r}) \ , \\
\overline{\mathfrak{C}}_{IJ}(\vec{r}, t) &= \sum_{\alpha\ell\vec{m}} \int_\omega e^{-i\omega t} H_{vv}(r, \omega, \alpha, \ell, \vec{m}) \ \left[\mathscr{D}_I \mathbb{V}_{J\alpha\ell\vec{m}}(\vec{r}) - \mathscr{D}_J \mathbb{V}_{I\alpha\ell\vec{m}}(\vec{r})\right] \ ,
\end{aligned} \tag{C.56}$$

where the components are given by

$$\begin{aligned}
H_v &= \partial_r \Phi_{_B} = \int_{\vec{r}_0} \partial_r \mathbb{G}_B(r, r_0; \omega, \ell) \mathbb{V}_J^{\alpha\ell\vec{m}*}(\hat{r}_0) J^J(\vec{r}_0, \omega) \ , \\
H_{vv} &= \Phi_{_B} = \int_{\vec{r}_0} \mathbb{G}_B(r, r_0; \omega, \ell) \mathbb{V}_J^{\alpha\ell\vec{m}*}(\hat{r}_0) J^J(\vec{r}_0, \omega) \ , \\
E_v &= i\omega \ \Phi_{_B} = i\omega \int_{\vec{r}_0} \mathbb{G}_B(r, r_0; \omega, \ell) \mathbb{V}_J^{\alpha\ell\vec{m}*}(\hat{r}_0) J^J(\vec{r}_0, \omega) \ .
\end{aligned} \tag{C.57}$$

Note that apart from the magnetic fields, we have an induced electric field when toroidal currents are time-dependent. These time-varying electric and magnetic fields sustain each other as they escape the source and propagate far away as radiation.

Fields outside sources

To get a handle on the structure of multipole radiation, we will now focus on fields outside the sources. We will generalise our definition of static magnetic multipole moment in Eq.(C.16) as follows:

$$\mathcal{J}^B(\omega, \alpha, \ell, \vec{m}) \equiv \frac{1}{2\nu} \int_{\vec{r}_0} r_0^{\nu - \frac{d}{2} + 2} {}_0F_1\left[1 + \nu, -\frac{\omega^2 r_0^2}{4}\right] \mathbb{V}_I^{\alpha\ell\vec{m}*}(\hat{r}_0) J^I(\vec{r}_0, \omega) \ . \tag{C.58}$$

It is clear from our general solution that this is the magnetic moment that determines the fields outside currents. The magnetic Debye field outside the currents is given by

$$\Phi_B^{\text{Out}} = G_{_B}^{\text{Out}}(r,\omega,\ell)\ \mathcal{J}^B(\omega,\alpha,\ell,\vec{m}) = \frac{\theta_{\nu-\frac{1}{2}}(-i\omega r)}{\theta_{\nu-\frac{1}{2}}(0)}\frac{e^{i\omega r}}{r^{\ell+d-3}}\mathcal{J}^B(\omega,\alpha,\ell,\vec{m})\ . \tag{C.59}$$

The value of field strength components are given explicitly in terms of $\mathcal{J}^B$ by

$$\begin{aligned}
H_{vv}^{\text{Out}} &= G_{_B}^{\text{Out}}(r,\omega,\ell)\ \mathcal{J}^B(\omega,\alpha,\ell,\vec{m}) = \frac{\theta_{\nu-\frac{1}{2}}(-i\omega r)}{\theta_{\nu-\frac{1}{2}}(0)}\frac{e^{i\omega r}}{r^{\ell+d-3}}\mathcal{J}^B(\omega,\alpha,\ell,\vec{m}),\\
H_{v}^{\text{Out}} &= -\frac{(\ell+d-3)}{r}G_{_B}^{\text{Out}}(r,\omega,\ell)\ \mathcal{J}^B(\omega,\alpha,\ell,\vec{m})\\
&\quad + \frac{\omega^2}{2\ell+d-4}G_{_B}^{\text{Out}}(r,\omega,\ell-1)\ \mathcal{J}^B(\omega,\alpha,\ell,\vec{m})\\
&= -(\ell+d-3)\frac{\theta_{\nu-\frac{1}{2}}(-i\omega r)}{\theta_{\nu-\frac{1}{2}}(0)}\frac{e^{i\omega r}}{r^{\ell+d-2}}\ \mathcal{J}^B(\omega,\alpha,\ell,\vec{m})\\
&\quad + \frac{\omega^2}{2\ell+d-4}\frac{\theta_{\nu-\frac{3}{2}}(-i\omega r)}{\theta_{\nu-\frac{3}{2}}(0)}\frac{e^{i\omega r}}{r^{\ell+d-4}}\ \mathcal{J}^B(\omega,\alpha,\ell,\vec{m}),\\
E_{v}^{\text{Out}} &= i\omega G_{_B}^{\text{Out}}(r,\omega,\ell)\ \mathcal{J}^B(\omega,\alpha,\ell,\vec{m}) = i\omega\frac{\theta_{\nu-\frac{1}{2}}(-i\omega r)}{\theta_{\nu-\frac{1}{2}}(0)}\frac{e^{i\omega r}}{r^{\ell+d-3}}\mathcal{J}^B(\omega,\alpha,\ell,\vec{m})\ .
\end{aligned} \tag{C.60}$$

Here we have used Eq.(C.46) for evaluating $\partial_r G_{_B}^{\text{Out}}$. The large r asymptotics can be worked out using Eq.(C.43):

$$\begin{aligned}
H_{vv}^{\text{Rad}} &= \Phi_{_B}^{\text{Rad}} = \frac{(-i\omega)^{\nu-\frac{1}{2}}}{(2\nu-2)!!}\frac{e^{i\omega r}}{r^{\frac{d-3}{2}}}\ \mathcal{J}^B(\omega,\alpha,\ell,\vec{m}),\\
H_{v}^{\text{Rad}} &= \partial_r\Phi_{_B}^{\text{Rad}} = -\frac{(-i\omega)^{\nu+\frac{1}{2}}}{(2\nu-2)!!}\frac{e^{i\omega r}}{r^{\frac{d-3}{2}}}\mathcal{J}^B(\omega,\alpha,\ell,\vec{m}),\\
E_{v}^{\text{Rad}} &= i\omega\Phi_{_B}^{\text{Rad}} = -\frac{(-i\omega)^{\nu+\frac{1}{2}}}{(2\nu-2)!!}\frac{e^{i\omega r}}{r^{\frac{d-3}{2}}}\mathcal{J}^B(\omega,\alpha,\ell,\vec{m}).
\end{aligned} \tag{C.61}$$

We remind the reader that this holds even when d is even, provided the double factorial for even integers is defined recursively with the convention that $0!! \equiv \sqrt{\frac{2}{\pi}}$. We will see later how these asymptotics get modified in dS (3.46).

We next turn to the description in terms of cartesian STF tensors. The STF version of magnetic moment is still defined by Eq.(C.20) but now generalised to arbitrary frequency,i.e.,

$$\frac{1}{\ell!}\left[{}^{B}\mathcal{Q}(\omega)\right]_{k<i_1 i_2\dots i_\ell>} x^{i_1}\dots x^{i_\ell} \equiv \mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|\left(\frac{\partial x_k}{\partial \vartheta_I}\right)\sum_{\vec{m}\alpha}\mathcal{J}^B(\omega,\alpha,\ell,\vec{m})r^{\ell-1}\mathbb{V}^I_{\alpha\ell\vec{m}}(\hat{r})\ . \tag{C.62}$$

Repeating the same logic as in magnetostatics, we can give a direct expression in terms of the vector STF projector:

$${}^{B}\mathcal{Q}_{k<i_1 i_2\dots i_\ell>}(\omega) \equiv (\Pi^V_{kj})^{<i_1 i_2\dots i_\ell>}_{<j_1 j_2\dots j_\ell>}\int_{\vec{r}_0} x_0^{j_1}\dots x_0^{j_\ell}{}_0F_1\left[1+\nu,-\frac{\omega^2 r_0^2}{4}\right]J^j(\vec{r}_0,\omega)\ . \tag{C.63}$$

The tensor structure appearing here is exactly identical to that seen in statics (e.g., see (C.31)). The main difference in the time-dependent situation is the smearing due to time-delays: using Eq.(C.35), we can write

$${}^{B}\mathcal{Q}_{k<i_1 i_2\dots i_\ell>} = (\Pi^V_{kj})^{<i_1 i_2\dots i_\ell>}_{<j_1 j_2\dots j_\ell>}\int_{-1}^{1}dz\ \delta_{\nu-\frac{1}{2}}(z)\int_{\vec{r}_0} x_0^{j_1}\dots x_0^{j_\ell}e^{i\omega z r_0}J^j(\vec{r}_0,\omega)\ . \tag{C.64}$$

where

$$\delta_{\nu-\frac{1}{2}}(z) \equiv \frac{\Gamma(1+\nu)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\nu\right)}(1-z^2)^{\nu-\frac{1}{2}} = \frac{(2\nu)!!}{2^{\nu+\frac{1}{2}}\left(\nu-\frac{1}{2}\right)!}(1-z^2)^{\nu-\frac{1}{2}} \tag{C.65}$$

with $\nu \equiv \ell + \frac{d}{2} - 1$[5]. The interpretation in terms of time-delays is more transparent in the time-domain where the above equation becomes

$${}^{B}\overline{\mathcal{Q}}_{k<i_1 i_2\dots i_\ell>}(t) = (\Pi^V_{kj})^{<i_1 i_2\dots i_\ell>}_{<j_1 j_2\dots j_\ell>}\int_{\vec{r}_0} x_0^{j_1}\dots x_0^{j_\ell}\int_{-1}^{1}dz\ \delta_{\nu-\frac{1}{2}}(z)\bar{J}^j(\vec{r}_0,t-zr_0)\ . \tag{C.66}$$

The time delay above is further compounded in fields by the standard retardation effect, i.e., the field depends on ${}^{B}\overline{\mathcal{Q}}_{k<i_1 i_2\dots i_\ell>}(t-r)$ and hence on $\bar{J}^j(\vec{r}_0,t-r-zr_0)$. Thus, for a source of size R spread around the origin, we get a time delay seen by a detector ranges from $r-R$ (in the near end of the source) to $r+R$ (in the far end of the source). In an expanding universe, there is a further effect due to redshifts, which have to be correctly taken into account while defining multipole moments of cosmologically big sources.

With these comments, let us return to the task at hand: in terms of the magnetic

[5] For even d, we work with the convention that $0!! = \sqrt{\frac{2}{\pi}}$.

multipole tensor, the vector potential/EM fields outside the sources can be written as[6]

$$\begin{aligned}
\mathcal{V}_k^{\rm Out}(\vec{r},\omega) &= \sum_\ell \frac{G_{_B}^{\rm Out}(r,\omega,\ell)}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\,{}^B\mathcal{Q}_{k<i_1i_2\ldots i_\ell>}\frac{x^{i_1}\ldots x^{i_\ell}}{r^{\ell+1}}\ , \\
\mathcal{C}_{kt}^{\rm Out}(\vec{r},\omega) &= i\omega\sum_\ell \frac{G_{_B}^{\rm Out}(r,\omega,\ell)}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\,{}^B\mathcal{Q}_{k<i_1i_2\ldots i_\ell>}\frac{x^{i_1}\ldots x^{i_\ell}}{r^{\ell+1}}\ , \\
\mathcal{C}_{jk}^{\rm Out}(\vec{r},\omega) &= {\rm Anti}_{jk}\sum_\ell \frac{G_{_B}^{\rm Out}(r,\omega,\ell)}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\,{}^B\mathcal{Q}_{j<i_1i_2\ldots i_\ell>}[(2\ell+d-2)x^kx^{i_\ell}-r^2\ell\delta^{ki_\ell}]\frac{x^{i_1}\ldots x^{i_{\ell-1}}}{r^{\ell+3}} \\
&\quad -\omega^2{\rm Anti}_{jk}\sum_\ell \frac{G_{_B}^{\rm Out}(r,\omega,\ell-1)}{\ell!\mathcal{N}_{d,\ell-2}|\mathbb{S}^{d-1}|}\,{}^B\mathcal{Q}_{j<i_1i_2\ldots i_\ell>}\frac{x^kx^{i_1}\ldots x^{i_\ell}}{r^{\ell+2}}\ .
\end{aligned} \tag{C.67}$$

The field strengths here can also be derived by direct cartesian differentiation (using Eq.(C.47) when needed). The multipole vector potential given here can be rewritten using Eq.(C.48) as

$$\mathcal{V}_k^{\rm Out}(\vec{r},\omega) = \sum_\ell \frac{(-1)^\ell}{\ell!}\partial^{i_1}\partial^{i_2}\ldots\partial^{i_\ell}\left\{{}^B\mathcal{Q}_{k<i_1i_2\ldots i_\ell>}\frac{G_{_B}^{\rm Out}(r,\omega,\ell=0)}{(d-2)|\mathbb{S}^{d-1}|r}\right\}\ . \tag{C.68}$$

For odd d, we can write

$$\frac{G_{_B}^{\rm Out}(r,\omega,\ell=0)}{(d-2)|\mathbb{S}^{d-1}|r} = \frac{\theta_{\frac{d-3}{2}}(-i\omega r)}{(d-2)!!}\frac{e^{i\omega r}}{|\mathbb{S}^{d-1}|r^{d-2}}\ . \tag{C.69}$$

The $e^{i\omega r}$ factor gives the standard retardation time-delay. Moving to time-domain, we then have a simple statement in $d=3$: the vector potential of a magnetic multipole is obtained by multiplying the retarded STF magnetic tensor with the Couloumb potential, followed by repeated differentiation. For odd $d>3$, we should apply an additional differential operator that depends on r and with maximum $\frac{d-3}{2}$ time-derivatives

$$\frac{\theta_{\frac{d-3}{2}}(r\partial_t)}{(d-4)!!}\ , \tag{C.70}$$

before the repeated differentiation [161]. Next, the large r asymptotics of both the vector

[6]Here we use the notation ${\rm Anti}_{jk}[T_{jk}]\equiv T_{jk}-T_{kj}$ for the anti-symmetrisation operator.

potential as well as the field strengths can be obtained via Eq.(C.43). We get

$$\begin{aligned}
\mathcal{V}_k^{\rm Rad}(\vec{r},\omega) &= \frac{e^{i\omega r}}{(d-2)!!|\mathbb{S}^{d-1}|r^{\frac{d-1}{2}}}\sum_\ell \frac{(-i\omega)^{\nu-\frac{1}{2}}}{\ell!}{}^B\mathcal{Q}_{k<i_1 i_2\ldots i_\ell>}n^{i_1}\ldots n^{i_\ell} \ , \\
\mathcal{C}_{kt}^{\rm Rad}(\vec{r},\omega) &= -\frac{e^{i\omega r}}{(d-2)!!|\mathbb{S}^{d-1}|r^{\frac{d-1}{2}}}\sum_\ell \frac{(-i\omega)^{\nu+\frac{1}{2}}}{\ell!}{}^B\mathcal{Q}_{k<i_1 i_2\ldots i_\ell>}n^{i_1}\ldots n^{i_\ell} \ , \\
\mathcal{C}_{jk}^{\rm Rad}(\vec{r},\omega) &= \frac{e^{i\omega r}}{(d-2)!!|\mathbb{S}^{d-1}|r^{\frac{d-1}{2}}}\times \text{Anti}_{jk}\left\{n^k\sum_\ell \frac{(-i\omega)^{\nu+\frac{1}{2}}}{\ell!}\ {}^B\mathcal{Q}_{j<i_1 i_2\ldots i_\ell>}n^{i_1}\ldots n^{i_\ell}\ \right\} \ ,
\end{aligned} \tag{C.71}$$

where we have used the notation $n^i \equiv \frac{x^i}{r}$. The infinite sum appearing in the field strengths is the EM waveform or the *light vector*.

C.3 Multipole expansion in statics II : poloidal currents and charges

Magnetic fields due to poloidal currents

We can go further and generalise to any time-independent, divergence-free current distribution. Such a current distribution, which *not* toroidal (i.e., not purely tangential to the sphere directions), is said to be a *poloidal* current distribution. A poloidal current can be expanded as

$$\bar{J}^r = \sum_{\ell\vec{m}} \bar{J}_P(r,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \quad \bar{J}^I = \sum_{\ell\vec{m}} \bar{J}_Q(r,\ell,\vec{m})\gamma^{IJ}\mathscr{D}_J\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \tag{C.72}$$

where $\mathscr{Y}_{\ell\vec{m}}(\hat{r})$ are orthonormal scalar spherical harmonics (SSHs) on $\mathbb{S}^{d-1}$. They satisfy

$$\left[\mathscr{D}^2+\ell(\ell+d-2)\right]\mathscr{Y}_{\ell\vec{m}} = 0 \ , \quad \int_{\mathbb{S}^{d-1}} \mathscr{Y}^*_{\ell'\vec{m}'}\mathscr{Y}_{\ell\vec{m}} = \delta_{\ell'\ell}\delta_{\vec{m}'\vec{m}} \ . \tag{C.73}$$

We want to determine the vector potential and magnetic field due to such a poloidal current distribution. With some hindsight, we will abandon the Coulomb gauge and use instead a gauge where the vector potential is purely radial. The vector potential and magnetic field are then of the form

$$\begin{aligned}\overline{\mathcal{V}}_r &\equiv -\sum_{\ell\vec{m}} \overline{H}_s(r,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \quad \overline{\mathcal{V}}_I = 0 \ , \\ \overline{\mathcal{C}}_{rI} &= \sum_{\ell\vec{m}} \overline{H}_s(r,\ell,\vec{m})\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \quad \overline{\mathcal{C}}_{IJ} = 0 \ .\end{aligned} \tag{C.74}$$

Using the sourced Maxwell equations[7], we obtain the following relation between the currents and the magnetic field:

$$\begin{aligned}\bar{J}^r &= -\sum_{\ell\vec{m}} \frac{\ell(\ell+d-2)}{r^2}\overline{H}_s(r,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\ \bar{J}^I &= -\sum_{\ell\vec{m}} \frac{1}{r^{d-1}}\partial_r[r^{d-3}\overline{H}_s(r,\ell,\vec{m})]\gamma^{IJ}\mathscr{D}_J\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ ,\end{aligned} \tag{C.76}$$

These two equations are not independent: they are related by current conservation, viz.,

$$\frac{1}{r^{d-1}}\frac{\partial}{\partial r}(r^{d-1}\bar{J}^r) + \mathscr{D}_I\bar{J}^I = 0 \ . \tag{C.77}$$

Thus, it is enough to invert the first equation. Using the orthonormality of scalar spherical harmonics, we can write

$$\begin{aligned}\overline{H}_s(r,\ell,\vec{m}) &= -\frac{r^2}{\ell(\ell+d-2)}\int_{\hat{r}\in\mathbb{S}^{d-1}} \mathscr{Y}^*_{\ell\vec{m}}(\hat{r})\bar{J}^r(\vec{r}) \\ &= -\frac{1}{\ell(\ell+d-2)}\frac{1}{r^{d-3}}\int_{\vec{r}_0} \mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\delta(r-r_0)\bar{J}^r(\vec{r}_0) \ .\end{aligned} \tag{C.78}$$

A few comments are in order: the result we have derived is valid if there is no $\ell = 0$ component (i.e., there is no spherically symmetric component) in $\bar{J}^r$. A bit of thought shows that this has to be true: a spherically symmetric radial current is inconsistent with charge conservation in the static limit. The next comment is about the locality in the radial direction: we see that if the poloidal current is confined within a radius R, its magnetic field also never extends beyond R. In particular, the far field multipole expansion we derived before does not get corrected by poloidal currents.

[7] They are of the form

$$-\gamma^{IJ}\mathscr{D}_I\overline{\mathcal{C}}_{Jr} = r^2\bar{J}^r \ , \quad \frac{1}{r^{d-1}}\partial_r[r^{d-3}\overline{\mathcal{C}}_{Ir}] + \frac{1}{r^4}\gamma^{JK}\mathscr{D}_K\overline{\mathcal{C}}_{IJ} = \gamma_{IJ}\bar{J}^J \ . \tag{C.75}$$

When we move to time-varying currents, we will see that $\overline{H}_s$ part of the magnetic field can in fact escape the currents and travel out as EM radiation. This suggests that, in the full dynamical situation, $\overline{H}_s$ satisfies a wave equation with a source. Such an equation should reduce to a Poisson-like equation in the static limit. To see how this works, we combine the two equations of Eq.(C.76) into a Poisson-like equation:

$$\begin{aligned} r^{d-3}\partial_r\left(r^{3-d}\partial_r[r^{d-3}\overline{H}_s]\right) - \frac{\ell(\ell+d-2)}{r^2}(r^{d-3}\overline{H}_s) \\ = \int_{\hat{r}\in\mathbb{S}^{d-1}} r^{d-3}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r})\left\{\bar{J}^r(\vec{r}) + \frac{1}{\ell(\ell+d-2)}\partial_r\left[r^2\mathscr{D}_I\bar{J}^I(\vec{r})\right]\right\} . \end{aligned} \tag{C.79}$$

As we did for toroidal currents, we can solve this equation by thinking of the source as made of spherical shells and then integrating. we end up with

$$\begin{aligned} r^{d-3}\overline{H}_s = -\int_{\vec{r}_0} \mathbb{G}_E(r,r_0,\ell)\ r_0^{1-d}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\left\{\bar{J}^r(\vec{r}_0) + \frac{1}{\ell(\ell+d-2)}\partial_{r_0}\left[r_0^2\mathscr{D}_I\bar{J}^I(\vec{r}_0)\right]\right\} \\ \text{with}\quad \mathbb{G}_E(r,r_0,\ell) \equiv \frac{1}{2\ell+d-2}\left\{\frac{r^{\ell+d-2}}{r_0^\ell}\Theta(r<r_0) + \frac{r_0^{\ell+d-2}}{r^\ell}\Theta(r>r_0)\right\} \end{aligned} \tag{C.80}$$

The locality in the radial direction can be seen by rewriting the source part of the integrand using the conservation of current:

$$\begin{aligned} \bar{J}^r(\vec{r}_0) + \frac{1}{\ell(\ell+d-2)}\partial_{r_0}\left[r_0^2\mathscr{D}_I\bar{J}^I(\vec{r}_0)\right] \\ = \bar{J}^r(\vec{r}_0) - \frac{1}{\ell(\ell+d-2)}\partial_{r_0}\left[r_0^{3-d}\partial_{r_0}(r_0^{d-1}\bar{J}^r(\vec{r}_0))\right] . \end{aligned} \tag{C.81}$$

The radial differential operator here is the same as the one in Eq.(C.79), and after integration by parts, it acts on the Green function to give a delta function. We will see later how this conclusion changes if the poloidal currents become time-dependent.

Electric fields due to static charges

We can repeat our magnetostatic analysis for electrostatics. Let us begin with a time-independent surface charge density $\bar{\sigma}(\hat{r})$ spread out on a thin spherical shell of radius R. Explicitly, we take a charge current density of the form

$$\bar{J}^t = \delta(r-R)\ \bar{\sigma}(\hat{r}) = \delta(r-R)\sum_{\ell\vec{m}}\bar{\sigma}_{\ell\vec{m}}(\hat{r}) , \tag{C.82}$$

where we have expanded out the charge density in terms of orthonormal Scalar Spherical Harmonics (SSHs) on $\mathbb{S}^{d-1}$ labelled by $\{\ell, \vec{m}\}$. Using such a decomposition, we can solve for the electric field due to each component and then add it up to get the final answer. Since the symmetry properties of each SSH under $SO(d)$ rotation is different, the scalar potential $\overline{\mathcal{V}}_t$ produced by $\bar{\sigma}_{\ell\vec{m}}$ should be proportional to $\bar{\sigma}_{\ell\vec{m}}$. We take an ansatz of the form

$$\overline{\mathcal{V}}_t = \sum_{\ell} f_\ell(r) \sum_{\vec{m}} \bar{\sigma}_{\ell\vec{m}}(\hat{r}) \ , \tag{C.83}$$

and impose the scalar Poisson equation

$$\frac{1}{r^{d-1}}\partial_r[r^{d-1}\overline{\mathcal{C}}_{rt}] + \frac{1}{r^2}\gamma^{JK}\mathscr{D}_K\overline{\mathcal{C}}_{It} = \frac{1}{r^{d-1}}\partial_r[r^{d-1}\partial_r\overline{\mathcal{V}}_t] + \frac{1}{r^2}\mathscr{D}^2\overline{\mathcal{V}}_t = \bar{J}^t \ . \tag{C.84}$$

Replacing $-\mathscr{D}^2$ by $\ell(\ell+d-2)$, we conclude that, away from the spherical shell, $f_\ell(r)$ should vary as r^ℓ or as $r^{-(\ell+d-2)}$. We should stitch together these two solutions continuously with an appropriate derivative discontinuity given by the charge density. We obtain the final answer

$$\overline{\mathcal{V}}_t = -\sum_{\ell\vec{m}} \frac{R\bar{\sigma}_{\ell\vec{m}}(\hat{r})}{2\ell+d-2}\left[\frac{r^\ell}{R^\ell}\Theta(r<R) + \frac{R^{\ell+d-2}}{r^{\ell+d-2}}\Theta(r>R)\right] \ . \tag{C.85}$$

In terms of the original data, we have

$$\begin{aligned}
\overline{\mathcal{V}}_t &= -\int_{\hat{r}_0\in\mathbb{S}^{d-1}}\sum_{\ell}\left[\frac{r^\ell}{R^\ell}\Theta(r<R) + \frac{R^{\ell+d-2}}{r^{\ell+d-2}}\Theta(r>R)\right]\frac{\sum_{\vec{m}}\mathscr{Y}_{\ell\vec{m}}(\hat{r})\mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)}{2\ell+d-2}R\bar{\sigma}_{\ell\vec{m}}(\hat{r}_0) \\
&= -\sum_{\ell}\left[\frac{r^\ell}{R^\ell}\Theta(r<R) + \frac{R^{\ell+d-2}}{r^{\ell+d-2}}\Theta(r>R)\right]\frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\int_{\hat{r}_0\in\mathbb{S}^{d-1}}\frac{\Pi^S_{d,\ell}(\hat{r}|\hat{r}_0)}{2\ell+d-2}R\bar{\sigma}_{\ell\vec{m}}(\hat{r}_0) \\
&= -\int_{\vec{r}_0}\sum_{\ell}\left\{\frac{r^\ell}{r_0^{\ell+d-2}}\Theta(r<r_0) + \frac{r_0^\ell}{r^{\ell+d-2}}\Theta(r>r_0)\right\}\frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\frac{\Pi^S_{d,\ell}(\hat{r}|\hat{r}_0)}{2\ell+d-2}\bar{J}^t(\vec{r}_0)
\end{aligned} \tag{C.86}$$

Here, we have used the SSH addition theorem at the second step. As we did in magnetostatics, we can expand the electrostatic potential as well as the electric fields in terms of spherical harmonics:

$$\begin{aligned}\overline{\mathcal{V}}_t &= \sum_{\ell\vec{m}} \overline{E}_s(r,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\ \overline{\mathcal{C}}_{rt} &\equiv \sum_{\ell\vec{m}} \overline{E}_r(r,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \quad \overline{\mathcal{C}}_{It} \equiv \sum_{\ell\vec{m}} \overline{E}_s(r,\ell,\vec{m})\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ ,\end{aligned} \tag{C.87}$$

with $\overline{E}_r = \partial_r \overline{E}_s$. The potential function then has a Green-function expression

$$\overline{E}_s = -\frac{1}{2\ell+d-2}\int_{\vec{r}_0}\left\{\frac{r^\ell}{r_0^{\ell+d-2}}\Theta(r<r_0) + \frac{r_0^\ell}{r^{\ell+d-2}}\Theta(r>r_0)\right\}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\bar{J}^t(\vec{r}_0) \ . \tag{C.88}$$

Multipole expansion outside the electric sources

Unlike the poloidal currents, the static charge distributions do give rise to fields outside them. The electric field outside the charges is then given by

$$\overline{E}_s^{\text{Out}}(r,\ell,\vec{m}) = -\frac{\overline{\mathcal{J}}^E(\ell,\vec{m})}{r^{\ell+d-2}} \ , \quad \overline{E}_r^{\text{Out}}(r,\ell,\vec{m}) = (\ell+d-2)\frac{\overline{\mathcal{J}}^E(\ell,\vec{m})}{r^{\ell+d-1}} \ , \tag{C.89}$$

where $\overline{\mathcal{J}}^E$ denotes the spherical electric multipole moments defined via

$$\overline{\mathcal{J}}^E(\ell,\vec{m}) \equiv \frac{1}{2\ell+d-2}\int_{\vec{r}_0} r_0^\ell \mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\bar{J}^t(\vec{r}_0) \ . \tag{C.90}$$

The corresponding cartesian multipole moment can be defined from the spherical moments via SSH addition theorem, viz.,

$$\frac{1}{\ell!}\ {}^E\overline{\mathcal{Q}}_{<i_1 i_2 \ldots i_\ell>} x^{i_1}\ldots x^{i_\ell} \equiv \mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|\sum_{\vec{m}} \overline{\mathcal{J}}^E(\ell,\vec{m}) r^\ell \mathscr{Y}_{\ell\vec{m}}(\hat{r}) = \int_{\vec{r}_0} \Pi^S(\vec{r}|\vec{r}_0)\bar{J}^t(\vec{r}_0) \ . \tag{C.91}$$

or equivalently, we have

$${}^E\overline{\mathcal{Q}}_{<i_1 i_2 \ldots i_\ell>} \equiv (\Pi^S)^{<i_1 i_2 \ldots i_\ell>}_{<j_1 j_2 \ldots j_\ell>}\int_{\vec{r}_0} x_0^{j_1}\ldots x_0^{j_\ell}\bar{J}^t(\vec{r}_0) \ . \tag{C.92}$$

The scalar potential and the cartesian components of the electric field take the form

$$
\begin{aligned}
\overline{\mathcal{V}}^{\text{Out}}_t &= -\sum_{\ell} \frac{1}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\,{}^{E}\overline{\mathcal{Q}}_{<i_1 i_2 \ldots i_\ell>} \frac{x^{i_1}\ldots x^{i_\ell}}{r^{2\ell+d-2}} \\
&= -\sum_{\ell} \frac{(-)^\ell}{\ell!} \partial^{i_1}\partial^{i_2}\ldots\partial^{i_\ell}\left\{\frac{{}^{E}\overline{\mathcal{Q}}_{<i_1 i_2 \ldots i_\ell>}}{(d-2)|\mathbb{S}^{d-1}|r^{d-2}}\right\} , \\
\overline{\mathcal{E}}^{\text{Out}}_{jt} &= \sum_{\ell} \frac{1}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\,{}^{E}\overline{\mathcal{Q}}_{<i_1 i_2 \ldots i_\ell>} [(2\ell+d-2)x^j x^{i_\ell} - r^2 \ell \delta^{j i_\ell}] \frac{x^{i_1}\ldots x^{i_{\ell-1}}}{r^{2\ell+d}} \ .
\end{aligned}
\tag{C.93}
$$

All this is completely analogous to our discussion of magnetostatics in STF language. As we did for magnetic STF moments, explicit expressions for the first few electric STF moments can be written down by contracting against a dummy variable:

$$
\begin{aligned}
{}^{E}\overline{\mathcal{Q}} &= \int_{\vec{r}} \bar{J}^t(\vec{r}) \ , \\
{}^{E}\overline{\mathcal{Q}}_{<i_1>}\kappa^{i_1} &= \int_{\vec{r}} (\kappa\cdot r)\ \bar{J}^t(\vec{r}) \ , \\
\frac{1}{2!}\,{}^{E}\overline{\mathcal{Q}}_{<i_1 i_2>}\kappa^{i_1}\kappa^{i_2} &= \int_{\vec{r}} \left[\frac{(\kappa\cdot r)^2}{2!} - \frac{\kappa^2 r^2}{2d}\right] \bar{J}^t(\vec{r}) \ , \\
\frac{1}{3!}\,{}^{E}\overline{\mathcal{Q}}_{<i_1 i_2 i_3>}\kappa^{i_1}\kappa^{i_2}\kappa^{i_3} &= \int_{\vec{r}} \left[\frac{(\kappa\cdot r)^3}{3!} - \frac{\kappa^2 r^2}{2(d+2)}(\kappa\cdot r)\right] \bar{J}^t(\vec{r}) \ , \\
\frac{1}{4!}\,{}^{E}\overline{\mathcal{Q}}_{<i_1 i_2 i_3 i_4>}\kappa^{i_1}\kappa^{i_2}\kappa^{i_3}\kappa^{i_4} &= \int_{\vec{r}} \left[\frac{(\kappa\cdot r)^4}{4!} - \frac{\kappa^2 r^2}{2(d+4)}\frac{(\kappa\cdot r)^2}{2!}\right. \\
&\qquad\qquad \left. + \frac{\kappa^4 r^4}{8(d+4)(d+2)}\right] \bar{J}^t(\vec{r}) \ , \\
\frac{1}{5!}\,{}^{E}\overline{\mathcal{Q}}_{<i_1 i_2 i_3 i_4 i_5>}\kappa^{i_1}\kappa^{i_2}\kappa^{i_3}\kappa^{i_4}\kappa^{i_5} &= \int_{\vec{r}} \left[\frac{(\kappa\cdot r)^5}{5!} - \frac{\kappa^2 r^2}{2(d+6)}\frac{(\kappa\cdot r)^3}{3!}\right. \\
&\qquad\qquad \left. + \frac{\kappa^4 r^4}{8(d+6)(d+4)}(\kappa\cdot r)\right] \bar{J}^t(\vec{r}) \ .
\end{aligned}
\tag{C.94}
$$

The corresponding STF tensors can be obtained by differentiating with respect to κ_i. Till $\ell = 4$, they are given as

$$\begin{aligned}
{}^{E}\overline{\mathcal{Q}} &= \int_{\vec{r}} \bar{J}^t(\vec{r}) \ , \\
{}^{E}\overline{\mathcal{Q}}_{<i_1>} &= \int_{\vec{r}} x^{i_1} \bar{J}^t(\vec{r}) \ , \\
{}^{E}\overline{\mathcal{Q}}_{<i_1 i_2>} &= \int_{\vec{r}} \left[x^{i_1} x^{i_2} - \frac{r^2}{d} \delta^{i_1 i_2} \right] \bar{J}^t(\vec{r}) \ , \\
{}^{E}\overline{\mathcal{Q}}_{<i_1 i_2 i_3>} &= \int_{\vec{r}} \left[x^{i_1} x^{i_2} x^{i_3} - \frac{r^2}{d+2} \left(x^{i_1} \delta^{i_2 i_3} + x^{i_2} \delta^{i_1 i_3} + x^{i_3} \delta^{i_1 i_2} \right) \right] \bar{J}^t(\vec{r}) \ ,
\end{aligned} \tag{C.95}$$

and

$$\begin{aligned}
{}^{E}\overline{\mathcal{Q}}_{<i_1 i_2 i_3 i_4>} = \int_{\vec{r}} \Bigg[& x^{i_1} x^{i_2} x^{i_3} x^{i_4} \\
& - \frac{r^2}{d+4} \Big(x^{i_1} x^{i_2} \delta^{i_3 i_4} + x^{i_1} x^{i_3} \delta^{i_2 i_4} + x^{i_1} x^{i_4} \delta^{i_2 i_3} \\
& \qquad + x^{i_2} x^{i_3} \delta^{i_1 i_4} + x^{i_2} x^{i_4} \delta^{i_1 i_3} + x^{i_3} x^{i_4} \delta^{i_1 i_2} \Big) \\
& + \frac{r^4}{(d+4)(d+2)} \left(\delta^{i_1 i_2} \delta^{i_3 i_4} + \delta^{i_1 i_3} \delta^{i_2 i_4} + \delta^{i_1 i_4} \delta^{i_2 i_3} \right) \Bigg] \bar{J}^t(\vec{r}) \ .
\end{aligned} \tag{C.96}$$

C.4 Electric multipole radiation

We will now turn to the problem of radiation from time-varying charge distributions and poloidal currents. Over and above the issues discussed above for the toroidal currents, there are some new subtleties which show up in this case.

First, in the dynamical setting, poloidal currents inevitably accompany changes in charge configurations, and the spacetime charge current $\bar{J}^\mu(\vec{r}, t)$ should satisfy the conservation equation

$$\partial_t \bar{J}^t + \frac{1}{r^{d-1}} \frac{\partial}{\partial r} (r^{d-1} \bar{J}^r) + \mathscr{D}_I \bar{J}^I = 0 \ . \tag{C.97}$$

Maxwell equations are mathematically consistent only if the sources obey this constraint. This means that to solve for the EM fields, we should characterise the class of currents consistent with charge conservation. Second, we need to deal with Gauss law constraint in this sector, i.e., one of the Maxwell equations serves to constrain the initial data of EM fields. Both these facts are intimately tied to gauge invariance in electromagnetism: since the gauge parameter is a scalar function, it is expandable into scalar spherical harmonics (SSHs), and its effect is visible in the scalar sector.

We will address the problem of conservation by imagining that the charge flow is described well by a time-varying electric polarisation field, i.e., we take

$$\begin{aligned} \bar{J}^t(\vec{r},t) &= -\frac{1}{r^{d-1}}\frac{\partial}{\partial r}[r^{d-1}\bar{P}^r(\vec{r},t)] - \mathscr{D}_I\bar{P}^I(\vec{r},t) \ , \\ \bar{J}^r(\vec{r},t) &= \partial_t\bar{P}^r(\vec{r},t) \ , \quad \bar{J}^I(\vec{r},t) = \partial_t\bar{P}^I(\vec{r},t) \ , \end{aligned} \tag{C.98}$$

which automatically satisfies the conservation equation. Such a polarisation field can always be defined by time integrating the current densities, i.e.,

$$\bar{P}^r(\vec{r},t) \equiv \int dt\ \bar{J}^r(\vec{r},t) \ , \quad \bar{P}^I(\vec{r},t) \equiv \int dt\ \bar{J}^I(\vec{r},t) \ . \tag{C.99}$$

Such polarisation fields also help simplify Gauss law inside charged matter: it becomes the statement of $\vec{E}+\vec{P}$ being divergence-free. This parametrisation is not without its subtleties, as we shall discuss in §§C.5. But, for now, we will take such time-varying polarisation fields are given and proceed.

We will solve the Maxwell equations by passing to the frequency domain and expanding all fields in terms of scalar spherical harmonics (SSHs). The expansion for the polarisation fields is

$$\begin{aligned} \bar{P}^r(\vec{r},t) &\equiv \sum_{\ell\vec{m}}\int_\omega e^{-i\omega t}J_1(r,\omega,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\ r^2\gamma_{IJ}\bar{P}^J(\vec{r},t) &\equiv \sum_{\ell\vec{m}}\int_\omega e^{-i\omega t}J_2(r,\omega,\ell,\vec{m})\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ . \end{aligned} \tag{C.100}$$

Using the same notation/gauge as in statics, we take the scalar/vector potential to be

$$\begin{aligned} \overline{\mathcal{V}}_t(\vec{r},t) &\equiv \sum_{\ell\vec{m}}\int_\omega e^{-i\omega t}E_s(r,\omega,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\ \overline{\mathcal{V}}_r(\vec{r},t) &\equiv -\sum_{\ell\vec{m}}\int_\omega e^{-i\omega t}H_s(r,\omega,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\ \overline{\mathcal{V}}_I(\vec{r},t) &= 0 \ . \end{aligned} \tag{C.101}$$

We aim to solve for E_s, H_s in terms of J_1 and J_2. To this end, we first compute the EM fields corresponding to the above potential:

$$\begin{aligned}
\overline{\mathcal{C}}_{rt}(\vec{r},t) &= \sum_{\ell\vec{m}} \int_\omega e^{-i\omega t} E_r(r,\omega,\ell,\vec{m})\ \mathscr{Y}_{\ell\vec{m}}(\hat{r})\ , \\
\overline{\mathcal{C}}_{It}(\vec{r},t) &= \sum_{\ell\vec{m}} \int_\omega e^{-i\omega t} E_s(r,\omega,\ell,\vec{m})\ \mathscr{D}_I \mathscr{Y}_{\ell\vec{m}}(\hat{r})\ , \\
\overline{\mathcal{C}}_{rI}(\vec{r},t) &= \sum_{\ell\vec{m}} \int_\omega e^{-i\omega t} H_s(r,\omega,\ell,\vec{m})\ \mathscr{D}_I \mathscr{Y}_{\ell\vec{m}}(\hat{r})\ , \\
\overline{\mathcal{C}}_{IJ}(\vec{r},t) &= 0\ ,
\end{aligned} \tag{C.102}$$

where $E_r = \partial_r E_s - i\omega H_s$. This relation giving E_r in terms of E_s and H_s can also be directly obtained from the Bianchi identity (or the unsourced Maxwell equations).

Next, we write down the sourced Maxwell equations:

$$\begin{aligned}
\frac{1}{r^{d-1}}\partial_r\left(r^{d-1}[E_r+J_1]\right) - \frac{\ell(\ell+d-2)}{r^2}[E_s+J_2] &= 0 \qquad \text{(t-Eqn)}\ , \\
-\frac{\ell(\ell+d-2)}{r^2}H_s + i\omega[E_r+J_1] &= 0 \qquad \text{(r-Eqn)}\ , \\
-\frac{1}{r^{d-3}}\partial_r\left(r^{d-3}H_s\right) + i\omega[E_s+J_2] &= 0 \qquad \text{(I-Eqn)}\ .
\end{aligned} \tag{C.103}$$

We note how the electric field always shows up in the $\vec{E}+\vec{P}$ combination. The above set of coupled ODEs can be solved by introducing the *electric Debye field* $\Phi_E(r,\omega,\ell,\vec{m})$ such that

$$E_r + J_1 = \frac{\ell(\ell+d-2)}{r^{d-1}}\Phi_E\ , \quad E_s + J_2 = \frac{1}{r^{d-3}}\partial_r\Phi_E\ , \quad H_s = \frac{i\omega}{r^{d-3}}\Phi_E\ . \tag{C.104}$$

The relation $E_r = \partial_r E_s - i\omega H_s$ then becomes the follow inhomogeneous Helmholtz equation for Φ_E:

$$-\frac{1}{r^{3-d}}\partial_r\left(r^{3-d}\partial_r\Phi_E\right) - \omega^2\Phi_E + \frac{\ell(\ell+d-2)}{r^2}\Phi_E = r^{d-3}\left[J_1 - \partial_r J_2\right]\ . \tag{C.105}$$

From hereon, the procedure here is similar to the one adopted for the magnetic Debye field Φ_B. The above equation is solved by finding an appropriate Green function $\mathbb{G}_E$ such that

$$\Phi_E(r,\omega,\ell,\vec{m}) = \int_0^\infty dr_0\ \mathbb{G}_E(r,r_0;\omega,\ell)\left[J_1(r_0,\omega,\ell,\vec{m}) - \partial_{r_0}J_2(r_0,\omega,\ell,\vec{m})\right] \tag{C.106}$$

The Green function $\mathbb{G}_E$ then obeys

$$-\partial_r[r^{3-d}\partial_r\mathbb{G}_E] - \omega^2 r^{3-d}\mathbb{G}_E + \ell(\ell+d-2)r^{1-d}\mathbb{G}_E = \delta(r-r_0) \ , \tag{C.107}$$

The magnetic Helmholtz equation in Eq.(C.53) can be mapped to the electric Helmholtz equation through the replacements

$$d \mapsto 6-d \ , \quad \ell \mapsto \ell+d-3 \ , \quad \nu \mapsto \nu \tag{C.108}$$

where $\nu \equiv \ell + \frac{d}{2} - 1$. Through such a replacement, the results we derived in the magnetic case can be recycled here. Given this, we will be content in stating just the final results in what follows.

Homogeneous spherical waves

The homogenous solutions are as follows: the solution regular everywhere is

$$r^{\nu+\frac{d}{2}-1}{}_0F_1\left[1+\nu, -\frac{\omega^2r^2}{4}\right] = \Gamma(1+\nu)\left(\frac{\omega}{2}\right)^{-\nu} r^{\frac{d}{2}-1}J_\nu(\omega r) \ , \tag{C.109}$$

and this solution plays the role of the time-delay smearing function in the electric case. The solution that is regular everywhere except the origin is given by

$$\begin{aligned}\frac{1}{r^{\nu-\frac{d}{2}+1}}&\left\{{}_0F_1\left[1-\nu, -\frac{\omega^2r^2}{4}\right] - \frac{\pi\cot\nu\pi}{\Gamma(\nu)\Gamma(1+\nu)}\left(\frac{\omega r}{2}\right)^{2\nu} {}_0F_1\left[1+\nu, -\frac{\omega^2r^2}{4}\right]\right\}\\ &= -\left(\frac{\omega}{2}\right)^{\nu}\frac{\pi \ r^{\frac{d}{2}-1}}{\Gamma(\nu)}Y_\nu(\omega r) \ .\end{aligned} \tag{C.110}$$

The outgoing/ingoing solutions are

$$\begin{aligned}\frac{1}{r^{\nu-\frac{d}{2}+1}}&\left\{{}_0F_1\left[1-\nu, -\frac{\omega^2r^2}{4}\right]\right.\\ &\left.\pm(1\pm i\cot\nu\pi)\frac{2\pi i}{\Gamma(\nu)^2}\frac{1}{2\nu}\left(\frac{\omega r}{2}\right)^{2\nu} {}_0F_1\left[1+\nu, -\frac{\omega^2r^2}{4}\right]\right\}\mathbb{V}_I^{\alpha\ell\vec{m}}\\ &= \pm i\left(\frac{\omega}{2}\right)^{\nu}\frac{\pi \ r^{\frac{d}{2}-1}}{\Gamma(\nu)}H_\nu^{\pm}(\omega r)\mathbb{V}_I^{\alpha\ell\vec{m}}\end{aligned} \tag{C.111}$$

As in the magnetic case, when d is even and ν is a positive integer, these expressions become indeterminate, and we should take a limit. The radial part of the outgoing waves is

$$G^{\text{Out}}_{E}(r,\omega,\ell) \equiv \frac{1}{r^{\nu-\frac{d}{2}+1}}\Bigg\{ {}_0F_1\left[1-\nu,-\frac{\omega^2r^2}{4}\right] \\ +(1+i\cot\nu\pi)\frac{2\pi i}{\Gamma(\nu)^2}\frac{1}{2\nu}\left(\frac{\omega r}{2}\right)^{2\nu}\ {}_0F_1\left[1+\nu,-\frac{\omega^2r^2}{4}\right]\Bigg\} \ . \tag{C.112}$$

When ν is a half-integer, the function G^{Out}_{E} can be expressed in terms of reverse Bessel polynomials as

$$G^{\text{Out}}_{E}(r,\omega,\ell) = \frac{\theta_{\nu-\frac{1}{2}}(-i\omega r)}{\theta_{\nu-\frac{1}{2}}(0)}\frac{e^{i\omega r}}{r^{\nu-\frac{d}{2}+1}} \qquad \text{for} \quad \nu\in\mathbb{Z}+\frac{1}{2}\ , \tag{C.113}$$

whose large r asymptotic are given by

$$G^{\text{Out}}_{E}(r,\omega,\ell) \to \mathcal{N}_{d,\ell-1}\frac{(-i\omega)^{\nu-\frac{1}{2}}}{(d-2)!!}\frac{e^{i\omega r}}{r^{\frac{3-d}{2}}} \qquad \text{as} \quad r\to\infty\ . \tag{C.114}$$

The raising and lowering relations become

$$\begin{aligned} -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{G^{\text{Out}}_{E}(r,\omega,\ell)}{\mathcal{N}_{d,\ell-1}r^{\ell+d-2}}\right] &= \frac{G^{\text{Out}}_{E}(r,\omega,\ell+1)}{\mathcal{N}_{d,\ell}r^{\ell+d-1}}\ , \\ \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{G^{\text{Out}}_{E}(r,\omega,\ell)}{\mathcal{N}_{d,\ell-1}\ r^{-\ell}}\right] &= \omega^2\frac{G^{\text{Out}}_{E}(r,\omega,\ell-1)}{\mathcal{N}_{d,\ell-2}\ r^{1-\ell}}\ , \end{aligned} \tag{C.115}$$

and the derivative of G^{Out}_{E} is

$$\begin{aligned} \partial_r G^{\text{Out}}_{E}(r,\omega,\ell) &= -\frac{\ell}{r}G^{\text{Out}}_{E}(r,\omega,\ell)\ +\frac{\omega^2}{2\ell+d-4}G^{\text{Out}}_{E}(r,\omega,\ell-1) \\ &= -\ell\ G^{\text{Out}}_{E}(r,\omega,\ell+1) + \frac{\ell+d-2}{2\ell+d-2}\frac{\omega^2}{2\ell+d-4}G^{\text{Out}}_{E}(r,\omega,\ell-1)\ . \end{aligned} \tag{C.116}$$

The cartesian version of raising/lowering relations is

$$-\partial_i\left[\frac{G^{\text{Out}}_{E}(r,\omega,\ell)}{\mathcal{N}_{d,\ell-1}r^{\ell+d-2}}\right] = x^i\frac{G^{\text{Out}}_{E}(r,\omega,\ell+1)}{\mathcal{N}_{d,\ell}r^{\ell+d-1}}\ , \quad \partial_i\left[\frac{G^{\text{Out}}_{E}(r,\omega,\ell)}{\mathcal{N}_{d,\ell-1}\ r^{-\ell}}\right] = \omega^2 x^i\frac{G^{\text{Out}}_{@}(r,\omega,\ell-1)}{\mathcal{N}_{d,\ell-2}\ r^{1-\ell}}\ , \tag{C.117}$$

and the electric analog of Eq.(C.48) is given by

$$(-1)^\ell \partial^{<i_1}\partial^{i_2}\dots\partial^{i_\ell>}\left\{\frac{G_E^{\text{Out}}(r,\omega,\ell=0)}{(d-2)r^{d-2}}\right\} = \frac{G_E^{\text{Out}}(r,\omega,\ell)}{\mathcal{N}_{d,\ell-1}r^{\ell+d-2}}x^{<i_1}\dots x^{i_\ell>} \ . \tag{C.118}$$

Green function for electric radiation

The Green function $\mathbb{G}_E$ can be constructed by stitching together the homogeneous solutions continuously but with an appropriate discontinuity in its derivative. We have

$$\begin{aligned}\mathbb{G}_E(r,r_0;\omega,\ell) &= \frac{1}{2\nu}\frac{r_<^{\nu+\frac{d}{2}-1}}{r_>^{\nu-\frac{d}{2}+1}}\,{}_0F_1\left[1+\nu,-\frac{\omega^2 r_<^2}{4}\right] \\ &\times\left\{{}_0F_1\left[1-\nu,-\frac{\omega^2 r_>^2}{4}\right]\right. \\ &\qquad\left.+(1+i\cot\nu\pi)\frac{2\pi i}{\Gamma(\nu)^2}\frac{1}{2\nu}\left(\frac{\omega r_>}{2}\right)^{2\nu}\ {}_0F_1\left[1+\nu,-\frac{\omega^2 r_>^2}{4}\right]\right\} \\ &= \frac{1}{2\nu}r_<^{\nu+\frac{d}{2}-1}\,{}_0F_1\left[1+\nu,-\frac{\omega^2 r_<^2}{4}\right]G_E^{\text{Out}}(r_>,\omega,\ell) \ ,\end{aligned} \tag{C.119}$$

where we have defined:

$$r_> \equiv \text{Max}(r,r_0) \ , \quad r_< \equiv \text{Min}(r,r_0) \ . \tag{C.120}$$

We can compute the the corresponding electric/magnetic field components as

$$\begin{aligned}E_s + J_2 &= \frac{1}{r^{d-3}}\partial_r\Phi_E \\ &= \frac{1}{r^{d-3}}\int_0^\infty dr_0\ \partial_r\mathbb{G}_E(r,r_0;\omega,\ell)\left[J_1(r_0,\omega,\ell,\vec{m}) - \partial_{r_0}J_2(r_0,\omega,\ell,\vec{m})\right] \ , \\ E_r + J_1 &= \frac{\ell(\ell+d-2)}{r^{d-1}}\Phi_E \\ &= \frac{\ell(\ell+d-2)}{r^{d-1}}\int_0^\infty dr_0\ \mathbb{G}_E(r,r_0;\omega,\ell)\left[J_1(r_0,\omega,\ell,\vec{m}) - \partial_{r_0}J_2(r_0,\omega,\ell,\vec{m})\right] \ , \\ H_s &= \frac{i\omega}{r^{d-3}}\Phi_E \\ &= \frac{i\omega}{r^{d-3}}\int_0^\infty dr_0\ \mathbb{G}_E(r,r_0;\omega,\ell)\left[J_1(r_0,\omega,\ell,\vec{m}) - \partial_{r_0}J_2(r_0,\omega,\ell,\vec{m})\right] \ .\end{aligned} \tag{C.121}$$

These time-varying electric and magnetic fields sustain each other and propagate outwards as electric multipole radiation.

Fields outside sources

We will now turn to fields outside the sources, determined by electric multipole moments. For $\ell \neq 0$, we will define a dynamic version of the electric multipole moment via

$$\begin{aligned}\mathcal{J}^E(\omega,\ell,\vec{m}) \equiv \frac{\ell}{2\nu}\int_0^\infty dr_0\ r_0^{\nu+\frac{d}{2}-1}\,{}_0F_1\left[1+\nu,-\frac{\omega^2 r_0^2}{4}\right] \\ \times\left\{J_1(r_0,\omega,\ell,\vec{m})-\partial_{r_0}J_2(r_0,\omega,\ell,\vec{m})\right\}\ .\end{aligned} \tag{C.122}$$

We will justify this definition and generalise it to the $\ell = 0$ case later. While this looks very different from the electric multipole moments defined in the static case, we will later see that it reduces to them in the appropriate limit.

The electric Debye field outside the sources is given by

$$\Phi_E^{\text{Out}} = \frac{1}{\ell}G_E^{\text{Out}}(r,\omega,\ell)\ \mathcal{J}^E(\omega,\ell,\vec{m}) = \frac{1}{\ell}\frac{\theta_{\nu-\frac{1}{2}}(-i\omega r)}{\theta_{\nu-\frac{1}{2}}(0)}\frac{e^{i\omega r}}{r^\ell}\mathcal{J}^E(\omega,\ell,\vec{m})\ . \tag{C.123}$$

The corresponding field strength components are

$$\begin{aligned}E_r^{\text{Out}} &= \frac{\ell+d-2}{r^{d-1}}G_E^{\text{Out}}(r,\omega,\ell)\ \mathcal{J}^E(\omega,\ell,\vec{m}) = (\ell+d-2)\frac{\theta_{\nu-\frac{1}{2}}(-i\omega r)}{\theta_{\nu-\frac{1}{2}}(0)}\frac{e^{i\omega r}}{r^{\ell+d-1}}\mathcal{J}^E(\omega,\ell,\vec{m}),\\ E_s^{\text{Out}} &= -\frac{1}{r^{d-2}}G_E^{\text{Out}}(r,\omega,\ell)\ \mathcal{J}^E(\omega,\ell,\vec{m}) + \frac{1}{r^{d-3}}\frac{\omega^2}{\ell(2\ell+d-4)}G_E^{\text{Out}}(r,\omega,\ell-1)\ \mathcal{J}^E(\omega,\ell,\vec{m})\\ &= -\frac{\theta_{\nu-\frac{1}{2}}(-i\omega r)}{\theta_{\nu-\frac{1}{2}}(0)}\frac{e^{i\omega r}}{r^{\ell+d-2}}\mathcal{J}^E(\omega,\ell,\vec{m}) + \frac{\omega^2}{\ell(2\ell+d-4)}\frac{\theta_{\nu-\frac{3}{2}}(-i\omega r)}{\theta_{\nu-\frac{3}{2}}(0)}\frac{e^{i\omega r}}{r^{\ell+d-4}}\mathcal{J}^E(\omega,\ell,\vec{m}),\\ H_s^{\text{Out}} &= \frac{i\omega}{r^{d-3}}\frac{1}{\ell}G_E^{\text{Out}}(r,\omega,\ell)\ \mathcal{J}^E(\omega,\ell,\vec{m}) = \frac{i\omega}{\ell}\frac{\theta_{\nu-\frac{1}{2}}(-i\omega r)}{\theta_{\nu-\frac{1}{2}}(0)}\frac{e^{i\omega r}}{r^{\ell+d-3}}\mathcal{J}^E(\omega,\ell,\vec{m}).\end{aligned} \tag{C.124}$$

Here we have used Eq.(C.116) for evaluating $\partial_r G_E^{\text{Out}}$. One check of our multipole moment definition is that it reduces to the correct static expressions in $\omega \to 0$ limit, i.e., we get

$$\overline{E}_s^{\text{Out}}(r,\ell,\vec{m}) = -\frac{\overline{\mathcal{J}}^E(\ell,\vec{m})}{r^{\ell+d-2}}\ ,\qquad \overline{E}_r^{\text{Out}}(r,\ell,\vec{m}) = (\ell+d-2)\frac{\overline{\mathcal{J}}^E(\ell,\vec{m})}{r^{\ell+d-1}}\ . \tag{C.125}$$

The radiative parts work out to be

$$\begin{aligned}
\Phi_E^{\text{Rad}} &= \frac{1}{\ell}\frac{(-i\omega)^{\nu-\frac{1}{2}}}{(2\nu-2)!!}\frac{e^{i\omega r}}{r^{\frac{3-d}{2}}}\mathcal{J}^E(\omega,\ell,\vec{m}) \ , \\
E_r^{\text{Rad}} &= (\ell+d-2)\frac{(-i\omega)^{\nu-\frac{1}{2}}}{(2\nu-2)!!}\frac{e^{i\omega r}}{r^{\frac{d+1}{2}}}\ \mathcal{J}^E(\omega,\ell,\vec{m}), \\
E_s^{\text{Rad}} &= -\frac{1}{\ell}\frac{(-i\omega)^{\nu+\frac{1}{2}}}{(2\nu-2)!!}\frac{e^{i\omega r}}{r^{\frac{d-3}{2}}}\mathcal{J}^E(\omega,\ell,\vec{m}), \\
H_s^{\text{Rad}} &= -\frac{1}{\ell}\frac{(-i\omega)^{\nu+\frac{1}{2}}}{(2\nu-2)!!}\frac{e^{i\omega r}}{r^{\frac{d-3}{2}}}\mathcal{J}^E(\omega,\ell,\vec{m}),
\end{aligned} \tag{C.126}$$

where we have used Eq.(C.114). Note the faster fall-off of the radial electric field: this is consistent with the expectation that, at large r, EM fields can be thought of as transverse plane waves travelling outwards radially.

We now turn to the formulation in terms of cartesian STF tensors. We convert the orthonormal electric moments to STF electric moments by Eq.(C.91) generalised to arbiitrary frequency, i.e.,

$$\frac{1}{\ell!}\ [{}^E\mathcal{Q}(\omega)]_{<i_1 i_2 \dots i_\ell>}\ x^{i_1}\dots x^{i_\ell} \equiv \mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|\sum_{\vec{m}}\mathcal{J}^E(\omega,\ell,\vec{m})\ r^\ell \mathscr{Y}_{\ell\vec{m}}(\hat{r})\ . \tag{C.127}$$

The cartesian form of the scalar/vector potential can be obtained by substituting $\{E_s^{\text{Out}}, H_s^{\text{Out}}\}$ into Eq.(C.101), and converting everything to cartesian coordinates. We end up with

$$\begin{aligned}
\mathcal{V}_t^{\text{Out}}(\vec{r},\omega) &= -\sum_\ell \frac{G_E^{\text{Out}}(r,\omega,\ell)}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\,{}^E\mathcal{Q}_{<i_1 i_2\dots i_\ell>}\frac{x^{i_1}\dots x^{i_\ell}}{r^{\ell+d-2}} \\
&\qquad + \omega^2\sum_{\ell>0}\frac{1}{\ell}\frac{G_E^{\text{Out}}(r,\omega,\ell-1)}{\ell!\mathcal{N}_{d,\ell-2}|\mathbb{S}^{d-1}|}\ {}^E\mathcal{Q}_{<i_1 i_2\dots i_\ell>}\frac{x^{i_1}\dots x^{i_\ell}}{r^{\ell+d-3}}\ , \\
\mathcal{V}_k^{\text{Out}}(\vec{r},\omega) &= -i\omega\sum_{\ell>0}\frac{1}{\ell}\frac{G_E^{\text{Out}}(r,\omega,\ell)}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\,{}^E\mathcal{Q}_{<i_1 i_2\dots i_\ell>}\frac{x^k x^{i_1}\dots x^{i_\ell}}{r^{\ell+d-2}}\ .
\end{aligned} \tag{C.128}$$

The field strengths are given by [8]

$$\begin{aligned}
\mathcal{C}^{\text{Out}}_{kt}(\vec{r},\omega) &= \sum_{\ell} \frac{G^{\text{Out}}_{E}(r,\omega,\ell)}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\,{}^{E}\mathcal{Q}_{<i_1 i_2\ldots i_\ell>}[(2\ell+d-2)x^k x^{i_\ell} - r^2\ell\delta^{k i_\ell}]\frac{x^{i_1}\ldots x^{i_{\ell-1}}}{r^{\ell+d}} \\
&\quad - \omega^2 \sum_{\ell} \frac{G^{\text{Out}}_{E}(r,\omega,\ell-1)}{\ell!\mathcal{N}_{d,\ell-2}|\mathbb{S}^{d-1}|}\,{}^{E}\mathcal{Q}_{<i_1 i_2\ldots i_\ell>}[x^k x^{i_\ell} - r^2\delta^{k i_\ell}]\frac{x^{i_1}\ldots x^{i_{\ell-1}}}{r^{\ell+d-1}} \ , \\
\mathcal{C}^{\text{Out}}_{jk}(\vec{r},\omega) &= \text{Anti}_{jk}\sum_{\ell}(-i\omega)\frac{G^{\text{Out}}_{E}(r,\omega,\ell)}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\,{}^{E}\mathcal{Q}_{<j i_1 i_2\ldots i_{\ell-1}>}\frac{x^k x^{i_1}\ldots x^{i_{\ell-1}}}{r^{\ell+d-2}} \ .
\end{aligned} \tag{C.129}$$

These expressions can be derived by converting the spherical components to cartesian components or by directly differentiating the potentials (using Eq.(C.117) when necessary). Using Eq.(C.114), the large r asymptotics of the potentials work out to be

$$\begin{aligned}
\mathcal{V}^{\text{Rad}}_{t}(\vec{r},\omega) &= -\frac{e^{i\omega r}}{(d-2)!!|\mathbb{S}^{d-1}|r^{\frac{d-3}{2}}}\sum_{\ell>0}\frac{1}{\ell}\frac{(-i\omega)^{\nu+\frac{1}{2}}}{\ell!}\,{}^{E}\mathcal{Q}_{<i_1 i_2\ldots i_\ell>}n^{i_1}\ldots n^{i_\ell} \ , \\
\mathcal{V}^{\text{Rad}}_{k}(\vec{r},\omega) &= \frac{e^{i\omega r}}{(d-2)!!|\mathbb{S}^{d-1}|r^{\frac{d-3}{2}}}\sum_{\ell>0}\frac{1}{\ell}\frac{(-i\omega)^{\nu+\frac{1}{2}}}{\ell!}\,{}^{E}\mathcal{Q}_{<i_1 i_2\ldots i_\ell>}n^k n^{i_1}\ldots n^{i_\ell} \ ,
\end{aligned} \tag{C.130}$$

where we have used the notation $n^i \equiv \frac{x^i}{r}$. The fall-off here is slower than what might have been naively expected, e.g., in $d=3$, the potentials tend to a non-zero angle-dependent constant as $r\to\infty$ instead of becoming zero. The corresponding field strengths, however, have the correct asymptotic fall-off, viz.,

$$\begin{aligned}
\mathcal{C}^{\text{Rad}}_{kt}(\vec{r},\omega) &= -\frac{e^{i\omega r}}{(d-2)!!|\mathbb{S}^{d-1}|r^{\frac{d-1}{2}}}\sum_{\ell}\frac{(-i\omega)^{\nu+\frac{1}{2}}}{\ell!}\,{}^{E}\mathcal{Q}_{<i_1 i_2\ldots i_\ell>}[\delta^{k i_\ell} - n^k n^{i_\ell}]n^{i_1}\ldots n^{i_{\ell-1}} \ , \\
\mathcal{C}^{\text{Rad}}_{jk}(\vec{r},\omega) &= \frac{e^{i\omega r}}{(d-2)!!|\mathbb{S}^{d-1}|r^{\frac{d-1}{2}}}\times\text{Anti}_{jk}\left\{n^k\sum_{\ell}\frac{(-i\omega)^{\nu+\frac{1}{2}}}{\ell!}\,{}^{E}\mathcal{Q}_{<j i_1 i_2\ldots i_{\ell-1}>}n^{i_1}\ldots n^{i_{\ell-1}}\right\} \ .
\end{aligned} \tag{C.131}$$

This suggests that there should be a (large) gauge transformation that brings the potentials to the naively expected fall-offs. Such a gauge transformation can in fact be presented explicitly. Consider the large gauge transformation that removes all the $G^{\text{Out}}_{E}(r,\omega,\ell-1)$

[8]Here we use the notation $\text{Anti}_{jk}[T_{jk}] \equiv T_{jk} - T_{kj}$ for the anti-symmetrisation operator.

terms from the scalar potential. The new potentials are then given by

$$\begin{aligned}\mathcal{V}_t^{\text{Out,New}}(\vec{r},\omega) &= -\sum_{\ell} \frac{G_E^{\text{Out}}(r,\omega,\ell)}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\,{}^E\mathcal{Q}_{<i_1 i_2\ldots i_\ell>}\frac{x^{i_1}\ldots x^{i_\ell}}{r^{\ell+d-2}}\ ,\\ \mathcal{V}_k^{\text{Out,New}}(\vec{r},\omega) &= -i\omega\sum_{\ell>0}\frac{1}{\ell}\frac{G_E^{\text{Out}}(r,\omega,\ell)}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\,{}^E\mathcal{Q}_{<i_1 i_2\ldots i_\ell>}\frac{x^k x^{i_1}\ldots x^{i_\ell}}{r^{\ell+d-2}}\\ &\quad+\partial_k\Big\{-i\omega\sum_{\ell>0}\frac{1}{\ell}\frac{G_E^{\text{Out}}(r,\omega,\ell-1)}{\ell!\mathcal{N}_{d,\ell-2}|\mathbb{S}^{d-1}|}\ {}^E\mathcal{Q}_{<i_1 i_2\ldots i_\ell>}\frac{x^{i_1}\ldots x^{i_\ell}}{r^{\ell+d-3}}\Big\}\\ &= -i\omega\sum_{\ell>0}\frac{G_E^{\text{Out}}(r,\omega,\ell-1)}{\ell!\mathcal{N}_{d,\ell-2}|\mathbb{S}^{d-1}|}\ {}^E\mathcal{Q}_{<k i_1 i_2\ldots i_{\ell-1}>}\frac{x^{i_1}\ldots x^{i_{\ell-1}}}{r^{\ell+d-3}}\ .\end{aligned} \tag{C.132}$$

In the last step, we have evaluated the cartesian derivative using Eq.(C.117). We can work out the large r behaviour for these new potentials using Eq.(C.114):

$$\begin{aligned}\mathcal{V}_t^{\text{Rad,New}}(\vec{r},\omega) &= -\frac{e^{i\omega r}}{(d-2)!!|\mathbb{S}^{d-1}|r^{\frac{d-1}{2}}}\sum_{\ell}\frac{(-i\omega)^{\nu-\frac{1}{2}}}{\ell!}\ {}^E\mathcal{Q}_{<i_1 i_2\ldots i_\ell>}n^{i_1}\ldots n^{i_\ell}\ ,\\ \mathcal{V}_k^{\text{Rad,New}}(\vec{r},\omega) &= \frac{e^{i\omega r}}{(d-2)!!|\mathbb{S}^{d-1}|r^{\frac{d-1}{2}}}\sum_{\ell>0}\frac{(-i\omega)^{\nu-\frac{1}{2}}}{\ell!}\ {}^E\mathcal{Q}_{<k i_1 i_2\ldots i_{\ell-1}>}n^{i_1}\ldots n^{i_{\ell-1}}\ .\end{aligned} \tag{C.133}$$

These agree with what is expected. Further, potentials in this new gauge also have a nice repeated STF derivative representation (see Eq.(C.118)):

$$\begin{aligned}\mathcal{V}_t^{\text{Out,New}}(\vec{r},\omega) &= -\sum_{\ell}\frac{(-)^\ell}{\ell!}\partial^{i_1}\partial^{i_2}\ldots\partial^{i_\ell}\Big\{\frac{{}^E\mathcal{Q}_{<i_1 i_2\ldots i_\ell>}G_E^{\text{Out}}(r,\omega,\ell=0)}{(d-2)|\mathbb{S}^{d-1}|r^{d-2}}\Big\}\ ,\\ \mathcal{V}_k^{\text{Out,New}}(\vec{r},\omega) &= -i\omega\sum_{\ell>0}\frac{(-)^{\ell-1}}{\ell!}\partial^{i_1}\partial^{i_2}\ldots\partial^{i_{\ell-1}}\Big\{\frac{{}^E\mathcal{Q}_{<k i_1 i_2\ldots i_{\ell-1}>}G_E^{\text{Out}}(r,\omega,\ell=0)}{(d-2)|\mathbb{S}^{d-1}|r^{d-2}}\Big\}\ .\end{aligned} \tag{C.134}$$

The gauge transformation which gives the correct fall-off can also be performed on the full solution in the spherical coordinates. The new potentials are given by

$$\begin{aligned}\overline{\mathcal{V}}_t^{\text{New}}(\vec{r},t) &\equiv \sum_{\ell\vec{m}}\int_\omega e^{-i\omega t}[E_s(r,\omega,\ell,\vec{m})-i\omega\Lambda_s(r,\omega,\ell,\vec{m})]\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ ,\\ \overline{\mathcal{V}}_r^{\text{New}}(\vec{r},t) &\equiv -\sum_{\ell\vec{m}}\int_\omega e^{-i\omega t}[H_s(r,\omega,\ell,\vec{m})-\partial_r\Lambda_s(r,\omega,\ell,\vec{m})]\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ ,\\ \overline{\mathcal{V}}_I^{\text{New}}(\vec{r},t) &\equiv \sum_{\ell\vec{m}}\int_\omega e^{-i\omega t}\ \Lambda_s(r,\omega,\ell,\vec{m})\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ ,\end{aligned} \tag{C.135}$$

where the gauge transformation function is

$$\Lambda_s(r,\omega,\ell,\vec{m}) = \frac{1}{i\omega r^{\ell+d-3}}\frac{\partial}{\partial r}\Big\{r^\ell \Phi_E(r,\omega,\ell,\vec{m})\Big\} \ . \tag{C.136}$$

We can simplify the new potential using Eq.(C.105) to get

$$\begin{aligned}
\overline{\mathcal{V}}_t^{\text{New}}(\vec{r},t) &\equiv -\sum_{\ell\vec{m}}\int_\omega e^{-i\omega t}\left[\frac{\ell}{r^{d-2}}\Phi_E(r,\omega,\ell,\vec{m}) + J_1(r,\omega,\ell,\vec{m})\right]\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\
\overline{\mathcal{V}}_r^{\text{New}}(\vec{r},t) &\equiv \sum_{\ell\vec{m}}\int_\omega \frac{e^{-i\omega t}}{i\omega}\Bigg[\frac{\ell}{r^{\ell+d-2}}\frac{\partial}{\partial r}\Big\{r^\ell\Phi_E(r,\omega,\ell,\vec{m})\Big\} \\
&\qquad\qquad - J_1(r,\omega,\ell,\vec{m}) + \partial_r J_2(r,\omega,\ell,\vec{m})\Bigg]\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\
\overline{\mathcal{V}}_I^{\text{New}}(\vec{r},t) &\equiv \sum_{\ell\vec{m}}\int_\omega \frac{e^{-i\omega t}}{i\omega}\ \frac{1}{r^{\ell+d-3}}\frac{\partial}{\partial r}\Big\{r^\ell\Phi_E(r,\omega,\ell,\vec{m})\Big\}\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ .
\end{aligned} \tag{C.137}$$

C.5 $\ell = 0$ and electrostatic limit

In our discussion of electric multipole radiation till now, we have avoided discussing the static limit, i.e., the limit as $\omega \to 0$. We should expect the static limit to recover our results on static poloidal currents/charges (see §§C.3). Unfortunately, this is not easy to see right away from the expressions we have seen till now. The key complication is the charge conservation that relates *time-derivative* of charge density to the currents. In practice, this means that various factors of ω might be introduced or removed using charge conservation. Some care is thus required in *how* we take the $\omega \to 0$ limit.

Another related complication is the applicability of our expressions to the $\ell = 0$ electric multipole. Many formulae in our discussion of electric multipole radiation have $\frac{1}{\ell}$ factors, and just setting $\ell = 0$ does not work. The issue here is again charge conservation: the $\ell = 0$ electric multipole is just the total electric charge, and it cannot have any time variation. In the frequency domain, this means that $\ell = 0$ mode always comes with a delta function $\delta(\omega)$, which should then be dealt with some care. The aim of this subsection is twofold: first, we will rewrite the expressions for $\{E_r, E_s, H_s\}$ in the last subsection in terms of (Fourier transforms of) charge and current densities. Next goal is to describe the $\ell = 0$ and $\omega \to 0$ limits of such expressions.

In the last subsection, we parameterised the charge/current densities as (See Eq.(C.98) and Eq.(C.100))

$$\begin{aligned}\bar{J}^t(\vec{r},t) &= -\frac{1}{r^{d-1}}\frac{\partial}{\partial r}[r^{d-1}\bar{P}^r(\vec{r},t)] - \mathscr{D}_I\bar{P}^I(\vec{r},t)\ ,\\ \bar{J}^r(\vec{r},t) &= \partial_t\bar{P}^r(\vec{r},t)\ ,\quad \bar{J}^I(\vec{r},t) = \partial_t\bar{P}^I(\vec{r},t)\ ,\end{aligned} \tag{C.138}$$

with the electric polarisation fields expanded as

$$\begin{aligned}\bar{P}^r(\vec{r},t) &\equiv \sum_{\ell\vec{m}}\int_\omega e^{-i\omega t}J_1(r,\omega,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ ,\\ r^2\gamma_{IJ}\bar{P}^J(\vec{r},t) &\equiv \sum_{\ell\vec{m}}\int_\omega e^{-i\omega t}J_2(r,\omega,\ell,\vec{m})\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ .\end{aligned} \tag{C.139}$$

These equations are equivalent to parameterising the *Fourier transforms* of charge and current densities as

$$\begin{aligned}J^t(\vec{r},\omega) &= \sum_{\ell\vec{m}}\Big\{-\frac{1}{r^{d-1}}\frac{\partial}{\partial r}[r^{d-1}J_1(r,\omega,\ell,\vec{m})] + \frac{\ell(\ell+d-2)}{r^2}J_2(r,\omega,\ell,\vec{m})\Big\}\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ ,\\ J^r(\vec{r},\omega) &= -i\omega\sum_{\ell\vec{m}}J_1(r,\omega,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ ,\\ r^2\gamma_{IK}J^K(\vec{r},\omega) &= -i\omega\sum_{\ell\vec{m}}J_2(r,\omega,\ell,\vec{m})\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ .\end{aligned} \tag{C.140}$$

For what follows, it is crucial to note that the SSH sum in the last equation has no $\ell = 0$ contribution. Another observation is that the $\ell = 0$ component of $J_2(r,\omega,\ell,\vec{m})$ never enters these expressions. Without loss of generality, we can thus set $J_2(r,\omega,\ell,\vec{m})|_{\ell=0} = 0$ and assume that $J^I(\vec{r},\omega)$ has no spherically symmetric component.

We will now turn to writing the EM fields in terms of charge/current densities instead of $\{J_1, J_2\}$. To this end, we use the above relations to derive the following identity

$$\begin{aligned}&i\omega r^2J^r(\vec{r},\omega) - \partial_r[r^2J^t(\vec{r},\omega)]\\ &= \sum_{\ell\vec{m}}r^{3-d}\mathscr{Y}_{\ell\vec{m}}(\hat{r})\Big\{\frac{1}{r^{3-d}}\frac{\partial}{\partial r}\left(r^{3-d}\frac{\partial}{\partial r}\right) + \omega^2 - \frac{\ell(\ell+d-2)}{r^2}\Big\}[r^{d-1}J_1(r,\omega,\ell,\vec{m})]\ ,\\ &\quad+\sum_{\ell\vec{m}}\ell(\ell+d-2)[J_1(r,\omega,\ell,\vec{m}) - \partial_rJ_2(r,\omega,\ell,\vec{m})]\mathscr{Y}_{\ell\vec{m}}(\hat{r})\ .\end{aligned} \tag{C.141}$$

The combination here is chosen such that the differential operator in the first line of

RHS is the one defining the electric Green function $\mathbb{G}_E$ (See Eq.(C.107)). In the last line of RHS, we recognise the $\{J_1, J_2\}$ source for the electric Debye field Φ_E in Eq.(C.106) . Using these facts, we can then write

$$\begin{aligned}&\frac{1}{r^{d-1}}\int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} \mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\ \mathbb{G}_E(r,r_0;\omega,\ell)\left\{i\omega r_0^2 J^r(\vec{r}_0,\omega) - \partial_{r_0}[r_0^2 J^t(\vec{r}_0,\omega)]\right\}\\ &= -J_1(r,\omega,\ell,\vec{m}) + \frac{\ell(\ell+d-2)}{r^{d-1}}\Phi_E(r,\omega,\ell,\vec{m})\ .\end{aligned} \tag{C.142}$$

We recognise here the combination that defines E_r (c.f. Eq.(C.121)), i.e.,

$$\begin{aligned}E_r(r,\omega,\ell,\vec{m}) = \frac{1}{r^{d-1}}\int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} &\mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\ \mathbb{G}_E(r,r_0;\omega,\ell)\\ &\times\left\{i\omega r_0^2 J^r(\vec{r}_0,\omega) - \partial_{r_0}[r_0^2 J^t(\vec{r}_0,\omega)]\right\}\ .\end{aligned} \tag{C.143}$$

One corollary is a formula for electric multipole moment directly in terms of charges or currents: we use Eq.(C.119) to evaluate $\mathbb{G}_E$ outside the sources and compare the result against Eq.(C.124). This yields

$$\begin{aligned}\mathfrak{J}^E(\omega,\ell,\vec{m}) \equiv \frac{1}{2\nu(\ell+d-2)}\int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} & r_0^{\nu+\frac{d}{2}-1}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\ {}_0F_1\left[1+\nu, -\frac{\omega^2 r_0^2}{4}\right]\\ \times\left\{i\omega r_0^2 J^r(\vec{r}_0,\omega) - \partial_{r_0}[r_0^2 J^t(\vec{r}_0,\omega)]\right\}&\ .\end{aligned} \tag{C.144}$$

This generalises our earlier definition in Eq.(C.122) to general source distributions. As a bonus, we now have an expression where both $\ell = 0$ and $\omega \to 0$ limits can be taken and be seen to give a non-zero electric moment, as expected. In fact, the static limit coincides with the electrostatics definition in §§C.3 as can be seen from

$$\begin{aligned}\mathfrak{J}^E(\omega=0,\ell,\vec{m}) &= -\frac{1}{2\nu(\ell+d-2)}\int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} r_0^{\ell+d-2}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\ \partial_{r_0}[r_0^2 J^t(\vec{r}_0,\omega)]\\ &= \frac{1}{2\nu}\int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} r_0^{\ell+d-1}\mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\ J^t(\vec{r}_0,\omega=0)\ .\end{aligned} \tag{C.145}$$

The last line follows via integration by parts. The above expression for electric multipole moment can also be converted into an STF moment via

$$\begin{aligned}
&\frac{1}{\ell!}\ {}^{E}\mathcal{Q}(\omega)_{<i_1 i_2 \ldots i_\ell>} x^{i_1} \ldots x^{i_\ell} \\
&\quad \equiv \mathcal{N}_{d,\ell-1} |\mathbb{S}^{d-1}| \sum_{\vec{m}} \mathcal{J}^{E}(\omega,\ell,\vec{m})\ r^{\ell} \mathscr{Y}_{\ell\vec{m}}(\hat{r}) \\
&\quad = \frac{1}{\ell+d-2} \int_{\vec{r}_0} \Pi^{S}(\vec{r}|\vec{r}_0)\ {}_0F_1\left[1+\nu, -\frac{\omega^2 r_0^2}{4}\right] \left\{ i\omega r_0 J^r(\vec{r}_0,\omega) - \frac{1}{r_0}\partial_{r_0}[r_0^2 J^t(\vec{r}_0,\omega)] \right\} .
\end{aligned} \tag{C.146}$$

Stripping off the x^i's on both sides, we get the STF electric multipole tensor as

$$\begin{aligned}
{}^{E}\mathcal{Q}(\omega)_{<i_1 i_2 \ldots i_\ell>} = \frac{(\Pi^S)^{<i_1\ldots i_\ell>}_{<j_1\ldots j_\ell>}}{\ell+d-2} \int_{\vec{r}} x^{j_1} \ldots x^{j_\ell}\ {}_0F_1\left[1+\nu, -\frac{\omega^2 r^2}{4}\right] \\
\times \left\{ i\omega r J^r(\vec{r},\omega) - \frac{1}{r}\partial_r[r^2 J^t(\vec{r},\omega)] \right\} .
\end{aligned} \tag{C.147}$$

We now turn to how the magnetostatics of the poloidal currents is recovered from our expressions. We will do this by relating the current combination that sources the poloidal magnetic field (Eq.(C.80)) to the source of electric Debye field, viz.,

$$\begin{aligned}
\int_{\hat{r}\in\mathbb{S}^{d-1}} \mathscr{Y}^{*}_{\ell\vec{m}}(\hat{r}) \Big\{ J^r(\vec{r},\omega) + \frac{1}{\ell(\ell+d-2)} \partial_r \left[r^2 \mathscr{D}_I J^I(\vec{r},\omega) \right] \Big\} \\
= -i\omega [J_1(r,\omega,\ell,\vec{m}) - \partial_r J_2(r,\omega,\ell,\vec{m})] .
\end{aligned} \tag{C.148}$$

This allows us to rewrite H_s in Eq.(C.121) in the form

$$\begin{aligned}
H_s = -\frac{1}{r^{d-3}} \int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} \mathscr{Y}^{*}_{\ell\vec{m}}(\hat{r}_0)\ \mathbb{G}_E(r,r_0;\omega,\ell) \\
\times \left\{ J^r(\vec{r}_0,\omega) + \frac{1}{\ell(\ell+d-2)} \partial_{r_0} \left[r_0^2 \mathscr{D}_I J^I(\vec{r}_0,\omega) \right] \right\} .
\end{aligned} \tag{C.149}$$

This form then has a straightforward static limit where it reduces to Eq.(C.80). If we account for the fact that J^I has no $\ell = 0$ component, the above expression also has a finite $\ell = 0$ limit. The outside fields in Eq.(C.124) also work out provided

$$\begin{aligned}
0 = i\omega\ \mathcal{J}^{E}(\omega,\ell,\vec{m}) + \frac{\ell}{2\nu} \int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} \mathscr{Y}^{*}_{\ell\vec{m}}(\hat{r}_0)\ r_0^{\nu+\frac{d}{2}-1} {}_0F_1\left[1+\nu, -\frac{\omega^2 r_0^2}{4}\right] \\
\times \left\{ J^r(\vec{r}_0,\omega) + \frac{1}{\ell(\ell+d-2)} \partial_{r_0} \left[r_0^2 \mathscr{D}_I J^I(\vec{r}_0,\omega) \right] \right\} .
\end{aligned} \tag{C.150}$$

This seems to give a new expression for the electric multipole moment that is different from Eq.(C.144). But, using the conservation equation (in frequency domain)

$$i\omega J^t = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}(r^{d-1}J^r) + \mathscr{D}_I J^I \ , \tag{C.151}$$

the difference between the two $\mathcal{J}^E$ definitions can be shown to be proportional to

$$\begin{aligned}&\int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} \mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\ r_0^{\nu+\frac{d}{2}-1}\,{}_0F_1\left[1+\nu, -\frac{\omega^2 r_0^2}{4}\right] \\ &\quad\times r_0^{3-d}\left\{-\frac{1}{r_0^{3-d}}\frac{\partial}{\partial r_0}\left(r_0^{3-d}\frac{\partial}{\partial r_0}\right) - \omega^2 + \frac{\ell(\ell+d-2)}{r_0^2}\right\}[r_0^{d-1}J^r(\vec{r}_0,\omega)] \ .\end{aligned} \tag{C.152}$$

This expression is zero since the derivative operator in the second line can be shifted onto its homogeneous solution in the first line via integration by parts. Finally, we can get an expression for E_s in terms of charge and current densities by using

$$\begin{aligned}\partial_r E_s &= E_r + i\omega H_s \\ &= \int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} \mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\ \mathbb{G}_E(r, r_0; \omega, \ell) \\ &\qquad \times \left\{ i\omega\frac{r_0^2 - r^2}{r^{d-1}} J^r(\vec{r}_0,\omega) - \partial_{r_0}\left[\frac{r_0^2}{r^{d-1}}J^t(\vec{r}_0,\omega) + i\omega\frac{r_0^2}{r^{d-3}}\frac{\mathscr{D}_I J^I(\vec{r}_0,\omega)}{\ell(\ell+d-2)}\right]\right\} \ .\end{aligned} \tag{C.153}$$

The last line follows from Eqs.(C.143) and (C.149). As $r\to\infty$ the field $E_s \to H_s$ (see Eq.(C.126)), and we can integrate the above equation to obtain

$$\begin{aligned}E_s = H_s(r\to\infty) &+ \int_\infty^r dr_1 \int_0^\infty dr_0 \int_{\hat{r}_0\in\mathbb{S}^{d-1}} \mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_0)\ \mathbb{G}_E(r_1, r_0; \omega, \ell) \\ &\times \left\{ i\omega\frac{r_0^2 - r_1^2}{r_1^{d-1}} J^r(\vec{r}_0,\omega) - \partial_{r_0}\left[\frac{r_0^2}{r_1^{d-1}}J^t(\vec{r}_0,\omega) + i\omega\frac{r_0^2}{r_1^{d-3}}\frac{\mathscr{D}_I J^I(\vec{r}_0,\omega)}{\ell(\ell+d-2)}\right]\right\}.\end{aligned} \tag{C.154}$$

In this form, we can easily take $\omega\to 0$ as well as $\ell = 0$ limits (provided we remember that J^I has no $\ell = 0$ component). Note that, for $d > 3$, we have $H_s(r\to\infty) = 0$, and we can drop the first term entirely.

C.6 Radiative power loss (Larmor's formula)

Before moving to the influence phase, it is instructive to generalise the textbook description of radiative power loss in EM to arbitrary dimensions. The power carried away by the radiation can be computed using the EM energy-momentum tensor:

$$ {}^{(\mathrm{EM})}\overline{T}^{\mu\nu} = \overline{\mathcal{C}}^{\mu\alpha}\overline{\mathcal{C}}_{\alpha}{}^{\nu} - \frac{1}{4}\eta^{\mu\nu}\overline{\mathcal{C}}_{\alpha\beta}\overline{\mathcal{C}}^{\alpha\beta} \ . \tag{C.155} $$

The energy flux through a sphere at radial coordinate r enclosing the origin is then the sphere integral of T_t^r (i.e., the radial component of the Poynting vector in the frequency domain). We are interested in the energy flux at large r, which can be computed using radiative fields given in Eqs.(C.126) and (C.61). We obtain

$$ \begin{aligned} \mathcal{P}(\omega) &\equiv \lim_{r\to\infty} \int_{\mathbb{S}_r^{d-1}} r^{d-1}\ {}^{(\mathrm{EM})}T_t{}^r(r,\omega,\Omega) \\ &= \lim_{r\to\infty} r^{d-3}\left[\sum_{\ell\vec{m}} \ell(\ell+d-2)E_s^* H_s + \sum_{\alpha\ell\vec{m}} E_v^* H_v\right] \\ &= \sum_{\ell\vec{m}} \frac{\ell+d-2}{\ell}\frac{\omega^{2\nu+1}}{[(2\nu-2)!!]^2}|\mathcal{J}^E(\omega,\ell,\vec{m})|^2 + \sum_{\alpha\ell\vec{m}} \frac{\omega^{2\nu+1}}{[(2\nu-2)!!]^2}|\mathcal{J}^B(\omega,\alpha,\ell,\vec{m})|^2 \ , \end{aligned} \tag{C.156} $$

where we have used the orthonormality of scalar/vector spherical harmonics to perform the sphere integrals. The sum over ℓ ranges from $\ell = 1$ to $\ell = \infty$, since the monopole moment at $\ell = 0$ (the total electric charge) is always time-independent and does not result in radiation.

When the number of spatial dimensions d is odd, the number $\nu \equiv \ell + \frac{d}{2} - 1$ is a half-integer, and the power loss $\mathcal{P}(\omega)$ is an even function of ω. This means that the power loss is always non-negative and is invariant under time-reversal, i.e., time-reversing the charges/currents still results in an irreversible loss of energy into radiation.

The situation is qualitatively different when d is even. We remind the reader that the radiative fields of Eqs.(C.126) and (C.61) as well as the power loss formula Eq.(C.156) are still valid with an appropriate definition of double factorials. The main difference now is that the power loss above can be reversed by time-reversing the charges/currents. *Such a reversible change in energy can then be absorbed into a redefinition of energy.* Physically, when d is even, the radiation lingers on around the source, and its back reaction serves to *renormalise* the source properties without any dissipative effects. We will see below that Eq.(C.156) should really be interpreted as a beta function in classical EM.

Before proceeding, we would like to recast Eq.(C.156) in terms of cartesian multipole tensors. Using Eq.(C.144), we have

$$\sum_{\vec{m}} |\mathcal{J}^E(\omega,\ell,\vec{m})|^2 = \frac{1}{[2\nu(\ell+d-2)]^2}$$
$$\times \int_0^\infty dr_1 \int_{\hat{r}_1\in\mathbb{S}^{d-1}} r_1^{\nu+\frac{d}{2}-1}\,{}_0F_1\left[1+\nu, -\frac{\omega^2 r_1^2}{4}\right]$$
$$\times \int_0^\infty dr_2 \int_{\hat{r}_2\in\mathbb{S}^{d-1}} r_2^{\nu+\frac{d}{2}-1}\,{}_0F_1\left[1+\nu, -\frac{\omega^2 r_2^2}{4}\right] \tag{C.157}$$
$$\times \left\{i\omega r_1^2 J^r(\vec{r}_1,\omega) - \partial_{r_1}[r_1^2 J^t(\vec{r}_1,\omega)]\right\}^* \left\{i\omega r_2^2 J^r(\vec{r}_2,\omega) - \partial_{r_2}[r_2^2 J^t(\vec{r}_2,\omega)]\right\}$$
$$\times \sum_{\vec{m}} \mathscr{Y}_{\ell\vec{m}}(\hat{r}_1)\ \mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_2) \quad .$$

The sum appearing in the last line can be performed by invoking the SSH addition theorem (Eq.(A.54)), and the answer factorised via symmmetry/idempotence of the SSH projector:

$$\sum_{\vec{m}} \mathscr{Y}_{\ell\vec{m}}(\hat{r}_1)\ \mathscr{Y}^*_{\ell\vec{m}}(\hat{r}_2) = \frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\ \Pi^S(\hat{r}_1|\hat{r}_2)_{d,\ell}$$
$$= \frac{1}{\ell!\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\ (\Pi^S)^{<i_1\ldots i_\ell>}_{<j_1\ldots j_\ell>}\hat{r}_1^{j_1}\ldots\hat{r}_1^{j_\ell}\ (\Pi^S)^{<i_1\ldots i_\ell>}_{<k_1\ldots k_\ell>}\hat{r}_2^{k_1}\ldots\hat{r}_2^{k_\ell}\ . \tag{C.158}$$

In the next step, we use the definition of STF electric moment in Eq.(C.147) to write

$$\sum_{\vec{m}} |\mathcal{J}^E(\omega,\ell,\vec{m})|^2 = \frac{1}{(2\nu)^2\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\frac{1}{\ell!}\ \left[{}^E\mathcal{Q}(\omega)_{<i_1\ldots i_\ell>}\right]^*\ \left[{}^E\mathcal{Q}(\omega)_{<i_1\ldots i_\ell>}\right]\ . \tag{C.159}$$

A similar derivation can be given for the magnetic moment. From Eq.(C.58), we have

$$\sum_{\alpha\vec{m}}|\mathcal{J}^B(\omega,\alpha,\ell,\vec{m})|^2$$
$$= \frac{1}{(2\nu)^2}\int_{\vec{r}_1} r_1^{\nu-\frac{d}{2}+2}\,{}_0F_1\left[1+\nu,-\frac{\omega^2 r_1^2}{4}\right]\int_{\vec{r}_2} r_2^{\nu-\frac{d}{2}+2}\,{}_0F_1\left[1+\nu,-\frac{\omega^2 r_2^2}{4}\right] \tag{C.160}$$
$$\times \left\{J^I(\vec{r}_1,\omega)\right\}^*\left\{J^J(\vec{r}_1,\omega)\right\}\sum_{\alpha\vec{m}} \mathbb{V}_I^{\alpha\ell\vec{m}}(\hat{r}_1)\ \mathbb{V}_J^{\alpha\ell\vec{m}*}(\hat{r}_2) \quad .$$

We can relate the spherical components of the currents to cartesian ones by writing $J^I(\vec{r},\omega) = J^i(\vec{r},\omega)\frac{\partial\vartheta^I}{\partial x^i}$. Applying the VSH addition theorem as well as the symmetry/idempotence of the VSH projector, we get

$$\begin{aligned}
&\frac{\partial\vartheta_1^I}{\partial x_1^i}\frac{\partial\vartheta_2^J}{\partial x_2^j}\sum_{\alpha\vec{m}}\mathbb{V}_I^{\alpha\ell\vec{m}}(\hat{r}_1)\ \mathbb{V}_J^{\alpha\ell\vec{m}*}(\hat{r}_2) \\
&\qquad=\frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|r_1r_2}\ \Pi^V_{ij}(\hat{r}_1|\hat{r}_2) \\
&\qquad=\frac{1}{\ell!\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|r_1r_2}\ (\Pi^V)^{k<k_1\ldots k_\ell>}_{i<i_1\ldots i_\ell>}\hat{r}_1^{i_1}\ldots\hat{r}_1^{i_\ell}\ (\Pi^V)^{k<k_1\ldots k_\ell>}_{j<j_1\ldots j_\ell>}\hat{r}_2^{j_1}\ldots\hat{r}_2^{j_\ell}\ .
\end{aligned} \tag{C.161}$$

We can then use the definition of ${}^B\mathcal{Q}$ given in Eq.(C.63) to write

$$\sum_{\alpha\vec{m}}|\mathcal{J}^B(\omega,\alpha,\ell,\vec{m})|^2=\frac{1}{(2\nu)^2\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}\frac{1}{\ell!}\ \left[{}^B\mathcal{Q}(\omega)_{k<k_1\ldots k_\ell>}\right]^*\ \left[{}^B\mathcal{Q}(\omega)_{k<k_1\ldots k_\ell>}\right]\ . \tag{C.162}$$

Putting these results together and using the explicit formula for $\mathcal{N}_{d,\ell}$, we can rewrite the power loss formula in Eq.(C.156) entirely in terms of STF moments:

$$\begin{aligned}
\mathcal{P}(\omega)&=\frac{1}{(d-2)!!|\mathbb{S}^{d-1}|}\sum_{\ell=1}^{\infty}\frac{\ell+d-2}{\ell}\frac{\omega^{2\ell+d-1}}{(2\ell+d-2)!!}\ \frac{1}{\ell!}\ \left[{}^E\mathcal{Q}(\omega)_{<i_1\ldots i_\ell>}\right]^*\ \left[{}^E\mathcal{Q}(\omega)_{<i_1\ldots i_\ell>}\right] \\
&\quad+\frac{1}{(d-2)!!|\mathbb{S}^{d-1}|}\sum_{\ell=1}^{\infty}\frac{\omega^{2\ell+d-1}}{(2\ell+d-2)!!}\ \frac{1}{\ell!}\ \left[{}^B\mathcal{Q}(\omega)_{i<i_1\ldots i_\ell>}\right]^*\ \left[{}^B\mathcal{Q}(\omega)_{i<i_1\ldots i_\ell>}\right]\ .
\end{aligned} \tag{C.163}$$

For odd d, the case where there is a dissipative power loss, the above expression can be rewritten in the following suggestive form:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathcal{P}(\omega) \\
&\quad=\int_0^{\infty}\omega\ \frac{\omega^{d-1}d\omega}{(2\pi)^d2\omega}\sum_{\ell=1}^{\infty}\frac{\ell+d-2}{\ell}\frac{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}{\ell!}\ \left[\omega^\ell\ {}^E\mathcal{Q}(\omega)_{<i_1\ldots i_\ell>}\right]^*\ \left[\omega^\ell\ {}^E\mathcal{Q}(\omega)_{<i_1\ldots i_\ell>}\right] \\
&\quad\quad+\int_0^{\infty}\omega\ \frac{\omega^{d-1}d\omega}{(2\pi)^d2\omega}\sum_{\ell=1}^{\infty}\frac{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}{\ell!}\ \left[\omega^\ell\ {}^B\mathcal{Q}(\omega)_{i<i_1\ldots i_\ell>}\right]^*\ \left[\omega^\ell\ {}^B\mathcal{Q}(\omega)_{i<i_1\ldots i_\ell>}\right]\ .
\end{aligned} \tag{C.164}$$

We recognise in front the Lorentz-invariant phase-space integral for a photon of energy ω, as well as another factor of ω, indicating that we are computing its energy. Since this is power loss, the remaining factor should be interpreted as the production rate of photon by a given multipole moment. This can be made even more explicit if we recognise each term in the sum as the inner product on the sphere for SSHs and VSHs: See Eq.(A.55) and Eq.(B.53). We then have the following integral representations:

$$\begin{aligned}
&\int_{\hat{k}\in\mathbb{S}^{d-1}}\left[\frac{1}{\ell!}{}^{E}\mathcal{Q}(\omega)_{<i_1\ldots i_\ell>}\hat{k}^{i_1}\ldots\hat{k}^{i_\ell}\right]^*\left[\frac{1}{\ell!}\,{}^{E}\mathcal{Q}(\omega)_{<j_1\ldots j_\ell>}\hat{k}^{j_1}\ldots\hat{k}^{j_\ell}\right]\\
&\qquad=\frac{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}{\ell!}\ \left[{}^{E}\mathcal{Q}(\omega)_{<i_1\ldots i_\ell>}\right]^*\ \left[{}^{E}\mathcal{Q}(\omega)_{<i_1\ldots i_\ell>}\right]\ ,\\
&\int_{\hat{k}\in\mathbb{S}^{d-1}}\left[\frac{1}{\ell!}{}^{E}\mathcal{Q}(\omega)_{<pi_1\ldots i_{\ell-1}>}\hat{k}^{i_1}\ldots\hat{k}^{i_{\ell-1}}\right]^*\left[\frac{1}{\ell!}\,{}^{E}\mathcal{Q}(\omega)_{<pj_1\ldots j_{\ell-1}>}\hat{k}^{j_1}\ldots\hat{k}^{j_{\ell-1}}\right]\\
&\qquad=\frac{1}{\ell^2}\frac{\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}{(\ell-1)!}\ \left[{}^{E}\mathcal{Q}(\omega)_{<i_1\ldots i_\ell>}\right]^*\ \left[{}^{E}\mathcal{Q}(\omega)_{<i_1\ldots i_\ell>}\right]\ ,\\
&\int_{\hat{k}\in\mathbb{S}^{d-1}}\left[\frac{1}{\ell!}{}^{B}\mathcal{Q}(\omega)_{p<i_1\ldots i_\ell>}\hat{k}^{i_1}\ldots\hat{k}^{i_\ell}\right]^*\left[\frac{1}{\ell!}\,{}^{B}\mathcal{Q}(\omega)_{p<j_1\ldots j_\ell>}\hat{k}^{j_1}\ldots\hat{k}^{j_\ell}\right]\\
&\qquad=\frac{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|}{\ell!}\ \left[{}^{B}\mathcal{Q}(\omega)_{i<i_1\ldots i_\ell>}\right]^*\ \left[{}^{B}\mathcal{Q}(\omega)_{i<i_1\ldots i_\ell>}\right]\ .
\end{aligned}\tag{C.165}$$

These, along with the identity

$$\frac{1}{\ell^2}\frac{\mathcal{N}_{d,\ell-1}}{(\ell-1)!}-\frac{\mathcal{N}_{d,\ell}}{\ell!}=\frac{\ell+d-2}{\ell}\frac{\mathcal{N}_{d,\ell}}{\ell!}\tag{C.166}$$

allows us to write the power loss as the Lorentz-invariant phase-space integral for the with momentum $\vec{k}$ and energy $\omega_k\equiv|\vec{k}|$, i.e.,

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathcal{P}(\omega)=&\int\frac{d^dk}{(2\pi)^d2\omega_k}\times\omega_k(\delta^{pq}-\hat{k}^p\hat{k}^q)\\
&\times\left\{\sum_{\ell=1}^{\infty}\left[\frac{\omega_k}{\ell!}{}^{E}\mathcal{Q}(\omega_k)_{<pi_1\ldots i_{\ell-1}>}k^{i_1}\ldots k^{i_{\ell-1}}\right]^*\left[\frac{\omega_k}{\ell!}\,{}^{E}\mathcal{Q}(\omega_k)_{<qj_1\ldots j_{\ell-1}>}k^{j_1}\ldots k^{j_{\ell-1}}\right]\right.\\
&\left.+\sum_{\ell=1}^{\infty}\left[\frac{1}{\ell!}{}^{B}\mathcal{Q}(\omega_k)_{p<i_1\ldots i_\ell>}k^{i_1}\ldots k^{i_\ell}\right]^*\left[\frac{1}{\ell!}\,{}^{B}\mathcal{Q}(\omega_k)_{q<j_1\ldots j_\ell>}k^{j_1}\ldots k^{j_\ell}\right]\right\}.
\end{aligned}\tag{C.167}$$

Here we have used the fact that ${}^{B}\mathcal{Q}_{p<i_1\ldots i_\ell>}\hat{k}^pk^{i_1}\ldots k^{i_\ell}=0$ due to the transversality. The factor $(\delta^{pq}-\hat{k}^p\hat{k}^q)$ is the polarisation sum, summing over all transverse polarisations of the photon. The power loss written above corresponds to the following photon emission amplitudes by the multipoles:

$$i(-i)^{\nu-\frac{1}{2}}\frac{\varepsilon^{p*}(\vec{k})}{\ell!}{}^{E}\mathcal{Q}(\omega_k)_{<pi_1\ldots i_{\ell-1}>}\omega_kk^{i_1}\ldots k^{i_{\ell-1}}\ ,\ i(-i)^{\nu-\frac{1}{2}}\frac{\varepsilon^{p*}(\vec{k})}{\ell!}{}^{B}\mathcal{Q}(\omega_k)_{p<i_1\ldots i_\ell>}k^{i_1}\ldots k^{i_\ell}\ .\tag{C.168}$$

Here $\varepsilon^p(\vec{k})$ is the polarisation for the photon with momentum $\vec{k}$ and energy $\omega_k \equiv |\vec{k}|$, and we have fixed the overall phase by comparing against radiative fields in Eq.(C.133) and Eq.(C.71).

C.7 EM influence phase in Flat Spacetime

We will now turn to the description of radiation reaction in flat spacetime. Our goal here is to get some sort of effective action that captures the effect of radiation on charge/current sources. Since radiation carries away energy in some cases (not always: see below), what we need is an action that can describe dissipation. This implies that the correct language here is that of *influence phase* ala Feynman-Vernon [56], i.e., an action that doubles the system degrees of freedom to allow us to describe the evolution of density matrices, its decoherence and dissipation into the environment. As described by Feynman-Vernon, the influence phase is computed by doing a path integral over the doubled environment (here the EM fields) coupled to a doubled system(here the charges/currents), with a specific in-in boundary condition on the environment fields. As emphasised by us in [101], such in-in boundary conditions are naturally implemented on the dS-SK geometry built by connecting two static patches at the future horizon. In the subsequent appendices, we will show that this works also for electromagnetism in dS. Coming back to the current topic of flat space EM, there is no such simple geometric construction: the in-in boundary conditions have to be imposed by hand.[9] For $d = 3$, the reader can find such an analysis in [184–186]. Given that we will be presenting a detailed derivation of the influence phase in the dS case, we will be content here with a brief sketch.

We remind the reader that the current density can be parametrised in terms of two

[9]Strictly speaking, flat spacetime EM path integral also has a variety of infrared subtleties. We will ignore them in what follows.

functions such that it solves the conservation equations in the following manner:

$$
\begin{aligned}
J^t(\vec{r},\omega) &= \sum_{\ell\vec{m}}\Big\{-\frac{1}{r^{d-1}}\frac{\partial}{\partial r}[r^{d-1}J_1(r,\omega,\ell,\vec{m})] + \frac{\ell(\ell+d-2)}{r^2}J_2(r,\omega,\ell,\vec{m})\Big\}\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\
J^r(\vec{r},\omega) &= -i\omega\sum_{\ell\vec{m}} J_1(r,\omega,\ell,\vec{m})\mathscr{Y}_{\ell\vec{m}}(\hat{r}) \ , \\
r^2\gamma_{IK}J^K(\vec{r},\omega) &= -i\omega\sum_{\ell\vec{m}} J_2(r,\omega,\ell,\vec{m})\mathscr{D}_I\mathscr{Y}_{\ell\vec{m}}(\hat{r}) + r^2\gamma_{IK}\sum_{\alpha\ell\vec{m}} J_V(r,\omega,\alpha,\ell,\vec{m})\mathbb{V}^K_{\alpha\ell\vec{m}} \ .
\end{aligned}
\tag{C.169}
$$

We will now consider two copies of these currents(J_L and J_R), which can be independently specified for all cases except in the $\ell = 0$ case.

$$
\begin{aligned}
&\int_\omega\int dr\int dr_0 \\
&\Big\{\sum_{\ell\vec{m}}\ell(\ell+d-2)\,[J_1(r,\omega,\ell,\vec{m}) - \partial_r J_2(r,\omega,\ell,\vec{m})]^*_D\,\mathbb{G}_E(r,r_0,\omega,\ell) \\
&\qquad\times[J_1(r_0,\omega,\ell,\vec{m}) - \partial_{r_0}J_2(r_0,\omega,\ell,\vec{m})]_A \\
&\qquad+\sum_{\alpha\ell\vec{m}}[J_V(r,\omega,\alpha,\ell,\vec{m})]^*_D\mathbb{G}_B(r,r_0,\omega,\ell)J_V(r_0,\omega,\alpha,\ell,\vec{m})_A\Big\}
\end{aligned}
\tag{C.170}
$$

For odd values of d, $\cot(\nu\pi)$ is zero, and the renormalised action is given by:

$$
\begin{aligned}
&\sum_{\ell\vec{m}}\ell(\ell+d-2)\int_\omega\frac{\pi i}{2\Gamma(\nu+1)^2}\left(\frac{\omega}{2}\right)^{2\nu} \\
&\qquad\times\int dr_0\ r_0^{\nu+\frac{d}{2}-1}\ {}_0F_1\left[1+\nu,-\frac{\omega^2r_0^2}{4}\right][\partial_{r_0}J_2(r_0,\omega) - J_1(r_0,\omega)]^*_D \\
&\qquad\times\int dr\ r^{\nu+\frac{d}{2}-1}\ {}_0F_1\left[1+\nu,-\frac{\omega^2r^2}{4}\right][\partial_r J_2(r,\omega) - J_1(r,\omega)]_A
\end{aligned}
\tag{C.171}
$$

This action for even values of d gets multiplied by a factor of $(1+i\cot(\nu\pi))$ and hence diverges. Further counterterms are needed for regularisation. These counterterms can be computed by expanding the action around $\nu = n \in \mathbb{Z}$. Consider:

$$
\begin{aligned}
(1+i\cot\pi\nu)\frac{2\pi i}{\Gamma(\nu)^2}\left(\frac{\omega}{2H}\right)^{2\nu} &= \frac{1}{\Gamma(n)^2}\left(\frac{\omega}{2H}\right)^{2n} \\
&\quad\times\left\{\frac{2}{\nu-n} - 4\psi^{(0)}(n) + \ln\left(\frac{\omega}{2H}\right)^4 + O(\nu-n)\right\} \ .
\end{aligned}
\tag{C.172}
$$

where H is the renormalisation scale. Following the same modified minimal subtraction scheme proposed in [101], we will counterterm the first two terms in the RHS. Using this scheme, the RR action becomes:

$$\begin{aligned}\sum_{\ell\vec{m}} \ell(\ell+d-2) \int_{\omega} \frac{1}{4\Gamma(\nu+1)^2} \left(\frac{\omega}{2}\right)^{2\nu} \ln\left(\frac{\omega^4}{H^4}\right) \\ \times \int dr_0 \; r_0^{\nu+\frac{d}{2}-1} \; {}_0F_1\left[1+\nu, -\frac{\omega^2 r_0^2}{4}\right] \left[\partial_{r_0} J_2(r_0,\omega) - J_1(r_0,\omega)\right]_D^* \\ \times \int dr \; r^{\nu+\frac{d}{2}-1} \; {}_0F_1\left[1+\nu, -\frac{\omega^2 r^2}{4}\right] \left[\partial_r J_2(r,\omega) - J_1(r,\omega)\right]_A\end{aligned} \quad \text{(C.173)}$$

A similar argument gives us the corresponding action for the magnetic parity action. We have:

$$S_{RR}^{\text{vector, bare}} = \sum_{\ell\vec{m}} \int_{\omega} \int r^{d-1} dr \int r_0^{d-1} dr_0 J_{VD}^*(r_0,w,\ell,\vec{m}) J_{VA}(r_0,w,\ell,\vec{m}) \mathbb{G}_B(r,r_0;\ell,\vec{m}) \quad \text{(C.174)}$$

For odd values of d, the dissipative part of the action is given by:

$$\begin{aligned}\sum_{\ell\vec{m}} \int_{\omega} \frac{\pi i}{2\Gamma(\nu+1)^2} \left(\frac{\omega}{2}\right)^{2\nu} \\ \times \int dr_0 \; r_0^{\nu+\frac{d}{2}+1} \; {}_0F_1\left[1+\nu, -\frac{\omega^2 r_0^2}{4}\right] J_{VD}^*(r_0,w,\ell,\vec{m}) \\ \times \int dr \; r^{\nu+\frac{d}{2}+1} \; {}_0F_1\left[1+\nu, -\frac{\omega^2 r^2}{4}\right] J_{VA}^*(r_0,w,\ell,\vec{m})\end{aligned} \quad \text{(C.175)}$$

Counterterming away the extra divergence for the case of even d in the same way as for the electric sector, we obtain the action:

$$\begin{aligned}\sum_{\ell\vec{m}} \ell(\ell+d-2) \int_{\omega} \frac{1}{4\Gamma(\nu+1)^2} \left(\frac{\omega}{2}\right)^{2\nu} \ln\left(\frac{\omega^4}{H^4}\right) \\ \times \int dr_0 \; r_0^{\nu+\frac{d}{2}+1} \; {}_0F_1\left[1+\nu, -\frac{\omega^2 r_0^2}{4}\right] J_{VD}^*(r_0,w,\ell,\vec{m}) \\ \times \int dr \; r^{\nu+\frac{d}{2}+1} \; {}_0F_1\left[1+\nu, -\frac{\omega^2 r^2}{4}\right] J_{VA}^*(r_0,w,\ell,\vec{m})\end{aligned} \quad \text{(C.176)}$$

Given this reduced boundary action, we can write down an SK action for the EM radiation reaction as:

$$S_{\rm RR} = \sum_{\ell\vec{m}} \frac{\ell+d-2}{\ell} \int_\omega \frac{2\pi i}{\Gamma(\nu)^2} \left(\frac{\omega}{2}\right)^{2\nu} \left\{ \mathfrak{J}_E^{D*}\mathfrak{J}_E^{A} + \frac{\ell}{\ell+d-2}\mathfrak{J}_B^{D*}\mathfrak{J}_B^{A} \right\} \tag{C.177}$$

where we define:

$$\begin{aligned}
\mathfrak{J}^E_A(\omega,\ell,\vec{m}) &\equiv \frac{\ell}{2\nu}\int dr \left[J_2^A\ \partial_r \left\{ r^{\nu+\frac{d}{2}-1}\ {}_0F_1\left[1+\nu, -\frac{\omega^2 r^2}{4}\right]\right\} \right. \\
&\qquad\qquad \left. + J_1^A\ r^{\nu+\frac{d}{2}-1}\ {}_0F_1\left[1+\nu,-\frac{\omega^2r^2}{4}\right]\right] , \\
\mathfrak{J}^E_D(\omega,\ell,\vec{m}) &\equiv \frac{\ell}{2\nu}\int dr \left[J_2^D\ \partial_r \left\{ r^{\nu+\frac{d}{2}-1}\ {}_0F_1\left[1+\nu, -\frac{\omega^2 r^2}{4}\right]\right\} \right. \\
&\qquad\qquad \left. + J_1^D\ r^{\nu+\frac{d}{2}-1}\ {}_0F_1\left[1+\nu,-\frac{\omega^2r^2}{4}\right]\right] , \\
\mathfrak{J}^B_A(\omega,\alpha,\ell,\vec{m}) &\equiv \frac{1}{2\nu}\int dr \left\{ J_V^A(\alpha,\ell,\vec{m}) r^{\nu+\frac{d}{2}+1} {}_0F_1\left[1+\nu,-\frac{\omega^2r^2}{4}\right] \right\} , \\
\mathfrak{J}^B_D(\omega,\alpha,\ell,\vec{m}) &\equiv \frac{1}{2\nu}\int dr \left\{ J_V^D(\alpha,\ell,\vec{m}) r^{\nu+\frac{d}{2}+1} {}_0F_1\left[1+\nu,-\frac{\omega^2r^2}{4}\right] \right\}
\end{aligned} \tag{C.178}$$

Later, we will see that the influence phase obtained in de Sitter reduces to Eq.(C.177) in the flat space limit.

To compute the radiation reaction of a single particle, it is much more convenient to work with STF scalar harmonics. Corresponding to appendix A3 of [101], one can rewrite the RR action in terms of STF moments $\mathcal{Q}^{<i_1\ldots i_\ell>}(\omega)$ as:

$$\begin{aligned}
{}^E\mathcal{Q}_A^{<i_1\ldots i_\ell>}(\omega) \equiv\ & \frac{1}{(\ell+d-2)} (\Pi^S)^{<i_1 i_2\ldots i_\ell>}_{<j_1 j_2 \ldots j_\ell>} \int d^dx\ x^{j_1}x^{j_2}\ldots x^{j_\ell} \\
& \times \left[\frac{1}{r^{\ell+d-3}}\partial_r \left\{ r^{\ell+d-2}\ {}_0F_1\left[1+\nu,-\frac{r^2\omega^2}{4}\right]\right\} J_A^t(\vec{r},\omega) \right. \\
& \qquad \left. + i\omega r\ {}_0F_1\left[1+\nu,-\frac{r^2\omega^2}{4}\right] J_A^r(\vec{r},\omega) \right] ,
\end{aligned} \tag{C.179}$$

$$
\begin{aligned}
{}^{E}\mathcal{Q}_D^{<i_1\ldots i_\ell>}(\omega) \equiv \frac{1}{(\ell+d-2)}(\Pi^S)^{<i_1 i_2 \ldots i_\ell>}_{<j_1 j_2 \ldots j_\ell>} \int d^d x\; x^{j_1} x^{j_2} \ldots x^{j_\ell} \\
\times \left[\frac{1}{r^{\ell+d-3}} \partial_r \left\{ r^{\ell+d-2}\; {}_0F_1\left[1+\nu, -\frac{r^2\omega^2}{4}\right] \right\} J_D^t(\vec{r},\omega) \right. \\
\left. + i\omega r\; {}_0F_1\left[1+\nu, -\frac{r^2\omega^2}{4}\right] J_D^r(\vec{r},\omega) \right] .
\end{aligned}
\tag{C.180}
$$

In terms of these STF multipole moments, we can write the RR action as follows:

$$
S^E_{\mathrm{RR}} = \sum_\ell \frac{\ell+d-2}{\ell} \int \frac{d\omega}{2\pi} \frac{\pi i}{2\Gamma(\nu+1)^2} \left(\frac{\omega}{2}\right)^{2\nu} \frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|} \frac{1}{\ell!}\, {}^{E}\mathcal{Q}_D^{*<i_1 i_2 \ldots i_\ell>}\, {}^{E}\mathcal{Q}^A_{<i_1 i_2 \ldots i_\ell>} .
\tag{C.181}
$$

The magnetic multipole moment can be written in terms of the vector STF projector:

$$
\begin{aligned}
{}^{B}\mathcal{Q}_A^{i<i_1\ldots i_\ell>} \equiv (\Pi^V)^{i<i_1 i_2 \ldots i_\ell>}_{j<j_1 j_2 \ldots j_\ell>} \int d^d x\; x^{j_1} x^{j_2} \ldots x^{j_\ell}\; {}_0F_1\left[1+\nu, -\frac{r^2\omega^2}{4}\right] J_A^j \\
{}^{B}\mathcal{Q}_D^{i<i_1\ldots i_\ell>} \equiv (\Pi^V)^{i<i_1 i_2 \ldots i_\ell>}_{j<j_1 j_2 \ldots j_\ell>} \int d^d x\; x^{j_1} x^{j_2} \ldots x^{j_\ell}\; {}_0F_1\left[1+\nu, -\frac{r^2\omega^2}{4}\right] J_D^j
\end{aligned}
\tag{C.182}
$$

Using the vector addition theorem (B.58), we can show that the magnetic part of the influence phase can be written in terms of the STF multipole moments as:

$$
S^B_{\mathrm{RR}} = \sum_\ell \int \frac{d\omega}{2\pi} \frac{\pi i}{2\Gamma(\nu+1)^2} \left(\frac{\omega}{2}\right)^{2\nu} \frac{1}{\mathcal{N}_{d,\ell}|\mathbb{S}^{d-1}|} \frac{1}{\ell!}\, {}^{B}\mathcal{Q}_{D,STF}^{*i<i_1 i_2 \ldots i_\ell>}\; {}^{B}\mathcal{Q}^{A,STF}_{i<i_1 i_2 \ldots i_\ell>} .
\tag{C.183}
$$

C.8 Multipole expansion in $d = 3$

In $d = 3$, we can trade the magnetic field 2-form for an axial vector field $\bar{B}_i \equiv \frac{1}{2}\varepsilon_{ijk}\overline{\mathfrak{C}}_{jk}$. It is more convenient to deal with vectors than tensors, and the electric-magnetic duality is easier to see. We will describe here how the multipole expansion and radiation reaction formulae can be recast to make EM duality manifest.

At the level of orthonormal spherical harmonics, in $d = 3$, all vector spherical harmonics can be replaced with the toroidal operator acting on scalar spherical harmonics, viz.,

$$\mathbb{V}_I^{1\ell m}(\hat{r}) = \frac{1}{\sqrt{\ell(\ell+1)}}\varepsilon_{IJ}\mathscr{D}^J\mathscr{Y}_{\ell m}(\hat{r}) \ , \tag{C.184}$$

where the spherical indices are raised using the unit sphere metric. However, if we do this, we must contend with the irrational factors of $\sqrt{\ell(\ell+1)}$ everywhere in the spherical harmonic expansions. We will instead use the following strategy, motivated primarily by EM duality. We use a rescaled basis of VSHs[10]

$$\mathbb{U}_I^{\ell m}(\hat{r}) \equiv \sqrt{\frac{\ell+1}{\ell}}\mathbb{V}_I^{1\ell m}(\hat{r}) = \frac{1}{\ell}\varepsilon_{IJ}\mathscr{D}^J\mathscr{Y}_{\ell m}(\hat{r}) \ . \tag{C.186}$$

Similarly, we scale the components appearing in the vector spherical harmonic expansions. We will add a $\vee$ symbol to all scaled components to avoid confusion. This rescaling factor should also be included in orthonormality, addition theorem, etc. A rough thumb rule is to replace all occurrences of $\mathbb{V}_I$s in our formulae with $\mathbb{U}_I$s, but replace $\mathbb{V}_I^*$ with $\frac{\ell}{\ell+1}\mathbb{U}_I^*$ (apart from adding $\vee$ symbol to the components). We will see an example of this below.

Toroidal magnetostatics in $d = 3$

Let us first see how this works in the simpler setting of magnetostatics: the generalisation to the full dynamical situation is straightforward.

[10] In what follows, it is useful to remember that

$$d\vartheta^I\mathbb{U}_I^{\ell m}(\hat{r}) = \frac{1}{\ell}\left(\frac{d\vartheta}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \sin\vartheta\ d\varphi\ \frac{\partial}{\partial\vartheta}\right)Y_{\ell m}(\hat{r}) = \overrightarrow{dr}\cdot\left\{-\frac{\hat{e}_r}{\ell}\times\vec{\nabla}Y_{\ell m}(\hat{r})\right\} \ . \tag{C.185}$$

The last expression gives a cartesian form for our rescaled VSH.

In terms of the rescaled VSH, the magnetostatic expansion in Eq.(C.13) becomes

$$\begin{aligned}
&\overline{\mathcal{V}}_r = 0 \ , \quad \overline{\mathcal{V}}_I \equiv \sum_{\ell m} \overline{\Phi}^{\vee}_{_B}(r,\ell,m)\mathbb{U}^{\ell m}_I(\hat{r}) \ , \\
&\overline{\mathcal{C}}_{rI} \equiv \sum_{\ell m} \overline{H}^{\vee}_v(r,\alpha,\ell,\vec{m})\mathbb{U}^{\ell m}_I(\hat{r}) = \varepsilon_{_{IJ}} \sum_{\ell m} \frac{1}{\ell}\overline{H}^{\vee}_v(r,\ell,m)\mathscr{D}^J\mathscr{Y}_{\ell m}(\hat{r}) \ , \\
&\overline{\mathcal{C}}_{IJ} \equiv \sum_{\ell\vec{m}\alpha} \overline{H}^{\vee}_{vv}(r,\ell,m)\mathscr{D}_{[I}\mathbb{U}^{\ell m}_{J]}(\hat{r}) = \varepsilon_{_{IJ}} \sum_{\ell m}(\ell+1)\overline{H}^{\vee}_{vv}(r,\ell,m)\mathscr{Y}_{\ell m}(\hat{r}) \ ,
\end{aligned} \tag{C.187}$$

where, in the last step, we have used the identity

$$\mathscr{D}_{[I}\mathbb{U}^{\ell m}_{J]}(\hat{r}) = \varepsilon_{_{IJ}}\varepsilon^{MN}\mathscr{D}_M\mathbb{U}^{\ell m}_N = (\ell+1)\mathscr{Y}_{\ell m}(\hat{r})\ \varepsilon_{_{IJ}} \ . \tag{C.188}$$

The vector magnetic field is obtained by stripping off the $\varepsilon_{_{rIJ}} = r^2\varepsilon_{_{IJ}}$ factors in $\overline{\mathcal{C}}_{ij}$ and then lowering the indices. The spherical/radial components of the vector magnetic field are

$$\begin{aligned}
&\overline{B}_I = r^2\gamma_{IJ} \times \frac{1}{r^2}\sum_{\ell m}\frac{1}{\ell}\overline{H}^{\vee}_v(r,\ell,m)\mathscr{D}^J\mathscr{Y}_{\ell m}(\hat{r}) = \sum_{\ell m}\frac{1}{\ell}\overline{H}^{\vee}_v(r,\ell,m)\mathscr{D}_I\mathscr{Y}_{\ell m}(\hat{r}) \ , \\
&\overline{B}_r = \sum_{\ell m}\frac{\ell+1}{r^2}\overline{H}^{\vee}_{vv}(r,\ell,m)\mathscr{Y}_{\ell m}(\hat{r}) \ .
\end{aligned} \tag{C.189}$$

The expression for the Debye field in Eq.(C.15) becomes

$$\overline{\Phi}^{\vee}_{_B}(r,\ell,m) \equiv \frac{\ell}{\ell+1}\int_{\vec{r}_0}\mathbb{G}_B(r,r_0;\ell)\mathbb{U}^{\ell m*}_J(\hat{r}_0)\bar{J}^J(\vec{r}_0) = \sqrt{\frac{\ell}{\ell+1}}\overline{\Phi}_{_B}(r,\alpha=1,\ell,m) \ . \tag{C.190}$$

In the first step, there is a rescaling pre-factor $\frac{\ell}{\ell+1}$ which multiplies $\mathbb{U}^{\ell m*}_J$ in accordance with the thumb rule mentioned above. Similar factors appear in the definition of the magnetic moment:

$$\overline{\mathcal{J}}^{\vee B}(\ell,m) \equiv \frac{1}{2\ell+1}\frac{\ell}{\ell+1}\int_{\vec{r}_0} r_0^{\ell+1}\mathbb{U}^{\ell m*}_J(\hat{r}_0)\bar{J}^J(\vec{r}_0) = \sqrt{\frac{\ell}{\ell+1}}\overline{\mathcal{J}}^B(\alpha=1,\ell,m) \ . \tag{C.191}$$

The magnetic Debye field and the magnetic field components outside the sources take the form

$$
\begin{aligned}
\overline{\Phi}^{\vee,\text{Out}}_{B}(r,\ell,m) &= \frac{\overline{\mathcal{J}}^{\vee B}(\ell,m)}{r^{\ell}} \ , \\
\frac{1}{\ell}\overline{H}^{\vee\text{Out}}_{v}(r,\ell,m) &= -\frac{\overline{\mathcal{J}}^{\vee B}(\ell,m)}{r^{\ell+1}} \ , \qquad \frac{\ell+1}{r^2}\overline{H}^{\vee\text{Out}}_{vv}(r,\ell,m) = (\ell+1)\frac{\overline{\mathcal{J}}^{\vee B}(\ell,m)}{r^{\ell+2}} \ .
\end{aligned}
\tag{C.192}
$$

This should be compared against the electrostatic expressions written in terms of electric multipole moments:

$$
\begin{aligned}
\overline{\Phi}^{\vee,\text{Out}}_{E}(r,\ell,\vec{m}) &= \frac{\overline{\mathcal{J}}^{E}(\ell,\vec{m})}{r^{\ell}} \ , \\
\overline{E}^{\text{Out}}_{s}(r,\ell,\vec{m}) &= -\frac{\overline{\mathcal{J}}^{E}(\ell,\vec{m})}{r^{\ell+d-2}} \ , \qquad \overline{E}^{\text{Out}}_{r}(r,\ell,\vec{m}) = (\ell+d-2)\frac{\overline{\mathcal{J}}^{E}(\ell,\vec{m})}{r^{\ell+d-1}} \ .
\end{aligned}
\tag{C.193}
$$

We see that the forms of the outside electrostatic vs magnetostatic fields agree in $d=3$. In this normalisation, EM duality acts by mapping electric to magnetic fields and $\overline{\mathcal{J}}^{E}$ to $\overline{\mathcal{J}}^{\vee B}$. As to the cartesian moments, in analogy with electrostatic multipole tensors in Eq.(C.91), we define[11]

$$
\begin{aligned}
\frac{1}{\ell!}\ {}^{B}\overline{\mathbb{Q}}^{\vee}_{<i_1 i_2\dots i_\ell>}x^{i_1}\dots x^{i_\ell} &\equiv \frac{4\pi}{(2\ell-1)!!}\sum_m \overline{\mathcal{J}}^{\vee B}(\ell,m)r^{\ell}\mathscr{Y}_{\ell m}(\hat{r}) \\
&= \frac{1}{\ell+1}\int_{\vec{r}_0}[\vec{r}_0\times\vec{\mathcal{J}}(\vec{r}_0)]\cdot\vec{\nabla}_0\Pi^{S}(\vec{r}|\vec{r}_0)_{d=3,\ell} \ .
\end{aligned}
\tag{C.194}
$$

In the second line, we have used the expression for $\overline{\mathcal{J}}^{\vee B}$ in Eq.(C.191), the explicit form of $\mathbb{U}_I$, as well as the SSH addition theorem. The integral appearing here can be evaluated as

$$
\int_{\vec{r}_0}[\vec{r}_0\times\vec{\mathcal{J}}(\vec{r}_0)]\cdot\vec{\nabla}_0\Pi^{S}(\vec{\kappa}|\vec{r}_0) = \frac{\ell}{\ell!}\kappa^{i_1}\dots\kappa^{i_\ell}(\Pi^{S})^{<i_1 i_2\dots i_\ell>}_{<j_1 j_2\dots j_\ell>}\int_{\vec{r}_0}[\vec{r}_0\times\vec{\mathcal{J}}(\vec{r}_0)]^{j_1}x_0^{j_2}\dots x_0^{j_\ell} \ ,
\tag{C.195}
$$

thus yielding a direct cartesian definition

$$
{}^{B}\overline{\mathbb{Q}}^{\vee}_{<i_1 i_2\dots i_\ell>} \equiv \frac{\ell}{\ell+1}\int_{\vec{r}_0}[\vec{r}_0\times\vec{\mathcal{J}}(\vec{r}_0)]^{<i_1}x_0^{i_2}\dots x_0^{i_\ell>} \ .
\tag{C.196}
$$

By construction, the normalisation here is fixed to ensure that EM duality maps the STF

[11] We use the result that $\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}| = \frac{4\pi}{(2\ell-1)!!}$ for $d=3$.

tensors ${}^E\overline{\mathcal{Q}}$ to ${}^B\overline{\mathcal{Q}}^\vee$. The outside vector potential/magnetic field can be written in terms of this STF magnetic moment as

$$\begin{aligned}\overline{\mathcal{V}}_k^{\rm Out} &= -\varepsilon_{kij}x^i\sum_\ell {}^B\overline{\mathcal{Q}}^\vee_{<ji_1\ldots i_{\ell-1}>}\frac{x^{i_1}\ldots x^{i_{\ell-1}}}{4\pi r^{2\ell+1}}\frac{(2\ell-1)!!}{\ell!} \\ &= -\varepsilon_{kij}\sum_\ell\frac{(-)^\ell}{\ell!}\partial^i\partial^{i_1}\partial^{i_2}\ldots\partial^{i_{\ell-1}}\left\{\frac{{}^B\overline{\mathcal{Q}}^\vee_{<ji_1\ldots i_{\ell-1}>}}{4\pi r}\right\}, \\ \overline{B}_j^{\rm Out} &= \sum_\ell {}^B\overline{\mathcal{Q}}^\vee_{<i_1i_2\ldots i_\ell>}[(2\ell+1)x^jx^{i_\ell}-r^2\ell\delta^{ji_\ell}]\frac{x^{i_1}\ldots x^{i_{\ell-1}}}{4\pi r^{2\ell+3}}\frac{(2\ell-1)!!}{\ell!}\ .\end{aligned} \tag{C.197}$$

As expected from EM duality, the magnetic field here has the same form as the outside electrostatic field in Eq.(C.93). The description in terms of the magnetic moment ${}^B\overline{\mathcal{Q}}^\vee$ can be related to ones in terms of ${}^B\overline{\mathcal{Q}}$ as follows: first, we begin by rewriting the VSH projector as

$$\Pi^V_{ij}(\vec{\kappa}|\vec{r})|_{d=3} = \frac{[\vec{\kappa}\times\vec{\nabla}_\kappa]_i[\vec{r}\times\vec{\nabla}]_j}{\ell(\ell+1)}\Pi^S(\vec{\kappa}|\vec{r})|_{d=3}\ . \tag{C.198}$$

This follows from the fact that, in $d=3$, all the VSHs can be obtained by applying a single toroidal operator $[\vec{r}\times\vec{\nabla}]$ on SSHs. To get orthonormal VSHs, we need to divide by a factor $\sqrt{\ell(\ell+1)}$. This means that the VSH addition theorem in $d=3$ can be obtained by sandwiching the SSH addition theorem between two toroidal operators [163–165]. The relation between the two addition theorems then yields the above relation between the projectors.

In the next step, we write

$$\begin{aligned}\frac{1}{\ell!}{}^B\overline{\mathcal{Q}}_{i<i_1i_2\ldots i_\ell>}\kappa^{i_1}\ldots\kappa^{i_\ell} &\equiv \int_{\vec{r}_0}\Pi^V_{ij}(\vec{\kappa}|\vec{r}_0)\bar{J}^j(\vec{r}_0) \\ &= -\frac{1}{\ell}[\vec{\kappa}\times\vec{\nabla}_\kappa]_i\frac{1}{\ell+1}\int_{\vec{r}_0}[\vec{r}_0\times\vec{\bar{J}}(\vec{r}_0)]\cdot\vec{\nabla}_0\Pi^S(\vec{\kappa}|\vec{r}_0)\ .\end{aligned} \tag{C.199}$$

In RHS, we recognise the integral that defines the ${}^B\overline{\mathcal{Q}}^\vee$. Stripping off the κ^i's, we get the relation between the two kinds of moments as

$${}^B\overline{\mathcal{Q}}_{i<i_1i_2\ldots i_\ell>} = \frac{1}{\ell}\sum_{p=1}^{\ell}{}^B\overline{\mathcal{Q}}^\vee_{<ji_1\ldots\underline{i_p}\ldots i_\ell>}\ \varepsilon_{iji_p}\ . \tag{C.200}$$

Here, the underlining on the i_p index indicates that it should be dropped, i.e., the indices of ${}^{B}\overline{\mathbb{Q}}^{\vee}$ are $ji_1i_2\ldots i_{p-1}i_{p+1}\ldots i_\ell$. We can invert this relation by rewriting Eq.(C.198) as

$$[\vec{\kappa}\times\vec{\nabla}_\kappa]_i\Pi^V_{ij}(\vec{\kappa}|\vec{r})|_{d=3} = -[\vec{r}\times\vec{\nabla}]_j\Pi^S(\vec{\kappa}|\vec{r})|_{d=3} \ . \tag{C.201}$$

This is equivalent to the tensorial relation

$$\sum_{p=1}^{\ell}\varepsilon_{ii_pn}(\Pi^V)^{i<ni_1\ldots\underline{i_p}\ldots i_\ell>}_{j<j_1j_2\ldots j_\ell>} = -\sum_{p=1}^{\ell}\varepsilon_{jj_pn}(\Pi^S)^{<i_1i_2\ldots i_\ell>}_{<nj_1\ldots\underline{j_p}\ldots j_\ell>} \ . \tag{C.202}$$

Multiplying by $x_0^{j_1}x_0^{j_2}\ldots x_0^{j_\ell}\bar{J}^n(\vec{r}_0)$ and integrating, we get

$${}^{B}\overline{\mathbb{Q}}^{\vee}_{<i_1i_2\ldots i_\ell>} = -\frac{1}{\ell+1}\sum_{p=1}^{\ell}{}^{B}\overline{\mathbb{Q}}_{i<ji_1\ldots\underline{i_p}\ldots i_\ell>}\ \varepsilon_{iji_p} \ . \tag{C.203}$$

The relations Eq.(C.200) and Eq.(C.203) show that same information is contained in the STF tensors ${}^{B}\overline{\mathbb{Q}}$ and ${}^{B}\overline{\mathbb{Q}}^{\vee}$.

Multipole radiation in $d = 3$

We will now move to describing the full radiative fields in $d = 3$ outside the sources. The decomposition of EM fields in terms of spherical harmonics has the form

$$\begin{aligned}
E_I(\vec{r},\omega) &= \sum_{\ell m} E_s(r,\omega,\ell,m)\mathscr{D}_I\mathscr{Y}_{\ell m}(\hat{r}) + \sum_{\ell m} E^{\vee}_v(r,\omega,\ell,m)\mathbb{U}^{\ell m}_I(\hat{r}) \ , \\
E_r(\vec{r},\omega) &= \sum_{\ell m} E_r(r,\omega,\ell,m)\mathscr{Y}_{\ell m}(\hat{r}) \ , \\
B_I(\vec{r},\omega) &= \sum_{\ell m}\frac{1}{\ell}H^{\vee}_v(r,\omega,\ell,m)\mathscr{D}_I\mathscr{Y}_{\ell m}(\hat{r}) - \sum_{\ell m}\ell\ H_s(r,\omega,\ell,m)\mathbb{U}^{\ell m}_I(\hat{r}) \ , \\
B_r(\vec{r},\omega) &= \sum_{\ell m}\frac{\ell+1}{r^2}H^{\vee}_{vv}(r,\omega,\ell,m)\mathscr{Y}_{\ell m}(\hat{r}) \ .
\end{aligned} \tag{C.204}$$

We can write this succinctly as

$$
\begin{aligned}
\vec{E}(\vec{r},\omega) &= \sum_{\ell m}\Big[E_r(r,\omega,\ell,m)\mathscr{Y}_{\ell m}\hat{e}_r + E_s(r,\omega,\ell,m)\ \vec{\nabla}\mathscr{Y}_{\ell m} + E_v^{\vee}(r,\omega,\ell,m)\ \vec{\mathbb{U}}_{\ell m}\Big]\ ,\\
\vec{B}(\vec{r},\omega) &= \sum_{\ell m}\Big[\frac{\ell+1}{r^2}H_{vv}^{\vee}(r,\omega,\ell,m)\mathscr{Y}_{\ell m}\hat{e}_r\\
&\qquad + \frac{1}{\ell}H_v^{\vee}(r,\omega,\ell,m)\ \vec{\nabla}\mathscr{Y}_{\ell m} - \ell H_s(r,\omega,\ell,m)\ \vec{\mathbb{U}}_{\ell m}\Big]\ ,
\end{aligned}
\tag{C.205}
$$

where we have introduced

$$
\vec{\mathbb{U}}_{\ell m} \equiv -\frac{1}{\ell}\hat{e}_r \times \vec{\nabla}\mathscr{Y}_{\ell m}\ , \quad \vec{\nabla}\mathscr{Y}_{\ell m} = \ell\ \hat{e}_r \times \vec{\mathbb{U}}_{\ell m}\ . \tag{C.206}
$$

With this understanding, much of our description of EM fields in this section continue to hold true. Explicit expressions for outside fields in terms of multipole moments can be written down using Eq.(C.60) and Eq.(C.124). We have

$$
\begin{aligned}
\vec{E}^{\rm Out}(\vec{r},\omega) &= \sum_{\ell m} G^{\rm Out}_{E}(r,\omega,\ell)\ \mathcal{J}^{E}(\omega,\ell,m)\left[\frac{\ell+1}{r^2}\mathscr{Y}_{\ell m}\hat{e}_r - \frac{1}{r}\ \vec{\nabla}\mathscr{Y}_{\ell m}\right]\\
&+ \sum_{\ell m}\frac{\omega^2}{\ell(2\ell-1)}G^{\rm Out}_{E}(r,\omega,\ell-1)\ \mathcal{J}^{E}(\omega,\ell,m)\vec{\nabla}\mathscr{Y}_{\ell m}\\
&+ i\omega\ \sum_{\ell m} G^{\rm Out}_{B}(r,\omega,\ell)\ \mathcal{J}^{\vee B}(\omega,\ell,m)\ \vec{\mathbb{U}}_{\ell m}\ ,\\
\vec{B}^{\rm Out}(\vec{r},\omega) &= \sum_{\ell m} G^{\rm Out}_{B}(r,\omega,\ell)\ \mathcal{J}^{\vee B}(\omega,\ell,m)\left[\frac{\ell+1}{r^2}\mathscr{Y}_{\ell m}\hat{e}_r - \frac{1}{r}\ \vec{\nabla}\mathscr{Y}_{\ell m}\right]\\
&+ \sum_{\ell m}\frac{\omega^2}{\ell(2\ell-1)}G^{\rm Out}_{B}(r,\omega,\ell-1)\ \mathcal{J}^{\vee B}(\omega,\ell,m)\vec{\nabla}\mathscr{Y}_{\ell m}\\
&- i\omega\ \sum_{\ell m} G^{\rm Out}_{E}(r,\omega,\ell)\ \mathcal{J}^{E}(\omega,\ell,m)\ \vec{\mathbb{U}}_{\ell m}\ .
\end{aligned}
\tag{C.207}
$$

In $d=3$, the electric outgoing fields are the same as the magnetic ones, i.e.,

$$
G^{\rm Out}_{E}(r,\omega,\ell) = G^{\rm Out}_{B}(r,\omega,\ell) \equiv r^{-\ell}{}_0F_1\left[\frac{1}{2}-\ell, -\frac{\omega^2 r^2}{4}\right] = \frac{\theta_\ell(-i\omega r)}{(2\ell-1)!!}\frac{e^{i\omega r}}{r^\ell}\ , \tag{C.208}
$$

where the $\theta_\ell(z)$ are the reverse Bessel polynomials (see Table C.1 for explicit expressions). We remind the reader that the above function is essentially the outgoing, spherical Hankel function up to a frequency-dependent normalisation factor:

$$ {}_0F_1\left[\frac{1}{2}-\ell,-\frac{\omega^2r^2}{4}\right]=\frac{i(\omega r)^\ell}{(2\ell-1)!!}h_\ell^{(1)}(\omega r)\ . \tag{C.209} $$

The above expressions for outgoing EM waves are consistent with EM duality, which maps

$$ (\vec{E},\mathcal{J}^E)\mapsto(\vec{B},\mathcal{J}^{\vee B})\ ,\quad (\vec{B},\mathcal{J}^{\vee B})\mapsto(-\vec{E},-\mathcal{J}^E)\ . \tag{C.210} $$

The explicit expression for the spherical multipole moments are given by

$$ \begin{aligned} \mathcal{J}^E(\omega,\ell,\vec{m})&\equiv\frac{1}{(2\ell+1)(\ell+1)}\int_{\vec{r}\in\mathbb{R}^3} r^{\ell-1}\mathscr{Y}^*_{\ell m}(\hat{r})\ {}_0F_1\left[\ell+\frac{3}{2},-\frac{\omega^2r^2}{4}\right]\\ &\qquad\times\left\{i\omega r^2J^r(\vec{r},\omega)-\partial_r[r^2J^t(\vec{r},\omega)]\right\}\ ,\\ \mathcal{J}^{\vee B}(\omega,\ell,m)&\equiv\frac{1}{2\ell+1}\frac{\ell}{\ell+1}\int_{\vec{r}\in\mathbb{R}^3} r^{\ell+1}\mathbb{U}_I^{\ell m*}(\hat{r})_0F_1\left[\ell+\frac{3}{2},-\frac{\omega^2r^2}{4}\right]J^I(\vec{r},\omega)\\ &=\frac{1}{(2\ell+1)(\ell+1)}\int_{\vec{r}_0\in\mathbb{R}^3} r^\ell{}_0F_1\left[\ell+\frac{3}{2},-\frac{\omega^2r^2}{4}\right][\vec{r}\times\vec{J}(\vec{r},\omega)]\cdot\vec{\nabla}\mathscr{Y}_{\ell m}(\hat{r})\ . \end{aligned} \tag{C.211} $$

Here, we have specialised the electric moment of Eq.(C.144) to $d=3$, and we have generalised the magnetic moment in Eq.(C.191) to dynamical situations. The ${}_0F_1$ is the time-delay smearing function and is essentially the spherical Bessel function up to a normalisation:

$$ {}_0F_1\left[\ell+\frac{3}{2},-\frac{\omega^2r^2}{4}\right]=\frac{(2\ell+1)!!}{(\omega r)^\ell}j_\ell(\omega r)=\frac{(2\ell+1)!!}{2^\ell\ell!}\int_{-1}^{1}(1-z^2)^\ell\ e^{\pm i\omega zr}\ . \tag{C.212} $$

At large r, far away from sources, the EM fields given in Eq.(C.207) become

$$ \begin{aligned} \vec{E}_{\rm Rad}(\vec{r},\omega)&=-e^{i\omega r}\sum_{\ell m}\frac{(-i\omega)^{\ell+1}}{\ell(2\ell-1)!!}\left[\mathcal{J}^E(\omega,\ell,m)\ \vec{\nabla}\mathscr{Y}_{\ell m}-\mathcal{J}^{\vee B}(\omega,\ell,m)\ \hat{e}_r\times\vec{\nabla}\mathscr{Y}_{\ell m}\right]\ ,\\ \vec{B}_{\rm Rad}(\vec{r},\omega)&=-e^{i\omega r}\sum_{\ell m}\frac{(-i\omega)^{\ell+1}}{\ell(2\ell-1)!!}\left[\mathcal{J}^E(\omega,\ell,m)\ \hat{e}_r\times\vec{\nabla}\mathscr{Y}_{\ell m}\ +\mathcal{J}^{\vee B}(\omega,\ell,m)\ \vec{\nabla}\mathscr{Y}_{\ell m}\right]\ . \end{aligned} \tag{C.213} $$

Here the sum runs over $\ell\geq1$, and we recognise in these formulae the transverse, radially outgoing EM waves with $\vec{B}_{\rm Rad}=\hat{e}_r\times\vec{E}_{\rm Rad}$. The cartesian forms of these equations can be obtained by using the definitions

$$\begin{aligned}\sum_m \mathcal{J}^E(\omega,\ell,m)\mathscr{Y}_{\ell m}(\hat{r}) &\equiv \frac{(2\ell-1)!!}{\ell!}\ {}^E\mathcal{Q}(\omega)_{<i_1 i_2 \dots i_\ell>}\frac{x^{i_1}\dots x^{i_\ell}}{4\pi r^\ell}\ ,\\ \sum_m \mathcal{J}^{\vee B}(\omega,\ell,m)\mathscr{Y}_{\ell m}(\hat{r}) &\equiv \frac{(2\ell-1)!!}{\ell!}\ {}^B\mathcal{Q}^\vee(\omega)_{<i_1 i_2 \dots i_\ell>}\frac{x^{i_1}\dots x^{i_\ell}}{4\pi r^\ell}\ .\end{aligned} \tag{C.214}$$

These generalise the relation between spherical and cartesian STF moments beyond statics. The explicit expression for the cartesian moments are given by

$$\begin{aligned}{}^E\mathcal{Q}(\omega)_{<i_1 i_2 \dots i_\ell>} &\equiv \frac{1}{\ell+1}\int_{\vec{r}\in\mathbb{R}^3} x^{<i_1}x^{i_2}\dots x^{i_\ell>}\ {}_0F_1\left[\ell+\frac{3}{2},-\frac{\omega^2 r^2}{4}\right]\\ &\qquad\times\left\{i\omega r J^r(\vec{r},\omega)-\frac{1}{r}\partial_r[r^2 J^t(\vec{r},\omega)]\right\}\ ,\\ {}^B\mathcal{Q}^\vee(\omega)_{<i_1 i_2 \dots i_\ell>} &= \frac{\ell}{\ell+1}\int_{\vec{r}_0\in\mathbb{R}^3} {}_0F_1\left[\ell+\frac{3}{2},-\frac{\omega^2 r^2}{4}\right][\vec{r}\times\vec{J}(\vec{r},\omega)]^{<i_1}x^{i_2}\dots x^{i_\ell>}\ .\end{aligned} \tag{C.215}$$

It is then straightforward to see that Eq.(C.200) and Eq.(C.203) also generalise to time-dependent situations.

The cartesian components of the EM fields can be worked out by directly substituting the above definitions into Eq.(C.207). Alternately, we can take the cartesian Electric fields in general dimensions (i.e. the sum of electric fields in Eq.(C.129) and Eq.(C.129)), and specialise to $d=3$. In either case, once the electric field has been figured out, the magnetic field expressions follow from EM duality. The EM fields outside the sources evaluate to

$$\begin{aligned}E_k^{\rm Out}(\vec{r},\omega) &= e^{i\omega r}\sum_\ell \frac{\theta_\ell(-i\omega r)}{\ell!}{}^E\mathcal{Q}(\omega)_{<i_1 i_2 \dots i_\ell>}[(2\ell+1)n^k n^{i_\ell}-\ell\delta^{k i_\ell}]\frac{n^{i_1}\dots n^{i_{\ell-1}}}{4\pi r^{\ell+2}}\\ &\quad+\omega^2 e^{i\omega r}\sum_\ell \frac{\theta_{\ell-1}(-i\omega r)}{\ell!}{}^E\mathcal{Q}(\omega)_{<i_1 i_2 \dots i_\ell>}[\delta^{k i_\ell}-n^k n^{i_\ell}]\frac{n^{i_1}\dots n^{i_{\ell-1}}}{4\pi r^{\ell}}\\ &\quad+i\omega e^{i\omega r}\varepsilon_{k j i_1}\sum_\ell \frac{\theta_\ell(-i\omega r)}{\ell!}{}^B\mathcal{Q}^\vee(\omega)_{<j i_2 \dots i_\ell>}\frac{n^{i_1}\dots n^{i_\ell}}{4\pi r^{\ell+1}},\\ B_k^{\rm Out}(\vec{r},\omega) &= e^{i\omega r}\sum_\ell \frac{\theta_\ell(-i\omega r)}{\ell!}{}^B\mathcal{Q}^\vee(\omega)_{<i_1 i_2 \dots i_\ell>}[(2\ell+1)n^k n^{i_\ell}-\ell\delta^{k i_\ell}]\frac{n^{i_1}\dots n^{i_{\ell-1}}}{4\pi r^{\ell+2}}\\ &\quad+\omega^2 e^{i\omega r}\sum_\ell \frac{\theta_{\ell-1}(-i\omega r)}{\ell!}{}^B\mathcal{Q}^\vee(\omega)_{<i_1 i_2 \dots i_\ell>}[\delta^{k i_\ell}-n^k n^{i_\ell}]\frac{n^{i_1}\dots n^{i_{\ell-1}}}{4\pi r^{\ell}}\\ &\quad-i\omega e^{i\omega r}\varepsilon_{k j i_1}\sum_\ell \frac{\theta_\ell(-i\omega r)}{\ell!}{}^E\mathcal{Q}(\omega)_{<j i_2 \dots i_\ell>}\frac{n^{i_1}\dots n^{i_\ell}}{4\pi r^{\ell+1}}\ .\end{aligned} \tag{C.216}$$

Here we have used a shorthand $n^i \equiv \frac{x^i}{r}$. The reader might recognise the form appearing in the first line of the fields from the EM fields outside static multipoles. Only the second and the third line in each field contribute to the radiative part. These are again manifestly duality invariant if we also take

$$ {}^E\mathcal{Q} \mapsto {}^B\mathcal{Q}^\vee \ , \quad {}^B\mathcal{Q}^\vee \mapsto -{}^E\mathcal{Q} \ . \tag{C.217} $$

With some relabelling of indices, the radiative EM fields can be cast into the following form

$$ \begin{aligned} E_k^{\mathrm{Rad}}(\vec{r},\omega) &= -\frac{e^{i\omega r}}{4\pi r}\sum_\ell \frac{(-i\omega)^{\ell+1}}{\ell!} \\ &\quad \times \left\{ {}^E\mathcal{Q}(\omega)_{<i_1 i_2 \ldots i_\ell>}[\delta^{k i_1} - n^k n^{i_1}] + \varepsilon_{k j i_1}\, {}^B\mathcal{Q}^\vee(\omega)_{<j i_2 \ldots i_\ell>}\ n^{i_1} \right\} n^{i_2} \ldots n^{i_\ell} \\ B_k^{\mathrm{Rad}}(\vec{r},\omega) &= -\frac{e^{i\omega r}}{4\pi r}\sum_\ell \frac{(-i\omega)^{\ell+1}}{\ell!} \\ &\quad \times \left\{ {}^B\mathcal{Q}^\vee(\omega)_{<i_1 i_2 \ldots i_\ell>}[\delta^{k i_1} - n^k n^{i_1}] - \varepsilon_{k j i_1}\, {}^E\mathcal{Q}(\omega)_{<j i_2 \ldots i_\ell>}\ n^{i_1} \right\} n^{i_2} \ldots n^{i_\ell}. \end{aligned} \tag{C.218} $$

The scalar and vector potentials corresponding to the above EM fields can be written down by choosing a gauge. Combining Eqs.(C.67) and (C.132), we can write

$$ \begin{aligned} \mathcal{V}_t^{\mathrm{Out,New}}(\vec{r},\omega) &= -\sum_\ell \frac{G_E^{\mathrm{Out}}(r,\omega,\ell)}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\, {}^E\mathcal{Q}(\omega)_{<i_1 i_2 \ldots i_\ell>} \frac{x^{i_1}\ldots x^{i_\ell}}{r^{\ell+d-2}} \ , \\ \mathcal{V}_k^{\mathrm{Out,New}}(\vec{r},\omega) &= -i\omega \sum_{\ell>0} \frac{G_E^{\mathrm{Out}}(r,\omega,\ell-1)}{\ell!\mathcal{N}_{d,\ell-2}|\mathbb{S}^{d-1}|}\ {}^E\mathcal{Q}(\omega)_{<k i_1 i_2 \ldots i_{\ell-1}>} \frac{x^{i_1}\ldots x^{i_{\ell-1}}}{r^{\ell+d-3}} \\ &\quad + \sum_{\ell>0} \frac{G_B^{\mathrm{Out}}(r,\omega,\ell)}{\ell!\mathcal{N}_{d,\ell-1}|\mathbb{S}^{d-1}|}\, {}^B\mathcal{Q}(\omega)_{k<i_1 i_2 \ldots i_\ell>} \frac{x^{i_1}\ldots x^{i_\ell}}{4\pi r^{\ell+1}} \ . \end{aligned} \tag{C.219} $$

Specialising to $d = 3$ and rewriting the magnetic moment part accordingly, we obtain

$$\begin{aligned}
\mathcal{V}_t^{\text{Out,New}}(\vec{r},\omega) &= -e^{i\omega r}\sum_{\ell}\frac{\theta_\ell(-i\omega r)}{\ell!}\,{}^{E}\mathcal{Q}(\omega)_{<i_1 i_2 \dots i_\ell>}\frac{n^{i_1}\dots n^{i_\ell}}{4\pi r^{\ell+1}}\ , \\
\mathcal{V}_k^{\text{Out,New}}(\vec{r},\omega) &= -i\omega e^{i\omega r}\sum_{\ell>0}\frac{\theta_{\ell-1}(-i\omega r)}{\ell!}\ {}^{E}\mathcal{Q}(\omega)_{<k i_1 i_2 \dots i_{\ell-1}>}\frac{n^{i_1}\dots n^{i_{\ell-1}}}{4\pi r^{\ell}} \\
&\quad + e^{i\omega r}\varepsilon_{kji_1}\sum_{\ell>0}\frac{\theta_\ell(-i\omega r)}{\ell!}\,{}^{B}\mathcal{Q}^{\vee}(\omega)_{<j i_2 \dots i_\ell>}\frac{n^{i_1}\dots n^{i_\ell}}{r^{\ell+1}}\ .
\end{aligned} \tag{C.220}$$

These outgoing solutions and their time-reversed counterparts are useful in quantising electrodynamics in spherical coordinates. Far away from the sources, we get the radiative fields

$$\begin{aligned}
\mathcal{V}_t^{\text{Rad,New}}(\vec{r},\omega) &= -\frac{e^{i\omega r}}{4\pi r}\sum_{\ell}\frac{(-i\omega)^\ell}{\ell!}\,{}^{E}\mathcal{Q}(\omega)_{<i_1 i_2 \dots i_\ell>}n^{i_1}\dots n^{i_\ell}\ , \\
\mathcal{V}_k^{\text{Rad,New}}(\vec{r},\omega) &= \frac{e^{i\omega r}}{4\pi r}\sum_{\ell>0}\frac{(-i\omega)^\ell}{\ell!}\Bigg\{{}^{E}\mathcal{Q}(\omega)_{<k i_1 i_2 \dots i_{\ell-1}>} \\
&\qquad + \varepsilon_{kji_\ell}n_{i_\ell}\,{}^{B}\mathcal{Q}^{\vee}(\omega)_{<j i_1 \dots i_{\ell-1}>}\Bigg\}n^{i_1}\dots n^{i_{\ell-1}}\ .
\end{aligned} \tag{C.221}$$

Power loss in $d = 3$

We will now show how the power loss formulae in general dimension can be cast into familiar forms when $d = 3$. The main point is to rewrite the magnetic multipole power loss to make the EM duality manifest. In terms of spherical multipoles, this is easy: we start with the power radiated in terms of our multipole moments is

$$\mathcal{P}(\omega) = \sum_{\ell\vec{m}}\frac{2\pi}{\Gamma(\nu)^2}\frac{\omega^{2\nu+1}}{2^{2\nu}}\frac{\ell+d-2}{\ell}|\mathcal{J}^E(\omega,\ell,\vec{m})|^2 + \sum_{\alpha\ell\vec{m}}\frac{2\pi}{\Gamma(\nu)^2}\frac{\omega^{2\nu+1}}{2^{2\nu}}|\mathcal{J}^B(\omega,\alpha,\ell,\vec{m})|^2, \tag{C.222}$$

and rewrite it in $d = 3$ using Eq.(C.191) to get

$$\mathcal{P}(\omega)_{d=3} = \sum_{\ell m}\frac{\omega^{2\ell+2}}{[(2\ell-1)!!]^2}\frac{\ell+1}{\ell}\left[|\mathcal{J}^E(\omega,\ell,m)|^2 + |\mathcal{J}^{\vee B}(\omega,\ell,m)|^2\right]\ . \tag{C.223}$$

This expression is now manifestly invariant under EM duality.

Power loss in terms of STF moments requires a bit more work: the $d = 3$ version of

Eq.(C.163) is

$$\mathcal{P}(\omega)_{d=3} = \frac{1}{4\pi}\sum_{\ell=1}^{\infty}\frac{\ell+1}{\ell}\frac{\omega^{2\ell+2}}{(2\ell+1)!!}\ \frac{1}{\ell!}\ \left[{}^E\mathcal{Q}(\omega)_{<i_1\dots i_\ell>}\right]^*\ \left[{}^E\mathcal{Q}(\omega)_{<i_1\dots i_\ell>}\right] \\ + \frac{1}{4\pi}\sum_{\ell=1}^{\infty}\frac{\omega^{2\ell+2}}{(2\ell+1)!!}\ \frac{1}{\ell!}\ \left[{}^B\mathcal{Q}(\omega)_{k<i_1\dots i_\ell>}\right]^*\ \left[{}^B\mathcal{Q}(\omega)_{k<i_1\dots i_\ell>}\right] \tag{C.224}$$

We want to rewrite this formula in terms of ${}^B\mathcal{Q}^\vee$ tensor. Using Eq.(C.200), we can write

$$\frac{1}{\ell!}[{}^B\mathcal{Q}_{k<i_1 i_2\dots i_\ell>}]^*[{}^B\mathcal{Q}^{k<i_1 i_2\dots i_\ell>}]|_{d=3} \\ = \frac{1}{\ell!}\left[\frac{1}{\ell}\sum_{p=1}^{\ell}\varepsilon_{kqi_p}\,{}^B\mathcal{Q}^{\vee<qi_1 i_2\dots i_{\underline{p}}\dots i_{\ell-1}>}\right]^*\left[\frac{1}{\ell}\sum_{p'=1}^{\ell}\varepsilon_{kq'i_{p'}}\,{}^B\mathcal{Q}^\vee_{<q'i_1 i_2\dots i_{\underline{p'}}\dots i_{\ell-1}>}\right] . \tag{C.225}$$

In this product of sums made of ℓ^2 terms, we get ℓ terms with $p = p'$ and $\ell(\ell-1)$ terms with $p \neq p'$. Here every $p \neq p'$ term evaluates to

$$\varepsilon_{kq_1 i_1}\varepsilon_{kq_2 i_2}[{}^B\mathcal{Q}^{\vee<q_1 i_2 i_3\dots i_\ell>}]^*[{}^B\mathcal{Q}^\vee_{<q_2 i_1 i_3\dots i_\ell>}] \\ = (\delta_{q_1 q_2}\delta_{i_1 i_2} - \delta_{q_1 i_2}\delta_{q_2 i_1})[{}^B\mathcal{Q}^{\vee<q_1 i_2 i_3\dots i_\ell>}]^*[{}^B\mathcal{Q}^\vee_{<q_2 i_1 i_3\dots i_\ell>}] \\ = [{}^B\mathcal{Q}^{\vee<i_1 i_2 i_3\dots i_\ell>}][{}^B\mathcal{Q}^\vee_{<i_1 i_2 i_3\dots i_\ell>}] . \tag{C.226}$$

Here, the second term evaluates to zero because of the trace-free condition on ${}^B\mathcal{Q}^\vee$. Every $p = p'$ term evaluates to

$$\varepsilon_{kq_1 i_1}\varepsilon_{kq_2 i_1}[{}^B\mathcal{Q}^{\vee<q_1 i_2 i_3\dots i_\ell>}]^*[{}^B\mathcal{Q}^\vee_{<q_2 i_2 i_3\dots i_\ell>}] = 2\ [{}^B\mathcal{Q}^{\vee<i_1 i_2 i_3\dots i_\ell>}]^*[{}^B\mathcal{Q}^\vee_{<i_1 i_2 i_3\dots i_\ell>}] . \tag{C.227}$$

Adding up all the terms yields then a factor of $\ell(\ell-1)+2\ell = \ell(\ell+1)$. Thus, we have proved that

$$\frac{1}{\ell!}[{}^B\mathcal{Q}_{k<i_1 i_2\dots i_\ell>}]^*[{}^B\mathcal{Q}^{k<i_1 i_2\dots i_\ell>}]|_{d=3} = \frac{\ell+1}{\ell}\times\frac{1}{\ell!}[{}^B\mathcal{Q}^{\vee<i_1 i_2 i_3\dots i_\ell>}]^*[{}^B\mathcal{Q}^\vee_{<i_1 i_2 i_3\dots i_\ell>}] . \tag{C.228}$$

This identity can be used to recast the power loss formula in Eq.(C.224) into a duality invariant form:

$$\mathcal{P}(\omega)_{d=3} = \frac{1}{4\pi}\sum_{\ell=1}^{\infty}\frac{\ell+1}{\ell}\frac{\omega^{2\ell+2}}{(2\ell+1)!!} \times \frac{1}{\ell!}\left\{\left[{}^{E}\mathcal{Q}_{<i_1\ldots i_\ell>}\right]^* \ \left[{}^{E}\mathcal{Q}_{<i_1\ldots i_\ell>}\right] + \left[{}^{B}\mathcal{Q}^{\vee<i_1 i_2 i_3\ldots i_\ell>}\right]^*\left[{}^{B}\mathcal{Q}^{\vee}_{<i_1 i_2 i_3\ldots i_\ell>}\right]\right\} . \tag{C.229}$$

The first few terms here correspond to electric/magnetic dipole/quadrupole moments, i.e., if we set

$$ {}^{E}\mathcal{Q}_{i_1} \equiv d^E_{i_1} \ , \quad {}^{E}\mathcal{Q}_{<i_1 i_2>} \equiv q^E_{<i_1 i_2>} \ , \quad {}^{B}\mathcal{Q}^{\vee}_{k i_1} \equiv d^B_{i_1} \ , \quad {}^{B}\mathcal{Q}^{\vee}_{<i_1 i_2>} = q^B_{<i_1 i_2>} \ . \tag{C.230}$$

The power loss formula then takes the form

$$\begin{aligned}\mathcal{P}(\omega)_{d=3} &= \frac{\omega^4}{6\pi}\sum_i |d^E_i|^2 + \frac{\omega^6}{80\pi}\sum_{ij}|q^E_{<ij>}|^2 + \ldots \\ &+ \frac{\omega^4}{6\pi}\sum_i |d^B_i|^2 + \frac{\omega^6}{80\pi}\sum_{ij}|q^B_{<ij>}|^2 + \ldots\end{aligned} \tag{C.231}$$

This agrees with the Larmor formula quoted in standard textbooks.[12]

C.9 Comparison of normalisations

In our discussion of multipole expansions, we have chosen our multipole moment definitions uniformly across statics and radiation, and we have tried to use the simplest normalisations consistent with electric-magnetic duality. This is consistent with modern treatments of gravitational multipole expansions based on STF tensors. Unfortunately, our notations differ from popular textbooks on electromagnetism where *radiative* multipole moments are treated very differently from *static* multipole moments. This fact complicates the comparison of our expression with those available in standard EM textbooks like that of Jackson [114] and Zangwill [178]. Our normalisations also differ slightly from papers on STF multipole expansion [151, 160, 162]. Given this bewildering array of

[12]For example, Jackson defines his cartesian STF electric quadrupole moment to be three times our quadrupole moment, and his definition of power loss in frequency domain is half of ours due to Fourier transform conventions. So, in his textbook, he gives $2 \times 6\pi = 12\pi$ as the denominator for dipole power loss and $2 \times 3^2 \times 80\pi = 1440\pi$ as the denominator for quadrupole power loss.

existent normalisations for EM multipoles, we will conclude this appendix by providing the necessary dictionary for translation to the notations in such texts and papers. There are no new results here, and a reader disinterested in notational fine-print may safely skip what follows.

Our expressions can be converted to static moments appearing in textbooks via

$$\begin{aligned} \bar{\mathcal{J}}^E(\ell,m) &= \frac{1}{2\ell+1} q^{\text{Jackson}}_{\ell m} = \frac{1}{4\pi} A^{\text{Zangwill}}_{\ell m} = \frac{Q^{(e)\text{LBP}}_{\ell m}}{\sqrt{4\pi(2\ell+1)}} \ , \\ \bar{\mathcal{J}}^{\vee B}(\ell,m) &= \ell \frac{\mu^{\text{Jackson}}_{\ell m}}{\sqrt{4\pi(2\ell+1)}} = \frac{1}{2\ell+1} M^{\text{Zangwill}}_{\ell m} = \frac{Q^{(m)\text{LBP}}_{\ell m}}{\sqrt{4\pi(2\ell+1)}} \ . \end{aligned} \tag{C.232}$$

These factors can be figured out by comparing our static moments (i.e., Eqs.(C.90) and (C.191)) against the definitions given in these texts.[13] As indicated in the footnote, Jackson does not define a magnetostatic multipole moment in his text. Instead, in Problem(5.8b) involving axi-symmetric currents of the form $\vec{J} = J_\varphi(r,\theta)\hat{\varphi}$, Jackson defines

$$\mu^{\text{Jackson}}_{\ell,m=0} = -\frac{1}{\ell(\ell+1)} \int_{\mathbb{R}^3} r^\ell P^1_\ell(\cos\vartheta) \bar{J}_\varphi(r,\theta) = \frac{1}{\ell+1}\sqrt{\frac{4\pi}{2\ell+1}} \int_{\mathbb{R}^3} r^{\ell+1} \vec{\mathbb{U}}^*_{\ell,m=0}(\hat{r}) \cdot \bar{J}(\vec{r}) \ , \tag{C.233}$$

where we have used the fact that

$$\vec{r} \times \vec{\nabla} \mathscr{Y}_{\ell,m=0} = \hat{e}_\varphi \sqrt{\frac{2\ell+1}{4\pi}} P^1_\ell(\cos\vartheta) \ . \tag{C.234}$$

We have generalised Jackson's definition to general m to determine the relative normalisation quoted above. With this dictionary, we have checked that the multipole expansions for static fields given in these texts agree with our expressions.

Moving on to radiative multipole moments, the relative normalisations are given by[14]

$$\begin{aligned} \Lambda^{E,\text{Zangwill}}_{\ell m}(\omega) &= \frac{1}{\sqrt{\ell(\ell+1)}} a^{\text{Jackson}}_E(\omega,\ell,m) = -i\frac{\omega^{\ell+2}}{\ell(2\ell-1)!!} \mathcal{J}^E(\omega,\ell,m) \ , \\ \Lambda^{M,\text{Zangwill}}_{\ell m}(\omega) &= \frac{1}{\sqrt{\ell(\ell+1)}} a^{\text{Jackson}}_M(\omega,\ell,m) = i\frac{\omega^{\ell+2}}{\ell(2\ell-1)!!} \mathcal{J}^{\vee B}(\omega,\ell,m) \ . \end{aligned} \tag{C.235}$$

[13]The static moments of Jackson [114] are defined in JEq(4.3) and Problem(5.8b), that of Zangwill [178] are defined in ZEq.(4.87) and ZEq.(11.66), and of Lifshitz-Berestetskii-Pitaevskii [187] are given in LBPEq.(46.7) and LBPEq(47.3).

[14]This follows from comparing Zangwill's ZEq.(20.224),(20.225), as well as Jackson's JEq.(9.167), (9.168) against our Eq.(C.211).

Our normalisations are closer to that of Campbell-Macek-Morgan [175] with

$$Q^{\rm CMM}_{\ell m}(\omega) = (2\ell+1)\mathcal{J}^E(\omega,\ell,m)\ , \qquad M^{\rm CMM}_{\ell m}(\omega) = (2\ell+1)\mathcal{J}^{\vee B}(\omega,\ell,m)\ . \tag{C.236}$$

With this dictionary, we have checked that the multipole expansions for radiative fields as well as power loss given in these texts agree with our expressions.[15]

We now turn to STF multipole moments. In general d, the authors Amalberti-Larrouturou-Yang (ALY) [162] define STF electric and magnetic moments. Their definitions are related to ours by

$$^{E}\mathcal{Q}_{<i_1i_2\ldots i_\ell>} = I^{\rm ALY}_{<i_1i_2\ldots i_\ell>}\ , \quad {}^{B}\mathcal{Q}_{i<i_1i_2\ldots i_\ell>} = \frac{2\ell}{\ell+1} J^{\rm ALY}_{i<i_1i_2\ldots i_\ell>}\ . \tag{C.237}$$

These normalisations can be fixed by comparing the Fourier transform of ALYEq(3.17) against our definitions in Eq.(C.147) and Eq.(C.63). For the vector projector, ALY seem to use Eq.(B.89), but their formula seems to omit the final symmetrisation. The conversion rule quoted above assumes that such a symmetrisation is implicit in their expressions.[16]

The rest of the references dealing with STF moments are specific to $d=3$: Damour-Iyer [160] and Ross [151]. We claim that the relative normalisation factors are

$$\begin{aligned} ^{E}\mathcal{Q}_{<i_1i_2\ldots i_\ell>} &= Q^{\text{Damour-Iyer}}_{<i_1i_2\ldots i_\ell>} = I^{<i_1i_2\ldots i_\ell>}_{\rm Ross}\ , \\ ^{B}\mathcal{Q}^{\vee}_{<i_1i_2\ldots i_\ell>} &= \frac{\ell}{\ell+1} M^{\text{Damour-Iyer}}_{<i_1i_2\ldots i_\ell>} = \frac{\ell}{\ell+1} J^{<i_1i_2\ldots i_\ell>}_{\rm Ross}\ . \end{aligned} \tag{C.238}$$

Here, we have converted the time-domain expressions of [151, 160] into the frequency domain. The Damour-Iyer definitions in DIeq(4.18) of [160] can be Fourier transformed

[15]Note that power loss formulae in Jackson's chapter§9 are half of ours due to differences in Fourier transform conventions.

[16]The reader should note that ALY's anti-symmetric projection involves an additional factor of half compared to our conventions here.

to frequency domain as

$$
\begin{aligned}
Q^{\text{Damour-Iyer}}_{<i_1 i_2 \dots i_\ell>} &= \frac{1}{\ell+1}\int_{\mathbb{R}^3} x^{<i_1}x^{i_2}\dots x^{i_\ell>}\int\limits_{-1}^{1} dz\ \frac{(2\ell+1)!!}{2^{\ell+1}\ell!}(1-z^2)^\ell e^{-i\omega r z} \\
&\quad\times\left[(\ell+1-i\omega r z)J^t(\vec{r},\omega)+i\omega J^r(\vec{r},\omega)\right]\ , \\
M^{\text{Damour-Iyer}}_{<i_1 i_2 \dots i_\ell>} &= \frac{1}{\ell}\int_{\mathbb{R}^3}\int\limits_{-1}^{1} dz\ \frac{(2\ell+1)!!}{2^{\ell+1}\ell!}(1-z^2)^\ell e^{-i\omega r z}\ x^{<i_1}x^{i_2}\dots x^{i_\ell>}(\vec{r}\times\vec{\nabla})\cdot\vec{J}(\vec{r},\omega)\ .
\end{aligned}
\tag{C.239}
$$

To relate it to Eq.(C.215), we perform an integration by parts:

$$
\begin{aligned}
Q^{\text{Damour-Iyer}}_{<i_1 i_2 \dots i_\ell>} &= \frac{1}{\ell+1}\int_{\mathbb{R}^3} x^{<i_1}x^{i_2}\dots x^{i_\ell>}\int\limits_{-1}^{1} dz\ \frac{(2\ell+1)!!}{2^{\ell+1}\ell!}(1-z^2)^\ell e^{-i\omega r z} \\
&\quad\times\left\{i\omega r J^r(\vec{r},\omega)-\frac{1}{r}\partial_r[r^2 J^t(\vec{r},\omega)]\right\}\ , \\
M^{\text{Damour-Iyer}}_{<i_1 i_2 \dots i_\ell>} &= \int_{\mathbb{R}^3}\int\limits_{-1}^{1} dz\ \frac{(2\ell+1)!!}{2^{\ell+1}\ell!}(1-z^2)^\ell e^{-i\omega r z}\ [\vec{r}\times\vec{J}(\vec{r},\omega)]^{<i_1}x^{i_2}\dots x^{i_\ell>}\ .
\end{aligned}
\tag{C.240}
$$

Invoking Eq.(C.212), we then obtain the normalisations claimed above.

As for Ross [151], his magnetic moment expressions are directly of the form Eq.(C.212). After converting to frequency domain, his electric moment definition is

$$
\begin{aligned}
I(\omega)^{<i_1 i_2 \dots i_\ell>}_{\text{Ross}} &= \int_{\mathbb{R}^3}\Bigg\{{}_1F_2\left[\frac{\ell}{2}\ ;\frac{\ell}{2}+1\ ,\ell+\frac{3}{2}\ ;-\frac{\omega^2 r^2}{4}\right]J^t(\vec{r},\omega)\ x^{<i_1}x^{i_2}\dots x^{i_\ell>} \\
&- i\omega\frac{\ell}{(\ell+1)(\ell+2)}{}_1F_2\left[\frac{\ell}{2}+1\ ;\frac{\ell}{2}+2\ ,\ell+\frac{3}{2}\ ;-\frac{\omega^2 r^2}{4}\right] \\
&\quad\times\left\{r^2 J^{<i_\ell}(\vec{r},\omega)\ x^{<i_1}x^{i_2}\dots x^{i_{\ell-1}>}-x_k J^k(\vec{r},\omega)\ x^{<i_1}x^{i_2}\dots x^{i_\ell>}\right\}\Bigg\}\ .
\end{aligned}
\tag{C.241}
$$

From charge conservation, we have the following identity:

$$
\begin{aligned}
&-i\omega\int_{\mathbb{R}^3} J^0(\vec{r},\omega)\ r^{2p}x^{<i_1}x^{i_2}\dots x^{i_\ell>} = -\int_{\mathbb{R}^3}\partial_k J^k(\vec{r},\omega)\ r^{2p}x^{<i_1}x^{i_2}\dots x^{i_\ell>} \\
&= \int_{\mathbb{R}^3} r^{2p-2}\left[\ell\ J^{<i_\ell}(\vec{r},\omega)\ x^{<i_1}x^{i_2}\dots x^{i_{\ell-1}>}+2p\ x_k J^k(\vec{r},\omega)\ x^{<i_1}x^{i_2}\dots x^{i_\ell>}\right]
\end{aligned}
\tag{C.242}
$$

We use this to write

$$
\begin{aligned}
I(\omega)^{<i_1 i_2 \ldots i_\ell>}_{\text{Ross}} = \int_{\mathbb{R}^3} \Bigg\{ & {}_1F_2\left[\frac{\ell}{2} \, ; \frac{\ell}{2}+1 \, , \ell+\frac{3}{2} \, ; -\frac{\omega^2 r^2}{4}\right] J^t(\vec{r},\omega) \; x^{<i_1} x^{i_2} \ldots x^{i_\ell>} \\
- \frac{\omega^2 r^2}{(\ell+1)(\ell+2)} & {}_1F_2\left[\frac{\ell}{2}+1 \, ; \frac{\ell}{2}+2 \, , \ell+\frac{3}{2} \, ; -\frac{\omega^2 r^2}{4}\right] J^t(\vec{r},\omega) \; x^{<i_1} x^{i_2} \ldots x^{i_\ell>} \\
- \frac{1}{(\ell+1)} & \Big\{ {}_1F_2\left[\frac{\ell}{2}+1 \, ; \frac{\ell}{2}+2 \, , \ell+\frac{3}{2} \, ; -\frac{\omega^2 r^2}{4}\right] \\
& - \frac{\omega^2 r^2}{(\ell+4)(2\ell+3)} {}_1F_2\left[\frac{\ell}{2}+2 \, ; \frac{\ell}{2}+3 \, , \ell+\frac{5}{2} \, ; -\frac{\omega^2 r^2}{4}\right] \Big\} x_k J^k(\vec{r},\omega) \; x^{<i_1} x^{i_2} \ldots x^{i_\ell>} \Bigg\} .
\end{aligned}
\tag{C.243}
$$

The above expression can then be further simplified to

$$
\begin{aligned}
I(\omega)^{<i_1 i_2 \ldots i_\ell>}_{\text{Ross}} = & \frac{1}{\ell+1} \int_{\mathbb{R}^3} \Bigg\{ \frac{1}{r^\ell} \partial_r \left(r^{\ell+1} {}_0F_1\left[\ell+\frac{3}{2}, -\frac{\omega^2 r^2}{4}\right] \right) J^t(\vec{r},\omega) \; x^{<i_1} x^{i_2} \ldots x^{i_\ell>} \\
& + i\omega {}_0F_1\left[\ell+\frac{3}{2}, -\frac{\omega^2 r^2}{4}\right] x_k J^k(\vec{r},\omega) \; x^{<i_1} x^{i_2} \ldots x^{i_\ell>} \Bigg\} .
\end{aligned}
\tag{C.244}
$$

After an integration by parts, this matches with our definition. We have also checked that the power loss formula and the radiative fields given by Ross agrees with our expressions.

Dimensional analysis

This section provides a list of scaling dimensions of various physical quantities defined in our analysis for the reader's convenience. This allows for easy checks on the dimensional compatibility of all the equations. We work in units where the speed of light(c), the permittivity of vacuum(ϵ_0) and the reduced Planck's constant($\hbar$) are set to one. This leaves us a single scaling dimension, which we pick to be mass $[M]$.

For our de Sitter analysis, we will also work with the Hubble's constant, H, set to 1. To check the dimensional consistency of equations in the following sections, one should restore the H's, after which they can be checked against the above table.

Quantity	Quantity	Mass Dimension	Quantity	Mass Dimension
Debye potential	$[\Phi_E]$	$-\frac{d-3}{2}-1$	$[\Phi_B]$	$\frac{d-3}{2}-1$
EM field strength	$[E_r]$	$\frac{d-3}{2}+1$	$[H_{vv}]$	$\frac{d-3}{2}-1$
	$[E_s]$	$\frac{d-3}{2}$	$[H_s]$	$\frac{d-3}{2}$
	$[E_v]$	$\frac{d-3}{2}$	$[H_v]$	$\frac{d-3}{2}$
Gauge potential	$[\mathcal{V}_t]$	$\frac{d-3}{2}$	$[\mathcal{V}_r]$	$\frac{d-3}{2}$
			$[\mathcal{V}_I]$	$\frac{d-3}{2}-1$
Charge/current density	$[J^t]$	$\frac{d-3}{2}+2$	$[J^r]$	$\frac{d-3}{2}+2$
			$\left[J^I\right]$	$\frac{d-3}{2}+3$
Spherical multipole moments	$[\mathcal{J}_E]$	$-\frac{d-3}{2}-(\ell+1)$	$[\mathcal{J}_B]$	$-\frac{d-3}{2}-(\ell+1)$
STF multipole moments	$\left[{}^E\mathcal{Q}_{<i_1 i_2 \ldots i_\ell>}\right]$	$-\frac{d-3}{2}-(\ell+1)$	$\left[{}^B\mathcal{Q}_{k<i_1 i_2 \ldots i_\ell>}\right]$	$-\frac{d-3}{2}-(\ell+1)$
Outgoing waves	$\left[G_E^{\text{Out}}\right]$	ℓ	$\left[G_B^{\text{Out}}\right]$	$(d-3)+\ell$
EM Green fns.	$[\mathbb{G}_E]$	$-(d-3)-1$	$[\mathbb{G}_B]$	$(d-3)-1$

Table C.2: Mass dimensions of various quantities in the frequency domain. The electric quantities appear in the second and the third column, whereas the magnetic counterparts appear in the third and the fourth. The time domain versions are denoted by a tilde over the symbol, and their mass dimensions are 1 more than the dimensions quoted above.

Appendix D

Summary of the formulae for dS_4

In this appendix, we will summarise our results for both scalars and electromagnetic fields for the most relevant case of $d = 3$. This should facilitate comparison with existing literature in dS_4 for the readers. We will section them into designer scalar formulae and electromagnetism formulae.

D.1 Designer Scalar in dS_4

The parameters used in the scalar analysis, $\{\mathcal{N}, \nu, \mu\}$, have explicit dimensional dependence. Amongst these, the parameter ν carries the information of the mode's angular momentum: $\nu = \ell + \frac{d}{2} - 1$ which for the case of dS_4 becomes $\nu = \ell + \frac{1}{2}$. The other two parameters depend on the dimension according to the class of fields they encode. For the light fields, the parameters take the values as given in table D.1.

Table D.1: $\mathcal{N}, \mu$ values for different massless fields: massless Klein-Gordon scalar, electric and magnetic parity sectors of electromagnetism and linearised gravity.

	KG Sca	EM Mag	EM Elec	Grav Mag	Grav Elec
$\mathcal{N}$	2	0	0	-2	0
μ	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Since the equation of motion for these scalars is even in μ(2.5), we can explicitly see that both electric and magnetic modes have the same equation of motion in $d = 3$. The electric and magnetic parity gravitational perturbations(there is no extra tensor sector in

$d = 3$) also have the same μ dependence and only differ in the value of $\mathcal{N}$. This implies that the functional behaviour of their solutions is the same as that of the electromagnetic fields, only differing in their overall r powers. In fact, both electromagnetic and linearised gravity fields map to the conformally coupled scalar in $d = 3$ (this was also pointed out by [7]).

Let's begin with the retarded boundary to bulk(i.e. $r = 0$ to bulk) Green's function that satisfies the outgoing boundary conditions at the horizon:

$$\begin{aligned} G^{\text{Out}}_{\mathcal{N}} &= r^{-\ell-\frac{\mathcal{N}}{2}}(1+r)^{-i\omega} \\ &\times \left\{ {}_2F_1\left[\frac{\frac{1}{2}-\ell+\mu-i\omega}{2}, \frac{\frac{1}{2}-\ell-\mu-i\omega}{2}; \frac{1}{2}-\ell; r^2\right] \right. \\ &\left. -K_{\text{Out}}\frac{r^{2\ell+1}}{2\ell+1}\, {}_2F_1\left[\frac{\ell+\frac{3}{2}-\mu-i\omega}{2}, \frac{\ell+\frac{3}{2}+\mu-i\omega}{2}; \ell+\frac{3}{2}; r^2\right] \right\} . \end{aligned} \tag{D.1}$$

where K_{Out}, which encodes the radiation reaction kernel, is given by:

$$K_{\text{Out}} = (-1)^{\ell}\frac{2\pi}{\Gamma(\ell+\frac{1}{2})^2}\frac{\Gamma\left(\frac{\ell+\frac{3}{2}-\mu-i\omega}{2}\right)\Gamma\left(\frac{\ell+\frac{3}{2}+\mu-i\omega}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}-\ell+\mu-i\omega}{2}\right)\Gamma\left(\frac{\frac{1}{2}-\ell-\mu-i\omega}{2}\right)} . \tag{D.2}$$

Given these formulae, we can define the 'Schwarzschild time' boundary to bulk Green's functions using (2.90):

$$\begin{aligned} \Xi_n &= \frac{1}{2\ell+1}r^{\ell+1-\frac{\mathcal{N}}{2}}(1-r^2)^{-\frac{i\omega}{2}}\, {}_2F_1\left[\frac{\ell+\frac{3}{2}-\mu-i\omega}{2}, \frac{\ell+\frac{3}{2}+\mu-i\omega}{2}; \ell+\frac{3}{2}; r^2\right] , \\ \Xi_{nn} &= r^{-\ell-\frac{\mathcal{N}}{2}}(1-r^2)^{-\frac{i\omega}{2}}\, {}_2F_1\left[\frac{\frac{1}{2}-\ell+\mu-i\omega}{2}, \frac{\frac{1}{2}-\ell-\mu-i\omega}{2}; \frac{1}{2}-\ell; r^2\right] . \end{aligned} \tag{D.3}$$

Using these and (2.103), one can derive the multipole moments for localised sources on the dS-SK contour explicitly.

Finally, we will quote the full covariant ALD force for the massless Klein-Gordon scalar:

$$F^{\mu}_{\text{RR}} = \frac{1}{4\pi}\left[\frac{P^{\mu\nu}}{3!!}\left\{a^{(1)}_{\nu}\right\} - H^2\left\{v^{\mu}\right\}\right] , \tag{D.4}$$

where $P^{\mu\nu}$ is the projector perpendicular to the 4 velocity, $a^{(1)}_{\nu}$ is the derivative of the 4-acceleration and v^{μ} is the 4-velocity.

D.2 Electromagnetism in dS$_4$

In appendix C.8, we provide an analysis for 3+1-dimensional flat space electromagnetism. Much of the analysis is similar for the spherical harmonics decomposition, but the bulk-to-boundary propagators get modified for de Sitter.

The electric and magnetic sector boundary to bulk retarded Green's functions in 4-dimensional de Sitter are:

$$\begin{aligned}
G_E^{\text{Out}} &= r^{-\ell}(1+r)^{-i\omega} \\
&\quad \times \Big\{ {}_2F_1\left[\frac{-\ell-i\omega}{2}, \frac{1-\ell-i\omega}{2}; \frac{1}{2}-\ell; r^2\right] \\
&\qquad -K_E^{\text{Out}}(\omega,\nu)\frac{r^{2\ell+1}}{2\ell+1}\, {}_2F_1\left[\frac{2+\ell-i\omega}{2}, \frac{1+\ell-i\omega}{2}; \frac{3}{2}+\ell; r^2\right]\Big\} , \\
G_B^{\text{Out}} &= r^{-\ell}(1+r)^{-i\omega} \\
&\quad \times \Big\{ {}_2F_1\left[\frac{1-\ell-i\omega}{2}, \frac{-\ell-i\omega}{2}; \frac{1}{2}-\ell; r^2\right] \\
&\qquad -K_B^{\text{Out}}(\omega,\nu)\frac{r^{2\ell+1}}{2\ell+1}\, {}_2F_1\left[\frac{1+\ell-i\omega}{2}, \frac{2+\ell-i\omega}{2}; \frac{3}{2}+\ell; r^2\right]\Big\} .
\end{aligned} \tag{D.5}$$

where we have defined the corresponding 3+1-dimensional radiation reaction kernels as follows:

$$\begin{aligned}
K_E^{\text{Out}} &= (-1)^\ell \frac{2\pi}{\Gamma(\ell+\frac{1}{2})^2} \frac{\Gamma\left(\frac{\ell+2-i\omega}{2}\right)\Gamma\left(\frac{\ell+1-i\omega}{2}\right)}{\Gamma\left(\frac{1-\ell-i\omega}{2}\right)\Gamma\left(\frac{-\ell-i\omega}{2}\right)} , \\
K_B^{\text{Out}} &= (-1)^\ell \frac{2\pi}{\Gamma(\ell+\frac{1}{2})^2} \frac{\Gamma\left(\frac{\ell+1-i\omega}{2}\right)\Gamma\left(\frac{\ell+2-i\omega}{2}\right)}{\Gamma\left(\frac{-\ell-i\omega}{2}\right)\Gamma\left(\frac{1-\ell-i\omega}{2}\right)} .
\end{aligned} \tag{D.6}$$

Both of these are equal as a result of electric-magnetic duality in 4 spacetime dimensions.

The 'Schwarzschild time' normalisable modes required to define the electromagnetic radiative multipole moments are:

$$\begin{aligned}
\Xi_n^B(r,\omega,\ell) &= \frac{r^{\ell+1}}{2\ell+1}(1-r^2)^{-\frac{i\omega}{2}}\, {}_2F_1\left[\frac{\ell+1-i\omega}{2}, \frac{\ell+2-i\omega}{2}; \ell+\frac{3}{2}; r^2\right] , \\
\Xi_n^E(r,\omega,\ell) &= \frac{r^{\ell+1}}{2\ell+1}(1-r^2)^{-\frac{i\omega}{2}}\, {}_2F_1\left[\frac{\ell+2-i\omega}{2}, \frac{\ell+1-i\omega}{2}; \ell+\frac{3}{2}; r^2\right] .
\end{aligned} \tag{D.7}$$

These functions are again equal as a result of the electric-magnetic duality. Given these equations, one can obtain the corresponding dS-SK multipole moments $\mathcal{J}_E$ and $\mathcal{J}_B$ using equations (3.82) and (3.83).

Finally, we will quote the covariant Abraham-Lorentz-Dirac force in de Sitter:

$$F^{\mu}_{RR} = \frac{1}{4\pi}\frac{P^{\mu\nu}}{3!!}\Big\{ -2a^{(1)}_{\nu} \Big\} \ . \tag{D.8}$$

This takes the same form as in flat space, with the crucial difference being that the expressions are de Sitter covariant four vectors instead of Poincare covariant four vectors. This concludes our list of formulae specific to dS_4 that can be compared easily with other literature. In particular, [188] calculates the electromagnetic self-force in de Sitter for 4 spacetime dimensions.

Appendix E

Miscellany

This appendix contains some computations that are useful to the results presented in the thesis, but are mentioned here so as to make the main sections of the thesis easier to read.

E.1 dS-Bessel Polynomials

In this section, we will generalise the flat space reverse Bessel polynomials [103, 18.34], obtained in the study of outgoing radiation in $3+1$ dimensions, to a generic class of scalar fields in all even-dimensional de Sitter spacetimes. In chapter §2, we introduced a system of 'designer scalars' that are governed by the action:

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\; r^{\mathcal{N}+1-d}\Big\{(\partial\Phi_{\mathcal{N}})^2 + \frac{\Phi_{\mathcal{N}}^2}{4r^2}\left[(d+\mathcal{N}-3)(d-\mathcal{N}-1) \right. \\ \left. -r^2\left(4\mu^2-(\mathcal{N}+1)^2\right)\right]\Big\}\,. \tag{E.1}$$

where the $\mathcal{N}$ and μ parametrise the various scalar fields. The centrifugal and mass terms are chosen in the action such that for appropriate values of $\mathcal{N}$ and μ, one can obtain various scalar fields that are obtained in the study of massive Klein-Gordon fields, electromagnetism, and linearised gravity.

In even dimensional de Sitter spacetimes, for massless fields, i.e., when $4\mu^2 = (\mathcal{N}+1)^2$, one finds that the outgoing boundary to bulk propagator $G_{\mathcal{N}}^{\text{Out}}$ can be written in polynomials in ωr and rH. These polynomials are the de Sitter analogues of the reverse Bessel polynomials generalised to $d+1$ dimensions and reproduce them in the zero curvature limit. To make this explicit, we will write the designer scalar EOM as follows:

$$\frac{1}{r^{1-2\nu}}D_+\left(r^{1-2\nu}D_+\psi_{\mathcal{N}}\right)+\omega^2\psi_{\mathcal{N}}+H^2(1-r^2H^2)\left[\mu^2-(\nu-1)^2\right]\psi_{\mathcal{N}}=0\ . \tag{E.2}$$

where we have scaled the designer scalar with a power of r for convenience:

$$\Phi_{\mathcal{N}}(r,\omega,\ell)=r^{\frac{1-\mathcal{N}}{2}-\nu}\psi_{\mathcal{N}}(r,\omega,\ell)\ . \tag{E.3}$$

We remind the reader that $\nu=\ell+\frac{d}{2}-1$. One can write the solution for this equation in the following form:

$$\psi_{\mathcal{N}}=\sum_{n=0}^{\infty}(r^2H^2)^n\frac{\left(\frac{1-\mu-\nu}{2}\right)_n\left(\frac{1+\mu-\nu}{2}\right)_n}{\Gamma(n+1)\left(1-\nu\right)_n}\Theta_{\nu-\frac{1}{2}-n}(z)\ . \tag{E.4}$$

Here,

$$(a)_n=\prod_{k=0}^{k=n-1}(a-k)\ , \tag{E.5}$$

is the falling factorial and $\Theta_{\nu-\frac{1}{2}}$ satisfies the μ independent equation of motion:

$$\frac{1}{r^{1-2\nu}}D_+\left(r^{1-2\nu}D_+\Theta_{\nu-\frac{1}{2}}\right)+\omega^2\Theta_{\nu-\frac{1}{2}}=0\ . \tag{E.6}$$

Although this differential equation can be solved in terms of Hypegeometric functions, that form is not particularly illuminating to extract out the polynomial nature of the propagator. Instead, we will express the solutions in a Hubble expansion, which makes the polynomial nature explicit.

Table E.1: $\Theta_{\nu-\frac{1}{2}}$ for various values of ν $(H = 1)$

ν	$\Theta_{\nu-\frac{1}{2}}$
$\frac{1}{2}$	1
$\frac{3}{2}$	$1+z$
$\frac{5}{2}$	$3+3z+z^2+r^2z$
$\frac{7}{2}$	$15+15z+6z^2+z^3+r^2z\,(5+3z)+3r^4z$
$\frac{9}{2}$	$105+105z+45z^2+10z^3+z^4+r^2z\,(35+26z+6z^2)+r^4z\,(21+15z)+15r^6z$
$\frac{11}{2}$	$945+945z+420z^2+105z^3+15z^4+z^5+r^2z\,(315+255z+80z^2+10z^3)$ $+r^4z\,(189+170z+45z^2)+r^6z\,(135+105z)+105r^8z$
$\frac{13}{2}$	$10395+10395z+4725z^2+1260z^3+210z^4+21z^5+z^6$ $+r^2z\,(3465+2940z+1050z^2+190z^3+15z^4)$ $+r^4z\,(2079+2059z+750z^2+105z^3)+r^6z\,(1485+1470z+420z^2)$ $+r^8z\,(1155+945z)+945r^{10}z$

Taking $z = -i\omega r$, we can write the Hubble expanded solutions as:

$$\begin{aligned}
\Theta_\ell(z, Hr) &= \sum_{k=0}^{\ell} \frac{z^{\ell-k}}{2^k k!}\frac{(\ell+k)!}{(\ell-k)!} \\
&+ \frac{H^2r^2}{2!}\sum_{k=0}^{\ell-1}(\ell-1-k)\frac{z^{\ell-1-k}}{2^k k!}\frac{(\ell-1+k)!}{(\ell-1-k)!}\left\{\ell+\frac{1}{3}k\right\} \\
&+ \frac{H^4r^4}{4!}\sum_{k=0}^{\ell-2}(\ell-2-k)\frac{z^{\ell-2-k}}{2^k k!}\frac{(\ell-2+k)!}{(\ell-2-k)!} \\
&\quad\times\left\{\ell(\ell-1)(3\ell+3-k)-\frac{1}{15}k(k-1)(25\ell+5k-3)\right\} \\
&+ \frac{H^6r^6}{6!}\sum_{k=0}^{\ell-3}(\ell-3-k)\frac{z^{\ell-3-k}}{2^k k!}\frac{(\ell-3+k)!}{(\ell-3-k)!} \\
&\quad\times\Big\{15\ell(\ell^2-1)(\ell^2-4)+k\ell(\ell-1)[(42+25\ell-15\ell^2)-(12+10\ell)k] \\
&\qquad+\frac{1}{63}k(k-1)(k-2)[-2-441\ell+350\ell^2+7(-9+35\ell)k+35k^2]\Big\} \\
&+ O(H^8)\ .
\end{aligned} \tag{E.7}$$

We give explicit expressions for the retarded boundary to bulk propagators in tables E.2, E.3, E.4 and E.5.

Table E.2: $r^{\nu+\frac{d-4}{2}}G^{\text{Out}}$ for magnetic Debye potentials ($z=-i\omega r$, $H=1$).

$\mu=\frac{d}{2}-1$	$\ell=0$	$\ell=1$
$d=3$	1	$1+z$
$d=5$	$1+z+r^2$	$1+\frac{z^2}{3}+r^2+\frac{r^2z}{3}$
$d=7$	$1+z+\frac{z^2}{3}+r^2\left(z+\frac{2}{3}\right)+r^4$	$1+z+\frac{z^2}{15}(z+6)+\frac{r^2z}{15}(3z+5)+\frac{r^4z}{5}$

Table E.3: $r^{\nu+\frac{d-4}{2}}G^{\text{Out}}$ for magnetic Debye potentials ($z=-i\omega r$, $H=1$)

$\mu=\frac{d}{2}-1$	$\ell=2$
$d=3$	$1+z+\frac{z^2}{3}-\frac{r^2}{3}$
$d=5$	$1+z+\frac{2z^2}{5}+\frac{z^3}{15}-\frac{r^2}{15}\left(6+z-z^2\right)-\frac{r^4}{15}$
$d=7$	$1+z+\frac{3z^2}{7}+\frac{2z^3}{21}+\frac{z^4}{105}-\frac{r^2}{105}\left(45+10z+8z^2+3z^3\right)-\frac{r^4}{35}\left(3+z-z^2\right)-\frac{r^6}{35}$

Table E.4: $r^{\nu+\frac{2-d}{2}}G^{\text{Out}}$ for electric Debye potentials ($z=-i\omega r$, $H=1$).

$\mu=\frac{d}{2}-2$	$\ell=0$	$\ell=1$
$d=3$	1	$1+z$
$d=5$	$1+z$	$1+z+\frac{z^2}{3}-\frac{r^2}{3}$
$d=7$	$1+z+\frac{z^2}{3}+\frac{r^2z}{3}$	$1+z+\frac{z^2}{5}+\frac{2z^3}{15}-\frac{r^2}{15}(6+z-z^2)-\frac{r^4}{15}$

Table E.5: $r^{\nu+\frac{2-d}{2}}G^{\text{Out}}$ for electric Debye potentials ($z=-i\omega r$, $H=1$).

$\mu=\frac{d}{2}-2$	$\ell=2$
$d=3$	$1+z+\frac{z^2}{3}-\frac{r^2}{3}$
$d=5$	$1+z+\frac{2z^2}{5}+\frac{z^3}{15}-r^2(\frac{3}{5}+\frac{4z}{15})$
$d=7$	$1+z+\frac{3z^2}{7}+\frac{2z^3}{21}+\frac{z^4}{105}-\frac{r^2}{105}(75+40z+4z^2-z^3)-\frac{4r^4z}{105}$

E.2 Detweiler-Whiting decomposition of the retarded scalar Green function

We will begin by writing down the non-normalisable solution Ξ_{nn} and the normalisable solution Ξ_n in the Schwarzschild time valid for odd d (restoring all factors of H):

$$\begin{aligned}\Xi_{nn} &\equiv r^{-\nu-\frac{d}{2}+\frac{1}{2}(d-1-\mathcal{N})}(1-H^2r^2)^{-\frac{i\omega}{2H}}\\ &\quad\times {}_2F_1\left[\frac{1}{2}\left(1+\mu-\nu-\frac{i\omega}{H}\right),\frac{1}{2}\left(1-\mu-\nu-\frac{i\omega}{H}\right),1-\nu,H^2r^2\right]\ ,\\ \Xi_n &\equiv \frac{1}{2\nu}r^{\nu-\frac{d}{2}+1+\frac{1}{2}(d-1-\mathcal{N})}(1-H^2r^2)^{-\frac{i\omega}{2H}}\\ &\quad\times {}_2F_1\left[\frac{1}{2}\left(1+\mu+\nu-\frac{i\omega}{H}\right),\frac{1}{2}\left(1-\mu+\nu-\frac{i\omega}{H}\right),1+\nu,H^2r^2\right]\ ,\\ K_{\text{Out}} &\equiv 2\frac{\Gamma\left(\frac{1+\nu-\mu-\frac{i\omega}{H}}{2}\right)\Gamma\left(\frac{1+\nu+\mu-\frac{i\omega}{H}}{2}\right)\Gamma\left(1-\nu\right)}{\Gamma\left(\frac{1-\nu+\mu-\frac{i\omega}{H}}{2}\right)\Gamma\left(\frac{1-\nu-\mu-\frac{i\omega}{H}}{2}\right)\Gamma\left(\nu\right)}\ .\end{aligned} \tag{E.8}$$

We have also quoted above the retarded two-point function on the world line. The outgoing Green function can then be decomposed into $K_{\text{Out}}\ \Xi_n$ and Ξ_{nn}: we will now argue that this should be thought of as the regular/singular Green functions ala Deitweiler-Whiting(DW) [96] corresponding to dS spacetime.

The relation to DW decomposition is not prima facie clear, since DW formulated their rules for general curved spacetimes in the time domain, whereas the above expressions are quoted in the frequency domain. So, to substantiate our assertion, we need to Fourier transform the complicated expressions above into the time domain, and then show that the DW axioms are satisfied. Rather than do that exercise in general, we will content ourselves with showing how this works in the particular example of a massless scalar field in dS_4, whose DW decomposition is described in [97, 98].

The regular term for DW decomposition, in this case, was calculated by the authors of [97] in FLRW-like coordinates as

$$\begin{aligned}G_R = &\frac{\eta\eta'}{2|\mathbf{x}-\mathbf{x}'|}\left[\delta(\eta-\eta'-|\mathbf{x}-\mathbf{x}'|)-\delta(\eta-\eta'+|\mathbf{x}-\mathbf{x}'|)\right]\\ &+\frac{1}{2}\left[\theta(\eta-\eta'-|\mathbf{x}-\mathbf{x}'|)+\theta(\eta-\eta'+|\mathbf{x}-\mathbf{x}'|)\right]\ ,\end{aligned} \tag{E.9}$$

To check this expression against $K_{\text{Out}}\Xi_n$, we will convert it into static coordinates and then Fourier transform the result to the frequency domain.

The coordinate transformation between static and FLRW coordinates is given by

$$\eta = -\frac{e^{-Ht}}{\sqrt{1-r^2H^2}}\ , \quad \rho = \frac{re^{-Ht}}{\sqrt{1-r^2H^2}}\ . \tag{E.10}$$

We will assume the source to be at the origin $\rho' = 0$, so that only the $\ell = 0$ term survives by spherical symmetry. With this choice, G_R becomes

$$\begin{aligned} G_R = & \frac{e^{H(t-t')}}{2r} \\ & \times \left[\sqrt{\frac{1-Hr}{1+Hr}}\delta\left(t'-t-\frac{1}{H}\ln\left(\sqrt{\frac{1-Hr}{1+Hr}}\right)\right)\right. \\ & \left. -\sqrt{\frac{1+Hr}{1-Hr}}\delta\left(t'-t-\frac{1}{H}\ln\left(\sqrt{\frac{1+Hr}{1-Hr}}\right)\right)\right] \\ & + \frac{1}{2}\left[\theta\left(t'-t-\frac{1}{H}\ln\left(\sqrt{\frac{1-Hr}{1+Hr}}\right)\right)\right. \\ & \left. +\theta\left(t'-t-\frac{1}{H}\ln\left(\sqrt{\frac{1+Hr}{1-Hr}}\right)\right)\right]\ . \end{aligned} \tag{E.11}$$

This expression can be readily Fourier transformed with respect to $t - t'$ yielding

$$\widetilde{G}_R = \frac{1}{2r}\left[\left(\frac{1-Hr}{1+Hr}\right)^{-\frac{i\omega}{2H}} - \left(\frac{1+Hr}{1-Hr}\right)^{-\frac{i\omega}{2H}}\right] - \frac{H^2}{2i\omega}\left[\left(\frac{1-Hr}{1+Hr}\right)^{-\frac{i\omega}{2H}} + \left(\frac{1+Hr}{1-Hr}\right)^{-\frac{i\omega}{2H}}\right]\ . \tag{E.12}$$

Regularity near the origin is manifest in the frequency domain. Further, the above expression is also an odd function of the frequency ω, signalling that these terms encode the dissipation due to radiation reaction.

Similarly, we can consider the singular Green's function quoted in [97]:

$$\begin{aligned} G_S = & \frac{\eta\eta'}{2|\mathbf{x}-\mathbf{x}'|}\left[\delta(\eta-\eta'-|\mathbf{x}-\mathbf{x}'|)+\delta(\eta-\eta'+|\mathbf{x}-\mathbf{x}'|)\right] \\ & +\frac{1}{2}\left[\theta(\eta-\eta'-|\mathbf{x}-\mathbf{x}'|)-\theta(\eta-\eta'+|\mathbf{x}-\mathbf{x}'|)\right] \end{aligned} \tag{E.13}$$

whose Fourier transform at $\rho' = 0$ is

$$\tilde{G}_S = \frac{1}{2r}\left[\left(\frac{1-Hr}{1+Hr}\right)^{-\frac{i\omega}{2H}} + \left(\frac{1+Hr}{1-Hr}\right)^{-\frac{i\omega}{2H}}\right] - \frac{H^2}{2i\omega}\left[\left(\frac{1-Hr}{1+Hr}\right)^{-\frac{i\omega}{2H}} - \left(\frac{1+Hr}{1-Hr}\right)^{-\frac{i\omega}{2H}}\right] . \tag{E.14}$$

This expression has a $\sim \frac{1}{r}$ behaviour near the origin and is an even function of ω. The expressions in Eq.(E.12) and Eq.(E.14) can then be matched against $K_{\text{Out}}\ \Xi_n$ and Ξ_{nn} respectively. This is done by taking Eq.(E.8), setting $\mathcal{N} = d-1, \mu = \frac{d}{2}, \nu = \ell + \frac{d}{2} - 1$, and then taking the limit $d = 3$ and $\ell = 0$.

E.3 dS ALD forces in 10 and 12 spacetime dimensions

The scalar ALD forces in higher dimensions can be obtained using (2.139) and the following formulae:

$$\begin{aligned}
{}^0f_9^\mu \equiv \frac{P^{\mu\nu}}{9!!}\Bigg\{ & -a_\nu^{(7)} + 30\,(a\cdot a)\,a_\nu^{(5)} + 210\,(a\cdot a^{(1)})\,a_\nu^{(4)} + 378\,(a\cdot a^{(2)})\,a_\nu^{(3)} \\
& + 420\,(a\cdot a^{(3)})\,a_\nu^{(2)} + 300\,(a\cdot a^{(4)})\,a_\nu^{(1)} + 108\,(a\cdot a^{(5)})\,a_\nu + 336\,(a^{(1)}\cdot a^{(1)})\,a_\nu^{(3)} \\
& + 1050\,(a^{(1)}\cdot a^{(2)})\,a_\nu^{(2)} + 960\,(a^{(1)}\cdot a^{(3)})\,a_\nu^{(1)} + 420\,(a^{(1)}\cdot a^{(4)})\,a_\nu \\
& + 675\,(a^{(2)}\cdot a^{(2)})\,a_\nu^{(1)} + 756\,(a^{(2)}\cdot a^{(3)})\,a_\nu + O(a^5)\Bigg\} \\
& - H^2\frac{P^{\mu\nu}}{9!!}\Bigg\{a_\nu^{(5)} + 97\,(a\cdot a)\,a_\nu^{(3)} + 433\,(a\cdot a^{(1)})\,a_\nu^{(2)} + 408\,(a\cdot a^{(2)})\,a_\nu^{(1)} \\
& + 199\,(a\cdot a^{(3)})\,a_\nu + 339\,(a^{(1)}\cdot a^{(1)})\,a_\nu^{(1)} + 448\,(a^{(1)}\cdot a^{(2)})\,a_\nu + O(a^5)\Bigg\} \\
& + H^4\frac{P^{\mu\nu}}{9!!}\Bigg\{-a_\nu^{(3)} + 157\,(a\cdot a)\,a_\nu^{(1)} + 296\,(a\cdot a^{(1)})\,a_\nu\Bigg\} + O(H^6)\ ,
\end{aligned} \tag{E.15}$$

$$
\begin{aligned}
{}^{0}f^{\mu}_{11} \equiv \frac{P^{\mu\nu}}{11!!}\Big\{ & -a^{(9)}_{\nu} + 55\,(a\cdot a)\,a^{(7)}_{\nu} + 495\,(a\cdot a^{(1)})\,a^{(6)}_{\nu} + 1188\,(a\cdot a^{(2)})\,a^{(5)}_{\nu} \\
& + 1848\,(a\cdot a^{(3)})\,a^{(4)}_{\nu} + 1980\,(a\cdot a^{(4)})\,a^{(3)}_{\nu} + 1485\,(a\cdot a^{(5)})\,a^{(2)}_{\nu} \\
& + 770\,(a\cdot a^{(6)})\,a^{(1)}_{\nu} + 220\,(a\cdot a^{(7)})\,a_{\nu} + 1056\,(a^{(1)}\cdot a^{(1)})\,a^{(5)}_{\nu} \\
& + 4620\,(a^{(1)}\cdot a^{(2)})\,a^{(4)}_{\nu} + 6336\,(a^{(1)}\cdot a^{(3)})\,a^{(3)}_{\nu} + 5775\,(a^{(1)}\cdot a^{(4)})\,a^{(2)}_{\nu} \\
& + 3520\,(a^{(1)}\cdot a^{(5)})\,a^{(1)}_{\nu} + 1155\,(a^{(1)}\cdot a^{(6)})\,a_{\nu} + 4455\,(a^{(2)}\cdot a^{(2)})\,a^{(3)}_{\nu} \\
& + 10395\,(a^{(2)}\cdot a^{(3)})\,a^{(2)}_{\nu} + 7700\,(a^{(2)}\cdot a^{(4)})\,a^{(1)}_{\nu} + 2970\,(a^{(2)}\cdot a^{(5)})\,a_{\nu} \\
& + 4928\,(a^{(3)}\cdot a^{(3)})\,a^{(1)}_{\nu} + 4620\,(a^{(3)}\cdot a^{(4)})\,a_{\nu}\Big\} \\
& - H^2\frac{P^{\mu\nu}}{11!}\Big\{ a^{(7)}_{\nu} + 342\,(a\cdot a)\,a^{(5)}_{\nu} + 2294\,(a\cdot a^{(1)})\,a^{(4)}_{\nu} + 3826\,(a\cdot a^{(2)})\,a^{(3)}_{\nu} \\
& + 3737\,(a\cdot a^{(3)})\,a^{(2)}_{\nu} + 2066\,(a\cdot a^{(4)})\,a^{(1)}_{\nu} + 622\,(a\cdot a^{(5)})\,a_{\nu} + 3231\,(a^{(1)}\cdot a^{(1)})\,a^{(3)}_{\nu} \\
& + 8490\,(a^{(1)}\cdot a^{(2)})\,a^{(2)}_{\nu} + 5663\,(a^{(1)}\cdot a^{(3)})\,a^{(1)}_{\nu} + 1974\,(a^{(1)}\cdot a^{(4)})\,a_{\nu} \\
& + 3785\,(a^{(2)}\cdot a^{(2)})\,a^{(1)}_{\nu} + 3210\,(a^{(2)}\cdot a^{(3)})\,a_{\nu} + O(a^5)\Big\} \\
& - H^4\frac{P^{\mu\nu}}{11!!}\Big\{ a^{(5)}_{\nu} + 1340\,(a\cdot a)\,a^{(3)}_{\nu} + 6108\,(a\cdot a^{(1)})\,a^{(2)}_{\nu} + 6148\,(a\cdot a^{(2)})\,a^{(1)}_{\nu} \\
& + 2599\,(a\cdot a^{(3)})\,a_{\nu} + 5182\,(a^{(1)}\cdot a^{(1)})\,a^{(1)}_{\nu} + 5876\,(a^{(1)}\cdot a^{(2)})\,a_{\nu} + O(a^5)\Big\}\ .
\end{aligned}
\tag{E.16}
$$

Here $v^\mu = \frac{dx^\mu}{d\tau}$ is the proper velocity of the particle computed using dS metric, $a^\mu \equiv \frac{D^2x^\mu}{D\tau^2}$ is its proper acceleration, and $P^{\mu\nu} \equiv g^{\mu\nu} + v^\mu v^\nu$ is the transverse projector to the worldline. We use $a^{(k)}_\mu \equiv \frac{D^k a^\mu}{D\tau^k}$ to denote the proper-time derivatives of the acceleration. All the spacetime dot products are computed using dS metric.

Similarly, the electromagnetic ALD forces in dS_{10} and dS_{12} are given by:

$$
\begin{aligned}
f_9^\mu \equiv \frac{P^{\mu\nu}}{9!!} & \left\{ -8a_\nu^{(7)} + 132\ (a\cdot a)\ a_\nu^{(5)} + 924\ (a\cdot a^{(1)})\ a_\nu^{(4)} + 1512\ (a\cdot a^{(2)})\ a_\nu^{(3)} \right. \\
& +1470\ (a\cdot a^{(3)})\ a_\nu^{(2)} + 888\ (a\cdot a^{(4)})\ a_\nu^{(1)} + 324\ (a\cdot a^{(5)})\ a_\nu \\
& +1344\ (a^{(1)}\cdot a^{(1)})\ a_\nu^{(3)} + 3570\ (a^{(1)}\cdot a^{(2)})\ a_\nu^{(2)} + 2640\ (a^{(1)}\cdot a^{(3)})\ a_\nu^{(1)} \\
& \left. +1092\ (a^{(1)}\cdot a^{(4)})\ a_\nu + 1830\ (a^{(2)}\cdot a^{(2)})\ a_\nu^{(1)} + 1890\ (a^{(2)}\cdot a^{(3)})\ a_\nu + O(a^5) \right\} \\
- H^2\frac{P^{\mu\nu}}{9!!} & \left\{ 448a_\nu^{(5)} + 3488\ (a\cdot a)\ a_\nu^{(3)} + 17240\ (a\cdot a^{(1)})\ a_\nu^{(2)} + 18132\ (a\cdot a^{(2)})\ a_\nu^{(1)} \right. \\
& \left. +9944\ (a\cdot a^{(3)})\ a_\nu + 15424\ (a^{(1)}\cdot a^{(1)})\ a_\nu^{(1)} + 21980\ (a^{(1)}\cdot a^{(2)})\ a_\nu + O(a^5) \right\} \\
+ H^4\frac{P^{\mu\nu}}{9!!} & \left\{ -784a_\nu^{(3)} + 19286\ (a\cdot a)\ a_\nu^{(1)} + 53770\ (a\cdot a^{(1)})\ a_\nu \right\} + O(H^6)\ ,
\end{aligned}
\tag{E.17}
$$

$$
\begin{aligned}
f_{11}^\mu \equiv \frac{P^{\mu\nu}}{11!!} & \Big\{ -10a_\nu^{(9)} + 330(a\cdot a)a_\nu^{(7)} + 2970(a\cdot a^{(1)})a_\nu^{(6)} + 6600(a\cdot a^{(2)})a_\nu^{(5)} \\
& + 9240(a\cdot a^{(3)})a_\nu^{(4)} + 8712(a\cdot a^{(4)})a_\nu^{(3)} + 5610(a\cdot a^{(5)})a_\nu^{(2)} \\
& + 2420(a\cdot a^{(6)})a_\nu^{(1)} + 660(a\cdot a^{(7)})a_\nu \\
& + 5940(a^{(1)}\cdot a^{(1)})a_\nu^{(5)} + 23100(a^{(1)}\cdot a^{(2)})a_\nu^{(4)} + 27324(a^{(1)}\cdot a^{(3)})a_\nu^{(3)} \\
& + 20790(a^{(1)}\cdot a^{(4)})a_\nu^{(2)} + 10120(a^{(1)}\cdot a^{(5)})a_\nu^{(1)} + 2970(a^{(1)}\cdot a^{(6)})a_\nu \\
& + 19140(a^{(2)}\cdot a^{(2)})a_\nu^{(3)} + 36960(a^{(2)}\cdot a^{(3)})a_\nu^{(2)} + 21560(a^{(2)}\cdot a^{(4)})a_\nu^{(1)} \\
& + 7260(a^{(2)}\cdot a^{(5)})a_\nu + 13706(a^{(3)}\cdot a^{(3)})a_\nu^{(1)} + 11088(a^{(3)}\cdot a^{(4)})a_\nu \Big\} \\
+H^2\frac{P^{\mu\nu}}{11!} & \Big\{ -1200a_\nu^{(7)} + 21460(a\cdot a)a_\nu^{(5)} + 149660(a\cdot a^{(1)})a_\nu^{(4)} + 244048(a\cdot a^{(2)})a_\nu^{(3)} \\
& + 236030(a\cdot a^{(3)})a_\nu^{(2)} + 141308(a\cdot a^{(4)})a_\nu^{(1)} + 50660(a\cdot a^{(5)})a_\nu \\
& + 216186(a^{(1)}\cdot a^{(1)})a_\nu^{(3)} + 570000(a^{(1)}\cdot a^{(2)})a_\nu^{(2)} + 417254(a^{(1)}\cdot a^{(3)})a_\nu^{(1)} \\
& + 169692(a^{(1)}\cdot a^{(4)})a_\nu + 288650(a^{(2)}\cdot a^{(2)})a_\nu^{(1)} + 292932(a^{(2)}\cdot a^{(3)})a_\nu \\
& + O(a^5)\Big\} \\
-H^4\frac{P^{\mu\nu}}{11!!} & \Big\{ 43680a_\nu^{(5)} + 363860(a\cdot a)a_\nu^{(3)} + 1786540(a\cdot a^{(1)})a_\nu^{(2)} \\
& + 1864072(a\cdot a^{(2)})a_\nu^{(1)} + 1006750(a\cdot a^{(3)})a_\nu \\
& + 1581920(a^{(1)}\cdot a^{(1)})a_\nu^{(1)} + 2218698(a^{(1)}\cdot a^{(2)})a_\nu + O(a^5)\Big\} \\
& + O(H^6)\ .
\end{aligned}
\tag{E.18}
$$

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List of Publications

1. R. Loganayagam and **Omkar Shetye**. *Influence phase of a dS observer. Part I. Scalar exchange*, *JHEP* **01** (2024) 138. [2309.07290]

2. R. Loganayagam and **Omkar Shetye**. *Influence phase of a dS observer. Part II. Electromagnetism*, *JHEP* **08** (2025) 027. [2503.00135]