

Hydrodynamic Equations Across Scales: Turbulence, Chaos, and Collective Dynamics

A Thesis

Submitted to the
Tata Institute of Fundamental Research, Mumbai
Subject Board of Physics
for the degree of Doctor of Philosophy

by

Ritwik Mukherjee

International Centre for Theoretical Sciences
Tata Institute of Fundamental Research

July, 2026

Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Samriddhi Sankar Ray, at the International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bengaluru.



Ritwik Mukherjee

In my capacity as the formal supervisor of record of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.



Samriddhi Sankar Ray

Date: 15 July 2026

ACKNOWLEDGEMENTS

My teacher Sabyasachi Patra asked me long back, when I was in school, “*Ritwik, do you know what are said to be two of the nicest feelings?*” He paused, and then answered himself: *It is enjoying the presence of an awakened (tattvajña) person, and the ability to appreciate poetry.*” He then added, “*I think there is one more thing — gratitude, remembering what is being done for you.*” This has stayed with me ever since. I would like to take this moment to remember those who shaped, transformed, and accompanied my life.

The first and most natural place to begin is with my thesis advisor, Professor **Samriddhi Sankar Ray**. That he endured me — my food preferences, my complete indifference to cricket — for the entire duration of my PhD is something I will remain grateful for. When I first started my project with him, I had no clue of the subject; yet I somehow had a feeling that if I were to do a PhD at all, it could only be with this person. That instinct turned out to be one of the best I have had. Samriddhi, you looked after me with a care that often exceeded my own concern for myself. In the early days, when I had absolutely no idea how to write a simple code for numerics, you sat with me for hours, taught me, and helped me debug. Whenever I felt low or uncertain, you would call and deliver your magical lectures on *priorities in life* — and those vocal tonics always worked, even right before my synopsis when I was very nervous of my presentation. I am certain they will come back to me in difficult times ahead. You taught me how to write, and I am still trying to learn. You invested in my PhD work, my postdoc applications, and every other important thing — and I can safely say that you worked harder for my PhD than I did. I am still amazed at how you managed to do so much while keeping such calm. Your writing never fails to surprise me, I still do not understand how you write LaTeX in vim and by some magic it compiles! Our conversations, about science and mainly its significant digressions, were always something to look forward to. In the early days I could not quite appreciate your jokes, but over time I started to appreciate them. I had the good fortune of experiencing your cooking firsthand, the “work lunches” at your home were always a pleasure. I extend my warmest gratitude to Purbasha, whose warmth has always meant a great deal. It was always a delight with Buchi and Daaki around during my visits. Buchi, you deserve a special mention here: I know only one story by Roald Dahl, and it was told by you.

Professor **Abhishek Dhar** is one of the gentlest and most insightful people I have met; he has been a mentor to me since I joined ICTS, and will remain one. The way he thinks about a problem and arrives at its heart is something I continue to admire. Many times I went to him with a doubt, and he would immediately scratch his head and say — “*the answer should be this,*

right?” I would stand there with a completely clueless expression, and only after he worked through the long calculation would I finally understand. Dinners with him are always an equal pleasure.

Professor **Vishal Vasan** fundamentally changed how I think about PDEs; his course is among the finest I have taken, and every conversation with him feels like a different experience that I look forward to. Professor **Sthitadhi Roy**, despite his claim that I rarely come to ICTS, will know that we crossed paths more than a few times. Lunches at ICTS with Sthitadhi and Samriddhi were something which can only be experienced, never put into words. I must also thank Sthitadhi for the invaluable information on where to find *sparkling water* in Bangalore. Professor **Anupam Kundu** is a person of the highest intellectual rigor; his statistical mechanics course remains the best I have attended, and his command over long and difficult calculations continues to surprise me. Professor **Rama Govindarajan** is one of the warmest people to speak with. I am always amazed by her extraordinary depth of physical intuition. On the occasions I had the chance to discuss problems with her, the physical insight she brought was simply phenomenal. I also had the pleasure of working with Professor **Manas Kulkarni**, and am grateful for the experience. Towards the end of my PhD, I had the fortune of working with Professor **Subhro Bhattacharjee** — it was a genuinely pleasurable experience. Professor **Brato Chakrabarti** — an incredibly sharp mind, and someone who makes some of the finest presentation slides I have seen. Brato, we had a fantastic time in Kolkata; roaming around the city with you, and the warmth of your home, fantastic food, are things I look forward to again.

My time at ICTS would have been far less without the people I was lucky enough to call friends. The first person I met, while on my way to ICTS, was **Pandit Ji (Shridhar)** — at the Bangalore airport. Pandit ji, you are a gifted singer and a connoisseur of poor jokes; how you come up with them, preferably when Siddhartha is around, and how I cannot stop laughing at them, remains one of life’s great mysteries. Thank you for making this journey so much better — from breakfasts at *Trupthi Thindi* to dinners at Siddhartha’s place that kept me going. You introduced me to Dhruvad, and I cannot express how much that has shaped my taste in music. The long discussions on music, philosophy, and other very important controversial things are something I will always carry with me. **Vinay** is a talented musician whom I have attended the most concerts in Bangalore with. Together with Vinay, Siddhartha and Pandit ji, the trips to Ujwal — with our ritual detour at Chowdiah — are memories I will carry for a long time. Vinay, I truly admire your miserably failed attempts to put even the slightest sense of *taal* in my head. The music sessions on the terrace of Hostel 3, with Siddhartha and Pandit ji, are something I miss very much. We should do it again when Siddhartha visits us! **Mrinal** is a good soul with a fine instinct for *measure theory*. I am glad to see you working well with the ζ function, and grateful for the cooking sessions and for being such a good friend. I will always remember many (uninvited) cups of your special tea at your place. And I will never forget the two of us trying to drive the car up that hill — Neha, thank you for saving us that day! **Bikram** has been a constant through the entirety of my PhD. I have seen you going through phase transitions, just like me! I still firmly believe that you have spent more time abroad than I have. The *jhal muri* sessions will be missed.

Ashik — legend — we have had good times and a few trips, and more are certainly pending. **Avi**, it is always a pleasure to talk to you; your *awareness* and your collection of *information* on anything and everything happening at ICTS is remarkable. **Kaustubh**, it was genuinely difficult to sleep in Hostel 3 with your late-night video games and your voice carrying through the walls — I am glad we both eventually moved to Hostel 2. We had a fantastic time together, and our Kerala trip with Aiswarya, Mrinal, and Neha is something I will always remember. **Irshad**, the most annoying thing was your alarm, which managed to wake up the entire floor every single morning, apart from you — I have full sympathies for your current floormates. We share one thing in common: frequently going home. I am not sure who does it more often, but I suspect it is you. You have been a fantastic friend I will always remember you and your *topology assignment*. **Sandip da**, you have been a fantastic friend and I really like your plots and slides, thanks for the good time.

Some people arrive in your life and quietly alter the course of it; **Siddhartha** is one such person. I had the privilege of meeting him, becoming friends with him, and later of becoming his flatmate — and through him I learned about books, cooking, life, complex fluids, and a great many other things. It is safe to say that many of the phase transitions I went through during my PhD were initiated by him. Siddhartha, you have always been a guide and a go-to person. I have learnt a great deal of coding from you — whatever quality the plots I make today, it comes from what you have taught me. I am grateful for all the intellectual conversations on science, literature, poetry, philosophy, and music. I miss the *Mary Oliver* poetry reading sessions over some fancy tea. I was also fortunate to have the opportunity to work together on multiple projects. Another blessing of living in your flat, and even before that, was the *cooking*. You cook such amazing food! You were there to receive me at the airport when I first arrived in Nice, and I will never forget the company you provided through those three months. I had a very grand start on the first day in France; the experience was intense, and it continued for all three months. It is a pleasure whenever you visit ICTS and also visiting you. It was also you who introduced me to Professor Satyajit Ghosh — a person of incredible depth and wisdom, whose company feels like a blessing.

After Siddhartha, I had the fortune of sharing a flat with **Debojyoti Kundu** — thank you for being such a generous friend and flatmate; our elaborate cooking sessions and late-night discussions are held close to my heart. I had some very fun trips with you.

Divya, whenever I came to you with a problem — numerical or otherwise — I was always struck by the sharpness of your questions. You were part of our Chowdiah–Ujwal evenings, and those memories will always be close. **Aditya and Aditya — Sharma and Vijaykumar** — the elaborate dinners that Siddhartha hosted, with the two of you around, are something I will always remember warmly. **Sharat**, I was always amazed by your grasp of the subject and the rigor you bring to any project you take up; I tried to learn from you, and quickly understood that some things are simply beyond imitation. **Jitu**, your clarity on statistical mechanics is something I genuinely admire. I miss our almost-daily discussions on thermalization, integrability, and mixing, each one ending, of course, without a conclusion. I enjoyed going to movies with you. **Kunal**, I am equally amazed by your running skills and your fantastic slides. **Anikat**, it is

always a pleasure to talk and to work with you. **Santhiya**, photographer and artist — thank you for being such a good friend, and for the justified brutal criticisms of my colormaps in my plots. The dinners at yours and Vinay's place will always be close to my heart. **Neha**, we had lovely trips, films, and conversations on the bus — and the dinners at yours and Mrinal's place will not be forgotten either. **Aiswarya** it was a pleasure visiting your home, we had a very memorable trip. **Seema**, you have been a good friend, we had a fun time in Kolkata mainly in and around college streets. **Anwasha**, it was a pleasure working with you. It was not an easy task to get approval of the color maps from you! Hope to fight more on aesthetics while writing papers. **Aikya**, it was a genuine pleasure being your friend and working with you during your MS project. Your calculation skills and physical intuition are amazing — you are a truly gifted person, and a wonderful artist too.

Harshit, I learned a great deal of coding and numerics from you. One fine evening I spotted you at Prateek's room, and I discovered what a fun person you truly are. Some occasional but very *cultural* evenings at **Prateek Bhai Anand's** place were always a great time. **Ankush**, every time I stepped out of my hostel, there you were drinking tea — thank you for being such a good friend. **Rajarshi**, I am jealous of your coding skills — you are, to my knowledge, the only person in the history of humankind to write a code in MPI and then ask *ChatGPT* to make it serial. **Saumav**, we had such a fun time together, with some truly fantastic car trips and long discussions about your wild ideas on topology, helicity, Euler equations, and many other things. **Subhajit**, you have been both a friend and a mentor to me — you taught me Monte Carlo and molecular dynamics simulations, and I am grateful for everything. It is always a pleasure to visit you at DU. **Indranil**, you are truly a good human being at heart and one of the nicest friends I have. I will always remember the warm welcome at your home and the excellent food that came with it. Long discussions with you and Aikya were always super fun.

I have been fortunate in my collaborators and mentors beyond ICTS as well. I had the great fortune of meeting Professor **Dario Vincenzi** and, in time, of calling him a friend. I spent three months in Nice working with him, and if you have not met Dario, it is genuinely difficult to convey what kind of person he is. Dario, you told me at ICTS, before I left for Nice, that I should "*meditate on norms*" — and I only truly understood what you meant when I went to Nice and saw you actually meditating on norms. You are a magician when it comes to *norms* and *inequalities*. When I arrived, I was a complete novice, and yet you guided me with extraordinary gentleness and patience, never once showing frustration, always speaking with a warmth that is difficult to describe. You took care of me in Nice as one might care for a family member. The excursion to Bordighera, the late-night walks along the seaside, and various other *adventurous* activities — all of it will be remembered always. I am a great admirer of your writing; the thought you put into choosing even a single word for a sentence is something I have tried to learn from, and hope to keep learning. You are far more than a collaborator or a mentor for me. I look forward to meeting you again and again, whenever the opportunity comes.

I am grateful to be both a collaborator and a friend of Professor **Jérémie Bec**. The long discussions at INPHYNI during my stay in Nice — most of which I followed only partially — the lunches at *Chez Tonton*, and the excellent dinners at your home are all memories I hold

warmly. During your visits to India too, after long days of intense discussion, our evening outings, mostly at *The Druid Garden* are something I always look forward to. During my stay in Nice, I had the fortune of spending time with **Jason**. I had the chance on a few occasions to accompany you to a café, where you would work intensely for hours. Whatever I know about Itô and Stratonovich calculus is due to your fantastic lectures. The outings with you, Siddhartha, and Dario are memories I will cherish forever. I was also glad to have had the opportunity to interact and work with Professor **Takeshi Matsumoto**. Takeshi, our outings in Bangalore — especially the dinners at Ujwal — were a real pleasure, and our time together in Florence was something I will not forget. I am a great admirer of your *random thoughts*, and hope to see more of them soon.

I would like to express my gratitude to the staff of ICTS — the academic office, the administration, the security personnel, and the housekeeping team — all of whom made the time here warmer and easier than I can properly acknowledge. I am continually struck by how good and how efficient everyone is at what they do. **Nidhi, Ashwini, Jenny, Manjunatha, Madhulika** — you will always be remembered. **Irshad** has been the go-to person for any IT issues, always patient and always helpful. **Hemanta** deserves a special mention for his continuous support with the HPC clusters which played no small part in bringing this thesis to life. A special mention is **Mani akka** and **Manju akka**. For the last two years, Mani akka would come to the ICTS hostel for work, and every conversation with her felt like a breath of fresh air. Through them both I had the chance to share wonderful food at their home, and the warmth they showed me is something I will not forget.

I would like to speak of my teachers who shaped my early life. My mathematics teacher, **Debasish Mukhopadhyay**, is a polymath in the truest sense — I cannot recall a subject I did not learn from him. It was he who first made me believe that physics was something I could pursue, and who pointed me towards theoretical physics in particular. During my PhD, every time I went home I had long conversation with him regarding my work, he was curious about each and every aspect of my work. He is glad, I know, to see his once-nervous and thoroughly confused student finally writing a thesis. **Sabyasachi Patra**, it is rare to be in the company of a poet. Most days I would arrive to find a half-finished poem lying somewhere in the room, and class would begin with its recitation. My reading habits, and much of my taste in books, were shaped by him in ways I keep discovering. He introduced me to Indian philosophy, Indian logic, and Sanskrit literature, and I cannot imagine having pursued academia — let alone writing a thesis — without that formation. Sir, thank you for everything; you have truly shaped the way I think. I also extend my gratitude to my teachers **Shekhar Ghosh, Rabi Lochan Gorai, Sandip Ruidas, Swapan Rakshit, Radhagobinda Kar, Subhankar Ray**, and **Dhruba Banerjee**, from whom I learned more than I realised at the time.

I have always been lucky with my friends. I cannot imagine my college years without them — from the long discussions in the green zone to the chaos of the lab sessions, those are among the wildest and most cherished memories of my life. **Budhaditya**, your arguments changed the way I think about more things than I am willing to admit, and I have always loved your poems. **Sahil**, the long hours we spent at Jadavpur and College Street are fondly remembered,

and you helped me a great deal — especially with coding in MATLAB during my early days at ICTS. **Aniket**, all of us passed our courses largely because of you — thank you for attending the classes and taking notes on our behalf. You joined IISc before I joined ICTS, so when I arrived in Bangalore I already had one of my dearest friends waiting, which made the beginning far less daunting. **Puja** and **Samapti**, thank you for tolerating me for so long — and for still calling. **Snigdha**, thank you for being such a wonderful friend. **Pinaki**, I have always wondered when exactly you studied — I never once saw it happening, and yet somehow you always managed. Thanks for being a good friend.

Krishnendu, I am always amazed by your skill in mathematics, and even more by your writing and your book collection; you have inspired me quietly for a long time, and you taught me statistics. **Sanghapriya**, you have tolerated me for longer than most and remained a constant friend through it all. You are a gifted singer and artist, and you introduced me to a whole world of classical music. I have thoroughly enjoyed our intellectually enriched utterly nonsense conversations. **Saurav**, **Manas**, **Biplab** — you have been my friends for the longest stretch of my life, and a return to Bankura without seeing you would feel incomplete.

Haragourab da and **Madhurima di**, thank you for everything. I have spent long evenings in conversation with both of you, followed by excellent food cooked by Madhurima di, and I still look forward to it every time I go home. **Babu Kaku**, you were the first person to kindle in me the love of reading, and the evenings spent with you, with your amazing storytelling, are among the most beautiful I have known.

The most difficult part of this section is what remains. It is impossible to truly acknowledge my family in words, but I will attempt a few.

I am deeply privileged to have parents who supported me without condition and withstood me without complaint. **Ma**, **Baba** — without you, nothing would have been possible, let alone a PhD. There is nothing more I can say than: thank you for all the love, the blessings, and for everything.

I grew up in the company of my grandparents, **Dida** and **Dadu** (Dan), their love, their presence is beyond what words can hold. In my childhood, Dadu used to take me to school and make sure I sat in the last bench so I could sleep undisturbed, and would always be there to collect me afterwards. Dida taught me the alphabet. They are both glad, I know, to see me writing something that has — as Dida might note — rather a lot of alphabets in it. I grew up surrounded by my *masis* — **Kuchi Masi**, **Piya**, and **Masimani** — and my childhood would not have been what it was without them. The love and care I received from them is beyond what words can hold. Thank you for everything.

And finally, with my *pranams* to the lotus feet of Pagal Haranath and Kusumkumari Devi. *KusumHaranath*.

LIST OF FIGURES

1.1	Comparison of kinetic energy for different Re	4
1.2	Dissipation field	5
1.3	Schematic for Richardson Cascade	7
1.4	Schematic for Kolmogorov-Obukhov multiplicative cascade	12
1.5	Sensitivity to initial conditions in a chaotic dynamical system	16
1.6	Perturbation spread in fully developed turbulence	18
1.7	Schematic for attractor	24
1.8	Schematic for a lattice gas	29
2.1	Snapshot of dissipation field: Eulerian Versus Lagrangian	35
2.2	Multiplicative cascade schematic	37
2.3	Eulerian Statistics	38
2.4	Scale depended multipliers for the Eulerian case	42
2.5	Lagrangian statistics	44
2.6	Shematic for dynamical cascade	47
2.7	Pair separation	48
2.8	Evidence of modified bridge relation	49
3.1	Decorrelator growth and Re dependence	55
3.2	Re dependence of largest Lyapunov	57
3.3	Decorrelator growth in shell model	59
3.4	PDF of γ_3 and, $\hat{n}_i^2$ and shell wise decorrelator growth	66
4.1	Correlation of amplitude and orientation	69
4.2	Exponents for three-dimensional and two-dimensional turbulence	73
4.3	Comparison between turbulent versus synthetic dataset	77
4.4	Fractal geometry of zero set	78
4.5	Second order amplitude and geometric structure functions	82
4.6	Local slopes	83
4.7	PDF of increments	83
6.1	Free evolution in 1D	101
6.2	Free evolution in a finite background	106

6.3	Convergence for the finite background case	107
6.4	Relaxation dynamics	109
6.5	Verification of damping and oscillation time scales from theory	110
6.6	Valididty of perturbation calculations	110
6.7	Time evolution of 2D density field	113
6.8	Time evolution of radial density and velocity for 2D	114
6.A.1	Time evolution for skewed initial conditions	118
7.1.1	Quasi-1D model	122
7.3.1	Test of hydrodynamics	124
7.3.2	Phase diagram	125
7.4.1	Two-point correlations	126
7.6.1	MIPS in $2d$	128
7.7.1	Kymographs	129
7.D.1	Local equilibrium for the quasi-1D model	139
7.D.2	Local equilibrium for the strictly-1D model	140
7.D.3	Validity of noiseless hydrodynamics in the quasi-1D model	140
7.D.4	Breakdown of noiseless hydrodynamics in the strictly-1D model	141
7.F.1	Plot for f_0 and g_0	144
7.F.2	Metastability	145
7.G.1	Correlations in the homogeneous phase in quasi-1D model	147
7.G.2	Correlations in the homogeneous phase in strictly-1D model	148

LIST OF TABLES

2.1	Parameters of the direct numerical simulations	39
4.1	A summary of the equal-time scaling exponents in 3D	84
4.2	A summary of the equal-time scaling exponents in 2D	84
5.1	Large- Re estimates for the two-dimensional Navier–Stokes equations with $\dot{H}^s$ constant-in-time forcing.	89
7.1	Possible values of the various dynamical events for the quantities in (7.7) and corresponding rates.	131

LIST OF PUBLICATION

PUBLICATIONS INCLUDED IN THIS THESIS

6. *Eulerian vs Lagrangian intermittency in turbulence*
J. Bec*, **R. Mukherjee***, S. Mukherjee, and S. S. Ray
Manuscript in preparation
* Joint first authors
5. *Estimates for the 2D Navier–Stokes equations: the effects of forcing*
R. Mukherjee, J. D. Gibbon, and D. Vincenzi
[arXiv:2512.15188 \(2025\)](#)
4. *Geometric Intermittency in turbulence*
R. Mukherjee, S. Mukherjee, I. V. Kolokolov, V. V. Lebedev, T. Matsumoto, and S. S. Ray
[arXiv:2505.09538 \(2025\)](#)
3. *Understanding extreme dynamics and relaxation in quantum gases: a hydrodynamic approach*
R. Mukherjee, M. Kulkarni, A. Dhar, and S. S. Ray
[Physical Review A 113, 043305 \(2026\)](#)
2. *Intermittent fluctuations determine the nature of chaos in turbulence*
A. Banerjee*, **R. Mukherjee***, S. D. Murugan, S. Bhattacharjee, and S. S. Ray
[Physical Review Letters 136, 214001 \(2026\) \[Editors' Suggestion\]](#)
* Joint first authors
1. *Hydrodynamics of a hard-core active lattice gas*
R. Mukherjee, S. Saha, T. Sadhu, A. Dhar, and S. Sabhapandit
[Physical Review E 111, 024128 \(2025\)](#)

OTHER PUBLICATIONS

3. *Parity-dependent scaling of velocity-gradient correlations in turbulence*

A. Dey*, **R. Mukherjee***, A. Banerjee, and S. S. Ray

[arXiv:2605.19921](#)

* Joint first authors

2. *Reduction of Triadic Interactions Suppresses Intermittency and Anomalous Dissipation in Turbulence*

A. Kankaria*, **R. Mukherjee***, S. D. Murugan, M. Rosti, and S. S. Ray

[arXiv:2603.19180](#)

* Joint first authors

1. *Turbulent flows are not uniformly multifractal*

S. Mukherjee, S. D. Murugan, **R. Mukherjee**, and S. S. Ray

[Physical Review Letters 132, 184002 \(2024\)](#)

CONTENTS

Acknowledgements	iii
List of Figures	ix
List of Tables	xi
List of Publication	xii
Abstract	xvii
1 Introduction	1
1.1 Statistical Physics of Fully Developed Turbulence	2
1.2 Chaos and Predictability	15
1.3 Analysis Approach to the Navier–Stokes Equations	20
1.4 One-Dimensional Hydrodynamic Systems	25
1.4.1 Isentropic Euler Equations	26
1.4.2 Hydrodynamics of Lattice Gases	28
1.5 Overview	31
2 Intermittency of Turbulent Dissipation is Intrinsically Lagrangian	34
2.1 Introduction	34
2.2 Multifractality and Large Deviations	39
2.3 Eulerian Dissipation: Absence of a Scale-Invariant Spatial Cascade	40
2.4 Lagrangian Dissipation as a Temporal Multiplicative Cascade	45
2.5 Revisiting Bridge Relations: From Spatial Sampling to Temporal Inheritance	46
2.6 Conclusions	49
3 Intermittent Fluctuations Determine the Nature of Chaos in Turbulence	52
3.1 Introduction	52
3.2 Decorrelator for the Navier–Stokes	53
3.3 Reynolds dependence of largest Lyapunov: Effect of strain rate fluctuations	56
3.4 Test on a shell Model	58
3.A Direct Numerical Simulations	60
3.A.1 The Navier-Stokes Equation	60
3.A.2 The Gledzer-Ohkitani-Yamada (GOY) Model	61
3.B Derivation of the decorrelator from the Navier-Stokes equation and its limits	61

3.B.1	Evolution equation for $\phi(\mathbf{r}, t)$ and Φ	61
3.B.2	Short and long-time asymptotics of β_S	62
3.B.3	Short and long-time asymptotics of β_η	65
3.C	Decorrelator in the shell model for turbulence	65
4	Geometric Intermittency in Turbulence	67
4.1	Introduction	68
4.2	Geometric Observables in Homogeneous and Isotropic Turbulence	70
4.3	Generality of Geometric Multiscaling: Comparison with Gaussian Fields	72
4.4	Interpretation: Geometry, Correlations, and Saturation	74
4.5	Phenomenological Interpretation of Geometric Multiscaling	76
4.6	Conclusions	80
4.A	Scaling form for the second-order structure functions	82
4.B	Appendix B: Multiscaling Exponents and the Probability Distribution of Increments	83
5	Forcing Effects on Energy and Enstrophy Dissipation in Two-Dimensional Flows	85
5.1	Introduction	85
5.2	Definitions and previous two-dimensional results	88
5.3	Estimates for $\dot{H}^s$ forcing	92
5.3.1	The case $0 \leq s \leq 2$	92
5.3.2	The case $-1 \leq s \leq 0$	95
5.3.3	Time-dependent forcing	95
5.4	Concluding remarks	98
6	Extreme Dynamics and Relaxation of Quantum Gases	100
6.1	Introduction	100
6.2	Free evolution in one-dimensional case	104
6.2.1	Evolution in a vacuum	104
6.2.2	Evolution in a finite density background	106
6.3	Relaxation of a one-dimensional trapped gas	109
6.4	Two-dimensional case	113
6.5	Conclusions and Outlook	116
6.A	Direct Numerical Simulations and Robustness of Initial Conditions	118
7	Hydrodynamics of a Hard-Core Active Lattice Gas	120
7.1	Introduction	120
7.2	A quasi-1D model	122
7.3	A hydrodynamic description	123
7.4	Spinodal and binodal analysis	124
7.5	Two-point correlations	127
7.6	Generalization to d dimensions	127
7.7	A strictly 1D model	128
7.8	Conclusion	129
7.A	An action formulation for the quasi-1D model	130

7.A.1	The hydrodynamic limit	132
7.A.2	The fluctuating hydrodynamics equations	135
7.B	Generalization to d dimensions	136
7.B.1	Case I: Two-species generalization	136
7.B.2	Case II: $2d$ -species generalization	137
7.C	The strictly 1D model with exchange dynamics	138
7.D	Numerical evidence for local equilibrium	139
7.E	The spinodal analysis	142
7.F	The binodal analysis	143
7.F.1	First relation	144
7.F.2	Second relation	144
7.G	The two-point correlations in the stationary-state homogeneous phase	146
7.H	Details of the numerical simulations	151
8	Conclusion and Outlook	152
	Bibliography	155

ABSTRACT

Chapter 1 provides a comprehensive introduction to the physical, mathematical, and computational ideas underlying the work presented throughout this thesis, and outlines the broader context connecting turbulence, nonequilibrium statistical physics, and hydrodynamic modeling. Chapter 2 investigates dissipation statistics from both Eulerian and Lagrangian viewpoints using high-resolution direct numerical simulations and elucidates the sharp distinction between spatial and temporal statistics of dissipation. Chapter 3 studies chaos and predictability in turbulence through decorrelation dynamics. In particular, it uncovers the scaling of the Lyapunov exponent with the Reynolds number and the role of intermittency in determining the scaling exponent. In Chapter 4, we revisit velocity increment statistics using a geometric decomposition of the increment field. Furthermore, we make connections with synthetic turbulence and uncover geometric properties common to both. Chapter 5 shifts to two-dimensional Navier–Stokes turbulence and derives strict upper bounds for the energy and enstrophy dissipation rates as functions of the Reynolds number, without relying on phenomenological modeling assumptions, for different forcing regularities. The final part of the thesis focuses on hydrodynamic descriptions of nonequilibrium many-body systems. In Chapter 6, we study the expansion and relaxation dynamics of quantum gases using Euler hydrodynamics in one and two dimensions whereas Chapter 7 develops a fluctuating hydrodynamic theory for an active lattice gas with excluded-volume interactions exhibiting motility-induced phase separation. Taken together, the results of this thesis provide new perspectives on intermittency, chaos, and scaling in turbulence, while also developing hydrodynamic frameworks for complex nonequilibrium systems ranging from active matter to expanding quantum gases. Chapter 8 summarises the work in this Thesis and provides a brief perspective and further directions which could emanate from this Thesis.

INTRODUCTION

Nonlinear partial differential equations form the backbone of our theoretical understanding of coarse-grained dynamics in a wide range of physical systems. Among them, hydrodynamic equations occupy a central place: they describe fluid motion across a wide range of scales, from geophysical flows to active systems at microscopic scales. Their inherently nonlinear and multiscale structure often gives rise to complex emergent behavior and, in particular, robust scaling laws. By *scaling laws*, we mean power-law relationships of the form $y \sim x^\alpha$ between variables or parameters of the system. Despite their apparent simplicity, such laws encode deep connections to scale invariance, symmetry, and self-similarity.

The simplest route to scaling laws is through dimensional analysis. Even when the governing equations are not amenable to exact solutions, symmetry arguments and dimensional reasoning can often determine the scaling exponents without solving the equations themselves. This approach is remarkably powerful when it applies. There are, however, important exceptions. When self-similarity is only partial — when a dimensionless parameter formally tends to zero or infinity in some limit yet cannot be neglected because the limit is singular — dimensional analysis alone becomes insufficient. The scaling exponents remain well defined, but they carry information about the internal structure of the problem that dimensional reasoning cannot access. Such scaling laws are known as *anomalous scaling*, and are often described in the literature as *incomplete self-similarity* [1]. Indeed, this is the generic situation in many problems of physical interest. Understanding the mechanisms behind such scaling laws, and deriving them from first principles rather than merely measuring them, remains a major challenge in hydrodynamics, particularly for systems governed by the Navier–Stokes equations.

Turbulence is one of the most prominent examples of *anomalous scaling* [2, 3]. Although governed by the deterministic Navier–Stokes equations, turbulent flows exhibit complex, chaotic, and seemingly random motion across a vast range of spatial and temporal scales [2, 4]. The statistical physics approach has been one of the most successful frameworks for understanding this complexity. Kolmogorov’s 1941 theory [5, 6] provided the first quantitative statistical description

of turbulence and remains the natural starting point for modern theories. Real turbulent flows, however, deviate systematically from Kolmogorov's predictions, and understanding the origin, structure, and consequences of these deviations remains an open problem.

A complementary approach to hydrodynamic equations comes from mathematical analysis. Rather than describing typical behavior statistically, one asks what the equations rigorously imply: what bounds on quantities such as kinetic energy and energy dissipation can be derived directly from the Navier–Stokes equations, and how these bounds scale with parameters such as the Reynolds number. Such methods provide rigorous constraints on scaling laws, both in situations where statistical theories are successful [7, 8] and in regimes where phenomenological predictions are unavailable [9].

A third perspective approaches hydrodynamic equations through their mathematical structure and microscopic foundations, particularly in one-dimensional systems. Such models have long served as testing grounds for ideas that are difficult to access in three dimensions. The Burgers equation, for instance, shares the essential nonlinearity of Navier–Stokes while remaining analytically tractable [10]. These systems can be studied either from the top down, through Hamiltonian formulations, or from the bottom up by deriving continuum hydrodynamics from microscopic lattice models [11, 12]. Both approaches provide insight that is often inaccessible in the full three-dimensional problem.

This thesis studies hydrodynamic equations through all three of these perspectives. These approaches are not competing; rather, they address different questions and illuminate different aspects of nonlinear dynamics. The common thread throughout this thesis is the attempt to understand how nonlinear hydrodynamic equations generate complex multiscale behavior and scaling laws.

The Chapter is organized as follows. In Sec. 1.1, we review the statistical physics approach to fully developed turbulence, with particular emphasis on intermittency, multifractality, and anomalous scaling. In Sec. 1.2, we discuss chaos and predictability in turbulent flows and introduce the Lyapunov framework used later in the thesis. Section 1.3 outlines the mathematical-analysis approach to the Navier–Stokes equations, focusing on rigorous bounds and scaling estimates. Section 1.4 discusses one-dimensional hydrodynamic systems, including quantum gases and lattice-gas models, which form the basis for the final part of this thesis. Finally, in Sec. 1.5, we give a brief overview of the specific studies and chapters which form a part of this thesis.

1.1 STATISTICAL PHYSICS OF FULLY DEVELOPED TURBULENCE

The three-dimensional incompressible Navier–Stokes equations for a single component, unit density, Newtonian viscous fluid with velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure field $p(\mathbf{x}, t)$ are given by

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \quad (1.1)$$

$$\nabla \cdot \mathbf{u} = 0; \quad (1.2)$$

where, ν is kinematic viscosity and $\mathbf{f}$ is an external forcing. The Navier–Stokes equations are solved with initial condition $\mathbf{u}_0$ and appropriate boundary conditions, which we assume to be periodic throughout this thesis. So, $\mathbf{u}(\mathbf{x}, t) \in \mathbb{T}^3 \equiv [0, L]^3$. For simplicity, we take $L = 2\pi$ unless specified otherwise. Equation (1.1) can be recast in the dimensionless form by defining Reynolds number Re as

$$Re = \frac{UL}{\nu}, \quad (1.3)$$

where, typically L is taken as the large-length scale or the integral scale and the characteristic velocity U is typically defined as L^2 -norm of the velocity $U^2 = \langle |\mathbf{u}|^2 \rangle$. We study the Navier–Stokes equations in its *non-equilibrium steady state* where, energy is continuously injected at the large scales and balanced by the dissipation at the small scales. To make this precise, we derive the energy balance equation. The kinetic energy per unit mass is

$$E(t) = \frac{1}{2} \langle |\mathbf{u}|^2 \rangle \equiv \frac{1}{2L^3} \int_{\mathbb{T}^3} |\mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x}, \quad (1.4)$$

where angular brackets denote a spatial average over the periodic domain. To derive the energy equation, we take the inner product of Eq. (1.1) with $\mathbf{u}$ and average over the domain. The pressure term vanishes by incompressibility,

$$\langle \mathbf{u} \cdot \nabla p \rangle = \langle \nabla \cdot (p\mathbf{u}) \rangle = 0, \quad (1.5)$$

where the last equality follows from periodicity. The nonlinear term likewise contributes nothing to the energy budget, provided the solutions are smooth enough [13],

$$\langle \mathbf{u} \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u}] \rangle = \frac{1}{2} \langle \mathbf{u} \cdot \nabla |\mathbf{u}|^2 \rangle = \frac{1}{2} \langle \nabla \cdot (\mathbf{u} |\mathbf{u}|^2) \rangle = 0, \quad (1.6)$$

consistent with the notion that advection merely redistributes energy in space without creating or destroying it. The viscous term, integrated by parts, yields

$$\nu \langle \mathbf{u} \cdot \nabla^2 \mathbf{u} \rangle = -\nu \langle |\nabla \mathbf{u}|^2 \rangle \equiv -\varepsilon, \quad (1.7)$$

where we define the *energy dissipation rate* [2, 14]

$$\varepsilon(t) = \nu \langle |\nabla \mathbf{u}|^2 \rangle = 2\nu \langle S_{ij} S_{ij} \rangle, \quad (1.8)$$

with $S_{ij} = 1/2(\partial_i u_j + \partial_j u_i)$, known as the strain rate tensor. Assembling the terms, the energy equation reads

$$\frac{dE}{dt} = \mathcal{I} - \varepsilon, \quad (1.9)$$

where

$$\mathcal{I} = \langle \mathbf{f} \cdot \mathbf{u} \rangle \quad (1.10)$$

is the *energy injection rate*. Equation (1.9) encapsulates the competition between large-scale energy input and small-scale viscous dissipation. In a statistically stationary state, $dE/dt = 0$ on average, and we obtain

$$\langle \mathcal{I} \rangle = \langle \varepsilon \rangle, \quad (1.11)$$

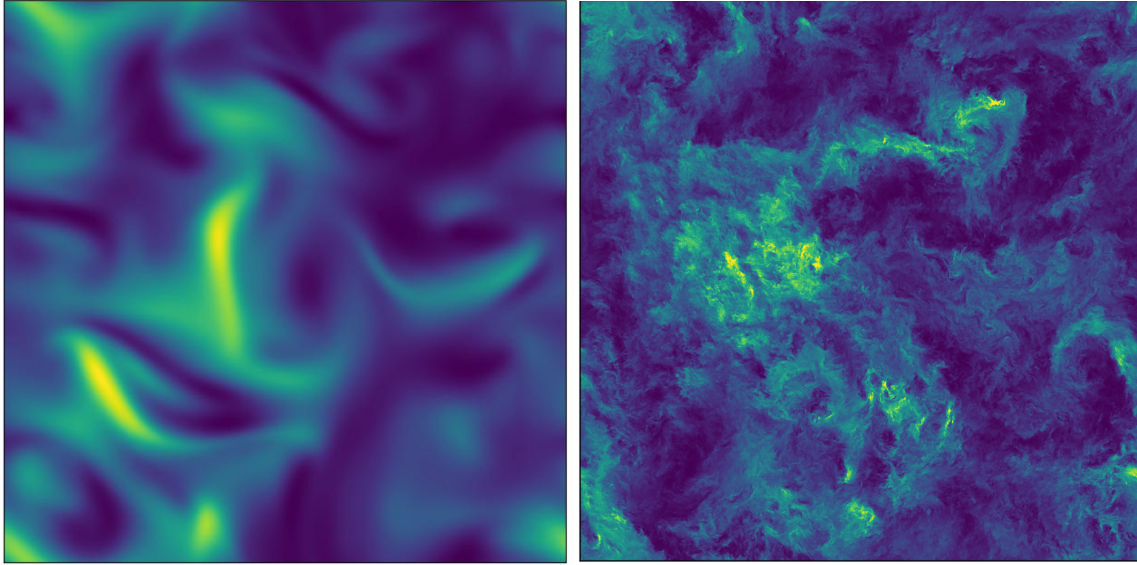


Figure 1.1. Left panel shows the kinetic energy density defined in Eq. (1.4) for $Re \sim 40$ and right shows the same for $Re \sim 40000$. We see the emergence of multiple scales and complex structures as the Re is increased.

which simply states that, in steady state, energy is injected at the large scales is statistically the same as dissipation. Throughout this thesis, all the measurements will be done on the nonequilibrium steady state where this balance is statistically valid and we are specifically interested in the $Re \rightarrow \infty$ limit, when the flow becomes *turbulent*. A superficial reading of Eq. (1.8) may suggest that ε goes to 0 as $Re \rightarrow \infty$ or $\nu \rightarrow 0$, but it has been shown that even in that limit dissipation persists and attains a finite value. This persistence of dissipation in the inviscid limit, the *dissipative anomaly*, is one of the most fundamental and still not fully understood facts about turbulence [2, 15–17]. We discuss dissipative anomaly briefly in a later section and it underpins the entire statistical theory developed in the chapters that follow.

Before developing the theoretical framework for a statistical theory, it is important to pause and ask: why does a statistical description become not merely useful, but *necessary*? The Navier–Stokes equations (1.1) are a (infinite dimensional) deterministic dynamical system. Given an initial condition $\mathbf{u}(\mathbf{x}, 0)$, the subsequent evolution of the flow is, in principle, uniquely determined. There is no randomness built into the equations themselves. Yet in the limit of small viscosity $\nu \rightarrow 0$, equivalently, at large Reynolds number, the solutions become extraordinarily complex. The velocity field develops structure across a vast range of spatial and temporal scales simultaneously, and the solutions become sensitive to initial conditions.

This is illustrated in Fig. 1.1, which shows snapshots of the kinetic energy field from two direct numerical simulations at different Reynolds numbers. At low Re (left), the energy field is smooth and varies on a single characteristic scale. At high Re (right), the field develops structure across many scales: large-scale sweeping motions coexist with fine-scale patches of intense activity. The intermittent character is more prominent in Fig. 1.2, which shows a snapshot of the dissipation field $\varepsilon(\mathbf{x}, t)$ alongside a one-dimensional cut through the domain. The cut reveals a signal that is quiescent for extended intervals, punctuated by sharp, intense bursts, a hallmark

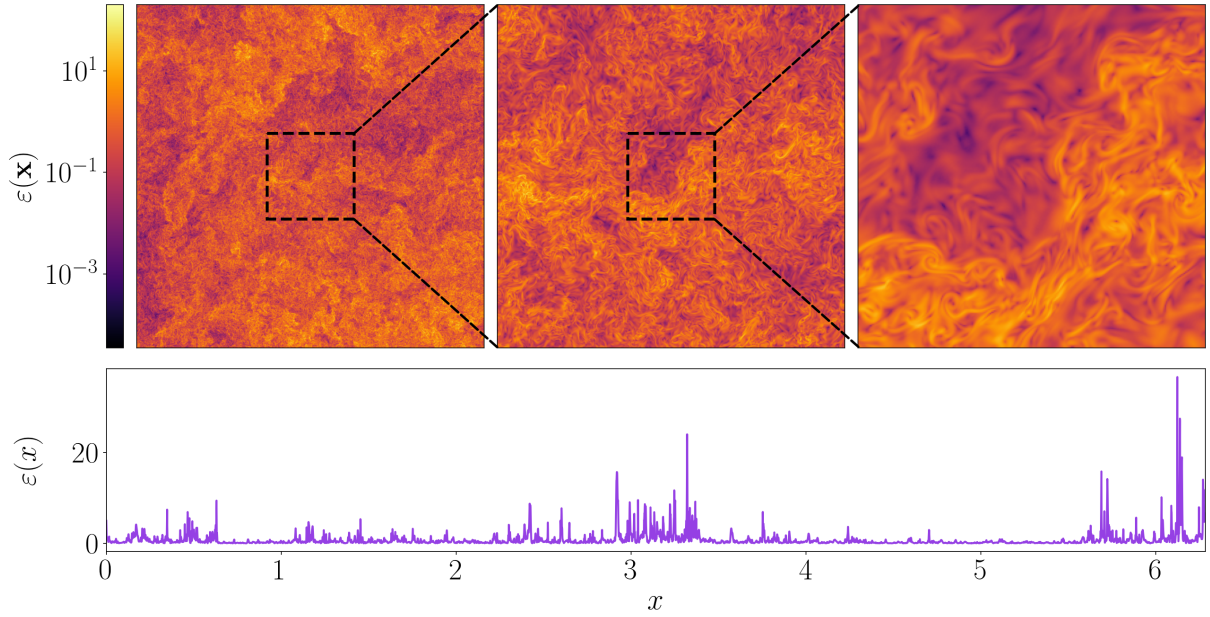


Figure 1.2. Pseudo-color plots show successively zoomed two-dimensional slices of the dissipation field (1.8) for fully developed three-dimensional turbulence. The line plot shows a one-dimensional cut of the same field, providing a visual representation of intermittency.

of intermittency, and the feature that makes a statistical description both necessary and difficult.

Tracking the exact trajectory of such a solution—resolving every filament, every fluctuation—is neither practical nor illuminating. No two realizations initialized from slightly different conditions will agree pointwise after a short time Δt , even as their statistical properties remain indistinguishable. In a probabilistic description, rather than asking *what is $\mathbf{u}(\mathbf{x}, t)$?*, we ask: *what are the probability distributions, moments, and correlations of these fields?* These quantities are reproducible, experimentally measurable, and, as we shall see, governed by scaling laws that reflect the physics of turbulent cascade. The recognition that statistical theories can explain the complex solutions of the Navier–Stokes equations, without requiring knowledge of any individual solution, is the foundational insight of the modern theory of turbulence.

One of the earliest attempts to use statistical physics ideas to understand turbulent flows originates from the foundational ideas of Richardson, Kolmogorov, and Obukhov. The first major breakthrough was the phenomenological theory by Kolmogorov in 1941 [5, 6]—K41, which postulates that the statistical properties within the *inertial range*—far from both the large-scale forcing L and the small-scale viscous dissipation η —are universal and independent of the specific details of the large-scale forcing, boundary effects and small scale dissipation.

K41 AND SELF-SIMILARITY

Figure 1.3 shows a schematic of the Richardson cascade picture (adapted from Ref. [2]) of turbulence, which states that energy is injected at large scales, generating large eddies that subsequently break into smaller ones, transferring energy to progressively smaller scales. Kolmogorov argued that far from the forcing scale and far from the dissipation scale, the physics is

independent of the particular fluid being studied and the mechanism by which it is forced.

At the non-equilibrium steady state, where energy injection rate equals flux equals dissipation rate, the statistics of fluctuations at a scale ℓ in the inertial range $L \gg \ell \gg \eta$ — with L the injection scale and η the dissipation scale — are governed by two parameters: the scale ℓ itself, and the mean energy dissipation rate $\bar{\varepsilon}$. Two-point statistics are studied via moments of the longitudinal velocity increment

$$\delta u_\ell \equiv [\mathbf{u}(\mathbf{x} + \boldsymbol{\ell}, t) - \mathbf{u}(\mathbf{x}, t)] \cdot \hat{\boldsymbol{\ell}}, \quad (1.12)$$

where $\hat{\boldsymbol{\ell}}$ is the unit vector along the separation $\boldsymbol{\ell}$.

A modern and precise formulation of these ideas [2] recasts Kolmogorov's original universality arguments in terms of *self-similarity*, and can be summarized by three core postulates:

- **Restoration of Symmetries:** Although symmetries such as homogeneity and isotropy are broken by the large-scale forcing mechanism, they are restored locally and statistically in the inertial range.
- **Self-Similarity:** The turbulent velocity field is globally self-similar, characterized by a unique Hölder exponent h . In the K41 framework, $h = 1/3$ everywhere in the fluid domain.
- **Dissipative Anomaly:** In the limit of vanishing kinematic viscosity ($\nu \rightarrow 0$), the mean energy dissipation rate approaches a finite, non-zero constant $\bar{\varepsilon} > 0$.

Based on these postulates and using dimensional analysis the p -th order longitudinal structure function:

$$S_p(\ell) = \langle (\delta u_\ell)^p \rangle \sim (\bar{\varepsilon} \ell)^{p/3} \sim \bar{\varepsilon}^{p/3} \ell^{\zeta_p}, \quad (1.13)$$

which follows by dimensional analysis from $[\bar{\varepsilon}] = L^2 T^{-3}$ and $[\delta u_\ell] = L T^{-1}$: the only combination of $\bar{\varepsilon}$ and ℓ with dimensions of velocity to the p -th power is $(\bar{\varepsilon} \ell)^{p/3}$.

The K41 framework is not purely dimensional, however. For $p = 3$, the scaling (1.13) is corroborated by an analytically exact result derived directly from the Navier–Stokes equations under the K41 symmetry assumptions alone, the Kármán–Howarth–Monin relation, which yields the *four-fifths law*:

$$\langle (\delta u_\ell)^3 \rangle = -\frac{4}{5} \bar{\varepsilon} \ell. \quad (1.14)$$

This is an exact result in turbulence theory, and it is derived from the Navier–Stokes equations by only assuming *local homogeneity*, which fixes the prefactor in (1.13) for $p = 3$ without any closure assumption and serve as a check for numerics or any new theory to understand scaling of moments of velocity difference.

While these assumptions yield the elegant linear scaling $\zeta_p = p/3$ (where $\langle (\delta u_\ell)^p \rangle \sim \ell^{\zeta_p}$), extensive experimental and high-resolution numerical measurements have conclusively shown systematic deviations from this linear prediction [3, 18–20]. This nonlinear dependence of ζ_p on p is known as *anomalous scaling* or *multiscaling*, a direct consequence of intense spatial fluctuations in the energy transfer process: *intermittency*.

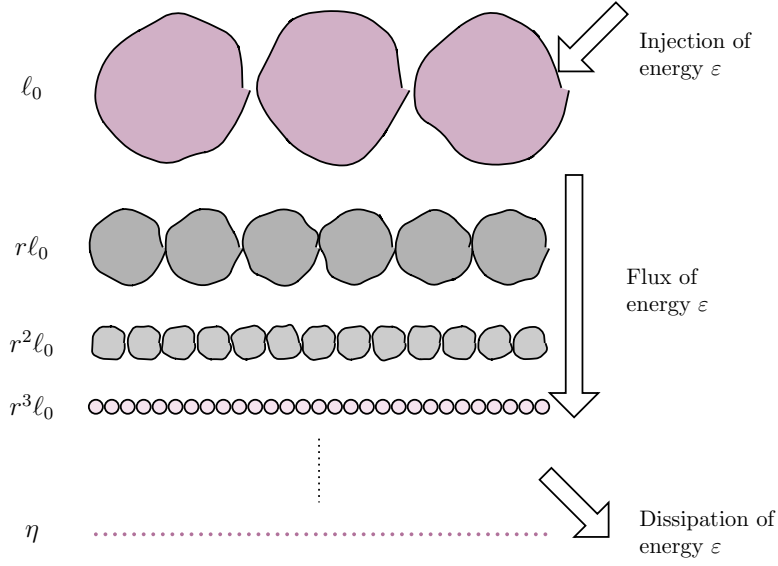


Figure 1.3. Schematic representation of the Richardson cascade; adapted from Ref. [2]). Energy is injected at large scales which flows through scales and dissipated at small scale. In this picture, larger eddies break into smaller eddies in a space filling way.

K62 AND THE REFINED SIMILARITY HYPOTHESIS

The first correction to K41 was proposed by Kolmogorov himself in 1962 [21] along with the Kolmogorov–Obukhov multiplicative cascade picture. In the previous K41 theory the central assumption was self-similarity of the energy transfer, in other words, the energy flow from bigger eddies to smaller eddies happen in an *uniform* way. Kolmogorov proposed a refined similarity approach, which implicitly assumed that the energy flux (and thus the dissipation) was *non-uniformly* distributed, or there are large-scale statistical fluctuations in the distribution of energy dissipation [2, 16] which cannot be ignored and can potentially break the K41 universality. Figure 1.4 shows a schematic for the Kolmogorov–Obukhov multiplicative cascade, which shows the random distribution of energy dissipation at different scales keeping the mean dissipation at all scales fixed. Kolmogorov therefore proposed that inertial-range statistics at scale ℓ are governed not by the global mean $\bar{\epsilon}$, but by the dissipation rate averaged locally over a ball $\mathcal{B}_\ell$ of size ℓ :

$$\epsilon_\ell(\mathbf{x}) = \frac{1}{\mathcal{B}_\ell} \int_{\mathcal{B}_\ell} d\mathbf{x}' \epsilon(\mathbf{x} + \mathbf{x}'). \quad (1.15)$$

Consequently, the Refined Similarity Hypothesis postulates that the structure functions should scale conditionally upon this local dissipation:

$$\langle (\delta u_\ell)^p \rangle \sim \langle (\epsilon_\ell \ell)^{p/3} \rangle = \langle \epsilon_\ell^{p/3} \rangle \ell^{p/3}. \quad (1.16)$$

More formally, Kolmogorov proposed *local universality*, and stated that one can define a local velocity scale $U_\ell = (\epsilon_\ell \ell)^{1/3}$ such that the normalized increment $\delta u_\ell / U_\ell$, conditioned on a sufficiently high local Reynolds number ($Re_\ell = \ell U_\ell / \nu \gg 1$), has a universal distribution, and statistically independent of both ℓ and ϵ_ℓ . These arguments

The theoretical challenge of determining the scaling exponents ζ_p is thus reduced to modeling the statistical moments of the coarse-grained dissipation, $\langle \varepsilon_\ell^p \rangle$. To achieve this, distribution of the fluctuations of coarse-grained dissipation needs to be known. Kolmogorov proposed a log-normal model based on three assumptions [16, 21]:

1. By construction, the mean is conserved across scales: $\langle \varepsilon_\ell \rangle = \bar{\varepsilon}$.
2. The logarithm of the coarse-grained dissipation, $\log \varepsilon_\ell$, follows a Gaussian (normal) distribution.
3. The variance of $\log \varepsilon_\ell$ grows logarithmically with the scale ratio:

$$\sigma_\ell^2 = \text{Var}[\log \varepsilon_\ell] = -2A - 2\mu_K \log(\ell/L), \quad (1.17)$$

where μ_K is the universal intermittency coefficient ≈ 0.13 , and A is a constant that is not important in the scaling laws we are interested in and L is the integral scale.

Under these assumptions, we can directly compute the moments $\langle \varepsilon_\ell^p \rangle$. Noting that $\langle \varepsilon_\ell^p \rangle$ is identically the moment generating function of the Gaussian random variable $\log \varepsilon_\ell$, evaluated at p :

$$\langle e^{p \log \varepsilon_\ell} \rangle = \exp \left(p \langle \log \varepsilon_\ell \rangle + \frac{1}{2} p^2 \sigma_\ell^2 \right). \quad (1.18)$$

Enforcing the mean conservation constraint ($\langle \varepsilon_\ell \rangle = \bar{\varepsilon}$) requires that

$$\langle \exp(\log \varepsilon_\ell) \rangle = \bar{\varepsilon} = \exp \left(\langle \log \varepsilon \rangle + \frac{1}{2} \sigma_\ell^2 \right) \quad (1.19)$$

Substituting this and Eq. (1.17) into Eq. (1.18) yields:

$$\langle \varepsilon_\ell^p \rangle = \bar{\varepsilon}^p \exp \left[\frac{1}{2} p(p-1) \sigma_\ell^2 \right] \propto \bar{\varepsilon}^p \left(\frac{\ell}{L} \right)^{-\mu_K p(p-1)}. \quad (1.20)$$

Finally, inserting this result into the RSH (Eq. 1.16), we obtain the K62 prediction for the anomalous velocity scaling exponents:

$$\zeta_p = \frac{p}{3} - \frac{\mu_K p}{3} \left(1 - \frac{p}{3} \right). \quad (1.21)$$

While this phenomenological log-normal derivation yields an explicit, parabolic correction to K41, its fundamental physical justification, and the framework to generalize beyond log-normality, is best understood through a Large Deviation Principle (LDP) applied to multiplicative cascades.

There are other models that explain the non-linear nature of ζ_p on p , one famous model is, She-Leveque [20, 22, 23] where a hierarchy of structures is assumed with the most singular structure as vortex filaments and corrections are shown to be of the form,

$$\zeta_p = \frac{p}{9} + 2 \left[1 - \left(\frac{2}{3} \right)^{p/3} \right]. \quad (1.22)$$

In the Kolmogorov refined similarity the statistics of coarse-grained dissipation is assumed to be log-normal. She-Leveque corrections can be arrived by taking the coarse-grained dissipation field to be Log-Poisson [22, 23].

LOCAL SELF-SIMILARITY AND MULTIFRACTAL THEORY

Probably the most modern theory of turbulence which rationalizes anomalous scaling comes from Parisi and Frisch in their famous multifractal model [2, 24, 25]. This model extends the refined similarity assumption to local-self similarity [2]. This is motivated from the symmetry of the scaling transformation of the Navier–Stokes equations in the $Re \rightarrow \infty$ limit.

One readily verifies that the Navier–Stokes equations in the limit $\nu \rightarrow 0$ remains invariant under the scaling transformation,

$$\mathbf{x} \rightarrow \lambda \mathbf{x} \quad , \quad \mathbf{u} \rightarrow \lambda^h \mathbf{u} \quad , \quad t \rightarrow t^{1-\lambda} t, \quad (1.23)$$

for any $\lambda > 0$ and any scaling exponent $h \in \mathbb{R}$. This is the *scaling group* of the Euler equations: the equations are invariant under the simultaneous rescaling of space by λ , time by λ^{1-h} , and velocity by λ^h , for an *arbitrary* value of h . Under this condition where the scaling exponent or the Hölder exponent can take any arbitrary values, the velocity differences scale as

$$\delta_\ell u \equiv [\mathbf{u}(\mathbf{x} + \boldsymbol{\ell}) - \mathbf{u}(\mathbf{x})] \cdot \hat{\boldsymbol{\ell}} \sim u_0 \left(\frac{\ell}{L} \right)^h. \quad (1.24)$$

For the Kolmogorov 1941 theory, $h = 1/3$ which refers to global self-similarity. Now, under the assumptions of local-self similarity, we say that the velocity field has a local Hölder exponent h and the relation $\delta_\ell u \sim \ell^h$ is valid with a local exponent $h(\mathbf{x})$ which characterizes the local regularity of the field. Large h corresponds to smooth regions, small h to singular, intense events. The K41 theory corresponds to the degenerate case $h(\mathbf{x}) = 1/3$ everywhere.

Now, where do these exponents live, or more precisely, what is the distribution of the set of points with Hölder exponent $\in [h, h + dh]$. Multifractal theory says that the set of points where $h(\mathbf{x}) \approx h$ forms a fractal set $\mathcal{S}(h)$ of fractal dimension $D(h)$ [2, 24]. The function $D(h)$, the *singularity spectrum*, encodes the geometry of flow corresponding to the Hölder h : it tells us that there are a continuous set of Hölder exponents corresponding to fractal dimension $D(h)$. Now, let us see how to compute the moments of velocity differences based on multifractal theory.

SCALING OF VELOCITY DIFFERENCE MOMENTS

The multifractal spectrum $D(h)$ is not directly observable from the velocity difference statistics, but it governs the scaling of the *structure functions*, which are. The p -th order longitudinal structure function is defined as

$$S_p(\ell) = \langle \delta_\ell u^p \rangle \sim \ell^{\zeta_p}, \quad \eta \ll \ell \ll L. \quad (1.25)$$

To compute ζ_p , we note that the contribution to the average from points with Hölder exponent h scales as ℓ^{hp} , weighted by the probability of finding such a point or probability of intersection of a ball of size ℓ with a fractal set of dimension $D(h)$. In $d = 3$ spatial dimensions, the number of inertial-range cells of size ℓ that is required to cover the fractal set $\mathcal{S}(h)$ is given by $(\ell/L)^{-D(h)}$, which immediately implies that the probability of intersection scales as $(\ell/L)^{3-D(h)}$. So the structure function receives a contribution

$$S_p(\ell) \sim \int \left(\frac{\ell}{L}\right)^{ph} \cdot \left(\frac{\ell}{L}\right)^{3-D(h)} d\mu(h) = \int \left(\frac{\ell}{L}\right)^{ph+3-D(h)} d\mu(h), \quad (1.26)$$

where, $d\mu(h)$ gives the weight of different exponents, whose exact form does not matter, as we will look in the limit $\ell/L \rightarrow 0$ where the dominant contribution of the integral can be approximated by a saddle point approximation. In the limit $\ell/L \rightarrow 0$, this integral is dominated by the value of h that minimizes the exponent, giving

$$\zeta_p = \inf_h [hp + 3 - D(h)]. \quad (1.27)$$

As we mentioned before, any theory must respect the four-fifths law which constraints $\zeta_3 = \inf_h [3h + 3 - D(h)] = 1$. Note that ζ_p is the Legendre transform of $D(h)$ [2]. Conversely, $D(h)$ can be recovered from the measured exponents ζ_p by the inverse Legendre transform,

$$D(h) = \inf_p [hp + 3 - \zeta_p]. \quad (1.28)$$

The K41 prediction $\zeta_p = p/3$ corresponds to the degenerate spectrum $D(1/3) = 3$, which is a monofractal spectrum. While the multifractal framework provides an elegant explanation for the observed multiscaling of the structure functions and quantitatively measures the systematic deviation, it does not a priori provide a way to measure $D(h)$. This is where, looking at the dissipation (coarse-grained dissipation) statistics is helpful and gives a concrete way to measure the multifractal spectrum. There are compelling reasons to recast the framework directly in terms of the coarse-grained dissipation rate ε_ℓ .

While a microscopic theory for the anomalous scaling in fully developed turbulence remains an open question, there has been success in explaining this for the passive scalar turbulence within the Kraichnan framework [26, 27].

MULTIFRACTAL DISSIPATION

Unlike the signed velocity increment δu_ℓ , the coarse-grained dissipation field $\varepsilon_\ell \geq 0$ is a *positive measure* on the fluid domain, and therefore admits, without modification, the full multifractal machinery developed by Halsey et al. [28] for characterizing the distribution and nature of singularities on strange attractors — namely, the generalized dimensions D_q and the singularity spectrum $f(\alpha)$. The bridge between the two descriptions is provided by Kolmogorov's Refined Similarity Hypothesis [21, 29]

$$\varepsilon_\ell \sim \frac{(\delta u_\ell)^3}{\ell}. \quad (1.29)$$

The multifractal framework suggests

$$\varepsilon_\ell \sim \ell^\alpha \quad (1.30)$$

with $\alpha = 3h - 1$ varying locally. Thus the coarse-grained dissipation in balls of size ℓ scales as ℓ^α , with the exponent α varying in space. In other words, the field is locally far from uniform and the variation of α quantifies this non-uniformity. Note that K41 implies $h = 1/3$, and hence a uniform dissipation $\alpha = 0$. Now, we proceed as before and compute moments of ε_ℓ which gives,

$$\langle \varepsilon_\ell^p \rangle = \int (\ell/L)^{3-f(\alpha)} (\ell/L)^{p\alpha} d\mu(\alpha) \sim \ell^{\gamma_p}, \quad (1.31)$$

with $\gamma_p = \inf_\alpha [p\alpha + 3 - f(\alpha)]$. This framework will serve as the theoretical backbone of the first part of this thesis.

MULTIPLICATIVE CASCADES AND LARGE DEVIATION THEORY

The multifractal theory that we have described so far is focused on geometry where singularities are assumed to exist with a distribution characterized by the fractal dimension [2]. There exist a natural interpretation of multifractal theory via the Large Deviation Principle. The mathematical underpinning of anomalous scaling and multifractal theory is deeply rooted in the theory of Large Deviations [30]. One way to understand multifractality using Large Deviation principle is through the Obukhov-Kolmogorov multiplicative cascade model. To build intuition for how Large Deviation Principle govern extreme fluctuations, it is instructive to first consider a toy model before applying the formalism to the multiplicative energy cascade.

LARGE DEVIATION AND COIN TOSS

Consider an experiment consisting of N independent tosses of a fair coin. Let the random variable $X_i \in \{0, 1\}$ represent the outcome of the i -th toss. The law of large numbers dictates that as $N \rightarrow \infty$, the empirical mean $(1/N) \sum_{i=1}^N X_i$ converges to the true expectation value, $\langle X \rangle = 1/2$.

Now, for finite but large N what is the probability of getting a value α . This is same as finding the probability of $n = \alpha N$ heads after N tosses, the probability of this specific outcome is given by the binomial distribution:

$$P(n) = \binom{N}{n} \left(\frac{1}{2}\right)^N. \quad (1.32)$$

Defining the observed mean as $\alpha = n/N$, we seek the probability $P(\alpha)$ of observing an empirical mean α that deviates from $1/2$. Using Stirling's approximation ($x! \approx x^x e^{-x}$) for large N , the probability density can be rewritten as:

$$\begin{aligned} P(\alpha) &\approx \frac{N^N e^{-N}}{(\alpha N)^{\alpha N} (N - \alpha N)^{N - \alpha N} e^{-N}} \left(\frac{1}{2}\right)^N \\ &= \frac{1}{\alpha^{\alpha N} (1 - \alpha)^{N(1 - \alpha)}} \left(\frac{1}{2}\right)^N \\ &= \exp \{ -N [\alpha \log(2\alpha) + (1 - \alpha) \log(2(1 - \alpha))] \}. \end{aligned} \quad (1.33)$$

This result can be cast into the canonical form of a Large Deviation Principle:

$$P(\alpha) \asymp \exp [-NI(\alpha)], \quad (1.34)$$

where $I(\alpha) = \alpha \log(2\alpha) + (1 - \alpha) \log(2(1 - \alpha))$ is the strictly convex *rate function* (or Cramér function). The function $I(\alpha)$ exhibits a global minimum of 0 at $\alpha = 1/2$, consistent with the law

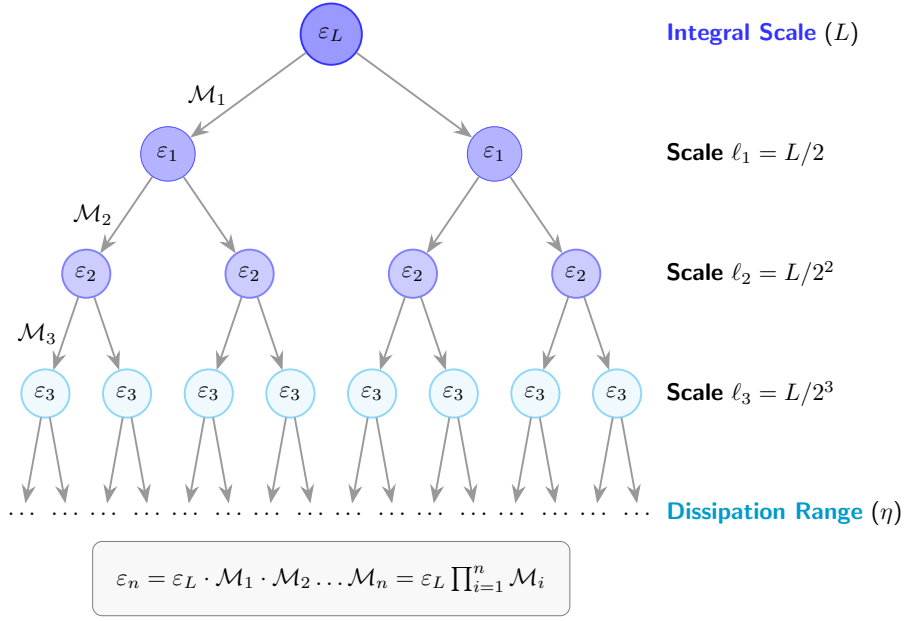


Figure 1.4. A schematic for the Kolmogorov-Obukhov multiplicative process.

of large numbers, but critically, it quantifies the exponential penalty for observing fluctuations away from the mean. The typical fluctuations are given by *central limit theorem* which is obtained by truncated the rate function at the quadratic order around its extrema.

THE MULTIPLICATIVE CASCADE AND LARGE DEVIATION THEORY

We now map this logic directly onto the turbulent dissipation field. Following the Kolmogorov-Obukhov multiplicative cascade process, the coarse-grained dissipation at an intermediate scale ℓ_n is formed by successive, random subdivisions of the large-scale energy flux ε_L . As illustrated in Fig. 1.4, this is represented as a product of dimensionless multipliers $\mathcal{M}_i$:

$$\varepsilon_{\ell_n} = \varepsilon_L \mathcal{M}_1 \mathcal{M}_2 \cdots \mathcal{M}_n = \varepsilon_L \prod_{i=1}^n \mathcal{M}_i. \quad (1.35)$$

Taking the logarithm and dividing by the cascade depth n yields an empirical mean:

$$\alpha \equiv \frac{1}{n} \log \left(\frac{\varepsilon_{\ell_n}}{\varepsilon_L} \right) = \frac{1}{n} \sum_{i=1}^n \log \mathcal{M}_i. \quad (1.36)$$

Assuming a dyadic scale subdivision ($\ell_n = L/2^n$), the cascade depth is proportional to the logarithmic scale ratio: $n = \log_2(L/\ell_n)$.

If we assume the multipliers $\mathcal{M}_i$ are independent and identically distributed (i.i.d.) random variables, Eq. (1.36) represents the sample mean of n independent variables. This allows us to invoke fundamental theorems from probability theory to describe the statistics of the coarse-grained dissipation [2]:

Law of Large Numbers (LLN): In the limit of infinite cascade depth ($n \rightarrow \infty$), the empirical mean converges almost surely to the expectation value of the multipliers. Defining the effective scaling exponent $\alpha = \frac{1}{n} \log(\varepsilon_{\ell_n}/\varepsilon_L)$, we have $\alpha \rightarrow \langle \log \mathcal{M} \rangle \equiv \alpha^*$.

Central Limit Theorem (CLT): For finite n , typical fluctuations around the mean α^* are of order $\mathcal{O}(1/\sqrt{n})$ and follow a Gaussian distribution. Specifically, the probability density function for small deviations behaves as $\mathcal{P}(\alpha \approx \alpha^*) \sim \exp[-n(\alpha - \alpha^*)^2/(2\sigma^2)]$, where $\sigma^2 = \text{Var}[\log \mathcal{M}]$.

Large deviation principle (LDP): While the CLT adequately describes typical fluctuations, the intermittent nature of turbulence is dominated by extreme, rare events (i.e., fluctuations of order $\mathcal{O}(1)$ away from the mean). To capture these tails, one must invoke the Large Deviation Principle (Cramér's Theorem) [30]. The LDP states that the probability density of observing an arbitrary exponent α decays exponentially with the cascade depth:

$$\mathcal{P}(\alpha) \asymp \exp[-nf(\alpha)], \quad (1.37)$$

where $f(\alpha)$ is the rate function (or Cramér function). Because $n \propto \log(L/\ell_n)$, this exponential decay in n translates directly to the power-law scaling of the moments of the coarse-grained dissipation [31].

LAGRANGIAN PICTURE AND MULTIFRACTAL DISSIPATION ALONG TRAJECTORIES

The Eulerian description of turbulence — tracking the velocity field from a fixed point in space — is the natural framework for both experiments and simulations. However, it is not the only natural framework, nor always the most illuminating one. Many physical processes in turbulence are inherently tied to the motion of fluid particles: the dispersion of pollutants, the aggregation of inertial particles, heat and mass transfer, and the dynamics of swimming microorganisms are all governed by what a fluid parcel experiences along its trajectory rather than what an observer at a fixed point measures. This motivates the *Lagrangian* description, in which one follows individual fluid particles as they are advected by the flow. The two descriptions are, of course, related — they describe the same physical flow — but the relationship between their respective statistics may be nontrivial. A Lagrangian particle does not sample the flow uniformly: it is swept preferentially through regions of intense activity, spending time in each region in proportion to the local eddy turnover time rather than uniformly in space. It is this *biased sampling* of the multifractal dissipation field that makes the Lagrangian picture both distinct from and complementary to its Eulerian counterpart, and that gives the bridge between the two its nontrivial structure.

So far, the multifractal description has been developed in the Eulerian frame, where one observes the velocity field at fixed points in space. A complementary and physically natural description follows fluid particles as they are advected by the flow — the *Lagrangian* frame. Along a particle trajectory $\mathbf{X}(t)$, the relevant velocity increment is

$$\delta_\tau v = v(t + \tau) - v(t), \quad (1.38)$$

where τ is a time lag. The Lagrangian structure functions are defined as

$$S_p^L(\tau) = \langle |\delta_\tau v|^p \rangle \sim \tau^{\zeta_p^L}, \quad (1.39)$$

and the K41 prediction for the Lagrangian exponents follows from dimensional analysis with $\langle \varepsilon \rangle$ and τ as the only relevant parameters, giving $\zeta_p^L = p/2$. Just as in the Eulerian case, real turbulent flows deviate from this prediction: the measured exponents ζ_p^L are a nonlinear function of p , with deviations growing with order, reflecting the same intermittent dissipation field responsible for Eulerian anomalous scaling. These Lagrangian exponents are related to their Eulerian counterparts ζ_p through linear bridge relations [32, 33].

The Eulerian–Lagrangian Bridge

Bridge relations connect Eulerian and Lagrangian multifractal properties by assuming that temporal statistics along particle trajectories can be inferred from spatial statistics of the flow through simple dimensional arguments. Such relations work reasonably well for velocity increments, for which empirical scaling exponents measured in the Eulerian and Lagrangian frames are broadly consistent with these mappings [32]. Their validity for energy dissipation, however, remains an open and far less explored question. The traditional argument underlying bridge relations relies on the assumption that Lagrangian tracers sample dissipation in a manner that faithfully reflects the local Eulerian structure of turbulence. Specifically, it is assumed that, over a time interval τ , a tracer remains within a ball $\mathcal{B}_\ell$ of spatial size ℓ and samples a dissipation field that is approximately constant over this region and time. The central assumption is that this time interval τ is given by the local eddy turnover time, $\tau_\ell = \varepsilon_\ell^{-1/3} \ell^{2/3}$, where ε_ℓ denotes the Eulerian dissipation coarse-grained over $\mathcal{B}_\ell$. One then writes

$$\tilde{\varepsilon}_\tau \sim \varepsilon_\ell \quad \text{for} \quad \tau \sim \tau_\ell = \varepsilon_\ell^{-1/3} \ell^{2/3}. \quad (1.40)$$

This relation implies that the statistics of $\tilde{\varepsilon}_\tau$ can be inferred from those of ε_ℓ , and conversely. Such relations have achieved notable success for velocity fluctuations and are often regarded as robust and broadly applicable across turbulent flows. Their extension to dissipation, however, is far less secure. In this case, bridge relations rely on assumptions about how Lagrangian tracers sample spatially coarse-grained Eulerian fields and about how local dissipation fluctuations are related to the eddy turnover dynamics experienced along particle trajectories.

Multifractal Spectrum of Lagrangian Dissipation

Similar to the Eulerian case where we computed moments of coarse-grained dissipation $\langle \varepsilon_\ell \rangle^p \sim \ell^{\gamma_p}$, with $\gamma_p = \inf_\alpha (p\alpha + 3 - f(\alpha))$, here we ask, how does the moments of coarse-grained dissipation $\langle \varepsilon_{\tau_\ell} \rangle^p$ scales with the coarse-graining time along the Lagrangian trajectory. Essential step of this calculation is to put $\langle \varepsilon_\ell \rangle^p$ and $\langle \varepsilon_{\tau_\ell} \rangle^p$ in the same footing and apply the bridge relation. To understand this more precisely we write the explicit steps,

$$\varepsilon_\ell = \int_{\mathcal{B}_\ell} \varepsilon_0(\mathbf{x}, t) \frac{d^3x}{|\mathcal{B}_\ell|} \approx \int_0^{\tau_\ell} \int_{\mathcal{B}_\ell} \varepsilon_0(\mathbf{x}, t) \frac{d^3x}{|\mathcal{B}_\ell|} \frac{dt}{\tau_\ell} \quad (1.41)$$

Now, we convert the space integral to an integral over the trajectories originating from the ball of size ℓ

$$\varepsilon_\ell \approx \int_{\mathcal{B}_\ell} \int_0^{\tau_\ell} \varepsilon_0(\mathbf{X}(t; x_0), t) \frac{dt d^3 x_0}{\tau_\ell |\mathcal{B}_\ell|} = \int_{\mathcal{B}_\ell} \tilde{\varepsilon}_{\tau_\ell} \frac{d^3 x_0}{|\mathcal{B}_\ell|}.$$

Now if we apply a mean-field like assumption that dissipation experienced by all the tracer in a given ball $\mathcal{B}_\ell$ is approximately same and drop the integral giving

$$\langle \varepsilon_\ell \rangle^p \approx \langle \varepsilon_{\tau_\ell} \rangle^p \quad (1.42)$$

For the Lagrangian case, the same multifractal measure applies, but now the coarse-graining time τ_ℓ depends on α through:

$$\tau_\ell \sim \ell^{(2-\alpha)/3}, \quad (1.43)$$

which can be inverted to express ℓ in terms of τ_ℓ and α :

$$\ell \sim \tau_\ell^{3/(2-\alpha)}. \quad (1.44)$$

Substituting, the coarse-grained dissipation and the measure weight become:

$$\varepsilon_{\tau_\ell} \sim \ell^\alpha \sim \tau_\ell^{3\alpha/(2-\alpha)}, \quad \ell^{3-f(\alpha)} \sim \tau_\ell^{3(3-f(\alpha))/(2-\alpha)}, \quad (1.45)$$

so that the moment integral becomes:

$$\langle \varepsilon_{\tau_\ell}^p \rangle \sim \int \tau_\ell^{3p\alpha/(2-\alpha)} \cdot \tau_\ell^{3(3-f(\alpha))/(2-\alpha)} d\alpha \quad (1.46)$$

Evaluating by saddle point as $\tau_\ell \rightarrow 0$, we obtain:

$$\langle \varepsilon_{\tau_\ell}^p \rangle \sim \tau_\ell^{\gamma_p^L}, \quad \gamma_p^L = \inf_\alpha \frac{3(p\alpha + 3 - f(\alpha))}{2 - \alpha}. \quad (1.47)$$

Equation (1.47) is the Lagrangian analogue of the Eulerian Legendre transform (1.27): it expresses the scaling of Lagrangian dissipation moments entirely in terms of the multifractal rate function $f(\alpha)$, which is in principle accessible from Eulerian measurements. The K41 prediction corresponds to $f(\alpha) = 3$ and $\alpha = 0$, giving $\gamma_p^L = p$ — a linear function, just as $\zeta_p^L = p/2$ is linear in p .

In Chapter 2 we will take a critical look at the theory, specifically the Large Deviation picture, that we have described so far and try to understand the Eulerian and Lagrangian dissipation in a unified way.

1.2 CHAOS AND PREDICTABILITY

A defining feature of fully developed turbulence is its chaotic nature. It is perhaps the most celebrated example of a chaotic many-body system. In contrast to other examples of chaotic, classical, many-body systems, turbulent flows have the distinction of being central across the natural world. The degree of chaos is quantified in a dynamical system using Lyapunov exponents.

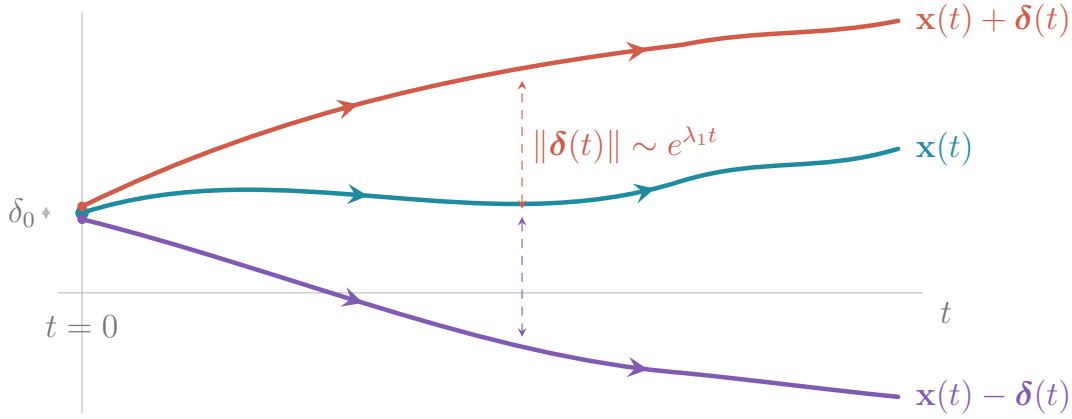


Figure 1.5. Two trajectories initialized within δ_0 of the reference trajectory $\mathbf{x}(t)$ diverge exponentially at rate λ_1 , the largest Lyapunov exponent.

Understanding chaos in the Navier–Stokes equations is not merely of mathematical interest – it is central to some of the most important open questions in fluid mechanics. A positive Lyapunov exponent sets an intrinsic *predictability* time for turbulent flows. Errors in initial conditions – whether from measurement noise or numerical discretizations – grow exponentially on a timescale λ_1^{-1} , placing a fundamental limit on weather forecasting and other applications that is entirely independent of computational power. The chaotic nature of a turbulent flow, directly controls the rate of *mixing*.

Consider a finite-dimensional dynamical system:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \quad (1.48)$$

Given two initially close trajectories $\mathbf{x}(t)$ and $\mathbf{x}(t) + \delta(t)$, the evolution of the infinitesimal perturbation $\delta(t)$ is governed by the linearized equation:

$$\dot{\delta} = \mathbf{J}(\mathbf{x}(t)) \delta, \quad (1.49)$$

where $\mathbf{J} = D\mathbf{F}(\mathbf{x})$ is the Jacobian of $\mathbf{F}$ evaluated along the reference trajectory. The Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ characterize the asymptotic average rates of growth or contraction of perturbations along the n principal directions in phase space. Formally, the largest Lyapunov exponent is defined as:

$$\lambda_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|\delta(t)\|}{\|\delta(0)\|}. \quad (1.50)$$

If the largest Lyapunov exponent is positive then the system is said to be chaotic. Nearby trajectories diverge exponentially in time, rendering long-time prediction impossible in practice, even when the equations of motion are known exactly. Figure 1.5 shows a schematic for an exponentially diverging trajectory.

This framework is well-established for finite-dimensional systems, but turbulence is governed by the Navier–Stokes equations, which define an infinite-dimensional dynamical system.

In this setting the phase space is replaced by a function space, and the familiar notion of a phase-space trajectory requires more care. A rigorous finite-dimensional description can still be recovered if the dynamics collapse onto a finite-dimensional attractor — this is known to happen for certain driven-dissipative PDEs, but establishing it requires proving the existence of absorbing sets and global attractors, most of which remains an open problem for the three-dimensional Navier–Stokes equations.

Whether turbulence effectively evolves on such a finite-dimensional attractor is therefore a central open question. In practice, two approaches are used to bypass this difficulty and estimate the degree of chaos directly. The first is the *Lagrangian approach*, where the flow is seeded with tracer particles and their separation is tracked over time to estimate the largest Lyapunov exponent. The second is the *Eulerian approach*, where a velocity field is perturbed infinitesimally and one tracks how the two fields diverge, or decorrelate, in time. In this thesis we take the Eulerian view and aim to understand the scaling laws that govern chaos in turbulent flows.

Turbulent flows are not merely statistically complex—they are chaotic in the sense of dynamical systems. As we have seen, turbulent flows are *multiscale* and hence admit the possibility of scale-dependent Lyapunov exponents [34, 35]. There had been a lot of work in the atmospheric community regarding scale-dependent Lyapunovs [36, 37], one intuition is weather is predictable at large length and time-scales but it is an extremely difficult task to predict weather at short time scales and locally. So, somehow the predictability depends on how much scale resolution we are talking about.

RUELLE’S ARGUMENT

The first systematic prediction for the scaling of the largest Lyapunov λ with Re was given by Ruelle [38], using dimensional arguments rooted in Kolmogorov’s theory. The key observation is that chaos is driven by the fastest, most intense velocity gradients in the flow—those at the smallest active scales. One therefore identifies λ with the inverse of the fastest timescale of the turbulent cascade.

In K41, three timescales are naturally associated with a scale ℓ in the inertial range. The eddy turnover time,

$$\tau_\ell = \frac{\ell}{\delta u_\ell}, \quad (1.51)$$

measures how long a structure of size ℓ takes to turn over, where δu_ℓ is the characteristic velocity difference at scale ℓ . The viscous diffusion time,

$$\tau_\ell^{\text{visc}} = \frac{\ell^2}{\nu}, \quad (1.52)$$

measures the timescale on which viscosity acts at scale ℓ . Finally, the large-scale turnover time is

$$\tau_L = \frac{u_{\text{rms}}^2}{\varepsilon}, \quad (1.53)$$

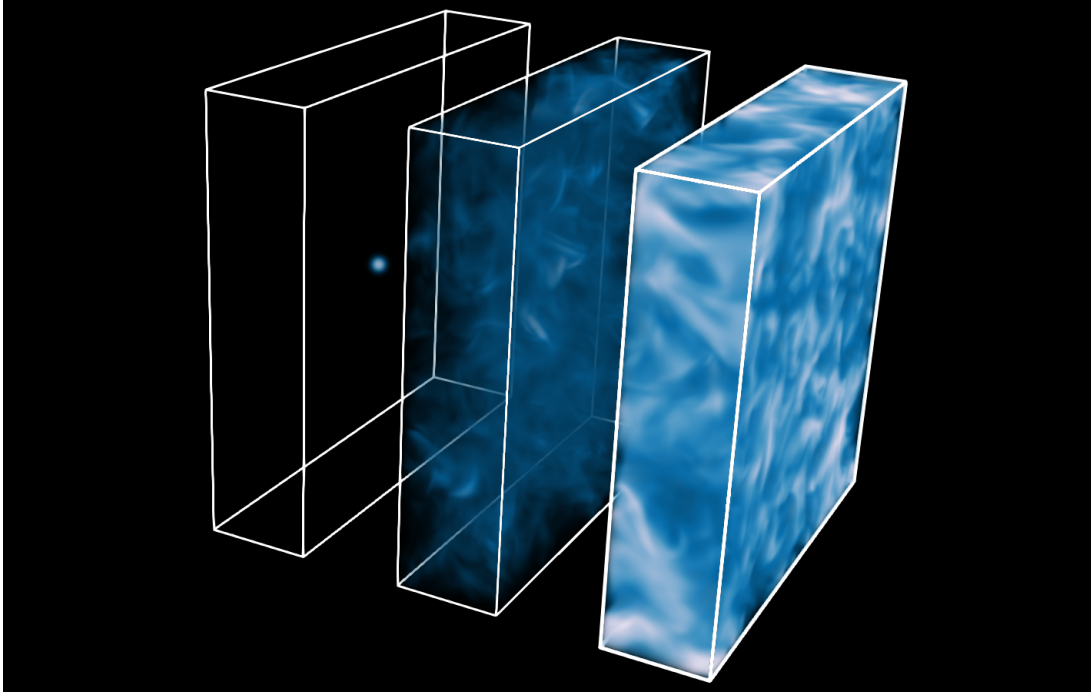


Figure 1.6. Evolution of an initially localized perturbation. We show snapshots of how an initially localized perturbation $|\delta\mathbf{x}(\mathbf{x}, t)|^2$ evolves spatio-temporally in a turbulent flow.

which sets the overall timescale of the energy-containing motions. The dissipation scale η is defined precisely as the scale at which these two timescales balance,

$$\tau_\ell|_{\ell=\eta} = \tau_\ell^{\text{visc}}|_{\ell=\eta}, \quad (1.54)$$

which, combined with the K41 scaling $\delta u_\ell \sim (\varepsilon \ell)^{1/3}$, yields the Kolmogorov scale

$$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}, \quad \frac{\eta}{L} \sim Re^{-3/4}. \quad (1.55)$$

Since λ is set by the fastest timescale, which occurs at the dissipation scale, Ruelle identified $\lambda \sim \tau_\eta^{-1}$ and by taking a ratio of τ_η and τ_L .

$$\lambda \sim \tau_L^{-1} \sqrt{Re}, \quad (1.56)$$

which follows from combining Eqs. (1.55) and (1.53). This is Ruelle's prediction: the largest Lyapunov exponent scales as $\lambda \sim Re^{1/2}$ in units of the integral time scale.

THE EFFECT OF INTERMITTENCY

Ruelle's argument, elegant as it is, rests on the K41 assumption that the velocity field scales uniformly with a single exponent $h = 1/3$ throughout the flow. In the presence of intermittency, the local scaling of velocity differences is not universal but depends on position, characterized by a fluctuating Hölder exponent $h(\mathbf{x})$ as discussed while discussing multifractality. The natural generalization is to replace the K41 scaling with the multifractal scaling,

$$\delta u_\ell \sim (L\varepsilon)^{1/3} \left(\frac{\ell}{L} \right)^h, \quad (1.57)$$

for a locally fluctuating exponent h . The dissipation scale is no longer a fixed length but becomes a *fluctuating* quantity that depends on h : equating τ_ℓ and τ_ℓ^{visc} with the scaling (1.57) yields [39]

$$\frac{\eta(h)}{L} \sim Re^{-1/(1+h)}, \quad (1.58)$$

which reduces to the K41 result $\eta/L \sim Re^{-3/4}$ when $h = 1/3$, as expected. Consequently, the Lyapunov exponent itself becomes a function of h [40],

$$\lambda(h) \sim \tau_{\eta(h)}^{-1} = \tau_L^{-1} Re^{(1-h)/(1+h)}, \quad (1.59)$$

and the observed, globally measured Lyapunov exponent reflects a nontrivial average over the multifractal measure of h values. Setting $h = 1/3$ in Eq. (1.59) recovers Ruelle's prediction (1.56), confirming that K41 is the special case of a more general intermittency-dependent scaling.

The central difficulty, however, is that it is not a priori clear how the effect of intermittency enters the *observed* scaling of λ with Re . The globally measured LLE is not simply $\lambda(h)$ evaluated at the most probable h : the exponential sensitivity of (1.59) to h , combined with the fractal geometry of the sets on which each h is realized, means that rare, intense events with $h < 1/3$ may dominate the Lyapunov scaling even though they occupy a negligible fraction of the volume. Establishing precisely how intermittency deforms the Ruelle prediction—and quantifying the departure from $\lambda \sim Re^{1/2}$ —requires going beyond dimensional arguments and engaging directly with the Navier–Stokes equations. This is the problem we take up in Chapter 3.

EULERIAN DESCRIPTION VIA DECORRELATORS

We now construct two, *nearly* identical, velocity fields $\mathbf{u}_0^A \equiv \mathbf{u}_{\text{steady}}$ and $\mathbf{u}_0^B = \mathbf{u}_0^A + \delta\mathbf{u}_0$, where the form of $\delta\mathbf{u}_0$ ensures that system B remains incompressible. We choose the amplitude of perturbation ϵ_0 sufficiently small [41] and localise it in space around $\mathbf{x}_0$; the precise choice of $\mathbf{x}_0$ is irrelevant and can be chosen to be the center of our domain.

We choose these fields — $\mathbf{u}_0^A$ and $\mathbf{u}_0^B$ — as initial conditions $t = 0$ for the Navier–Stokes equation, evolve them independently in time, and calculate the velocity difference $\delta\mathbf{u}(\mathbf{x}, t) \equiv \mathbf{u}^B(\mathbf{x}, t) - \mathbf{u}^A(\mathbf{x}, t)$ for $t > 0$. From such velocity differences, we are able to construct the decorrelator $\phi(\mathbf{x}, t) \equiv \frac{1}{2}|\delta\mathbf{u}(\mathbf{x}, t)|^2$ [42, 43]. It is easy to show, starting from the Navier–Stokes equation written separately for systems A and B, that the decorrelator follows an evolution (written in component form and sum over repeated indices assumed)

$$\frac{\partial\phi(\mathbf{x}, t)}{\partial t} = \partial_i W_i - \delta u_i S_{ij} \delta u_j + \nu \delta u_i \nabla^2 \delta u_i \quad (1.60)$$

with the strain-rate tensor $S_{ij} = \frac{1}{2}(\partial_i u_j^A + \partial_j u_i^A)$ and

$$W_i = (u_i^A + \delta u_i) \phi + \delta u_i \int d\mathbf{x}' G(\mathbf{x}, \mathbf{x}') [\partial_{jk}^2 (u_j^A \delta u_k + \delta u_j u_k^A + \delta u_j \delta u_k)], \quad (1.61)$$

where $G(\mathbf{x}, \mathbf{x}')$ is the Green's function obtained from the Poisson equation for pressure, and ν the coefficient of kinematic viscosity. Spatio-temporal spread of $\phi(\mathbf{x}, t)$ is shown in Fig. 1.6. We analyse this problem further in Chapter 3, using decorrelator we investigate the Reynolds-number

dependence of the maximal Lyapunov exponent. We show that the Lyapunov exponent scales with Reynolds number as $\lambda \sim Re^\alpha$, with an exponent $\alpha = 0.59 \pm 0.04$ exceeding the classical mean-field prediction. By explicitly separating the nonlinear strain and viscous contributions to decorrelator growth, we show that this departure is associated by intermittent fluctuations of the strain-rate tensor, which dominate the short-time growth of the infinitesimal perturbations over viscous damping. We also perform tests using a reduced shell model and finally show that the dynamical origin of chaotic divergence in turbulence is closely linked to intermittent strain-rate fluctuations.

1.3 ANALYSIS APPROACH TO THE NAVIER–STOKES EQUATIONS

So far, our approach to turbulence has been rooted in statistical physics. Indeed, the complexity and limited predictability of turbulent flows justify the use of statistical-physics tools to describe their dynamics, and statistical approaches have been successful in capturing key aspects of turbulence, offering explanations that align well with experimental observations [2, 44–46]. The Navier–Stokes equations, however, constitute a deterministic (infinite-dimensional) dynamical system. An important goal of mathematical analysis is therefore to recover the main facts of turbulence theory within a deterministic, assumption-free framework, without relying on phenomenological hypotheses such as the existence of a self-similar cascade or a clear separation between the large-scale forcing and the small-scale dissipation [13, 47–50]. When such assumptions are satisfied, the statistical framework effectively predicts scaling laws; when they are violated, for instance, due to forcing at intermediate scales, statistical predictions often lose their validity¹. Mathematical bounds derived directly from the Navier–Stokes equations, by contrast, do not rely on any such assumptions, and therefore provide a more general framework for computing quantities such as the energy dissipation rate even when no cascade picture is available.

One of the central questions in this programme is whether empirical observations such as the *dissipative anomaly*—the persistence of a finite energy-dissipation rate ε in the inviscid limit ($\nu \rightarrow 0$)—can be shown to be consistent with the Navier–Stokes equations without any additional assumptions. Two complementary lines of inquiry address this question, and we describe each in turn. The *first* is to derive rigorous bounds on the energy dissipation rate directly from the equations of motion. Mathematical results of this type generally take the form of bounds for the time average of Sobolev norms of Navier–Stokes weak solutions; interestingly, statistical predictions often correspond to the saturation of these bounds. The *second* is to establish the existence of weak solutions to the Euler equations and to determine whether they coincide with the $\nu \rightarrow 0$ limit of Navier–Stokes solutions.

¹For instance, if the Fourier spectrum of the forcing scales as a power of k – a setting appropriate for turbulent flows generated by fractal grids [51, 52] – the law of *finite energy dissipation* is violated and ε grows linearly with Re [53].

BOUNDS ON THE ENERGY DISSIPATION RATE

We consider the Navier–Stokes equations on a domain of characteristic scale ℓ , forced by a divergence-free body force $\mathbf{f}$, $\nabla \cdot \mathbf{f} = \mathbf{0}$. Two non-dimensional parameters characterise the system. The first is the Reynolds number, constructed from a characteristic velocity U and the domain scale,

$$Re = \frac{U\ell}{\nu}, \quad (1.62)$$

which measures the ratio of inertial to viscous forces. The second is the *Grashof number*,

$$Gr = \frac{\ell^3 \langle \|\mathbf{f}\|_2 \rangle}{\nu^2}, \quad (1.63)$$

where $\|\mathbf{f}\|_{L^2}$ is the L^2 -norm of $\mathbf{f}$, defined as $(1/\ell^2) \int_{\Omega} |\mathbf{f}|^2 \, d\mathbf{x}$, and $\langle \cdot \rangle$ denotes a long-time average. Crucially, Gr is constructed from the forcing amplitude alone and is therefore an *input* parameter, in contrast to Re , which is a response of the system and is known only a posteriori. This distinction matters: bounds expressed in terms of Gr are genuine a priori estimates, while those expressed in terms of Re require an additional step relating the two. The early estimates of this sort for the body-forced Navier–Stokes equations were indeed expressed in terms of Gr , introduced as a natural mathematical input parameter by Foias et al. [54]. However, experimental measurements and statistical theories are typically expressed in terms of Re , so a meaningful comparison requires converting the mathematical bounds to a Re -dependent form.

To derive such bounds, we take the L^2 inner product of the Navier–Stokes equation (1.1) with $\mathbf{u}$ and integrate over the domain, yielding the instantaneous energy equation,

$$\frac{1}{2} \frac{d}{dt} \int |\mathbf{u}|^2 \, d\mathbf{x} = -\nu \int |\nabla \mathbf{u}|^2 \, d\mathbf{x} + \int \mathbf{f} \cdot \mathbf{u} \, d\mathbf{x}. \quad (1.64)$$

Applying the Cauchy–Schwarz inequality to the forcing term,

$$\int \mathbf{f} \cdot \mathbf{u} \, d\mathbf{x} \leq \|\mathbf{f}\|_2 \|\mathbf{u}\|_2, \quad (1.65)$$

one obtains the energy inequality

$$\frac{1}{2} \frac{d}{dt} \int |\mathbf{u}|^2 \, d\mathbf{x} \leq -\nu \int |\nabla \mathbf{u}|^2 \, d\mathbf{x} + \|\mathbf{f}\|_2 \|\mathbf{u}\|_2. \quad (1.66)$$

Taking a time average over $[0, T]$ and applying Cauchy–Schwarz once more to separate the time-averaged product gives

$$\varepsilon \equiv \left\langle \nu \int |\nabla \mathbf{u}|^2 \, d\mathbf{x} \right\rangle_T \leq \langle \|\mathbf{u}\|_2 \|\mathbf{f}\|_2 \rangle_T + \mathcal{O}(T^{-1}) \leq \langle \|\mathbf{u}\|_2 \rangle_T \langle \|\mathbf{f}\|_2 \rangle_T + \mathcal{O}(T^{-1}), \quad (1.67)$$

where the $\mathcal{O}(T^{-1})$ term accounts for the boundary contribution from the time derivative and vanishes in the long-time limit.² Expressing $\langle \|\mathbf{u}\|_2 \rangle_T$ through U and $\langle \|\mathbf{f}\|_2 \rangle_T$ through Gr , one arrives at the primary bound,

$$\varepsilon \leq \nu^3 \ell^{-4} Re Gr + \mathcal{O}(T^{-1}). \quad (1.68)$$

²This is guaranteed only if the kinetic energy is uniformly bounded in time, which can be proved by applying Cauchy–Schwarz, Young’s inequality, and Gronwall’s lemma. See the later sections for a sketch of the proof.

This is a rigorous inequality: no assumptions are made about the nature of the flow, no closure hypotheses are invoked, and there is no restriction on Re .

To eliminate Gr in favour of Re , one requires an estimate relating the two. For periodic boundary conditions and square-integrable forcing, Doering and Foias [8] established

$$Gr \leq c_0(Re^2 + Re), \quad (1.69)$$

for a dimensionless constant c_0 . Substituting into (1.68) and retaining leading-order terms,

$$\varepsilon \leq \nu^3 \ell^{-4} (c_0 Re^3 + c_1 Re^2), \quad (1.70)$$

for constants c_0, c_1 . Forming the *normalised dissipation rate* $D = \varepsilon \ell / U^3$, the quantity most directly relevant to the dissipative anomaly, this becomes

$$D \equiv \frac{\varepsilon \ell}{U^3} \leq c_0 + c_1 Re^{-1}. \quad (1.71)$$

In the limit $Re \rightarrow \infty$, the bound on the normalised dissipation approaches a finite constant, entirely consistent with anomalous dissipation in fully developed turbulence. The laminar limit $Re \rightarrow 0$ is also captured: the characteristic strain rate scales as U/ℓ , so $\varepsilon \sim \nu(U/\ell)^2$, and normalising gives $\varepsilon \ell / U^3 \sim \nu / (U\ell) = Re^{-1}$, consistent with the second term in (1.71). It is striking that these scaling relations emerge as rigorous inequalities with no physical input whatsoever.

WEAK SOLUTIONS AND THE ONSAGER CONJECTURE

The dissipative anomaly itself hints at the existence of *irregular* solutions. One way to see this is through the Kolmogorov scaling assumption, which assigns a Hölder exponent h to the velocity increments, $\delta u \sim \ell^h$. In more mathematically precise terms, considering $|\langle (\delta u)^3 \rangle| \leq \langle |\delta u|^3 \rangle$, one can write

$$\langle |\delta u_\ell|^3 \rangle / \ell \leq C \ell^{3h-1}. \quad (1.72)$$

In the limit $\ell \rightarrow 0$, the energy dissipation vanishes unless $h \leq 1/3$. This suggests the existence of “rough” velocity fields for which $h \leq 1/3$ and the gradients are not classically defined. This observation is closely connected to the Onsager conjecture [15]: in such rough fields, classical derivatives in the Navier–Stokes equations fail because gradients can blow up as the scale ℓ approaches zero. Mathematically, solutions can still exist in a distributional sense, known as *weak solutions*. There has been considerable progress on this front [17, 55, 56], largely by moving the analysis into scale space [17, 57], which makes the mathematics regular enough to handle. Weak solutions thus allow a rigorous analysis of energy dissipation and non-smooth behavior that classical formulations cannot describe.

TWO-DIMENSIONAL TURBULENCE AND EFFECTS OF FORCING

We employ the analysis approach to establish rigorous bounds on dissipation in two-dimensional turbulence. In two dimensions, the Navier–Stokes equations admit two quadratic invariants for

strong solutions: energy and enstrophy. Enstrophy is defined as, $\nu \ell^{-2} \langle \int_{\Omega} \omega^2 dx \rangle = \nu \ell^{-2} \|\omega\|_{L^2}^2$, where ω is the vorticity $\omega = e_3 \cdot (\nabla \times \mathbf{u})$. Estimation of dissipation bound requires estimating two quantities, the energy dissipation rate, ε , and the enstrophy dissipation rate, χ . To define the enstrophy dissipation rate we need to vorticity equation corresponding to (1.1) and write down the enstrophy balance equation. Taking the curl of Eq. (1.1) yields the vorticity equation

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega + \phi, \quad (1.73)$$

where $\phi = e_3 \cdot (\nabla \times \mathbf{f})$. The enstrophy balance is obtained by multiplying Eq. (1.73) by ω and integrating over space

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \nu \|\nabla \omega\|_{L^2}^2 = \ell^{-2} \int_{\Omega} \omega \phi dx. \quad (1.74)$$

The presence of these two invariants alters the dynamics of the system. Specifically, the Kraichnan–Leith–Batchelor (KLB) theory posits that an inverse energy transfer toward large scales occurs simultaneously with a direct enstrophy cascade toward small scales (see Refs. [58, 59] for a review). This phenomenology implies that, in a forced state, ε vanishes as viscosity approaches zero, while the time-averaged enstrophy dissipation rate χ remains finite.

Similar to the three-dimensional case, mathematical estimates for ε and χ have traditionally been formulated in terms of the Grashof number Gr [60–62]; however, comparing these against experimental and numerical data requires converting them into a Re -dependent form. Rigorous upper bounds as a function of Re were derived by Alexakis and Doering [63] and Gibbon and Pavliotis [64] for periodic boundary conditions, provided the second derivative of forcing is square-integrable.

Regarding the forced case in two dimensions, existing Re -dependent estimates for ε and χ require forcing with a high degree of regularity³. In contrast to the three-dimensional setting, there has been, to our knowledge, no extension of such estimates to less regular forcing. Nevertheless, weak solutions are known to exist for sufficiently less-regular forcing⁴ and numerical simulations have shown that forcing regularity significantly influences turbulent flow behavior even in two dimensions [65, 66]. So, it is natural to ask if it is possible to extend the bounds on energy and enstrophy dissipation calculations to the full range of forcing regularity allowed by the theory of weak solutions. This is the precise question we will explore in Chapter 5.

DEGREES OF FREEDOM AND THE ATTRACTOR DIMENSION

A related quantity of interest is the number of *degrees of freedom* in turbulent flow. Landau and Lifshitz used dimensional analysis to give a heuristic account of this count. The idea is, if there is a minimum length scale ℓ_{Min} of the flow, usually identified by dimensional analysis, the number of boxes need to fill up the whole domain is proportional to ℓ_{Min}^{-d} , where d is the embedding dimension. So, a working definition of *degrees of freedom* is given by ℓ_{Min}^{-d} . In three dimensions the minimum scale is the *Kolmogorov dissipation scale* η , while in two-dimensional turbulence the analogous scale is the *Kraichnan scale*.

³At the least the second derivative of forcing should be square integrable, see [63, 64].

⁴In Chapter 5 we rigorously define what we mean by *forcing regularity*.

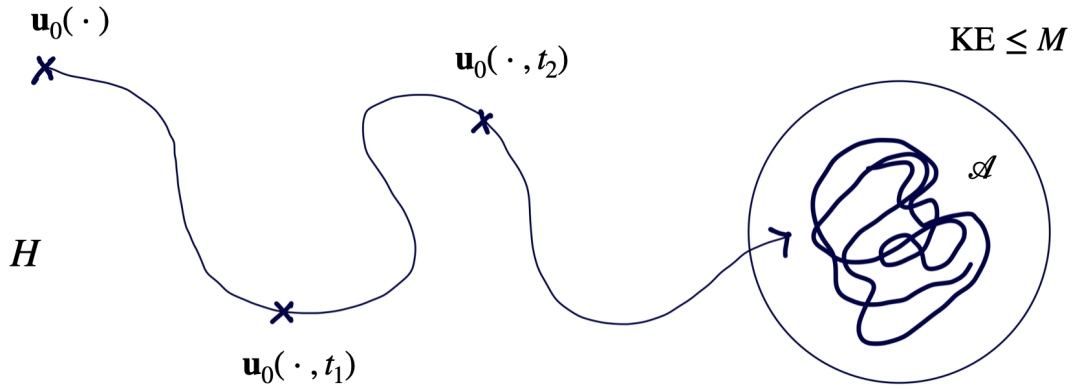


Figure 1.7. Full phase space is denoted by H and the bounded energy solutions are denoted by a ball M . The solutions asymptotically approaches $\mathcal{A}$. Figure adapted from Ref. [67].

Mathematically, an alternative and rigorous definition is provided by the dimension of the *global attractor*. Driven dissipative dynamical systems, often, despite being posed on an infinite-dimensional function space, exhibit a remarkable property in the presence of forcing and dissipation: their long-time dynamics collapse onto a finite-dimensional object. Let us try to understand this by sketching the steps. Given the Navier–Stokes equations (1.1), can we prove that the energy is bounded?

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 = \nu \|\nabla u\|_{L^2}^2 + \frac{1}{\ell^2} \int_{\Omega} f(x) \cdot u(x, t) dx. \quad (1.75)$$

Applying the Cauchy-Schwarz inequality $\ell^{-2} \int_{\Omega} f \cdot u dx \leq \ell^{-2} \|f\|_{L^2} \|\nabla u\|_{L^2}$ and Young's inequality to the forcing term gives

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 \leq -\frac{\nu}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{2\nu} \|f\|_{L^2}^2. \quad (1.76)$$

Then, the Poincaré inequality implies

$$\frac{d}{dt} \left(\frac{1}{2} \|u\|_{L^2}^2 \right) \leq -\frac{4\pi^2\nu}{\ell^2} \left(\frac{1}{2} \|u\|_{L^2}^2 \right) + \frac{1}{2\nu} \|f\|_{L^2}^2. \quad (1.77)$$

Finally Gronwall's lemma applied to this linear differential inequality gives

$$\frac{1}{2} \|u\|_{L^2}^2 \leq \frac{1}{2} \|u_0\|_{L^2}^2 \exp(-4\pi^2\nu t/\ell^2) + \frac{\ell^2}{8\pi^2\nu^2} \|f\|_{L^2}^2 (1 - \exp(-4\pi^2\nu t/\ell^2)). \quad (1.78)$$

Hence $\|u\|_{L^2}^2$ is uniformly bounded in time and in the long time limit

$$\limsup_{t \rightarrow \infty} \frac{1}{2} \int |\mathbf{u}|^2 dx \leq M : \frac{\ell^2}{8\pi^2\nu^2} \int |f|^2 dx \quad (1.79)$$

Figure 1.7, adapted from Ref. [67], illustrates this concept of a bound restricts the function space. These bounds guarantee that the long time dynamics settles down on a subset of H . If further bounds on enstrophy dissipation can be derived then one can restrict the phase-space further and finally to the *global attractor* $\mathcal{A} \subset H$.

$\mathcal{A}$ is a compact set on which the long-time dynamics is confined [68] (see also Refs. [69, 70]). Thus, asymptotically in time, any initial condition outside $\mathcal{A}$ approaches it, while any trajectory starting on $\mathcal{A}$ remains there for all time. The existence of such an attractor reflects the effectively finite-dimensional behavior of an otherwise infinite-dimensional dynamical system. Bounds for the fractal dimension of the attractor, $d_f(\mathcal{A})$, indeed provide a mathematically rigorous notion of the ‘degrees of freedom’ of the system (see also refs. [67, 71] for an informal introduction). Restricting to periodic boundary conditions – the setting considered here – Constantin, Foias, and Temam [72] were the first to obtain an estimate of $d_f(\mathcal{A})$ in terms of Gr (see also [73] for a simpler proof based on the vorticity rather than the velocity field).

$$d_L \leq c_1 Gr^{2/3} (1 + \log Gr)^{1/3} \quad (1.80)$$

Let us pause and try to understand this result intuitively. We recall that η_χ is the analogue of Kolmogorov scale η in two-dimensional turbulence, built using the viscosity ν and the space-time averaged viscous enstrophy dissipation $\chi = \nu \langle |\Delta \mathbf{u}|^2 \rangle$ via

$$\eta_\chi = \nu^{1/2} \chi^{-1/6}. \quad (1.81)$$

A classical estimate for the two-dimensional Navier–Stokes equations gives $\chi \leq \|f\|_{L^2}^2 L^{-2} \nu^{-1}$, from which one recovers $\eta_\chi/L \sim Gr^{-1/3}$. The bound (1.80) therefore agrees with the Kraichnan length scale to within logarithmic corrections, and does so on entirely rigorous grounds. Note that the scaling $d \sim \eta_\chi^{-2}$ for the two-dimensional attractor agrees with Landau-Lifshitz *degrees of freedom* within logarithmic corrections.

Later, Gibbon and Pavliotis [64] converted this estimate into a Re -dependent estimate under the more restrictive assumption of forcing (second derivative of forcing is square integrable) is given by

$$d_f(\mathcal{A}) \leq c_0 Re (1 + \ln Re)^{1/3}, \quad (1.82)$$

This bound is also consistent with the notion of *degrees of freedom*. Since η_χ is the smallest length scale in the system, we identified $d_f(\mathcal{A}) \sim (\ell/\eta_\chi)^2$. Comparing this interpretation of $d_f(\mathcal{A})$ with (1.82) yields an estimate of the enstrophy dissipation scale, $\eta_\chi \sim \ell Re^{-1/2}$, consistent with the KLB theory [58].

These rigorous results on attractor dimension, shown here, are for forcings which are highly regular but existence of attractor is proved for forcing with lower regularity [72]. In Chapter 5 we generalize these results for two-dimensional Navier–Stokes for the full forcing regularity.

1.4 ONE-DIMENSIONAL HYDRODYNAMIC SYSTEMS

The final two chapters of this thesis concern hydrodynamic models beyond the incompressible Navier–Stokes equations. We focus mainly on one-dimensional hydrodynamic systems and their possible extensions to two dimensions. One-dimensional models bypass many of the complications inherent to three-dimensional Navier–Stokes turbulence, and have long served as testing grounds for ideas that are difficult to access in full generality. The canonical example is

the Burgers equation,

$$\partial_t u + u \partial_x u = \nu \partial_x^2 u, \quad (1.83)$$

Its appeal lies in combining the essential nonlinearity of fluid motion with sufficient analytical tractability to admit exact solutions [10].

There are two natural routes to arriving at Burgers-like equations, and they reflect two complementary views. The first is a *top-down* approach: one begins with the Navier–Stokes equations which is essentially a force-balance equation and reduces to one dimension, discarding the incompressibility constraint. The inviscid ($\nu = 0$) Burgers equation can be obtained via a Hamiltonian framework as well [74]. The second is a *bottom-up* approach: one begins with a microscopic lattice model and derives an hydrodynamic description in the continuum limit. The lattice model corresponding to Burgers equation is the Weakly Asymmetric Simple Exclusion Process (WASEP) [11, 12], in which particles hop preferentially in one direction on a lattice subject to an exclusion constraint.

In last two chapters of this thesis we adopt both these perspectives. From the top-down, Hamiltonian viewpoint, we study the isentropic Euler equations in one dimension, which constitute the natural generalization of inviscid Burgers to a compressible fluid with an equation of state. From the bottom-up, lattice viewpoint, we study a specific two-species exclusion process, a generalization of WASEP to two conserved species, whose continuum limit yields a system of coupled Burgers-like equations.

While one-dimensional systems may appear to be a significant simplification, they arise naturally in several physical contexts: as effective descriptions of three-dimensional flows with imposed symmetry, as models of quasi-one-dimensional cold-atom systems through dimensional reduction, experiments of active systems through narrow channels, transport in roads, and as minimal models in which the essential nonlinear structure of hydrodynamics can be studied analytically and numerically with high precision.

1.4.1 ISENTROPIC EULER EQUATIONS

We construct the equations of motion from a Hamiltonian framework, following [74, 75]. Several key assumptions are made: The fluid is a single-component, Galilean invariant system with negligible entropy generation (hence isentropic) and local energy functional. Under these assumptions, a general Hamiltonian can be written as (we have ignored the quantum correction involving gradients of the density field ρ): Under these assumptions, the most general hydrodynamic Hamiltonian consistent with Galilean invariance and locality takes the form [75]

$$H = \int dx \left[\frac{\rho v^2}{2} + \rho \mathcal{E}(\rho) \right], \quad (1.84)$$

where $\mathcal{E}(\rho)$ is the internal energy per unit mass. The first is the kinetic energy, the second is the thermodynamic internal energy encoding the equation of state.

To extract the equations of motion, we equip the Hamiltonian with a Poisson bracket. The natural Poisson bracket for a one-dimensional fluid is [74, 75]

$$\{\rho(x), v(x')\} = \partial_x \delta(x - x'), \quad (1.85)$$

with all other brackets vanishing: $\{\rho, \rho\} = \{v, v\} = 0$. This is the noncanonical Poisson bracket of ideal hydrodynamics; it encodes the fact that ρ and ρv (not v alone) are the natural canonical variables of the fluid [74].

The equations of motion for any functional $\mathcal{F}[\rho, v]$ are then given by

$$\partial_t \mathcal{F} = \{\mathcal{F}, H\}, \quad (1.86)$$

from which the evolution equations for ρ and v follow by taking $\mathcal{F} = \rho(x)$ and $\mathcal{F} = v(x)$ respectively. The functional derivative of H with respect to v is

$$\frac{\delta H}{\delta v} = \rho v, \quad (1.87)$$

since v appears only in the kinetic energy term. Applying the bracket (1.85),

$$\partial_t \rho = \{\rho, H\} = -\partial_x \left(\frac{\delta H}{\delta v} \right) = -\partial_x(\rho v), \quad (1.88)$$

which gives the continuity equation,

$$\partial_t \rho + \partial_x(\rho v) = 0. \quad (1.89)$$

The functional derivative of H with respect to ρ requires more care, as ρ appears in all three terms of (1.84). Computing each contribution:

$$\frac{\delta}{\delta \rho} \int \frac{\rho v^2}{2} dx = \frac{v^2}{2}, \quad (1.90)$$

$$\frac{\delta}{\delta \rho} \int \rho \mathcal{E}(\rho) dx = \partial_\rho(\rho \mathcal{E}(\rho)) = \mathcal{E}(\rho) + \rho \mathcal{E}'(\rho) \equiv w(\rho), \quad (1.91)$$

where $w(\rho) = \partial_\rho(\rho \mathcal{E}(\rho))$ is the specific enthalpy, which plays the role of the chemical potential at zero temperature. Collecting terms, the full functional derivative is

$$\frac{\delta H}{\delta \rho} = \frac{v^2}{2} + w(\rho) \quad (1.92)$$

The equation of motion for v then follows from

$$\partial_t v = \{v, H\} = -\partial_x \left(\frac{\delta H}{\delta \rho} \right), \quad (1.93)$$

giving

$$\partial_t v + \partial_x \left(\frac{v^2}{2} + w \right) = 0. \quad (1.94)$$

The pressure $P(\rho)$ determined by the equation of state through $\partial_\rho P = \rho \partial_\rho w$ and the Standard isentropic Euler equations becomes,

$$\partial_t \rho + \partial_x(\rho v) = 0, \quad (1.95)$$

$$\partial_t v + v \partial_x v + \frac{1}{\rho} \partial_x P = 0. \quad (1.96)$$

The Hamiltonian framework describes ideal, non-dissipative flow. Dissipation is incorporated by supplementing the equations of motion with a Rayleigh dissipation functional,

$$F = \frac{1}{2} \int dx \, \eta_B(\rho) (\partial_x v)^2, \quad (1.97)$$

where $\eta_B(\rho)$ is the bulk viscosity of the one-dimensional fluid. The dissipative contribution to the momentum equation is obtained as $-\frac{1}{\rho} \frac{\delta F}{\delta v}$, yielding an additional term $\frac{1}{\rho} \partial_x (\eta_B \partial_x v)$ on the right-hand side of Eq. (1.94). The rate of energy dissipation is $2F$, which is positive definite for $\eta_B > 0$, ensuring thermodynamic consistency. In the chapters that follow, we consider both the purely Hamiltonian case ($\eta_B = 0$), relevant to the isentropic Euler equations and their connections to quantum gases, and the dissipative case, relevant to the active matter model.

A superficial reading of Eqs. (1.95)-(1.96) suggests that these equations are for a single-component, Galilean-invariant, *classical fluid*. This is true, however these equations can serve as a semiclassical description for *quantum gases* capturing the essential quantum nature. The essential quantum nature of its constituent bosons or fermions is effectively captured through a clever choice of the specific enthalpy $w(\rho) \propto \rho^\delta$. The exponent δ determines, in what follows, the kind of gas that our system models. For example, $\delta = 1$ recovers the Gross-Pitaevskii limit of the weakly coupled Bose gas; $\delta = 2/5$ describes the limiting case of a unitary Fermi gas.

This semiclassical description has served as useful models for quantum gases; however, as with Burgers equation, no closed-form analytical solution is available. We therefore take up this problem in Chapter 6, where we explore asymptotic solutions for an expanding gas that can be computed analytically for various scenarios and compare them against direct numerical simulations. We also explore the relaxation dynamics for the gases in a trapping potential. Finally, we extend the asymptotic scaling solutions to the symmetric two-dimensional case.

1.4.2 HYDRODYNAMICS OF LATTICE GASES

The hydrodynamic equations we have discussed so far — the Navier–Stokes equations, the isentropic Euler equations, the Burgers equation — are all arrived at through a top-down approach: one starts from continuum mechanics and writes down the equations governing the density and velocity fields at the macroscopic level. There is, however, a complementary *bottom-up* approach, in which one begins with a microscopic model of interacting particles on a lattice and derives the emergent hydrodynamic description in the continuum limit. This approach has several virtues. It makes the microscopic origin of each term in the hydrodynamic equation explicit: diffusion coefficients, drift velocities, and nonlinear interactions all emerge from specific features of the microscopic dynamics rather than being postulated at the outset. It also provides a natural framework for studying fluctuations beyond the deterministic hydrodynamic limit, since the microscopic model defines a well-posed stochastic process whose fluctuation statistics can be computed systematically. Lattice gas models have therefore served as an important testing ground for ideas in nonequilibrium statistical mechanics, and the connection between their microscopic dynamics and emergent hydrodynamics is by now well understood in several canonical cases [76].

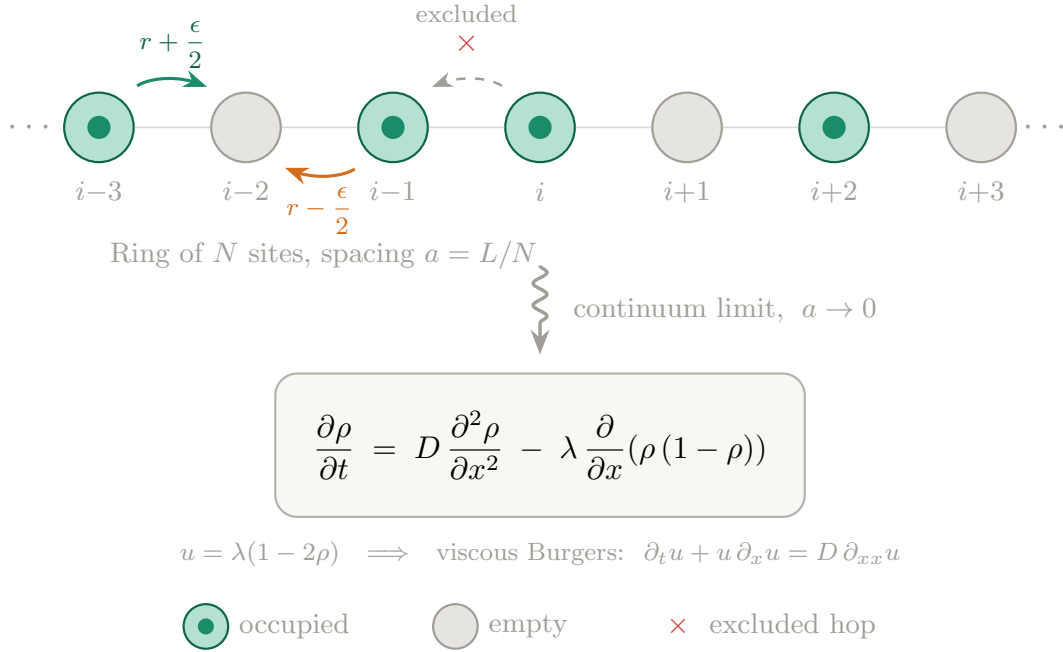


Figure 1.8. Schematic of the Weakly Asymmetric Simple Exclusion Process (WASEP) on a one-dimensional lattice. Occupied sites (teal) carry a particle; empty sites (gray) do not. A particle hops to the right at rate $r + \epsilon/2$ and to the left at rate $r - \epsilon/2$. The exclusion constraint forbids hopping onto an occupied site (dashed arrow, $\times$). In the continuum limit $a \rightarrow 0$, the mean-field density $\rho(x, t)$ satisfies the viscous Burgers equation.

To illustrate this bottom-up route concretely, we derive the viscous Burgers equation from the Weakly Asymmetric Simple Exclusion Process (WASEP) — the simplest lattice model whose hydrodynamic limit is a nonlinear conservation law. This derivation serves two purposes: it demonstrates the preliminary idea we use in Chapter 7, and it establishes the Burgers equation as a genuine emergent hydrodynamic description rather than merely a simplified model equation.

The WASEP Model

Particles hop on a ring of N sites with lattice spacing $a = L/N$, subject to an exclusion constraint, each site is occupied by at most one particle. The hopping is weakly asymmetric: the rate of jumping to the right is $p = r + \epsilon/2$ and to the left is $q = r - \epsilon/2$, where r is the symmetric part and $\epsilon \ll r$ is the weak asymmetry that drives the system out of equilibrium. The occupation number at site i is $n_i \in \{0, 1\}$.

Evolution Equation for the Mean Density

The evolution of the mean occupation $\langle n_i \rangle$ follows from the microscopic hopping rules as sketched in 1.8. A site i gains a particle from the left if site $i - 1$ is occupied and site i is empty, and from the right if site $i + 1$ is occupied and site i is empty; it loses a particle by the reverse

processes. Writing out all four contributions,

$$\begin{aligned} \frac{d}{dt} \langle n_i \rangle = & \left\langle \left(r + \frac{\epsilon}{2} \right) n_{i-1} (1 - n_i) + \left(r - \frac{\epsilon}{2} \right) n_{i+1} (1 - n_i) \right. \\ & \left. - \left(r + \frac{\epsilon}{2} \right) n_i (1 - n_{i+1}) - \left(r - \frac{\epsilon}{2} \right) n_i (1 - n_{i-1}) \right\rangle. \end{aligned} \quad (1.98)$$

For simplicity, we close this equation using a mean-field approximation⁵, factorizing two-point correlations as $\langle n_i n_{i\pm 1} \rangle = n_i n_{i\pm 1}$. Collecting symmetric and asymmetric contributions separately,

$$\frac{d}{dt} n_i = r(n_{i+1} + n_{i-1} - 2n_i) + \frac{\epsilon}{2} (n_{i-1} - 2n_i n_{i-1} - n_{i+1} + 2n_{i+1} n_i). \quad (1.99)$$

The first term is a discrete Laplacian. To identify the second term, we add and subtract n_i and $2n_i^2$ to rewrite the asymmetric part in terms of discrete gradients,

$$\begin{aligned} \frac{d}{dt} n_i &= r \frac{\partial^2 n_i}{\partial i^2} + \frac{\epsilon}{2} \left(-2 \frac{\partial n_i}{\partial i} + 4n_i \frac{\partial n_i}{\partial i} \right) \\ &= r \frac{\partial^2 n_i}{\partial i^2} - \epsilon \frac{\partial}{\partial i} (n_i - n_i^2). \end{aligned} \quad (1.100)$$

Continuum Limit

We pass to the continuum by introducing the physical coordinate $x = ia$ and identifying the continuum density field $\rho(x, t) \sim n_i$. Under this substitution, Eq. (1.100) becomes

$$\frac{\partial \rho}{\partial t} = ra^2 \frac{\partial^2 \rho}{\partial x^2} - a\epsilon \frac{\partial}{\partial x} (\rho(1 - \rho)), \quad (1.101)$$

Two hydrodynamic parameters emerge naturally from the microscopic rates and geometry: the diffusion constant

$$D = ra^2 = r \frac{L^2}{N^2}, \quad (1.102)$$

and the drift coefficient

$$\lambda = a\epsilon = \epsilon \frac{L}{N}, \quad (1.103)$$

so that Eq. (1.101) takes the compact form

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2} - \lambda \frac{\partial}{\partial x} (\rho(1 - \rho)). \quad (1.104)$$

This is the viscous Burgers equation in disguise. To see this explicitly, substitute $u = \lambda(1 - 2\rho)$, which gives

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2}, \quad (1.105)$$

the standard viscous Burgers equation, with D playing the role of viscosity. The nonlinear advection term $u \partial_x u$ arises entirely from the exclusion interaction: it is the macroscopic signature of the microscopic constraint that no two particles can occupy the same site. In the symmetric case $\epsilon = 0$, the drift vanishes and Eq. (1.104) reduces to the diffusion equation, as expected.

⁵This neglects all the effects of stochasticity in the microscopic model, this is only justified if the noise is subleading in the large system limit. We discuss these issues in details in Chapter 7

The WASEP derivation thus provides a microscopic justification for the Burgers equation that is entirely independent of the top-down Navier–Stokes reduction: the same continuum equation emerges from a simple exclusion process in the hydrodynamic limit. In Chapter 7, we generalize this construction to a two-species exclusion process, where the two conserved densities produce a system of coupled Burgers-like equations, and we study the resulting hydrodynamics both analytically and through Monte Carlo simulations of the underlying microscopic dynamics.

1.5 OVERVIEW

We begin this thesis by examining scaling laws that govern the statistics of coarse-grained dissipation field in fully developed turbulence. In Chapter 2 we perform a systematic comparison of Eulerian and Lagrangian statistics of energy dissipation. We show that, despite strong intermittency, Eulerian dissipation does not organize into a classical scale-invariant multiplicative cascade: its spatial correlations, coarse-grained statistics, and multiplier distributions are incompatible with multifractal or log-divisible descriptions. In sharp contrast, dissipation sampled along Lagrangian trajectories exhibits clear signatures of a temporal multiplicative cascade, including logarithmic time correlations, linear scaling of cumulants with the logarithm of the time scale, scale-invariant multipliers, and a well-defined Large Deviation principle. We further demonstrate that classical bridge relations fail for dissipation because they rely on a mean-field assumption that neglects fluctuations across trajectories. Instead, we propose an alternative interpretation in which Eulerian dissipation statistics emerge from temporal cascade dynamics along Lagrangian paths, with Eulerian multipliers encoding the scale-to-scale inheritance of fluctuations. These results identify dissipation intermittency as intrinsically Lagrangian in nature and call for a reassessment of cascade phenomenology formulated purely in Eulerian space.

In Chapter 3 we examine the scaling laws that characterize chaos in turbulent flows. We investigate the Reynolds-number dependence of the maximal Lyapunov exponent in fully developed turbulence, which quantifies the rate of chaotic divergence of nearby velocity fields. Using decorrelators constructed from infinitesimally perturbed flows, we find that the Lyapunov exponent scales with Reynolds number as $\lambda \sim Re^\alpha$, with an exponent $\alpha = 0.59 \pm 0.04$ exceeding the classical mean-field prediction. By explicitly separating the nonlinear strain and viscous contributions to decorrelator growth, we show that this departure is associated by intermittent fluctuations of the strain-rate tensor, which dominate the short-time growth of the infinitesimal perturbations over viscous damping. Direct numerical simulations of the Navier–Stokes equations and complementary tests using a reduced shell model yield consistent scaling behavior, indicating robustness within the frameworks considered. Our results show that the dynamical origin of chaotic divergence in turbulence is closely linked to intermittent strain-rate fluctuations. [77].

In Chapter 4 we revisit the Equal-time scaling exponents in fully developed turbulence, which typically exhibit non-anomalous scaling in the inverse cascade of two-dimensional turbulence and anomalous scaling in three dimensions. We demonstrate that multiscaling is

not confined to longitudinal scalar velocity increments, but also emerges in structure functions constructed from the magnitude and orientation of the velocity vector. This decomposition uncovers pronounced multiscaling in the two dimensional inverse cascade, which remains largely obscured when using conventional longitudinal structure functions. In particular, angle-based observables display strong departures from normal scaling together with signatures of saturation at high orders in both two- and three-dimensional turbulence. To investigate the extent to which these properties are specific to turbulence, we compare our results with synthetic Gaussian velocity fields possessing the same two-point statistics as turbulent flows. Remarkably, we find that much of the multiscaling phenomenology of the geometric observables survives even in correlated Gaussian fields that lack nonlinear cascade dynamics. This suggests that the dominant mechanism underlying the observed scaling originates not solely from turbulent intermittency, but more generally from the statistical structure of the observables themselves. We argue that the bounded and ratio-like nature of the angular increments enhances the role of correlations, kinematic constraints, and the geometry of low-velocity regions, thereby producing a class of multiscaling observables whose dominant scaling behavior can already emerge in correlated random fields. Our results therefore point to an observable-dependent interpretation of multiscaling in turbulence and clarify why certain classes of observables may be captured surprisingly well by synthetic or quasi-Gaussian descriptions despite the strongly non-Gaussian nature of turbulent flows [78].

Up to this point, the methods we use are more in line with statistical physics approaches to studying the three dimensional Navier–Stokes equations. In Chapter 5 we use tools from mathematical analysis to find estimates on the scaling laws for energy and enstrophy dissipation in two-dimensional turbulence. Mathematical estimates for the Navier–Stokes equations are traditionally expressed in terms of the Grashof number, which is a dimensionless measure of the magnitude of the forcing and hence a control parameter of the system. However, experimental measurements and statistical theories of turbulence are based on the Reynolds number. Thus, a meaningful comparison between mathematical and physical results requires a conversion of the mathematical estimates to a Reynolds-dependent form. In two dimensions, this was achieved for sufficiently smooth forcing. Nonetheless, numerical simulations have shown that the phenomenology of turbulence is sensitive to the degree of regularity of the forcing that sustains it. Therefore, we extend the available estimates for the energy and enstrophy dissipation rates as well as the attractor dimension to the full range of forcing regularity as allowed by the theory of weak solutions. The main result is the existence of three distinct regimes as a function of the regularity of the forcing [79].

In the final two chapters, we explore other hydrodynamic equations in one spatial dimension and their possible extension to two dimensions. In Chapter 6, we study scaling solutions of the isentropic Euler equations and their connections to quantum gases. The evolution of quantum gases, released from traps, is studied through hydrodynamics, both analytically and numerically, in one and two dimensions. In particular, we demonstrate the existence of long time self-similar solutions of the Euler equations, for the density and velocity fields, and derive the scaling exponents as well as the scaling functions. We find that the expanding gas develops a shock-like

front and the size of the cloud grows in time as a power law. We relate the associated exponent to that appearing in the corresponding equation of state of the quantum gas. Furthermore, we study the relaxation dynamics of a trapped quantum gas and show that the resulting steady state is in excellent agreement with that derived analytically. Our hydrodynamic approach is versatile and can be used to unravel several other far-from-equilibrium collective phenomena of extreme nature, relevant to the growing experimental interests in quantum gases. [80].

In Chapter 7 we work on the *bottom-up* approach to hydrodynamics. We work with a generalization of the WASEP model, as discussed in the introduction. We present a fluctuating hydrodynamic description of an active lattice gas model with excluded volume interactions that exhibits motility-induced phase separation under appropriate conditions. For quasi-one dimension and higher, stability analysis of the noiseless hydrodynamics gives quantitative bounds on the phase boundary of the motility-induced phase separation in terms of spinodal and binodal. Inclusion of the multiplicative noise in the fluctuating hydrodynamics describes the exponentially decaying two-point correlations in the stationary-state homogeneous phase. Our hydrodynamic description and theoretical predictions based on it are in excellent agreement with our Monte Carlo simulations and pseudospectral iteration of the hydrodynamics equations. Our construction of hydrodynamics for this model is not suitable in strictly one-dimension with single-file constraints, and we argue that this breakdown is associated with micro-phase separation. [81].

In Chapter 8 we provide a conclusion, open questions and outlook.

INTERMITTENCY OF TURBULENT DISSIPATION IS INTRINSICALLY LAGRANGIAN

Energy dissipation lies at the core of turbulence phenomenology, yet the statistical organization of its intermittent fluctuations remains unclear, particularly when comparing Eulerian and Lagrangian descriptions. Using direct numerical simulations of homogeneous isotropic turbulence with Lagrangian tracers at Taylor-scale Reynolds numbers up to $R_\lambda \simeq 1285$, we perform a systematic comparison of Eulerian and Lagrangian statistics of energy dissipation. We show that, despite strong intermittency, Eulerian dissipation does not organize into a classical scale-invariant multiplicative cascade: its spatial correlations, coarse-grained statistics, and multiplier distributions are incompatible with multifractal or log-divisible descriptions. In sharp contrast, dissipation sampled along Lagrangian trajectories exhibits clear signatures of a temporal multiplicative cascade, including logarithmic time correlations, linear scaling of cumulants with the logarithm of the time scale, scale-invariant multipliers, and a well-defined large-deviation principle. We further demonstrate that classical bridge relations fail for dissipation because they rely on a mean-field assumption that neglects fluctuations across trajectories. Instead, we propose an alternative interpretation in which Eulerian dissipation statistics emerge from temporal cascade dynamics along Lagrangian paths, with Eulerian multipliers encoding the scale-to-scale inheritance of fluctuations. These results identify dissipation intermittency as intrinsically Lagrangian in nature and call for a reassessment of cascade phenomenology formulated purely in Eulerian space¹.

2.1 INTRODUCTION

Turbulence research has long been driven by paradox. Time and again, a theoretical framework appears to capture the essential physics of turbulent motion—often through a universal law or a

¹This work is done in collaboration with Jérémie Bec, Siddhartha Mukherjee, and Samriddhi Sankar Ray.

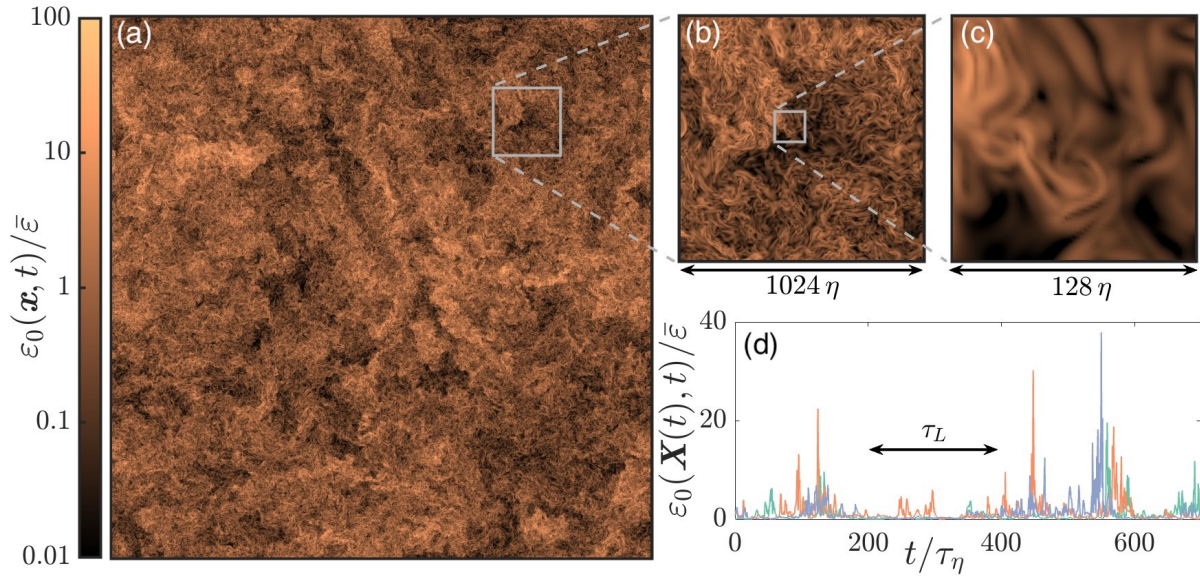


Figure 2.1. (a) Snapshots of the local Eulerian energy dissipation field $\varepsilon_0(\mathbf{x}, t)$ in a two-dimensional slice of the simulation at $R_\lambda = 730$, shown at increasing magnification corresponding to regions of linear size (b) 1024η and (c) 128η . Intense dissipation events appear highly localized and filamentary in space. (d) Dissipation sampled along different Lagrangian tracer trajectories, $\tilde{\varepsilon}_0(t) = \varepsilon_0(\mathbf{X}(t), t)$, shown for the same flow. Individual tracers experience strongly intermittent temporal fluctuations, illustrating the contrast between spatial and temporal organization of dissipation.

scaling principle—only for new experiments or numerical evidence to reveal subtle but significant departures from that picture. Kolmogorov’s 1941 theory [2] provides the most celebrated example. Its elegant dimensional arguments and prediction of universal inertial-range scaling form one of the cornerstones of modern turbulence theory. Yet high-resolution experiments and numerical simulations have revealed persistent intermittency, extreme fluctuations, and anomalous scaling [82, 83], demonstrating that turbulent fluctuations deviate significantly from simple self-similarity. These extreme events, which concentrate velocity gradients and dissipation into rare but intense structures, challenge the classical picture of a smooth scale-invariant cascade.

Comparable surprises have emerged from the Lagrangian viewpoint, which follows individual fluid particles along their trajectories rather than examining the flow at fixed spatial locations [84]. Advances in experimental particle tracking and large-scale numerical simulations have made it possible to probe turbulent dynamics from this perspective with unprecedented detail. Measurements of Lagrangian observables—such as velocity increments, accelerations, and temporal structure functions—reveal strong intermittency and extreme events that are often even more pronounced than suggested by Eulerian measurements alone [32]. Phenomena such as spontaneous stochasticity [85, 86], whereby initially coincident tracers separate unpredictably even in the absence of molecular diffusion, further illustrate the fundamentally stochastic character of turbulent transport and mixing. These observations suggest that intermittency may be closely tied to the dynamical history experienced by fluid elements as they sample the turbulent cascade.

Energy dissipation provides a particularly sharp lens through which these tensions become

visible. In fully developed turbulence, the dissipation field is neither uniform nor smooth but instead concentrates in rare and intense events that dominate the dynamics at the smallest scales [82, 87, 88]. This strong intermittency has long motivated multifractal descriptions of the dissipation field, originally formulated in an Eulerian framework through the statistics of spatially coarse-grained dissipation and their associated scaling exponents [2, 89]. Within this phenomenological picture, the refined similarity hypotheses of Kolmogorov and Obukhov [21, 90] relate local fluctuations of dissipation to the anomalous scaling of velocity increments, placing dissipation at the center of turbulence phenomenology [31, 89]. A complementary probabilistic formulation interprets the cascade through a large-deviation principle, which characterizes the statistics of rare but dynamically important fluctuations across scales [2, 91].

Despite its broad success, this framework leaves several fundamental questions unresolved. To what extent is intermittency truly encoded in the dissipation field itself? How robust is the multifractal interpretation when examined beyond idealized scaling assumptions? And more generally, how should dissipation statistics be interpreted when viewed from different observational frames, such as the Eulerian and Lagrangian descriptions of the flow?

Connections between these two perspectives are traditionally established through so-called bridge relations, which aim to infer temporal statistics along particle trajectories from spatial statistics of the flow using dimensional arguments [92–94]. Such relations have achieved notable success for velocity fluctuations and are often regarded as robust and broadly applicable across turbulent flows. Their extension to dissipation, however, is far less secure. In this case, bridge relations rely on assumptions about how Lagrangian tracers sample spatially coarse-grained Eulerian fields and about how local dissipation fluctuations are related to the eddy turnover dynamics experienced along particle trajectories.

This raises a more fundamental question about the direction in which information should flow between the two descriptions. If dissipation fluctuations experienced along trajectories are intrinsically intermittent and reflect the dynamical history of fluid elements, can Eulerian dissipation statistics truly be regarded as the primary object from which Lagrangian behavior should be inferred? Or should the relationship instead be inverted, with Lagrangian intermittency providing the dynamical foundation for understanding the organization of Eulerian dissipation structures?

In this paper we address these questions by performing a systematic comparison of Eulerian and Lagrangian statistics of energy dissipation in homogeneous isotropic turbulence. We show that dissipation intermittency exhibits fundamentally different statistical organizations in the two frames of observation. Dissipation sampled along Lagrangian trajectories displays clear signatures of a multiplicative cascade in time, including logarithmic correlations, scale-invariant multiplier statistics, and a large-deviation structure consistent with log-divisible processes. In contrast, the Eulerian coarse-grained dissipation field cannot be described within the standard multifractal or log-divisible framework: its statistics display a marked scale dependence that reflects a fundamentally different cascade structure. These findings suggest that dissipation intermittency is intrinsically Lagrangian in nature, and that Eulerian dissipation statistics should be interpreted as coarse projections of an underlying temporal cascade experienced along

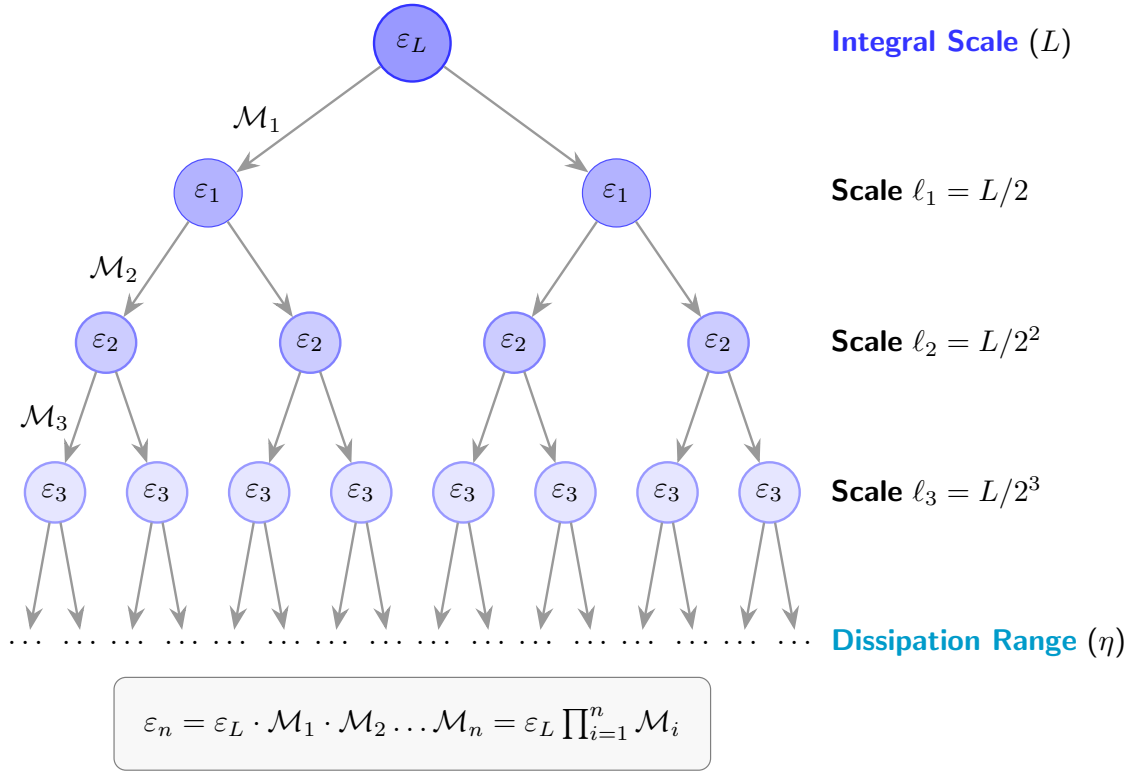


Figure 2.2. Schematic diagram showing a multiplicative cascade process.

particle trajectories.

A first visual impression of this contrast between spatial and trajectory-based views of dissipation is provided in Fig. 2.1, which illustrates the highly localized structure of the Eulerian dissipation field together with the strongly intermittent temporal fluctuations experienced along representative Lagrangian tracer trajectories.

To investigate these questions quantitatively, we analyze high-resolution direct numerical simulations of homogeneous isotropic turbulence with embedded Lagrangian tracer particles. The flow is governed by the three-dimensional incompressible Navier–Stokes equations. The velocity field is denoted by $v(x, t)$, and the local kinetic energy dissipation rate is $\varepsilon_0(x, t) = 2\nu S_{ij}S_{ij}$, where $S_{ij} = (\partial_j v_i + \partial_i v_j)/2$ is the rate-of-strain tensor and ν the kinematic viscosity.

At high Reynolds number, the dissipation field develops increasingly singular spatial structures and is therefore analyzed through its coarse-grained form $\varepsilon_\ell(x, t)$, defined as the average of ε_0 over a ball of size ℓ centered at x . By homogeneity, the mean dissipation rate $\bar{\varepsilon} = \langle \varepsilon_0 \rangle = \langle \varepsilon_\ell \rangle$ is independent of ℓ , where angular brackets denote time averaging, equivalent to ensemble averaging with respect to the invariant turbulent measure.

Within a multifractal description, the coarse-grained dissipation is expected to exhibit power-law scaling of its moments, $\langle \varepsilon_\ell^p \rangle \sim \ell^{\gamma_p}$, where γ_p is a nonlinear and concave function of p . To test this prediction and to compare Eulerian and Lagrangian statistics on equal footing, we perform direct numerical simulations using a pseudo-spectral method with resolutions up

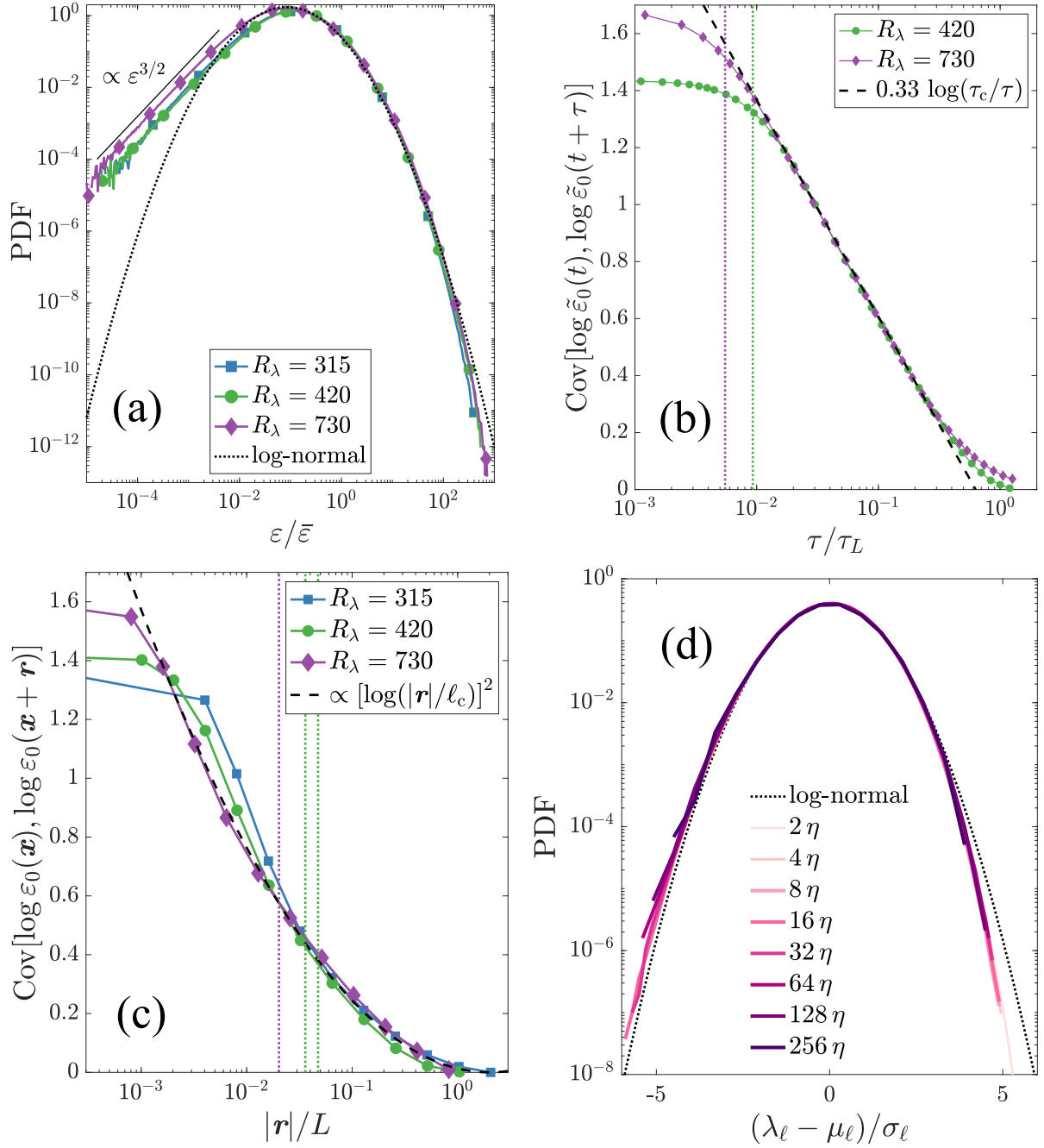


Figure 2.3. (a) Probability density functions of the local dissipation ε_0 for different Reynolds numbers; the dotted curve indicates a log-normal reference. (b) Spatial covariance of $\log \varepsilon_0$ as a function of separation r . The dashed line shows a quadratic logarithmic dependence $\propto [\log(|r|/\ell_c)]^2$, inconsistent with the logarithmic correlations expected from a scale-invariant multiplicative cascade. (c) Mean $\mu_\ell = \langle \log(\varepsilon_\ell/\bar{\varepsilon}) \rangle$ and variance σ_ℓ^2 of the logarithm of the coarse-grained dissipation as functions of the coarse-graining scale ℓ . Both exhibit a nonlinear dependence on $\log(\ell/L)$. (d) Centered and variance-normalized probability density functions of $\lambda_\ell = \log(\varepsilon_\ell/\bar{\varepsilon})$ for different ℓ , showing collapse onto a non-Gaussian master curve.

N^3	η	λ	L	τ_η	τ_L	R_λ
512^3	0.005	0.17	1.17	0.035	1.98	220
1024^3	0.0020	0.073	1.53	0.022	1.76	315
2048^3	0.0027	0.11	3.03	0.021	2.25	420
4096^3	0.00074	0.038	1.86	0.051	9.68	730
8192^3	0.00051	0.036	1.18	0.006	1.95	1285

Table 2.1. Parameters of the direct numerical simulations analyzed in this work. Here $\eta = \nu^{3/4}/\bar{\epsilon}^{1/4}$ is the Kolmogorov dissipative length scale, $\lambda = \sqrt{15\nu u_{\text{rms}}^2/\bar{\epsilon}}$ the Taylor microscale, and $L = u_{\text{rms}}^3/\bar{\epsilon}$ the integral length scale, with $u_{\text{rms}} = \sqrt{\langle |v|^2 \rangle}/3$. The corresponding time scales are the Kolmogorov time $\tau_\eta = \sqrt{\nu/\bar{\epsilon}}$ and the large-eddy turnover time $\tau_L = u_{\text{rms}}^2/\bar{\epsilon}$. The Taylor-scale Reynolds number is $R_\lambda = u_{\text{rms}}\lambda/\nu$. Together, these simulations span more than an order of magnitude in R_λ and provide the basis for the Eulerian and Lagrangian statistical analyses presented throughout the paper.

to 4096^3 grid points, corresponding to Taylor-scale Reynolds numbers up to $R_\lambda = 730$, and seed the flow with Lagrangian tracer particles. For the Eulerian measurements we additionally present results at $R_\lambda = 1285$ using publicly available data from the Johns Hopkins Turbulence Database (JHTD) [95–97].

2.2 MULTIFRACTALITY AND LARGE DEVIATIONS

The multifractal description of turbulence interprets intermittency as the statistical outcome of a multiplicative cascade across scales. Within the phenomenological framework introduced by Kolmogorov and Obukhov [2, 21, 90], the dissipation at a scale ℓ is obtained through successive refinements from the integral scale L . Denoting by ε_L the dissipation at the largest scale and by ε_{ℓ_n} the dissipation coarse-grained over a scale ℓ_n , the cascade is represented as a product of random multipliers,

$$\varepsilon_{\ell_n} = \varepsilon_L \mathcal{M}_1 \mathcal{M}_2 \cdots \mathcal{M}_n = \varepsilon_L \prod_{i=1}^n \mathcal{M}_i, \quad (2.1)$$

where $\mathcal{M}_i = \varepsilon_{\ell_i}/\varepsilon_{\ell_{i-1}}$ quantifies the transfer of dissipation fluctuations between successive scales (see Fig. 2.2). For simplicity we consider a dyadic cascade with $\ell_n = L/2^n$, so that the cascade depth is $n = \log_2(L/\ell_n)$.

Taking logarithms transforms this multiplicative process into an additive one,

$$\frac{1}{n} \log \left(\frac{\varepsilon_{\ell_n}}{\varepsilon_L} \right) = \frac{1}{n} \sum_{i=1}^n \log \mathcal{M}_i. \quad (2.2)$$

If the multipliers $\mathcal{M}_i$ are assumed to be independent and identically distributed random variables, the right-hand side of Eq. (2.2) corresponds to the empirical mean of n independent increments. Standard results from probability theory then describe the behavior of the coarse-grained dissipation across scales.

In the limit of large cascade depth, the empirical average converges to its expectation value, implying that the quantity $\alpha = \frac{1}{n} \log(\varepsilon_{\ell_n}/\varepsilon_L)$ approaches the mean increment $\langle \log \mathcal{M} \rangle \equiv \alpha^*$.

For finite n , typical fluctuations around this value are of order $n^{-1/2}$ and follow approximately Gaussian statistics with variance $\sigma^2 = \text{Var}[\log \mathcal{M}]$. However, turbulence is dominated by rare, intense events whose fluctuations remain of order unity even as n increases. The probability of observing such atypical values of α is therefore governed by a large-deviation principle [30],

$$\mathcal{P}(\alpha) \asymp \exp[n(3 - f(\alpha))], \quad (2.3)$$

where $f(\alpha)$ is the large-deviation rate function. In the geometric interpretation of multifractal theory, $3 - f(\alpha)$ corresponds to the scaling dimension (or co-dimension) of the set of points characterized by exponent α [31, 89].

Because the cascade depth grows logarithmically with the scale separation, $n \propto \log(L/\ell)$, this exponential dependence on n translates directly into power-law scaling of the coarse-grained dissipation moments,

$$\langle (\varepsilon_\ell / \varepsilon_L)^p \rangle \sim (\ell/L)^{\chi_p}, \quad (2.4)$$

where the exponents χ_p follow from a Legendre transform of the rate function, $\chi_p = \inf_\alpha (p\alpha + 3 - f(\alpha))$ [2, 31]. This relation forms the basis of the multifractal prediction that the moments of dissipation obey power-law scaling with a nonlinear dependence of χ_p on the order p .

The following section examines whether Eulerian dissipation statistics actually satisfy these predictions of a scale-invariant multiplicative cascade.

2.3 EULERIAN DISSIPATION: ABSENCE OF A SCALE-INVARIANT SPATIAL CASCADE

Before turning to detailed measurements, it is useful to clarify the logic of the analysis that follows. Classical multiplicative cascade models predict a specific statistical structure for Eulerian dissipation across scales. In particular, they imply three intertwined properties: logarithmic spatial correlations of the logarithm of dissipation, scale-invariant multiplier statistics, and power-law scaling of coarse-grained moments. Rather than testing these properties in isolation, we examine whether they form a consistent statistical organization across scales. The failure of any one of these ingredients already challenges the multiplicative cascade interpretation; their collective behavior provides a more stringent assessment.

We begin with the simplest observable: the one-point statistics of the Eulerian dissipation field. Figure 2.3(a) shows the distribution of the instantaneous dissipation ε_0 . At small values, the distribution follows a power-law behavior $\propto \varepsilon_0^{3/2}$ over more than two decades. This behavior has been reported previously and attributed to a chi-squared distribution arising from a quasi-Gaussian distribution of the rate-of-strain tensor near its zeros [95, 98]. At large values, the tail of the distribution is well approximated by a log-normal form, consistent with earlier experimental observations [16, 99]. While such log-normal tails naturally evoke multiplicative cascade dynamics, one-point statistics alone cannot establish the existence of such a process. This must be assessed through the scale dependence and correlation structure of the field.

A more stringent test of a multiplicative cascade concerns the spatial correlations of the logarithm of the dissipation field. In standard multifractal cascade models, the logarithm of

dissipation is expected to exhibit logarithmic spatial correlations over an inertial range, which constitutes a defining signature of multiplicative cascade processes. However, as shown in Fig. 2.3(b), the spatial covariance of the logarithmic dissipation,

$$\begin{aligned} \text{Cov}[\lambda_0(\mathbf{x}), \lambda_0(\mathbf{x} + \mathbf{r})] &= \langle \lambda_0(\mathbf{x}) \lambda_0(\mathbf{x} + \mathbf{r}) \rangle - \langle \lambda_0^2(\mathbf{x}) \rangle, \\ &\text{where } \lambda_0(\mathbf{x}) = \log[\varepsilon_0(\mathbf{x}, t) / \bar{\varepsilon}], \end{aligned} \quad (2.5)$$

shows no evidence of a clear logarithmic scaling regime as a function of the separation $|\mathbf{r}|$. This observation calls into question the widely used assumption that Eulerian dissipation is log-correlated in space [100]. Instead, the data are more consistent with a behavior of the form

$$\text{Cov}[\lambda_0(\mathbf{x}), \lambda_0(\mathbf{x} + \mathbf{r})] \propto [\log(|\mathbf{r}| / \ell_c)]^2, \quad (2.6)$$

suggesting a more intricate spatial organization than predicted by standard cascade models.

Deviations from the classical cascade picture become even more apparent when examining the statistics of the coarse-grained dissipation field ε_ℓ . As shown in Fig. 2.3(c), the mean μ_ℓ and variance σ_ℓ^2 of the logarithmic dissipation $\lambda_\ell = \log(\varepsilon_\ell / \bar{\varepsilon})$ do not exhibit the linear dependence on $\log(\ell / L)$ expected for a scale-invariant cascade [2, 16]. Instead, their scale dependence is well described by a quadratic behavior proportional to $[\log(\ell / L)]^2$. Such nonlinear scaling has been observed in cascade models that relax strict scale invariance [101, 102].

Despite this breakdown of classical scaling, a form of statistical self-similarity persists in the shape of the fluctuations of λ_ℓ around its mean. As shown in Fig. 2.3(d), the probability density functions of $\log(\varepsilon_\ell / \bar{\varepsilon})$ collapse onto a single master curve over more than two decades in ℓ when centered and rescaled to unit variance. This observation motivates the inertial-range scaling form

$$p_{\log(\varepsilon_\ell / \bar{\varepsilon})}(\lambda; \ell) \approx \frac{1}{\sigma_\ell} \Psi\left(\frac{\lambda - \mu_\ell}{\sigma_\ell}\right), \quad \eta \ll \ell \ll L, \quad (2.7)$$

where the scaling function Ψ is clearly non-Gaussian, as indicated by the dotted line in Fig. 2.3(d). In particular, the distribution exhibits a small but robust negative skewness $\kappa_3 = \langle (\lambda_\ell - \mu_\ell)^3 \rangle / \sigma_\ell^3 < 0$. For $R_\lambda = 730$ we measure $\kappa_3 \approx -0.085$, with no significant dependence on Reynolds number.

The departure from multifractal scaling can equivalently be expressed in terms of the moments of the coarse-grained dissipation. The collapse of the standardized distributions, Eq. (2.7), implies

$$\log \langle (\varepsilon_\ell / \bar{\varepsilon})^p \rangle \approx p \mu_\ell + \Phi(p \sigma_\ell), \quad (2.8)$$

where $\Phi(x) = \log \int e^{xy} \Psi(y) dy$ is the cumulant-generating function associated with Ψ . Writing $\Phi(x) = \sum_{n \geq 2} \kappa_n x^n / n!$ shows that the standardized cumulants κ_n fully determine the scaling behavior.

The constraint that the mean dissipation remain constant, $\langle \varepsilon_\ell \rangle = \bar{\varepsilon}$, implies $\mu_\ell \approx -\Phi(\sigma_\ell)$. As shown in Fig. 2.4(a), the scaling identity (2.8) is well supported by the numerical data. The function Φ shows at most weak Reynolds-number dependence and deviates clearly from the quadratic form $\Phi(x) \approx x^2 / 2$ expected for a log-normal cascade. Including the cubic correction $\kappa_3 x^3 / 6$ significantly improves the description, while higher-order terms contribute negligibly.

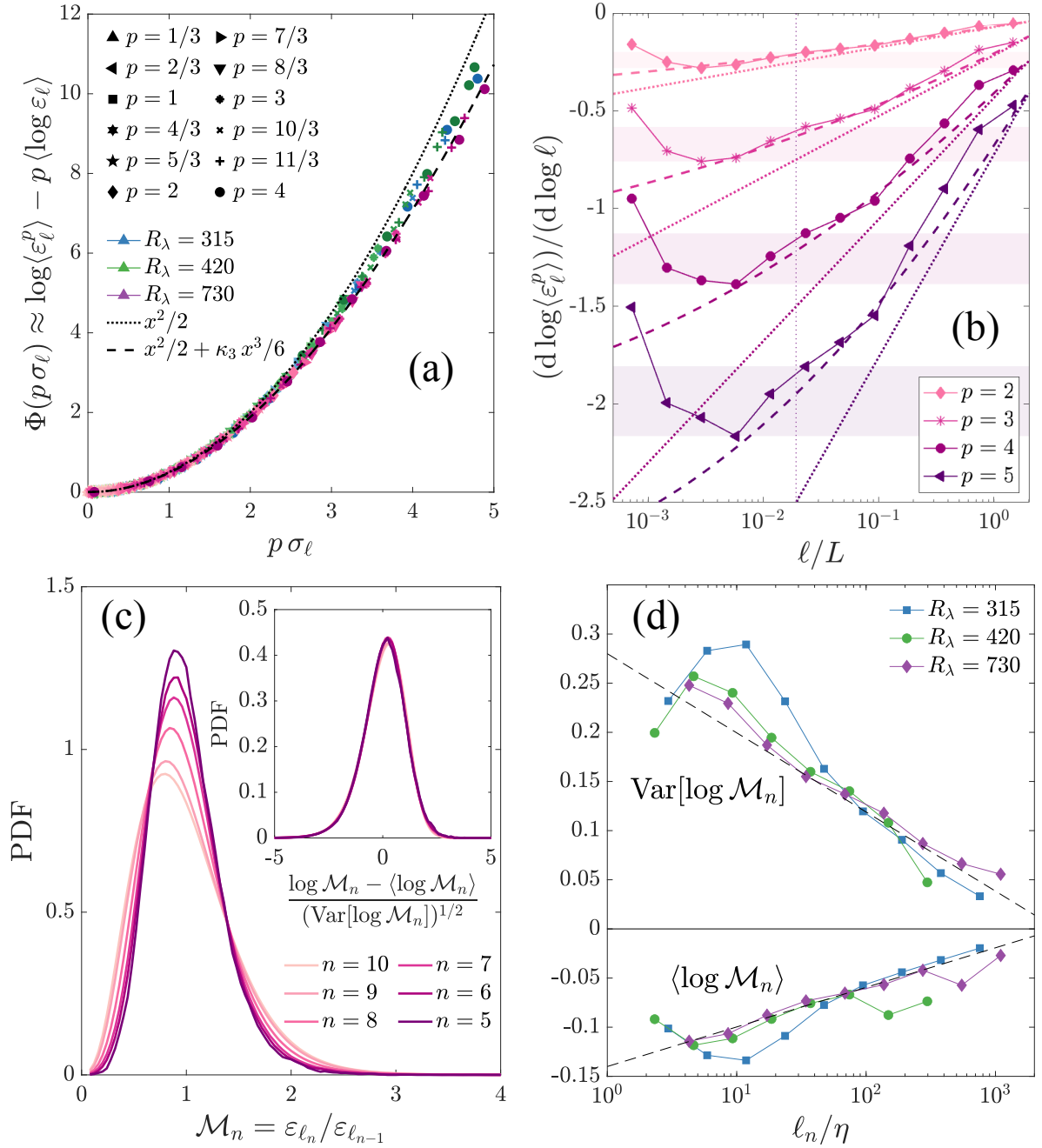


Figure 2.4. (a) Rescaled moments $\langle (\varepsilon_\ell / \bar{\varepsilon})^p \rangle$ plotted as functions of the rescaled variable $p\sigma_\ell$ for $\ell \in [2\eta, 2000\eta]$ and $p \leq 4$. The collapse supports the scaling relation (4) and deviates from the log-normal prediction (dotted line). (b) Local logarithmic derivatives of $\langle \varepsilon_\ell^p \rangle$ with respect to ℓ , showing the absence of extended power-law scaling ranges. (c) Probability density functions of Eulerian multipliers $M_n = \varepsilon_{\ell_n} / \varepsilon_{\ell_{n-1}}$ for a fixed scale ratio $\ell_n / \ell_{n-1} = 1/2$, demonstrating clear scale dependence. (d) Mean and variance of $\log M_n$ as functions of scale, revealing an approximately logarithmic growth and ruling out scale-invariant multiplier statistics.

The scaling relation (2.8) is incompatible with the classical multifractal prediction $\langle \varepsilon_\ell^p \rangle \propto \ell^{\gamma_p}$. Consistency with such power-law scaling would require the variance σ_ℓ^2 to scale linearly with $\log \ell$ and the generating function $\Phi(x)$ to be purely quadratic. Neither condition is satisfied by the data: the variance exhibits a quadratic dependence on $\log \ell$ (Fig. 2.3(c)), and Φ departs significantly from the quadratic form (Fig. 2.4(a)). We therefore conclude that Eulerian coarse-grained dissipation does not obey multifractal power-law scaling.

This conclusion is reinforced by a direct examination of the local slopes of the moments $\langle \varepsilon_\ell^p \rangle$, shown in Fig. 2.4(b). Except for a narrow interval around $\ell \approx 4 \times 10^{-3}L \approx 10\eta \approx 0.2\lambda$, no well-defined plateau emerges. Although averaging slopes over this interval yields exponents compatible with log-normal or log-Poisson models, this apparent agreement does not persist across scales. In the upper part of the inertial range ($\ell \gtrsim \lambda$), no consistent power-law behavior is observed.

One might attribute the absence of scaling to finite Reynolds number effects. However, the simulations analyzed here achieve a scale separation of approximately $\lambda/L \approx 50$, over which no clear scaling regime appears. Moreover, the behavior of the mean and variance of $\log \varepsilon_\ell$ (Fig. 2.3(c)), the collapse of its probability density functions (Fig. 2.3(d)), the moment scaling (Fig. 2.4(a)), and the corresponding local slopes (Fig. 2.4(b)) all fail to reveal any clear crossover near $\ell \approx \lambda$.

Further insight can be obtained by examining the statistics of the Eulerian multipliers $\mathcal{M}_n = \varepsilon_{\ell_n} / \varepsilon_{\ell_{n-1}}$ with $\ell_n / \ell_{n-1} = 1/2$. As shown in Fig. 2.4(c), the probability density functions of $\mathcal{M}_n$ vary significantly with the cascade level n , indicating that the multipliers are not scale-invariant. A collapse is obtained only after standardizing the distributions of $\log \mathcal{M}_n$ to zero mean and unit variance (inset of Fig. 2.4(c)). The mean and variance of $\log \mathcal{M}_n$, shown in Fig. 2.4(d), both vary approximately linearly with $\log \ell_n$ and exhibit a better collapse when scales are normalized by the Kolmogorov length η , indicating a residual viscous influence.

This scale dependence of the multipliers prevents the existence of a standard large-deviation principle for the Eulerian cascade. Writing

$$\log(\varepsilon_{\ell_n} / \varepsilon_L) = \sum_{i=1}^n \log \mathcal{M}_i$$

shows that the variance of the total logarithmic dissipation depends on both the multiplier variances and their cross-scale correlations,

$$\text{Var} \left[\log \frac{\varepsilon_{\ell_n}}{\varepsilon_L} \right] = \sum_{i=1}^n \text{Var}[\log \mathcal{M}_i] + \sum_{i \neq j} \text{Cov}[\log \mathcal{M}_i, \log \mathcal{M}_j]. \quad (2.9)$$

As shown in Fig. 2.4(d), the multiplier variance grows approximately linearly with the cascade index, $\text{Var}[\log \mathcal{M}_i] \propto i$, while the cross-scale covariances are negligible (inset of Fig. 2.4(d)). Summing these contributions yields

$$\text{Var} \left[\log \frac{\varepsilon_{\ell_n}}{\varepsilon_L} \right] \propto \sum_{i=1}^n i \approx n^2 \propto [\log(\ell_n / L)]^2. \quad (2.10)$$

Thus the quadratic scaling observed in the data arises naturally from the scale dependence of the Eulerian multipliers.

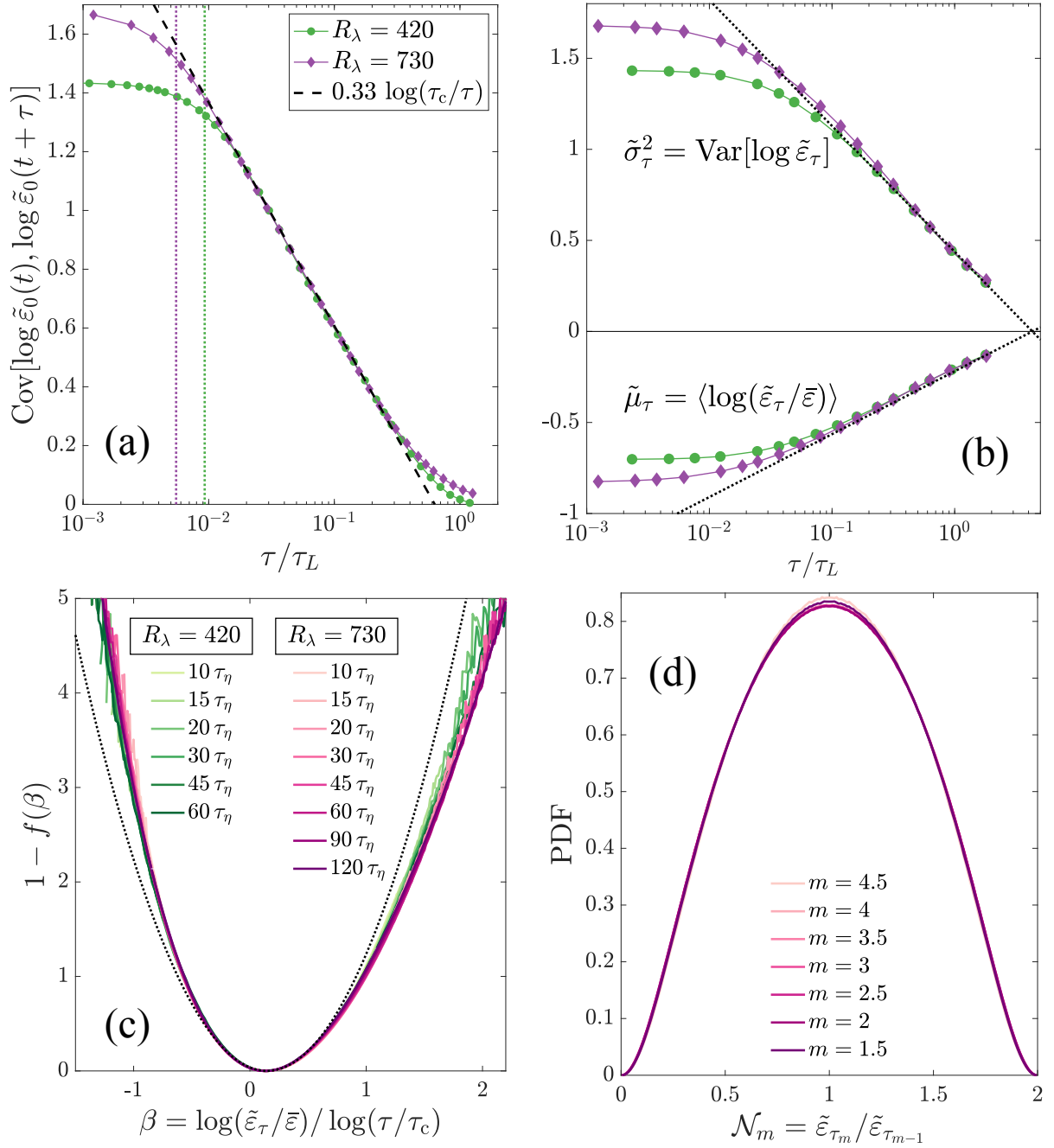


Figure 2.5. (a) Temporal covariance of $\log \tilde{\varepsilon}_0(t)$ as a function of time lag τ , showing clear logarithmic correlations (dashed line), in contrast to the Eulerian spatial correlations. (b) Mean $\tilde{\mu}_\tau$ and variance $\tilde{\sigma}_\tau^2$ of $\log(\tilde{\varepsilon}_\tau / \bar{\varepsilon})$ as functions of $\log(\tau / \tau_c)$, displaying linear scaling consistent with a scale-invariant multiplicative process in time. (c) Probability density functions of the coarse-grained Lagrangian dissipation $\tilde{\varepsilon}_\tau$, rescaled according to the large-deviation form (8), showing collapse across τ and Reynolds numbers. (d) Probability density functions of Lagrangian multipliers $N_m = \tilde{\varepsilon}_{\tau_m} / \tilde{\varepsilon}_{\tau_{m-1}}$, exhibiting approximate scale invariance.

Taken together, these results demonstrate that Eulerian dissipation, while strongly intermittent, does not organize into a classical multiplicative cascade across spatial scales. Logarithmic correlations, multifractal power-law scaling of moments, and scale-invariant multiplier statistics are all absent in a robust manner. Instead, Eulerian dissipation exhibits only a weaker form of scale invariance, limited to the standardized shape of its fluctuations.

This raises a question: should this intermittency arise more naturally in a Lagrangian and not in an Eulerian framework? Since dissipation is ultimately experienced along fluid trajectories, a description based solely on spatial coarse-graining may obscure dynamical structures that are organized in time rather than in space. This motivates a direct examination of dissipation intermittency from a Lagrangian perspective.

2.4 LAGRANGIAN DISSIPATION AS A TEMPORAL MULTIPLICATIVE CASCADE

In contrast to the absence of a spatial cascade in the Eulerian frame, the Lagrangian description reveals a remarkably simple statistical structure. Tracer particles are evolved according to the Lagrangian dynamics $d\mathbf{X}/dt = \mathbf{v}(\mathbf{X}, t)$, and the instantaneous dissipation sampled along their trajectories is defined as $\tilde{\varepsilon}_0(t) = \varepsilon_0(\mathbf{X}(t), t)$. By construction, the one-time probability distribution of $\tilde{\varepsilon}_0$ coincides with the single-point Eulerian distribution of ε_0 shown in Fig. 2.3(a). Differences between Eulerian and Lagrangian descriptions therefore arise not from single-point statistics but from the correlations and scale dependence of the fluctuations.

The two-time correlations of the logarithm of the Lagrangian dissipation, $\log \tilde{\varepsilon}_0(t)$, are shown in Fig. 2.5(a). A clear logarithmic decay with the time lag τ is observed, indicating that the Lagrangian dissipation behaves as a log-correlated stochastic process. This behavior stands in sharp contrast to the Eulerian dissipation field, whose logarithm does not exhibit logarithmic correlations in space. The Lagrangian dynamics therefore reveal a statistical structure that is absent in the Eulerian description.

To investigate the scale dependence of these fluctuations, we introduce the temporally coarse-grained Lagrangian dissipation,

$$\tilde{\varepsilon}_\tau(t) = \frac{1}{\tau} \int_t^{t+\tau} \tilde{\varepsilon}_0(t') dt'. \quad (2.11)$$

Further evidence that the Lagrangian dissipation is organized as a multiplicative process is provided by the statistics of $\log \tilde{\varepsilon}_\tau$. As shown in Fig. 2.5(b), both the mean and the variance of $\log \tilde{\varepsilon}_\tau$ vary linearly with $\log(\tau/\tau_c)$ over more than a decade in timescale. This behavior contrasts sharply with the quadratic dependence observed for the Eulerian coarse-grained dissipation and is consistent with fluctuations generated by a scale-invariant multiplicative process in time.

Such behavior naturally leads to a large-deviation description of the coarse-grained dissipation. In particular, the probability density function of $\tilde{\varepsilon}_\tau$ is expected to satisfy

$$\frac{\log p(\beta; \tau)}{\log(\tau/\tau_c)} \asymp f(\beta), \quad \beta = \frac{\log(\tilde{\varepsilon}_\tau/\bar{\varepsilon})}{\log(\tau/\tau_c)}, \quad (2.12)$$

where $f(\beta)$ is a concave rate function characterizing the fluctuations. This prediction is tested in Fig. 2.5(c), which shows the distributions of the coarse-grained dissipation $\tilde{\epsilon}_\tau$ for various time lags and Reynolds numbers. The collapse onto a universal curve when the data are expressed in terms of the rescaled variable β provides strong support for the large-deviation form (2.12).

To further probe the multiplicative structure of the cascade, we consider the Lagrangian multipliers

$$\mathcal{N}_m = \tilde{\epsilon}_{\tau_m} / \tilde{\epsilon}_{\tau_{m-1}}, \quad \tau_m = \tau_L / 2^m.$$

Figure 2.5(d) shows the distributions of these multipliers for different cascade levels m . The approximate collapse of the distributions across scales indicates that the multiplier statistics are nearly scale invariant, providing additional evidence that the Lagrangian dissipation is organized through a multiplicative cascade in time.

Taken together, these results indicate that the Lagrangian dissipation behaves as a *log-divisible process*. The large-deviation structure of $\log \tilde{\epsilon}_\tau$, the linear scaling of its mean and variance with $\log(\tau/\tau_c)$, and the approximate scale-invariance of the multiplier statistics all point toward a multiplicative cascade developing along particle trajectories. In such processes, dissipation fluctuations over a time interval can be viewed as the product of a sequence of random multipliers, implying that $\log \tilde{\epsilon}_\tau$ is a sum of statistically stationary, scale-dependent increments. This additive structure of the logarithm is consistent with infinite divisibility, a defining property of log-divisible processes. While the present observations do not uniquely identify the underlying stochastic law (e.g., log-normal, log-Poisson, or more general Lévy-driven models), they provide strong evidence that Lagrangian dissipation is governed by a multiplicative stochastic mechanism in time.

The emergence of a well-defined multiplicative cascade in the Lagrangian frame naturally raises the question of how these temporal statistics relate to spatial statistics measured in the Eulerian description. Classical bridge relations were introduced precisely to connect these two perspectives by mapping temporal fluctuations along trajectories to spatially coarse-grained quantities. In light of the strong Lagrangian cascade signatures observed here, it is therefore natural to re-examine the validity of these relations for dissipation.

2.5 REVISITING BRIDGE RELATIONS: FROM SPATIAL SAMPLING TO TEMPORAL INHERITANCE

Bridge relations connect Eulerian and Lagrangian multifractal properties by assuming that temporal statistics along particle trajectories can be inferred from spatial statistics of the flow through simple dimensional arguments. Such relations work reasonably well for velocity increments, for which empirical scaling exponents measured in the Eulerian and Lagrangian frames are broadly consistent with these mappings [32]. Their validity for energy dissipation, however, remains an open and far less explored question.

The traditional argument underlying bridge relations relies on the assumption that Lagrangian tracers sample dissipation in a manner that faithfully reflects the local Eulerian structure of turbulence. Specifically, it is assumed that, over a time interval τ , a tracer remains within

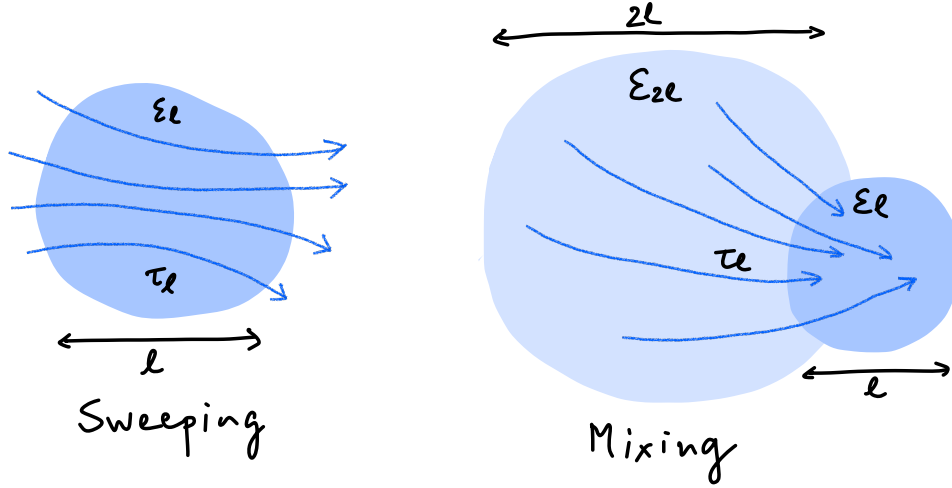


Figure 2.6. (a) Classical bridge-relation picture, in which a Lagrangian tracer samples the Eulerian coarse-grained dissipation ε_ℓ within a spatial region of size ℓ over a time τ_ℓ . (b) Lagrangian cascade picture proposed here, where tracers located within a region of size ℓ inherit dissipation fluctuations from larger scales at earlier times. In this interpretation, the relevant quantity linking Eulerian and Lagrangian statistics is the Eulerian multiplier $\varepsilon_\ell / \varepsilon_{2\ell}$ rather than ε_ℓ itself.

a ball $\mathcal{B}_\ell$ of spatial size ℓ and samples a dissipation field that is approximately constant over this region and time (see the sketch on the left-hand side of Fig. 2.6). The central assumption is that this time interval τ is given by the local eddy turnover time, $\tau_\ell = \varepsilon_\ell^{-1/3} \ell^{2/3}$, where ε_ℓ denotes the Eulerian dissipation coarse-grained over $\mathcal{B}_\ell$. One then writes

$$\tilde{\varepsilon}_\tau \sim \varepsilon_\ell \quad \text{for} \quad \tau \sim \tau_\ell = \varepsilon_\ell^{-1/3} \ell^{2/3}. \quad (2.13)$$

This relation implies that the statistics of $\tilde{\varepsilon}_\tau$ can be inferred from those of ε_ℓ , and conversely. However, this logic leads to an apparent inconsistency. If $\tilde{\varepsilon}_\tau$ behaves as a log-divisible (i.e., multiplicative) process, as supported by our Lagrangian analysis in the previous section, then the same property should be inherited by ε_ℓ under the mapping (2.13). By contrast, our Eulerian results in Sec. 2.3 demonstrate that ε_ℓ does not exhibit log-divisible statistics. This incompatibility indicates that the classical bridge relation fails when applied to dissipation.

To understand this more precisely, consider the expression for the Eulerian coarse-grained dissipation over a spatial ball $\mathcal{B}_\ell$:

$$\begin{aligned} \varepsilon_\ell &= \int_{\mathcal{B}_\ell} \varepsilon_0(\mathbf{x}, t) \frac{d^3x}{|\mathcal{B}_\ell|} \\ &\approx \int_0^{\tau_\ell} \int_{\mathcal{B}_\ell} \varepsilon_0(\mathbf{x}, t) \frac{d^3x}{|\mathcal{B}_\ell|} \frac{dt}{\tau_\ell} \\ &\approx \int_{\mathcal{B}_\ell} \int_0^{\tau_\ell} \varepsilon_0(\mathbf{X}(t; x_0), t) \frac{dt}{\tau_\ell} \frac{d^3x_0}{|\mathcal{B}_\ell|} \\ &= \int_{\mathcal{B}_\ell} \tilde{\varepsilon}_{\tau_\ell} \frac{d^3x_0}{|\mathcal{B}_\ell|}. \end{aligned} \quad (2.14)$$

In words, this relation shows that the Eulerian coarse-grained dissipation ε_ℓ can be expressed as an average, over all Lagrangian tracers initially located within $\mathcal{B}_\ell$, of the time-averaged

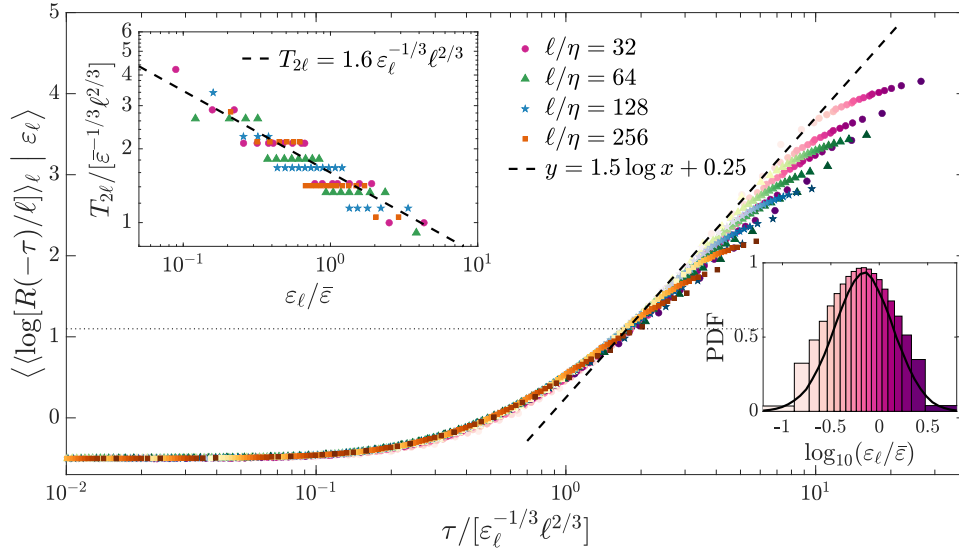


Figure 2.7. The conditional average of the logarithmic separation of tracer pairs that end within the same spatial region of size ℓ , conditioned on the local coarse-grained dissipation ε_ℓ . The dashed line indicates Richardson super-diffusive scaling. Insets show the conditioning procedure and the corresponding doubling times $T_{2\ell}$, consistent with the eddy-turnover estimate $T_{2\ell} \sim \tau_\ell = \varepsilon_\ell^{-1/3} \ell^{2/3}$. These results support the interpretation of dissipation fluctuations as dynamically inherited from larger scales.

Lagrangian dissipation $\tilde{\varepsilon}_{\tau_\ell}$ sampled along each trajectory. Thus, ε_ℓ and $\tilde{\varepsilon}_{\tau_\ell}$ are related only through this additional spatial averaging over tracer initial conditions. A direct identification $\varepsilon_\ell \sim \tilde{\varepsilon}_{\tau_\ell}$ would therefore require the Lagrangian dissipation to be nearly uniform across all tracers within $\mathcal{B}_\ell$. This amounts to a mean-field approximation that neglects fluctuations between trajectories. The breakdown of the classical bridge relation for dissipation, as demonstrated by our results, indicates that such fluctuations are essential and cannot be ignored.

In contrast to the classical bridge relation described above, an alternative picture views the Lagrangian coarse-grained dissipation as the outcome of a temporal multiplicative cascade developing along particle trajectories. In this scenario, dissipation fluctuations experienced by a tracer at time t over a time lag τ_ℓ are not determined by the dissipation within a single Eulerian ball of size ℓ . Instead, they arise from a sequence of interactions with larger eddies encountered at earlier times. For example, tracers located within a region of size ℓ at time t inherit dissipation levels that were imprinted by eddies of size 2ℓ at an earlier time $t - \tau_\ell$ (see the right-hand side of Fig. 2.6). This naturally leads to a cascade interpretation in the Lagrangian frame, in which fluctuations are transmitted from large to small temporal scales through multiplicative processes.

Within this perspective, it is not the Eulerian coarse-grained dissipation ε_ℓ itself, but rather the Eulerian multiplier $\mathcal{M} = \varepsilon_\ell/\varepsilon_{2\ell}$, that encodes the local refinement of dissipation experienced along trajectories and relates directly to the Lagrangian coarse-grained dissipation $\tilde{\varepsilon}_{\tau_\ell}$. Mathematically, the modified picture corresponding to Eq. (2.14) is

$$\frac{\varepsilon_\ell}{\varepsilon_{2\ell}} = \int_{\mathcal{B}_\ell} \frac{\tilde{\varepsilon}_{\tau_\ell}}{\varepsilon_{2\ell}} \frac{d^3x_0}{\mathcal{B}_\ell} \sim \tau_\ell^\beta \quad (2.15)$$

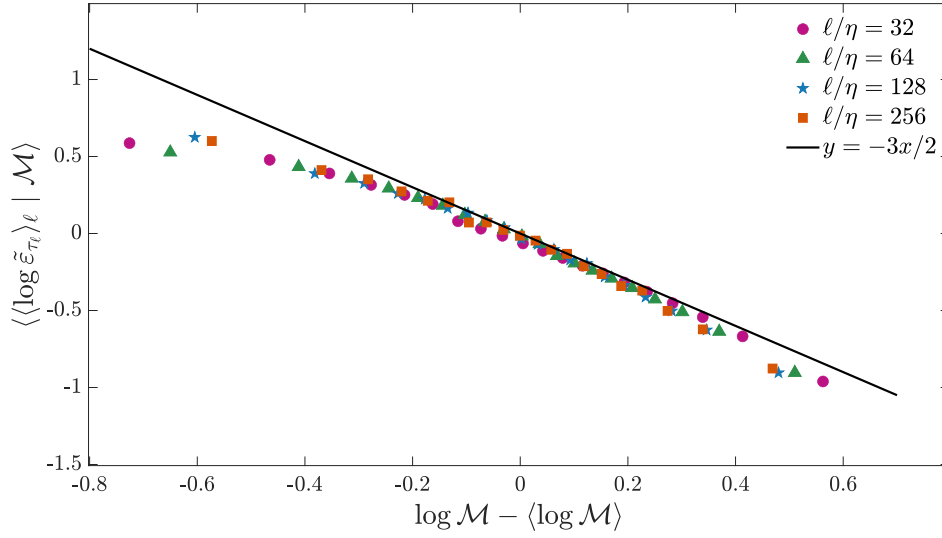


Figure 2.8. Conditional average of the logarithm of the coarse-grained Lagrangian dissipation $\tilde{\varepsilon}_{\tau_\ell}$, evaluated over the time scale $\tau_\ell = \varepsilon_\ell^{-1/3} \ell^{2/3}$, for tracers ending within a spatial region of size ℓ . Conditioning is performed on the local Eulerian multiplier $\mathcal{M} = \varepsilon_\ell / \varepsilon_{2\ell}$. The collapse across different ℓ demonstrates that Lagrangian dissipation fluctuations are directly linked to Eulerian scale-to-scale multipliers, supporting the proposed temporal-inheritance cascade picture.

This mechanism emphasizes temporal inheritance rather than instantaneous spatial sampling, and provides a coherent framework for interpreting the log-divisible statistics observed in the Lagrangian dissipation field.

Numerical experiments provide direct support for this picture. Figure 2.7 shows that tracers ending within a region of size ℓ and associated with a given coarse-grained dissipation ε_ℓ have followed separating trajectories over a characteristic doubling time $T_{2\ell} \sim \tau_\ell$, consistent with eddy-turnover scaling. This backward-in-time separation reflects the fact that tracers retain memory of their past embedding within larger eddies. Figure 2.8 further shows that the Lagrangian dissipation coarse-grained over τ_ℓ correlates with the Eulerian multiplier $\mathcal{M} = \varepsilon_\ell / \varepsilon_{2\ell}$, confirming that dissipation fluctuations experienced along trajectories are shaped by the cascade of Eulerian structures across scales. Moreover, data corresponding to different coarse-graining scales collapse onto a single curve, indicating the scale-independent, inertial-range nature of this relation. Taken together, these results support a causal, scale-local transfer of dissipation statistics consistent with a Lagrangian multiplicative process, rather than a direct mapping from Eulerian fields at a fixed spatial scale.

2.6 CONCLUSIONS

These results show that intermittency in turbulence cannot be understood solely from spatial statistics of the dissipation field, but instead reflects dynamical processes unfolding along Lagrangian trajectories. A central organizing idea in turbulence theory is that intermittency arises from a universal cascade process that can be expressed through multifractal statistics of

energy dissipation. Our results demonstrate that this picture is incomplete in a fundamental way. Energy dissipation does not possess a single, frame-independent cascade structure: its intermittent dynamics are organized primarily in time along Lagrangian trajectories rather than in space. The commonly assumed equivalence between Eulerian and Lagrangian dissipation statistics—mediated by bridge relations—fails precisely because dissipation fluctuations are not spatially self-similar in the Eulerian frame. In this sense, intermittency in turbulence is intrinsically Lagrangian in nature.

To reach this conclusion, we performed a systematic comparison of Eulerian and Lagrangian statistics of energy dissipation in homogeneous isotropic turbulence. By combining analyses of single-point distributions, correlation functions, coarse-grained fields, large-deviation properties, and multiplier statistics, we showed that the statistical organization of dissipation depends fundamentally on the frame of observation and cannot be captured by a single universal cascade description.

In the Eulerian frame, although the instantaneous dissipation exhibits approximately log-normal one-point statistics, its spatial correlations and coarse-grained properties do not support the existence of a scale-invariant multiplicative cascade. In particular, the absence of logarithmic spatial correlations, the nonlinear dependence of cumulants on scale, and the pronounced scale dependence of Eulerian multipliers collectively rule out a log-divisible or multifractal cascade interpretation of Eulerian dissipation in the inertial range.

In sharp contrast, dissipation sampled along Lagrangian trajectories displays all the defining signatures of a multiplicative cascade in time. These include logarithmic temporal correlations, linear scaling of the mean and variance of the logarithm of the coarse-grained dissipation with the logarithm of the time scale, a well-defined large-deviation principle, and approximately scale-invariant statistics of Lagrangian multipliers. Taken together, these results provide strong evidence that Lagrangian dissipation behaves as a log-divisible stochastic process consistent with a temporal multiplicative cascade.

Building on these findings, we revisited classical bridge relations that aim to connect Eulerian and Lagrangian dissipation statistics through dimensional arguments. We showed that such relations implicitly rely on a mean-field assumption, namely that dissipation fluctuations experienced by different Lagrangian trajectories within the same spatial region are effectively uniform. Our results demonstrate that this assumption is violated for dissipation, leading to a breakdown of the classical bridge relation. We proposed instead an alternative interpretation in which Lagrangian dissipation fluctuations arise from a temporal cascade inherited from larger Eulerian scales, with Eulerian multipliers encoding the scale-to-scale refinement of dissipation along particle trajectories. Numerical evidence supports this picture and highlights a causal, scale-local mechanism linking Eulerian and Lagrangian statistics without requiring a direct one-to-one mapping between coarse-grained fields.

More broadly, these results clarify long-standing ambiguities in the interpretation of dissipation intermittency and cascade phenomenology. While the Lagrangian description naturally reveals a multiplicative cascade structure in time, the Eulerian dissipation field does not admit an equivalent spatial cascade interpretation. This distinction suggests that Lagrangian statis-

tics provide a privileged framework for identifying the dynamical mechanisms underlying intermittency in turbulence.

Future work may explore how these conclusions extend to anisotropic or inhomogeneous flows, to other turbulent observables, or to stochastic models that explicitly incorporate temporal cascades along particle trajectories. More generally, the present results open the way toward turbulence models in which intermittency is formulated primarily in time along Lagrangian paths rather than inferred from spatial statistics alone.

INTERMITTENT FLUCTUATIONS DETERMINE THE NATURE OF CHAOS IN TURBULENCE

In this chapter, we investigate the Reynolds-number dependence of the maximal Lyapunov exponent in fully developed turbulence, which quantifies the rate of chaotic divergence of nearby velocity fields. Using decorrelators constructed from infinitesimally perturbed flows, we find that the Lyapunov exponent scales with Reynolds number as $\lambda \sim Re^\alpha$, with an exponent $\alpha = 0.59 \pm 0.04$ exceeding the classical mean-field prediction. By explicitly separating the nonlinear strain and viscous contributions to decorrelator growth, we show that this departure is associated by intermittent fluctuations of the strain-rate tensor, which dominate the short-time growth of the infinitesimal perturbations over viscous damping. Direct numerical simulations of the Navier–Stokes equations and complementary tests using a reduced shell model yield consistent scaling behavior, indicating robustness within the frameworks considered. Our results show that the dynamical origin of chaotic divergence in turbulence is closely linked to intermittent strain-rate fluctuations.

This chapter closely follows the work [77]¹.

3.1 INTRODUCTION

Fully developed, incompressible turbulence is perhaps the most celebrated example of a chaotic many body system. In contrast to other examples of chaotic, classical, many-body systems, turbulent flows have the distinction of being central across the natural world. Unsurprisingly, therefore, physicists working in problems ranging from astrophysics, atmospheric sciences, oceanography, and of course fluid dynamics have to factor in the underlying chaotic nature of

¹This work is done in collaboration with Aikya Banerjee, Sugan D. Murugan, Subhro Bhattacharjee, and Samriddhi Sankar Ray; part of the theoretical foundation of this study was reported in the MS Thesis of Aikya Banerjee.

such flows [103]. This ubiquity is not surprising: After all the Navier-Stokes equation underlies all such systems leading to solutions which are turbulent in the limit of small viscosities of such fluids [2].

While mathematical and engineering tools remain indispensable, the ideas of statistical physics provide the basis for much of our understanding of turbulence, especially in its most general form where the flow is homogeneous and isotropic. In particular, chaos along with the other fingerprint of fully developed turbulence, namely intermittency, and their dependence *inter alia* on the Reynolds number Re , forms a major challenge in a complete understanding of turbulence. The *degree* of chaos is usually quantified by the (largest) Lyapunov exponent λ of the flow. By using arguments tracing back to Kolmogorov's seminal work from 1941, Ruelle showed that $\lambda \propto \sqrt{Re}$ [38]. This is a consequence of associating the (largest) Lyapunov exponent with the inverse of the smallest time-scale of the flow $\tau_\eta \propto 1/\sqrt{Re}$, obtained most simply from phenomenological arguments, and thence the λ -scaling. More generally, assuming a Hölder exponent h characterising the flow, it is easy to show $\lambda \propto Re^{\frac{1-h}{1+h}}$ [40]; indeed taking the *monofractal*, Kolmogorov limit [5] of a unique $h = 1/3$ recovers the Ruelle scaling.

The result $\lambda \propto \sqrt{Re}$ implicitly assumes a unique dissipative scale characteristic of a monofractal flow. However, observational, experimental, and numerical evidence against this is overwhelming. Indeed, a rationalisation of turbulence rests on the multifractal approach [104] developed by Frisch and Parisi [24]. Adapting this model, which allows for a spread of Hölder exponents h in the flow, leads to the revised scaling obtained by Crisanti *et al.* $\lambda \propto Re^\alpha$, with $\alpha \approx 0.459$ [40].

This scaling exponent $\alpha \lesssim 0.5$ has been challenged in recent years. For example, Berera and Ho [105] have recently reported $\alpha = 0.53$. This lack of consensus between the theoretical estimate and measurements — as well as the lack of agreement between different simulations [34, 40, 105–109] — suggests that the origins of chaos in turbulence is far from settled.

But is there a way to address the microscopic origin of α from the Navier-Stokes equation itself and connect it to the intermittent aspect of fully developed turbulence? And what role does the non-locality (in length scale) of interactions — a defining feature of the Navier-Stokes equation — play? Let us emphasise right away that in what follows intermittency enters here only in this operational sense, through the statistics of velocity-gradient fluctuations relevant for perturbation growth, rather than as a subject of phenomenological characterization across scales.

3.2 DECORRELATOR FOR THE NAVIER-STOKES

In this paper we show that this is indeed possible through recently constructed ideas of *decorrelators* [110] — largely limited to Hamiltonian [42, 111, 112] systems in the context ergodicity, thermalisation and chaos — and adapt them for driven-dissipative, non-equilibrium systems such as turbulence. Consequently, by using a suitable mix of theory and numerical simulations, we uncover the competing effects of the viscous and nonlinear terms to show *exactly* how the chaos and the scaling $\lambda \sim Re^\alpha$ emerges along with what determines the value of $\alpha = 0.59 \pm 0.04$.

We also confirm the robustness of this result by using a dynamical systems approach to turbulence with the additional advantage of exploiting its nearest and next-nearest neighbour interactions to investigate short-time effects of locality which are absent in the Navier-Stokes equation. The objective of this work is not to establish the existence of intermittency or to characterize its multifractal structure, both of which are well established. Rather, we determine how fluctuations of the strain-rate field enter the short-time growth of infinitesimal perturbations and thereby control the Reynolds-number scaling of the maximal Lyapunov exponent.

We begin with the three-dimensional, incompressible, unit-density Navier-Stokes equation, driven by an external force to a non-equilibrium statistically steady state. The force $\mathbf{f}$ varies to ensure a constant energy injection [113] $\mathcal{I} = \langle \mathbf{f} \cdot \mathbf{u} \rangle \equiv \overline{\langle \epsilon \rangle}$, where the mean energy dissipation rate $\overline{\langle \epsilon \rangle}$ is the temporal $\overline{\cdot \cdot \cdot}$ and spatial $\langle \cdot \cdot \cdot \rangle$ average of the spatio-temporally fluctuating dissipation $\epsilon(\mathbf{x}, t)$ characteristic of such non-equilibrium steady states of fully developed turbulence with a statistically steady velocity field $\mathbf{u}_{\text{steady}}$. This follows from the familiar, instantaneous (spatially-averaged) kinetic energy E balance: $\frac{dE}{dt} = -\langle \epsilon \rangle + \mathcal{I}$ with a mean kinetic energy $E_0 = \overline{E}$.

We now construct two, *nearly* identical, velocity fields $\mathbf{u}_0^A \equiv \mathbf{u}_{\text{steady}}$ and $\mathbf{u}_0^B = \mathbf{u}_0^A + \delta \mathbf{u}_0$, where the form of $\delta \mathbf{u}_0$ ensures that system B remains incompressible. We choose the amplitude of perturbation ϵ_0 sufficiently small [41] and localise it in space around $\mathbf{x}_0$; the precise choice of $\mathbf{x}_0$ is irrelevant and can be chosen to be the center of our domain.

We choose these fields — $\mathbf{u}_0^A$ and $\mathbf{u}_0^B$ — as initial conditions $t = 0$ for the Navier-Stokes equation, evolve them independently in time, and calculate the velocity difference $\delta \mathbf{u}(\mathbf{x}, t) \equiv \mathbf{u}^B(\mathbf{x}, t) - \mathbf{u}^A(\mathbf{x}, t)$ for $t > 0$. From such velocity differences, we are able to construct the decorrelator $\phi(\mathbf{x}, t) \equiv \frac{1}{2} |\delta \mathbf{u}(\mathbf{x}, t)|^2$ [42, 43]. It is easy to show, starting from the Navier-Stokes equation written separately for systems A and B, that the decorrelator follows an evolution (written in component form and sum over repeated indices assumed)

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} = \partial_i W_i - \delta u_i S_{ij} \delta u_j + \nu \delta u_i \nabla^2 \delta u_i \quad (3.1)$$

with the strain-rate tensor $S_{ij} = \frac{1}{2} (\partial_i u_j^A + \partial_j u_i^A)$ and

$$W_i = (u_i^A + \delta u_i) \phi + \delta u_i \int d\mathbf{x}' G(\mathbf{x}, \mathbf{x}') [\partial_{jk}^2 (u_j^A \delta u_k + \delta u_j u_k^A + \delta u_j \delta u_k)]', \quad (3.2)$$

where $G(\mathbf{x}, \mathbf{x}')$ is the Green's function obtained from the Poisson equation for pressure, and ν the coefficient of kinematic viscosity [41].

A further simplification exploits the statistical isotropy and homogeneity of fully developed turbulence to construct, over the volume $\mathcal{V}$, the spatially averaged decorrelator $\Phi(t) \equiv \frac{1}{\mathcal{V}} \int d\mathbf{x} \phi(\mathbf{x}, t) = \langle \phi \rangle$. This leads to a simple cancellation of the divergence term in Eq. (3.1) and a resulting evolution equation for the integrated decorrelator:

$$\dot{\Phi} = \frac{d\Phi}{dt} = \beta_S + \beta_\eta, \quad (3.3)$$

where $\beta_S = -\langle \delta \mathbf{u} \cdot \mathbf{S} \cdot \delta \mathbf{u} \rangle$ and $\beta_\eta = \nu \langle \delta \mathbf{u} \cdot \nabla^2 \delta \mathbf{u} \rangle$ as the contributions from the non-linear strain and dissipative terms, respectively. There is one additional remark to make. In the derivations

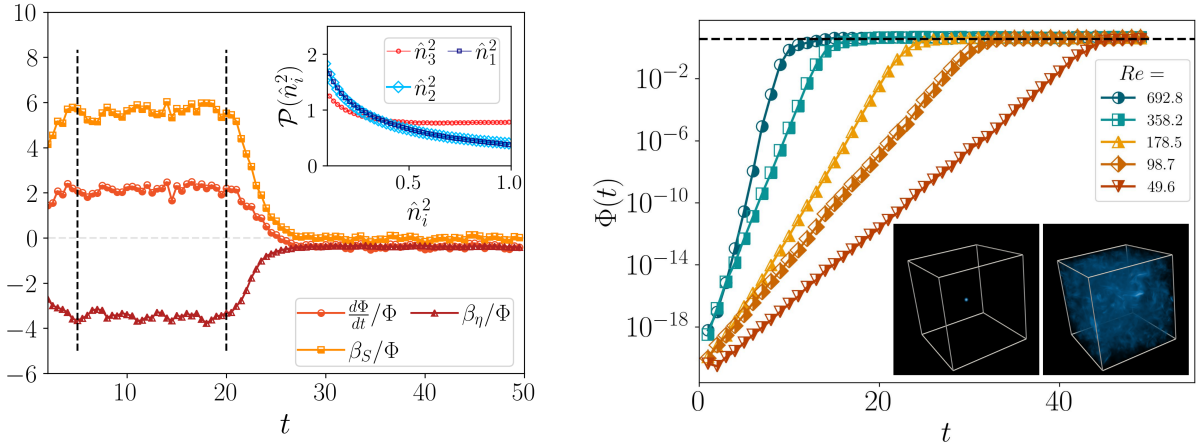


Figure 3.1. **Left panel** shows representative plots of the normalised spatially averaged decorrelator $\dot{\Phi}/\Phi$, β_S/Φ and β_η/Φ vs time for $Re = 178.5$. The pair of vertical dashed lines indicate the short-time exponential-growth phase. (Inset) Probability density function of the $\hat{n}_i^2$ direction cosines of $\mathbf{u}$ (at short times) showing a preferential growth of the difference field along the compressional $i = 3$ direction. **Right panel** shows Semilog plots of Φ vs t from DNSs for different Re showing an exponential regime followed by saturation at a value $2E_0$ (dashed horizontal line). The insets show two snapshots at (left) $t = 0$ and (right) $t = 12.5$ of the square of the difference field $|\delta \mathbf{u}|^2$ for $Re = 178.5$. (See Ref. [114] for the full evolution of $|\delta \mathbf{u}|^2$.)

of Eqs. (3.1)-(3.3) we have neglected the forcing terms on systems A and B. This is perfectly valid at short times but in the long time $t \rightarrow \infty$ limit it plays a critical role as we shall see below.

At short times we conjecture an exponential growth of the decorrelator: $\dot{\Phi}/\Phi \equiv \lambda$. We confirm this numerically by integrating Eq. (3.3) with an initial perturbation field $\delta \mathbf{u}$ and measurements of the strain-rate tensor $\mathbf{S}$ drawn from direct numerical simulations (DNSs) of the Navier-Stokes equation [41].

In Fig. 3.1 we show a representative plot of $\dot{\Phi}/\Phi$ vs time for $Re = 178.5$ displaying a clear plateau λ demarcated by a pair of vertical dashed lines. While this is not very surprising, given that turbulence is chaotic, it is far from obvious how to trace its origins in the structure of Eq. (3.3) and in particular how the competing effects of the strain and viscous terms conspire to give such exponential-growth phases. In Fig. 3.1 we plot β_S and β_η (compensated by Φ) and find strong evidence of their short-time exponential growth — the plateau between the vertical dashed lines — corresponding to Lyapunov exponents λ_S and λ_η , respectively. Furthermore, the dissipative effect of the viscous term $\lambda_\eta = \beta_\eta/\Phi \sim \mathcal{O}(1/\tau_\eta) < 0$ is compensated by the effect of the strain term $\lambda_S = \beta_S/\Phi > |\lambda_\eta|$, leading to an overall positive Lyapunov exponent $\lambda = \lambda_S + \lambda_\eta$.

3.3 REYNOLDS DEPENDENCE OF LARGEST LYAPUNOV: EFFECT OF STRAIN RATE FLUCTUATIONS

How do we understand the origins of $\lambda_S > |\lambda_\eta|$ and hence $\lambda > 0$? The answer lies in decomposing the strain term in the eigenbasis of the strain tensor $\delta \mathbf{u} \cdot \mathbf{S} \cdot \delta \mathbf{u} = \sum_{i=1}^3 \hat{n}_i^2 \gamma_i |\delta \mathbf{u}|^2$, with $\hat{n}_i$ direction cosines of $\mathbf{u}$ (along the eigendirections) and γ_i the eigenvalues. Thus for $\lambda > 0$, the overall negative sign on β_S implies that at short times, this sum must be dominated, even if marginally, by the statistics of the eigenvalue in the compressional direction, since $\gamma_3 < 0$, over the extensional ($\gamma_1 > 0$) and intermediate ($\gamma_2 \gtrsim 0$) directions. This bias for the compressional direction is confirmed by the marginally flatter distribution of $\hat{n}_3^2$, plotted in the inset of Fig. 3.1.

At such short times, a careful analysis of β_S shows [41] that $\beta_S \sim \langle \gamma_3 \rangle + \gamma_3^{\text{std}}$, where, $\gamma_3^{\text{std}} = \sqrt{\langle (\gamma_3 - \langle \gamma_3 \rangle)^2 \rangle}$. And hence, the Reynolds number dependence of λ_S must be dominated by the competing effects of the mean $\langle \gamma_3 \rangle$ eigenvalue and their intermittency-induced fluctuations γ_3^{std} .

Of course in the long time limit, $\beta_S \rightarrow 0$, all eigenvalues are sampled equally [41] leading to $\langle \delta \mathbf{u} \cdot \mathbf{S} \cdot \delta \mathbf{u} \rangle \propto \sum_{i=1}^3 \gamma_i = 0$ because of incompressibility. We see from Fig. 3.1 that this is indeed the case as the compensated β_S term rapidly falls to 0 for $t \gtrsim 25$.

Does the viscous term β_η also go to zero similarly at late times leading to $\dot{\Phi} = 0$? At such late times it is easy to show that [41] $\beta_\eta \rightarrow -2\overline{\langle \epsilon \rangle}$. Indeed, in Fig. 3.1 we do see a clear saturation of the normalised viscous term to a value consistent with $-2\overline{\langle \epsilon \rangle} \approx 1.6$ (corresponding to the simulations from which the rate of strain matrix is drawn). A trivial consequence of this is that within the framework of Eq. (3.3) $\dot{\Phi} = -2\overline{\langle \epsilon \rangle}$ (clearly seen in Fig. 3.1) and suggesting that the decorrelator never saturates. However this is a contradiction: By definition, the decorrelator is defined as the spatial average of the velocity difference of systems A and B, and hence at long times $\Phi \rightarrow 2E_0$ since the cross-correlator $\langle \mathbf{u}_A \cdot \mathbf{u}_B \rangle$ vanishes as the fields decorrelate as $t \rightarrow \infty$.

This contradiction is resolved by recalling that the derivation of Eq. (3.3) neglects the effective energy injection. While at short times, this is zero, at long times this contribution is no longer irrelevant. In fact we can show [41] that $\langle \delta \mathbf{f} \cdot \delta \mathbf{u} \rangle = 2\overline{\langle \epsilon \rangle}$ as $t \rightarrow \infty$ and thus compensates the viscous contribution $-2\overline{\langle \epsilon \rangle}$ [41] leading to $\dot{\Phi} = 0$.

How consistent are these ideas in actual measurements of the decorrelator in DNSs of the Navier-Stokes equation? We check for this by solving the incompressible Navier-Stokes equation on a 2π triply-periodic domain by using a pseudospectral method with a large-scale constant energy $\mathcal{I} = \overline{\langle \epsilon \rangle}$ [113] injection scheme to drive the flow and maintain it in non equilibrium statistically steady state with a mean (time averaged) energy E_0 ; see [41] for a detailed summary of the numerical scheme and the parameters used. By using the same protocol described earlier, we calculate the decorrelator Φ for several different values of the Reynolds number Re . In Fig. 3.1 we show representative snapshots at $t = 0$ (left inset) and $t = 12.5$ (right inset), corresponding to the initial and exponential-growth phases, respectively. (Ref. [114] shows the full evolution of $|\delta \mathbf{u}|^2$.)

These difference fields allow us to construct the spatially averaged decorrelator Φ which we plot, on a semilog scale, in the main panel of Fig. 3.1 as a function of time. We see a convincing

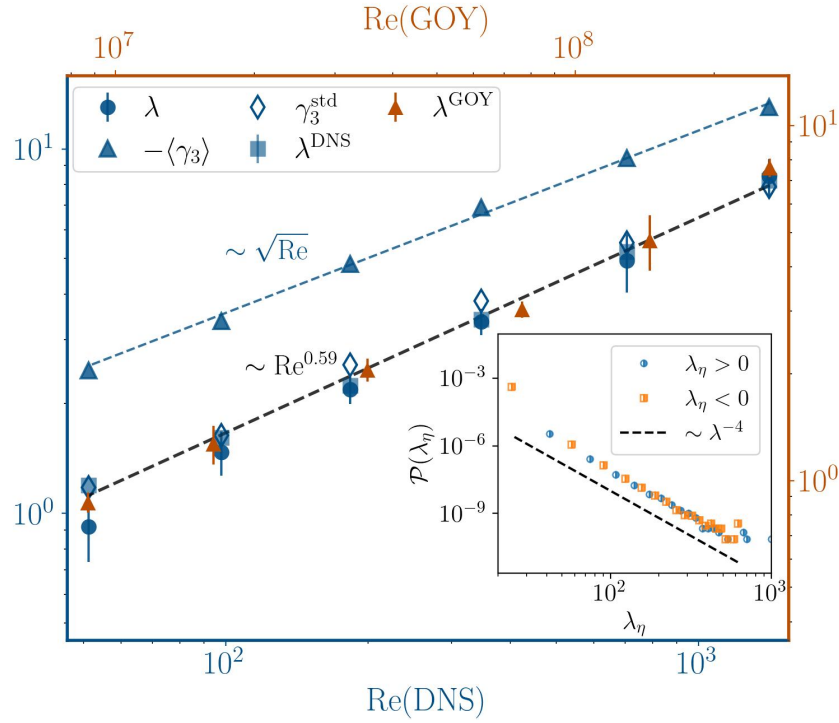


Figure 3.2. A loglog plot of λ , λ^{DNS} , γ_3^{std} and $-\langle\gamma_3\rangle$, obtained from various DNSs versus the Reynolds number Re_η (DNS) (bottom X axis), as well as the Lyapunov exponent obtained from the shell model λ^{GOY} (rescaled by a constant factor to overlap with the DNS data since the Reynolds number ranges are different) with the corresponding shell model Reynolds numbers $Re(\text{GOY})$ (top X axis). (Inset) A loglog plot of the probability density function $\mathcal{P}$ for λ_η showing a power-law $|\lambda_\eta|^{-4}$ tail.

exponential regime from which we can extract the Lyapunov exponent λ^{DNS} , followed by a saturation at a value approximately equal to $2E_0$, indicated by the dashed horizontal line, as our theory suggests. (We have confirmed from our data that the onset of the saturation of the decorrelators (see Figs. 3.1 and 3.1) is of the order of the inverse of the Lyapunov exponent.) We can now summarise these ideas in answering the question of how consistent are the various (largest) Lyapunov exponents — λ and λ^{DNS} — with each other given the different ways in which they are derived? In Fig. 3.2 we show a composite plot of λ and λ^{DNS} versus the Reynolds number Re_η (bottom x-axis). We estimate the Lyapunov exponents λ^{DNS} from a linear fit in the exponential-growth phase of the decorrelators (see Fig. 3.1); the errors on such fits yield the errorbars on λ^{DNS} . We find our errorbars smaller than the symbols sizes in Fig. 3.2. The errorbars for the exponent γ_3 are extracted likewise and are of the same order as those on λ^{DNS} . The exponent λ is obtained differently. We use plots of $\dot{\Phi}/\Phi$, such as the one shown in Fig. 3.1, to extract λ and its error bar as the mean and standard deviation of the plateau, respectively.

We find remarkable consistency (within error bars) between the different measurements of the Lyapunov exponents and they seem to follow a universal scaling: $\lambda^{\text{DNS}} = \lambda \sim Re^\alpha$, with $\alpha = 0.59 \pm 0.04$. The theory developed earlier suggests a connection between the Lyapunov exponent and the statistics of γ_3 with $\lambda_S \sim \langle\gamma_3\rangle + \gamma_3^{\text{std}}$ [41]. The departure of the measured Lyapunov exponents (Fig. 3.2) from a Kolmogorov-like mean field result $\lambda \sim \langle\gamma_3\rangle \sim \sqrt{Re}$, is a

clear indication that the deviation must stem from intermittent fluctuations and the dominant scaling in the statistics of γ_3 is due to $\gamma_3^{\text{std}} \sim Re^\alpha$. In Fig. 3.2 we show a plot of γ_3^{std} from our DNSs and find a near perfect agreement with the Lyapunov exponent confirming the robustness of this theory: $\lambda = \lambda^{\text{DNS}} = \gamma_3^{\text{std}} \sim Re^{0.59}$. Furthermore, we confirm that the mean eigenvalue $-\langle\gamma_3\rangle \sim \sqrt{Re}$ (Fig. 3.2) consistent with the simpler scaling analysis, not accounting for fluctuations, and at odds with the measured Lyapunov exponent.

It is important to observe that even at very short times, before the exponential growth phase, our results from the Navier-Stokes equation summarised in Fig. 3.1 do not suggest an initial power-law growth which has been seen in many local models of chaotic systems [115, 116]. Could this — as well as the somewhat self-similar evolution of the difference field (see Fig. 3.1, insets) — be a consequence of the essential non-local nature of the equation?

3.4 TEST ON A SHELL MODEL

To test this as well as the robustness of our conclusions, we use a phenomenological (cascade) model for turbulence namely the Gledzer-Ohkitani-Yamada (GOY) shell model [116–118] for complex dynamical variables u_n associated with a scalar, exponentially growing, wave-number $k_n = k_0 2^n$, with the shells numbers being integers $1 \leq n \leq N$ (see [41] for further details and caveats). We note that shell models lack explicit spatial structure and therefore cannot represent spatial intermittency or the full triadic geometry of the Navier–Stokes equations. Here they are used solely as reduced dynamical systems to test the robustness of Lyapunov scaling while keeping in mind that several studies have shown how shell models capture the effect of intermittency in measurements such as the scaling exponents of equal-time structure functions [2, 115].

In the statistically steady state [41], let the velocity *field* be denoted as u_n^A . We now construct a second field $u_n^B = u_n^A \forall n \neq n_p$ and $u_{n_p}^B = (1 + \epsilon_0)u_{n_p}^A$. Our choice of $n_p = 16$ ensures that the tiny perturbation (with $\epsilon_0 = 10^{-3}$) are always added at wavenumbers just before the onset of the dissipation scales for all the Reynolds number being considered. (We have checked that our results are robust to the specific choice of n_p or variations in ϵ_0 .)

We now use the nearly-identical fields u_n^A and u_n^B as initial conditions and evolve the GOY shell model independently for systems A and B and hence construct the shell model decorrelator $\phi_n(t) \equiv \langle |u_n^A - u_n^B|^2 / 2 \rangle$. The $\langle \cdot \rangle$ denote an ensemble averaging over independent realisations of the initial field u_n^A . While it is straightforward to write the evolution equation for ϕ_n [41], we find this is neither revealing nor analytically tractable. Instead, we choose to work with the numerically constructed ϕ_n and assess its behaviour from our simulations.

Given the local nature of interactions in the shell model [115], it is reasonable to expect that decorrelators such as ϕ_n may well have a self-similar short time growth followed by an eventual exponential rise with a saturation as $t \rightarrow \infty$. In the inset of Fig. 3.3 we show a loglog plot of the temporal evolution of ϕ_n for a small initial time window $0 \leq t \lesssim 0.5$ with a clear power-law range for $n < n_p$. This power-law eventually ($t \gtrsim 0.1$) gives way to the exponential-growth phase of decorrelators. The Lyapunov exponent characterising such shell-by-shell decorrelators

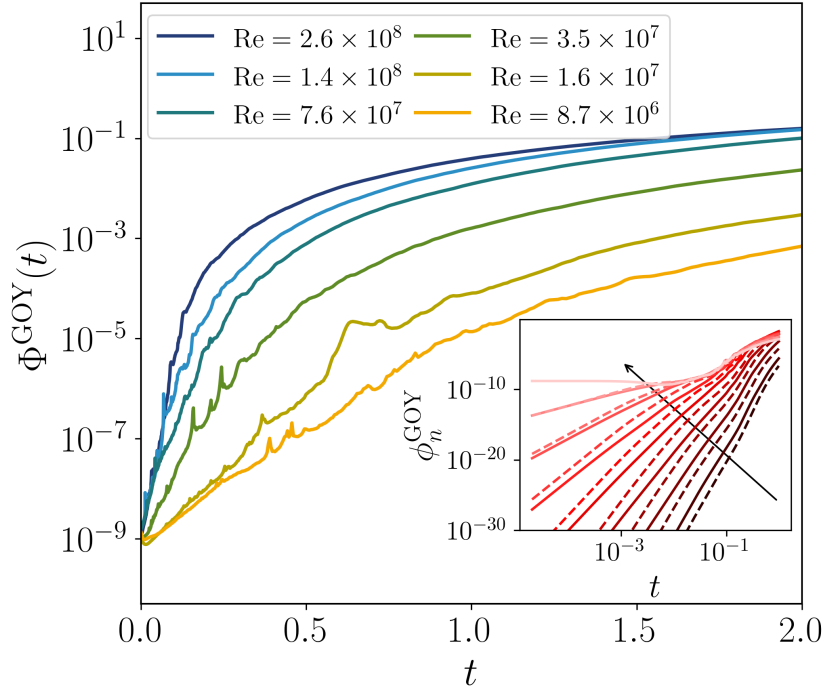


Figure 3.3. Semilog plot of the shell model decorrelator Φ^{GOY} vs t for different Re showing an exponential growth. (Inset) Loglog plot of the shell-specific decorrelator ϕ_n for different shells (the arrow indicates the direction $n = 1$ to $n = 16$ shells) at *very* early times shows a power-law growth.

ϕ_n are of course n -dependent. Hence we find it useful to construct the n -independent Lyapunov exponent λ^{GOY} via the integrated decorrelator $\Phi^{\text{GOY}} = \sum_1^N \phi_n$. In the main panel of Fig. 3.3 we show a semilog plot of the temporal evolution of Φ^{GOY} for different values of Re showing, yet again, a clear exponential growth, with a shell model Lyapunov exponent λ^{GOY} , followed by a saturation to values roughly corresponding to $2E_0$, with E_0 the statistically steady state energy in our shell model.

In Fig. 3.2, along with the results from the Navier-Stokes equation, we overlay λ^{GOY} versus the Reynolds number Re_y corresponding to the GOY shell model (top x-axis); the Lyapunov exponents are obtained from fits in the exponential-growth phase of the shell model decorrelators as shown in 3.3; the errors on such fits yield error bars which are smaller than the symbols sizes in Fig. 3.2. Clearly, comparing with the results from the Navier-Stokes equation, we find $\lambda^{\text{GOY}} = \lambda^{\text{DNS}} = \lambda \sim Re^\alpha$, with $\alpha = 0.59 \pm 0.04$. Remarkably, combining the shell model and DNS data Fig. 3.2 shows that our reported scaling holds for nearly 7 decades in Reynolds number.

In this work we have shown how recent ideas of decorrelators [42, 110–112, 119] provide a microscopic way to understand the origins of chaos in fully developed turbulence. Remarkably, these ideas are just as robust for simpler, cascade models of turbulence where locality [115] ensures an initial scale-invariant growth of the decorrelator. We find signatures of these fluctuations in one further aspect of the strain and viscous terms in Eq. (3.3) which connects with the idea of intermittency. The theory developed by us involve the *mean* exponents: λ_S and λ_η (and hence λ) are obtained from the spatial integrals over the corresponding terms. However, it is

possible to measure the *local* exponents $\lambda_S(\mathbf{x}) \equiv \frac{\delta \mathbf{u} \cdot \mathbf{S} \cdot \delta \mathbf{u}}{|\delta \mathbf{u}|^2/2}$ and $\lambda_\eta(\mathbf{x}) \equiv \frac{\nu \delta \mathbf{u} \cdot \nabla^2 \delta \mathbf{u}}{|\delta \mathbf{u}|^2/2}$ in the exponential-growth phase and thence their probability distribution functions $\mathcal{P}$. These distributions are found to have exponential (for $\lambda_S(\mathbf{x})$) or power-law (for $\lambda_\eta(\mathbf{x})$; see inset of Fig. 3.2) tails strongly suggestive of intermittency and its connections with the question of chaos in turbulence. Further, $\alpha > 1/2$ also suggests that *real* turbulence would enhance Lagrangian and scalar mixing by a significant shortening of the mixing time-scales $\sim 1/\lambda$ and increased stretching rates and hence ought to be factored in in modeling of such phenomena. We do not explore these ideas, as well as its implications for Lagrangian chaos [120, 121], any further here but leave it for a more detailed study of the statistics of β_S and β_η in future. Furthermore, because the decorrelator is a second-order quantity, the present analysis does not access the multifractal hierarchy of turbulent fluctuations; this limitation does not affect the conclusions regarding Lyapunov scaling.

3.A DIRECT NUMERICAL SIMULATIONS

3.A.1 THE NAVIER-STOKES EQUATION

We perform direct numerical simulations (DNSs), by using a pseudospectral method, of the three-dimensional (3D), incompressible, unit density Navier Stokes equation, with the pressure field P

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla P + \mathbf{f}. \quad (3.4)$$

on a periodic box of size $L = 2\pi$ with N^3 collocation points. We choose $N = 256$ and $N = 512$ (to check for numerical convergence) and vary the coefficient viscosity $10^{-3} \leq \nu \leq 32 \times 10^{-3}$ to obtain Reynolds numbers $50 \leq Re \leq 1400$. We use a second-order Adams-Bashforth for time-marching with a time step $\delta t = 5 \times 10^{-4}$ (for $N = 256$) and $\delta t = 4 \times 10^{-4}$ (for $N = 512$).

We initialise the flow with a random initial condition such that the initial energy spectrum $E(k) = A_0 k^2 \exp(-k^2/2k_0^2)$ with $A_0 = k_0 = 1$. The system is driven to a non-equilibrium steady state (NESS) through a large-scale forcing with a constant energy injection at large scales corresponding to wavenumber $1 \leq k_{\text{force}} \leq 2$.

This statistically steady velocity field $\mathbf{u}$ is taken as the initial field for system A: $\mathbf{u}_0^A \equiv \mathbf{u}$ and further define the initial condition for system B via $\mathbf{u}_0^B = \mathbf{u}_0^A + \delta \mathbf{u}_0$. We use a Gaussian perturbation $\delta \mathbf{u}_0(\mathbf{x}) = \epsilon_0 \exp\{-(x - x_0)^2/2\sigma_0^2\} \hat{\mathbf{x}}$ with $\epsilon_0 = 10^{-5}$, $\sigma_0 = 4dx$ and $x_0 = (\pi, \pi, \pi)$ at the center of the domain and which also satisfies the incompressibility constraint.

The nearly identical fields $\mathbf{u}_0^A$ and $\mathbf{u}_0^B$ are then used as initial conditions for the Navier-Stokes equation and evolved as before. The decorrelator is calculated from the evolution of such twin simulations as $\phi(\mathbf{x}, t) = |\mathbf{u}^B - \mathbf{u}^A|^2/2$ and thence $\Phi(t) = \int d\mathbf{x} \phi(\mathbf{x}, t)/\mathcal{V}$. The decorrelators are then used to estimate the Lyapunov exponents (obtained by fitting the initial exponential growth phase) and its dependence on the Reynolds number.

3.A.2 THE GLEDZER-OHKITANI-YAMADA (GOY) MODEL

The Gledzer-Ohkitani-Yamada or the GOY shell model

$$\frac{d}{dt}u_n = ik_n[u_{n+2}u_{n+1} - \frac{1}{4}u_{n+1}u_{n-1} - \frac{1}{8}u_{n-1}u_{n-2}]^* - \nu k_n^2 u_n + f \quad (3.5)$$

is one of several cascade models which mimic Navier-Stokes turbulence and are designed to achieve very high Reynolds number. In such models, the velocities u_n (with boundary conditions $u_{-1} = u_0 = u_{N+1} = u_{N+2} = 0$) associated with a scalar, exponentially growing, wave-number $k_n = k_0 2^n$, are complex, dynamical variables. The shell numbers range from $1 \leq n \leq N$; in our simulations we choose $N = 22$ and the coefficient of viscosity varies as $2 \times 10^{-7} \leq \nu \leq 64 \times 10^{-7}$ yielding Reynolds numbers in the range $8.7 \times 10^6 \leq Re \leq 2.6 \times 10^8$; the shell model, Reynolds number is simply defined as $Re = |u_{rms}|/k_0 \nu$. The equation is integrated numerically by using a second-order Runge-Kutta scheme with a time step $\delta t = 2 \times 10^{-5}$. The constant amplitude forcing $f = 0.1$, applied on the $n = 2$ shell, drives the system to a statistically steady state [122].

As with the strategy for the Navier-Stokes equation, we first obtain a non-equilibrium statistically steady state with an energy spectrum $E(k_n) \equiv |u_n|^2/k_n \sim k_n^{-5/3}$ over an inertial range of shells whose extent is determined by Re . We use this steady state velocity field to define the initial field for system A — $u_{n,0}^A$ and construct the initial condition for system B via $u_{n,0}^B = (1 + \epsilon_0)\delta_{n,n_p}u_{n,0}^A$. In all our simulations we have used $\epsilon_0 = 0.001$ and $n_p = 16$. Numerically it has been seen that the shell wise decorrelators first show a power-law rise later followed by the exponential-growth regime. The initial power law largely depends on the distance from the perturbed shell. Therefore, we add the perturbation in a fixed shell $n_p = 16$ for all values of Re .

A final remark and note of caution is necessary. Shell models are not derived directly from the Navier-Stokes equations and contain no explicit information about spatial structures [123]. However, they respect the symmetries and conservation laws of the Navier-Stokes equations and, remarkably, reproduce two-point statistics, including the scaling of structure functions with intermittency corrections.

3.B DERIVATION OF THE DECORRELATOR FROM THE NAVIER-STOKES EQUATION AND ITS LIMITS

3.B.1 EVOLUTION EQUATION FOR $\phi(\mathbf{r}, t)$ AND Φ

In this section we provide the derivation of the equation of motion of the decorrelator for the Navier-Stokes equation and in particular estimate the long and short time limits of β_S and β_η .

We begin by recalling that the pressure term in the Navier-Stokes equation can be rewritten as a Poisson equation

$$\nabla^2 P = -\partial_j u_i \partial_i u_j \quad (3.6)$$

by exploiting the incompressibility constraint. This Poisson equation underlines the non-local nature of turbulence.

It is useful, in what follows, to have a Green's function representation

$$P(\mathbf{x}) = - \int d\mathbf{x}' G(\mathbf{x}, \mathbf{x}') [\partial_j u_i \partial_i u_j]' \quad (3.7)$$

where the $'$ denotes the quantity evaluated at $\mathbf{x}'$.

This allows us to construct the evolution equation of the solenoidal velocity perturbation $\delta \mathbf{u} = \mathbf{u}^B - \mathbf{u}^A$ in component form

$$\frac{\partial \delta u_i}{\partial t} = -\delta u_j \partial_j u_i^A - u_j^B \partial_j \delta u_i - \partial_i \left(\int d\mathbf{x}' G(\mathbf{x}, \mathbf{x}') [\partial_{jk}^2 (u_j^A \delta u_k + \delta u_j u_k^A + \delta u_j \delta u_k)]' \right) + \nu \nabla^2 \delta u_i + \delta f_i \quad (3.8)$$

and hence, via a dot product of $\delta \mathbf{u}$ with Eq. (3.8), the equation for the decorrelator $\phi(\mathbf{r}, t)$:

$$\frac{\partial \phi}{\partial t} = -\delta u_i S_{ij} \delta u_j - \partial_i W_i + \nu \delta u_i \nabla^2 \delta u_i + \delta u_i \delta f_i. \quad (3.9)$$

Here, $S_{ij} = \frac{1}{2}(\partial_j u_i^A + \partial_i u_j^A)$ is the strain-rate tensor and $W_i = u_i^B \phi + \delta u_i \int d\mathbf{x}' G(\mathbf{x}, \mathbf{x}') [\partial_{jk}^2 (u_j^A \delta u_k + \delta u_j \delta u_k)]'$.

A spatial integration with $\Phi = \int d\mathbf{x} \phi$ now leads to the cancellation of the divergence term and hence

$$\frac{d\Phi}{dt} = -\langle \delta u_i S_{ij} \delta u_j \rangle + \nu \langle \delta u_i \nabla^2 \delta u_i \rangle + \langle \delta u_i \delta f_i \rangle \equiv \beta_S + \beta_\eta + \langle \delta u_i \delta f_i \rangle \quad (3.10)$$

the evolution equation for our spatially integrated decorrelator as discussed in the main text of the manuscript.

3.B.2 SHORT AND LONG-TIME ASYMPTOTICS OF β_S

At early times, $\langle \delta u_i \delta f_i \rangle = 0$ (since the forcing terms are nearly identical), the pre-saturation exponential growth of the decorrelator with the Lyapunov exponent $\lambda = \lambda^{\text{DNS}} \sim \lambda_S$ is controlled by the short time asymptotics of β_S . Hence, in what follows, we drop the contribution from β_η , to simplify presentation. (The β_η term is treated separately in the following subsection.)

We now expand $\delta u_i S_{ij} \delta u_j$ in the eigenbasis of the strain-rate tensor, with eigenvalues γ_i and the square of the direction cosines $\cos^2 \theta_i = \hat{n}_i^2 \in [0, 1]$, to obtain

$$\frac{\partial}{\partial t} \phi = -\phi(\mathbf{x}, t) \Gamma(\mathbf{x}, t) \quad (3.11)$$

with $\Gamma \equiv 2 \sum_{i=1}^3 \gamma_i \hat{n}_i^2$. By choosing an integrating factor via $\partial_t \Theta(\mathbf{x}, t) = \Gamma(\mathbf{x}, t)$, it follows:

$$\phi(\mathbf{x}, t) = \phi(\mathbf{x}, 0) \exp\{[\Theta(\mathbf{x}, 0) - \Theta(\mathbf{x}, t)]\}. \quad (3.12)$$

Furthermore, spatial averages $\langle \cdot \cdot \rangle = \frac{1}{V} \int_V d\mathbf{x}$ (over the whole domain $\mathcal{V} = (2\pi)^3$) lead to

$$\begin{aligned} \Phi(\tau) &= \langle \phi(\mathbf{x}, 0) \exp[\Theta(\mathbf{x}, 0) - \Theta(\mathbf{x}, t)] \rangle \\ &\leq \frac{f_{\max}}{\Phi(0)} \Phi(0) \langle \exp[\Theta(\mathbf{x}, 0) - \Theta(\mathbf{x}, t)] \rangle. \end{aligned} \quad (3.13)$$

In the last line, we have used the inequality that for any $g(\mathbf{x}), h(\mathbf{x}) \geq 0$, $\langle g(\mathbf{x})h(\mathbf{x}) \rangle \leq (\sup_{\mathbf{x}} g(\mathbf{x}))h(\mathbf{x})$. Here $f_{\max} = \sup_{\mathbf{x}} \phi(\mathbf{x}, 0)$ and $\Phi(0)$ are constants (independent of Re). Therefore, ignoring the

constant pre-factor we can write that $\Phi(t) \lesssim \Phi(0) \langle \exp[\Theta(\mathbf{x}, 0) - \Theta(\mathbf{x}, t)] \rangle$. Introducing the notation $\Delta\Theta = \Theta(\mathbf{x}, 0) - \Theta(\mathbf{x}, t)$, it thus follows trivially that —

$$\begin{aligned}
\Phi(\tau) &\lesssim \Phi(0) \langle \exp\{\Delta\Theta\} \rangle \\
&\lesssim \Phi(0) \left[\sum_{p=0}^{\infty} \frac{1}{p!} \langle \Delta\Theta^p \rangle \right] \\
&\lesssim \Phi(0) \exp \left\{ \left(\ln \left[\sum_{p=0}^{\infty} \frac{1}{p!} \langle \Delta\Theta^p \rangle \right] \right) \right\} \\
&\lesssim \Phi(0) \exp \left\{ \left(\ln \left[1 + \sum_{p=1}^{\infty} \frac{1}{p!} \langle \Delta\Theta^p \rangle \right] \right) \right\} \\
&\lesssim \Phi(0) \exp \left\{ \left[\sum_{p=1}^{\infty} \frac{1}{p!} \langle \Delta\Theta^p \rangle \right] \right\}
\end{aligned} \tag{3.14}$$

In the last step, we have used the inequality $\log(1+x) \leq x$ where, $x > -1$ ($\sum_p \langle \Delta\Theta^p \rangle / p!$ is positive in the growth phase because of the compressional bias). Considering terms up to $\mathcal{O}(\Delta\Theta^2)$, to account for corrections stemming from intermittent fluctuations [105]

$$\Phi(t) \lesssim \Phi(0) \exp \left[\langle \Delta\Theta \rangle + \frac{1}{2} \langle \Delta\Theta^2 \rangle \right] = \exp \left[\lambda_S^{(1)} t + \lambda_S^{(2)} t \right] \sim \exp \{ (\lambda_S t) \}. \tag{3.15}$$

The problem then reduces to evaluating $\lambda_S^{(1)}$ and $\lambda_S^{(2)}$ and then examine their Reynolds number dependence. The dominant contribution between the two will determine how λ_S , and in turn the Lyapunov exponent λ scales with Re .

Let us consider the first order term $\langle \Delta\Theta \rangle$ (and hence $\lambda_S^{(1)}$) in terms of Γ :

$$\begin{aligned}
\langle \Delta\Theta \rangle &= - \int_0^t dt_1 \langle \Gamma(\mathbf{x}, t_1) \rangle \\
&= -2 \int_0^t dt_1 \langle \gamma_i(\mathbf{x}, t_1) \hat{n}_i^2(\mathbf{x}, t_1) \rangle \\
&\leq -2 \int_0^t dt_1 \langle \gamma_3(\mathbf{x}, t_1) \rangle \\
&\leq -2 \langle \gamma_3 \rangle t
\end{aligned} \tag{3.16}$$

In the third step, we used the fact that $-\gamma_1, -\gamma_2 \leq -\gamma_3$ and $\sum_i \hat{n}_i^2 = 1$. Therefore, $-\sum_{i=1}^3 \gamma_i \hat{n}_i^2 \leq -(\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2) \gamma_3 \leq -\gamma_3$. For the ease of writing let us denote $-\langle \gamma_3 \rangle = \mu$ and $\gamma_3^{\text{std}} = \sigma$. Both μ and σ are positive definite. Thus, $\langle \Delta\Theta \rangle \leq \mu t$. Simple phenomenological arguments suggest that the mean eigenvalue is inversely proportional to the Kolmogorov time-scale and hence scales as $\sqrt{Re}$. Thus $\lambda_S^{(1)} \sim \sqrt{Re}$ and recovers the classical, mean-field result. We confirm this is indeed correct in Fig. 4 in the main text where $\langle |\gamma_3| \rangle$ is shown to scale perfectly with the theoretical prediction of $\sqrt{Re}$.

Let us now try to break down the second term $\langle \Delta \Theta^2 \rangle$. We solve in the following fashion —

$$\begin{aligned}
 \langle \Delta \Theta^2 \rangle &= 4 \left\langle \int_0^t dt_1 \int_0^t dt_2 \gamma_i(\mathbf{x}, t_1) \hat{n}_i^2(\mathbf{x}, t_1) \gamma_j(\mathbf{x}, t_2) \hat{n}_j^2(\mathbf{x}, t_2) \right\rangle \\
 &\leq 4 \int_0^t dt_1 \int_0^t dt_2 \left\langle |\gamma_i(\mathbf{x}, t_1) \hat{n}_i^2(\mathbf{x}, t_1) \gamma_j(\mathbf{x}, t_2) \hat{n}_j^2(\mathbf{x}, t_2)| \right\rangle \\
 &\leq 16 \int_0^t dt_1 \int_0^t dt_2 \langle |\gamma_3(\mathbf{x}, t_1)| |\gamma_3(\mathbf{x}, t_2)| \rangle \\
 &\lesssim 16 \langle \gamma_3^2 \rangle t_c t \\
 &\lesssim 16(\sigma^2 + \mu^2) t_c t
 \end{aligned} \tag{3.17}$$

In the second step we have used the triangle inequality and in the third step we used that $\sum_{i=1}^3 |\gamma_i \hat{n}_i^2| \leq |\gamma_2 + \gamma_3| \hat{n}_1^2 + |\gamma_2| \hat{n}_2^2 + |\gamma_3| \hat{n}_3^2 \leq |\gamma_2|(\hat{n}_1^2 + \hat{n}_2^2) + |\gamma_3|(\hat{n}_1^2 + \hat{n}_3^2) \leq |\gamma_2| \hat{n}_1^2 + |\gamma_3| \leq 2|\gamma_3|$ (since $|\gamma_2| \leq |\gamma_3|$ and $|\hat{n}_i^2| \leq 1$). Here, t_c is the characteristic auto-correlation time scale for the $\sim 1/\lambda$ as the time scale at which two systems decorrelates (predictability time) $\propto 1/\lambda$. Therefore, $\langle \Delta \Theta^2 \rangle \lesssim 16(\sigma^2 + \mu^2) \frac{t}{\lambda}$. So, $\lambda t \lesssim \mu t + 2(\mu^2 + \sigma^2)t/\lambda$ and therefore, $\lambda^2 - 2\mu\lambda - 8(\mu^2 + \sigma^2) \lesssim 0$.

From here it is just a matter of algebra (λ is positive) to show that —

$$\begin{aligned}
 \lambda &\lesssim \frac{2\mu + \sqrt{4\mu^2 + 32(\mu^2 + \sigma^2)}}{2} \\
 &\lesssim \mu + \sqrt{9\mu^2 + 8\sigma^2} \\
 &\lesssim \mu + \sqrt{9\mu^2 + 8\sigma^2 + 12\sqrt{2}\sigma\mu} \\
 &\lesssim 4\mu + 2\sqrt{2}\sigma
 \end{aligned} \tag{3.18}$$

Since, we are only interested in the general scaling relations (particularly in terms of the mean and fluctuation components), we drop the constant prefactors. Thus $\lambda \lesssim \mu + \sigma = \langle \gamma_3 \rangle + \gamma_3^{\text{std}}$. Hence the Reynolds number scaling of the Lyapunov exponent will be determined by the Reynolds number scaling of $\langle \gamma_3 \rangle$ and γ_3^{std} .

This is as far as we can go theoretically. We now turn to measurements of $\langle \gamma_3 \rangle$ and γ_3^{std} from our DNS(s) and find (Fig. 4, main text) that while $\lambda_S^{(1)} = \langle \gamma_3 \rangle \sim \sqrt{Re}$ (consistent with the mean field estimate), $\lambda_S^{(2)} = \gamma_3^{\text{std}} \sim Re^\alpha$, with $\alpha = 0.59 \pm 0.04$. Since $\alpha > 0.5$ (the mean field result), in the limit of large Reynolds numbers, the scaling of $\lambda_S^{(2)}$ dominates over the scaling $\lambda_S^{(1)}$ and hence $\lambda^{\text{DNS}} = \lambda \sim \lambda_S \sim \lambda_S^{(2)} \sim Re^\alpha$ as summarised in Fig. 4 of our manuscript. An alternative route to the same scaling laws can be obtained, for instance, via a cumulant expansion.

At long times, beyond the exponential-growth phase when the systems have started to completely decorrelate the forcing term can not be neglected. Thus as $t \rightarrow \infty$, we estimate $\langle \delta u_i \delta f_i \rangle = \langle u_i^B f_i^B \rangle + \langle u_i^A f_i^A \rangle - \langle (u_i^B f_i^A + u_i^A f_i^B) \rangle \rightarrow 2\overline{\langle \varepsilon \rangle}$. As we shall see below, this exactly compensates the contribution of the viscous term β_η at long times.

3.B.3 SHORT AND LONG-TIME ASYMPTOTICS OF β_η

The viscous term can be expanded as

$$\begin{aligned}\beta_\eta &= \nu \langle \delta u_i \nabla^2 \delta u_i \rangle = -\nu \langle \partial_j \delta u_i \partial_j \delta u_i \rangle \\ &= -\nu \langle (\partial_i u_j^B \partial_i u_j^B + \partial_i u_j^A \partial_i u_j^A - 2\partial_i u_j^B \partial_i u_j^A) \rangle \\ &= -4\nu \text{Tr}(S^2) + 2\nu \langle \partial_i u_j^A \partial_i u_j^B \rangle = 2\langle \epsilon \rangle + 2\nu \langle \partial_i u_j^A \partial_i u_j^B \rangle.\end{aligned}\quad (3.19)$$

We find at short times, in the exponential-growth phase, this term scales as $\beta_\eta = \mathcal{O}(1/\tau_\eta)$, where $\tau_\eta = \sqrt{\frac{\nu}{\langle \epsilon \rangle}}$ corresponds to the smallest Kolmogorov time scales. However, at long times $t \rightarrow \infty$, systems A and B decorrelate leading to $\langle \partial_i u_j^A \partial_i u_j^B \rangle \rightarrow 0$ and hence $\beta_\eta \sim -2\langle \epsilon \rangle$ (see Fig. 3.4(a) as well as the main text). This is easily understandable because, on average, dissipation is balanced by the energy injection derived above.

3.C DECORRELATOR IN THE SHELL MODEL FOR TURBULENCE

It is possible to construct an evolution equation for the decorrelator in our shell model but, as we shall see, its structure is less transparent than what we have seen for the Navier-Stokes equation.

By subtracting the GOY model equations for systems A and B we obtain the evolution of the shell-by-shell velocity difference $\delta u_n = u_n^B - u_n^A$:

$$\begin{aligned}\frac{d}{dt}\delta u_n &= ik_n [(\delta u_{n+2} u_{n+1}^A + u_{n+2}^A \delta u_{n+1} + \delta u_{n+2} \delta u_{n+1}) - \frac{\epsilon}{\lambda} (\delta u_{n+1} u_{n-1}^A + u_{n+1}^A \delta u_{n-1} + \delta u_{n+1} \delta u_{n-1}) \\ &\quad + \frac{(\epsilon - 1)}{\lambda^2} (\delta u_{n-2} u_{n-1}^A + u_{n-2}^A \delta u_{n-1} + \delta u_{n-2} \delta u_{n-1})]^* - \nu k_n^2 \delta u_n.\end{aligned}\quad (3.20)$$

From this by suitably multiplying with the complex conjugate field δu^* , the decorrelator obeys

$$\begin{aligned}\frac{d}{dt}\phi_n &= \frac{ik_n}{2} [(\delta u_{n+2} u_{n+1}^A \delta u_n + u_{n+2}^A \delta u_{n+1} \delta u_n + \delta u_{n+2} \delta u_{n+1} \delta u_n) - \frac{\epsilon}{\lambda} (\delta u_{n+1} \delta u_n u_{n-1}^A \\ &\quad + u_{n+1}^A \delta u_{n-1} \delta u_n + \delta u_{n+1} \delta u_n \delta u_{n-1}) + \frac{(\epsilon - 1)}{\lambda^2} (\delta u_{n-2} u_{n-1}^A \delta u_n + u_{n-2}^A \delta u_{n-1} \delta u_n \\ &\quad + \delta u_{n-2} \delta u_{n-1} \delta u_n)]^* + c.c. - 2\nu k_n^2 \phi_n.\end{aligned}\quad (3.21)$$

where c.c. means complex conjugate. A summation over all shells leads to simplifications in the evolution of the GOY model decorrelator Φ^{GOY}

$$\begin{aligned}\frac{d}{dt}\Phi^{\text{GOY}} &= \frac{l}{2} \sum k_n [\epsilon \delta u_n u_{n+1}^A \delta u_{n+2} + (1 - \epsilon) \delta u_n \delta u_{n+1} u_{n+2}^A - u_n^A \delta u_{n+1} \delta u_{n+2}]^* \\ &\quad + c.c - 2\nu \sum_n k_n^2 \phi_n\end{aligned}\quad (3.22)$$

Further progress is only possible through a numerical solution of the decorrelator equation as we report in the main text of the manuscript. The shell-wise decorrelator at very short times show a power-law growth with a wavenumber-dependent scaling exponent. In particular at long times when the decorrelator saturates, we find, unsurprisingly, $|\delta u_n|^2 \sim |u_n|^2 \sim k_n^{-2/3}$ as shown in Fig. 3.4(d).

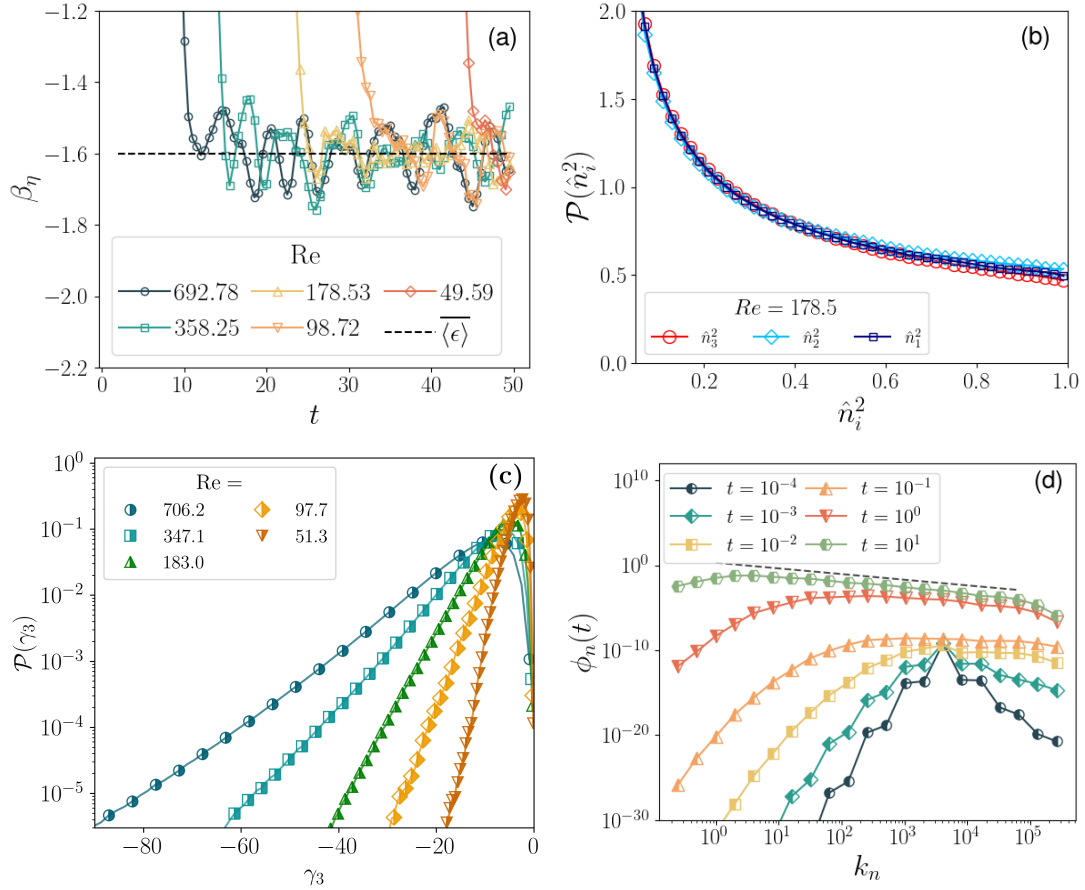


Figure 3.4. (a) Plots of β_η , at late times, for different Reynolds numbers, showing a saturation to $-2\overline{\epsilon}$ (black dotted, horizontal line). (b) A plot of the probability distribution of components of $\delta\mathbf{u}$ (for $Re = 358.2$), for the Navier-Stokes equation, along the three eigen vectors of the strain rate tensor at late times once the decorrelator saturates. Note that the normalisation used while computing the pdf is $\int \mathcal{P}(n_i^2) dn_i^2 = 1$. (c) The distribution of the compressional eigenvalues for different values of Re . Both the peak and the spread of the distribution scales with Re . (d) The evolution of ϕ_n vs k_n , for the GOY shell model, on a loglog scale at different times; the late time shows an asymptotic scaling corresponding to $k_n^{-2/3}$ indicated by the dashed line.

GEOMETRIC INTERMITTENCY IN TURBULENCE

Equal-time scaling exponents in fully developed turbulence typically exhibit non-anomalous scaling in the inverse cascade of two-dimensional (2D) turbulence and anomalous scaling in three dimensions. In this chapter we demonstrate that multiscaling is not confined to longitudinal scalar velocity increments, but also emerges in structure functions constructed from the magnitude and orientation of the velocity vector. This decomposition uncovers pronounced multiscaling in the 2D inverse cascade, which remains largely obscured when using conventional longitudinal structure functions. In particular, angle-based observables display strong departures from normal scaling together with signatures of saturation at high orders in both two- and three-dimensional turbulence. To investigate the extent to which these properties are specific to turbulence, we compare our results with synthetic Gaussian velocity fields possessing the same two-point statistics as turbulent flows. Remarkably, we find that much of the multiscaling phenomenology of the geometric observables survives even in correlated Gaussian fields that lack nonlinear cascade dynamics. This suggests that the dominant mechanism underlying the observed scaling originates not solely from turbulent intermittency, but more generally from the statistical structure of the observables themselves. We argue that the bounded and ratio-like nature of the angular increments enhances the role of correlations, kinematic constraints, and the geometry of low-velocity regions, thereby producing a class of multiscaling observables whose dominant scaling behaviour can already emerge in correlated random fields. Our results therefore point to an observable-dependent interpretation of multiscaling in turbulence and clarify why certain classes of observables may be captured surprisingly well by synthetic or quasi-Gaussian descriptions despite the strongly non-Gaussian nature of turbulent flows.

This chapter closely follows the work [78]¹.

¹This work is done in collaboration with Siddhartha Mukherjee, I. V. Kolokolov, V. V. Lebedev, Takeshi Matsumoto, and Samriddhi Sankar Ray.

4.1 INTRODUCTION

Kolmogorov's 1941 theory remains the peg on which our understanding of the statistics of non-equilibrium stationary states of three-dimensional (3D) fully developed, statistically homogeneous and isotropic turbulence rests [124]. It provides a statistical physics framework to interpret universality in turbulence through two-point correlation functions. Such p -th order correlation functions $S_p^L(r) = \langle (\delta u_r^L)^p \rangle$ — called longitudinal (hence the superscript L) structure functions in turbulence — are defined through the scalar velocity increment $\delta u_r^L \equiv \delta \mathbf{u}_r \cdot \hat{\mathbf{r}}$, where the $\delta \mathbf{u}_r$ is the (vector) velocity difference between points A and B separated by a distance vector $\mathbf{r}$ and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. The angular brackets $\langle \cdot \cdot \cdot \rangle$ correspond to an *ensemble* average, typically over space and time in the non-equilibrium, statistically, steady state. For separations within the inertial range of turbulence $\eta \ll r \ll \ell$, where η is the (small) dissipative scale and ℓ the large (integral scale) of turbulence, and where the flow is known, from experiments and numerical simulations, to be statistically homogeneous and isotropic, scale-invariance leads to $S_p^L(r) \propto r^{\zeta_p^L}$ [124]. The equal-time longitudinal exponents $\zeta_p^L = p/3$ scale linearly with p — the normal scaling form — follow from assumptions of homogeneity, isotropy and a finite energy dissipation rate ε (and hence flux), along with an ℓ independence, in the limit of high Reynolds number.

While such a normal scaling form for the third-order exponent yields $\zeta_3^L = 1$, a manifestation of a scale-independent energy flux and consistent with the Karman-Howarth relation [124], we know that the measured exponents deviate significantly from the $p/3$ scaling for $p \neq 3$. In particular, in the inertial range, $S_p^L(r) = C_p^L \left(\frac{\ell}{r}\right)^{p/3 - \zeta_p^L} (\varepsilon r)^{p/3}$ [124], where ζ_p^L is a convex, monotonically increasing function of p while satisfying the constraint $\zeta_3^L = 1$ [124]. This anomalous, multiscaling, rationalised through the Parisi-Frisch multifractal formalism [24], is a fundamental property of turbulence whose origins lie in the ubiquitous intermittent nature of turbulent flows.

At the same time, it is well known that not all statistical observables in turbulence are equally sensitive to intermittency. While high-order moments of velocity increments probe rare, intense fluctuations and therefore strongly deviate from Gaussian predictions, several lower-order or suitably constructed observables are often well captured by models that rely primarily on two-point statistics, such as quasi-Gaussian closures. This raises a natural and important question: to what extent does the manifestation of multiscaling depend on the choice of observable itself?

Intriguingly buried in the definition of the scalar increment δu_r^L are the tangled contributions coming from the statistics of the velocity amplitudes u_A and u_B as well as the cosines of the angles — $\cos \theta_A = \hat{\mathbf{u}}_A \cdot \hat{\mathbf{r}}$ and $\cos \theta_B = \hat{\mathbf{u}}_B \cdot \hat{\mathbf{r}}$ — the unit velocity vectors make with the unit vector between points A and B: For example, $S_2^L(r) = \langle u_A^2 \cos^2 \theta_A + u_B^2 \cos^2 \theta_B - 2u_A u_B \cos \theta_A \cos \theta_B \rangle$.

We confirm from numerical simulations that u_A^2 and $\cos^2 \theta_A$ are uncorrelated (likewise for point B) and hence $\langle u_A^2 \cos^2 \theta_A \rangle = \langle u_A^2 \rangle \langle \cos^2 \theta_A \rangle = c_0 v_{\text{rms}}^2$, where $c_0 = \langle \cos^2 \theta_A \rangle = 1/3$ follows trivially from a uniform distribution of orientations in 3D. But what about the correlations involving the velocity amplitudes and the cosine of the angles? Indeed, can the mixed correlation functions be simply written as products of correlations between the amplitudes and the angles?

Data from Direct Numerical Simulations (DNSs), whose details are discussed later, show

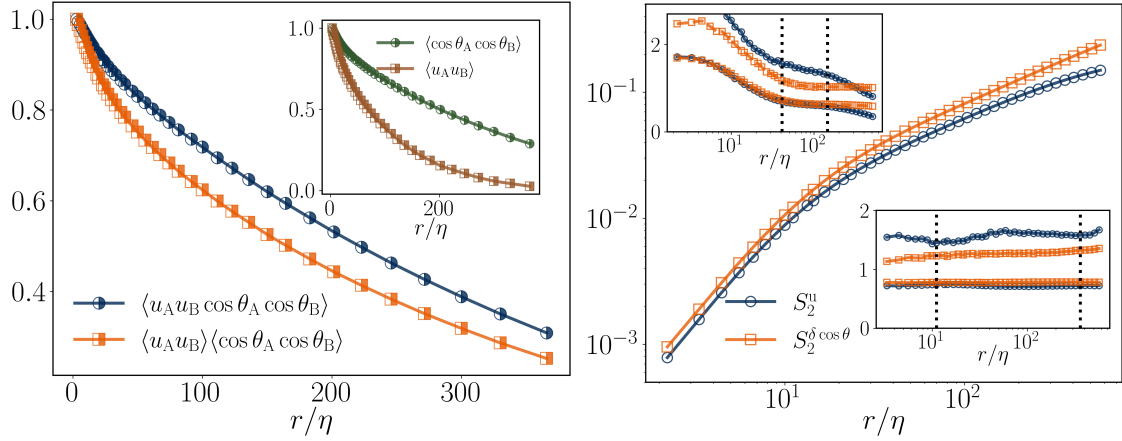


Figure 4.1. **Left panel:** A comparison of the mixed correlator $\langle 2u_A u_B \cos \theta_A \cos \theta_B \rangle$ and $\langle 2u_A u_B \rangle \langle \cos \theta_A \cos \theta_B \rangle$ as well as (inset) $\langle \cos \theta_A \cos \theta_B \rangle$ and $\langle u_A u_B \rangle$, normalised to 1 at $r = 0$, in three-dimensional turbulence. **Right panel:** Log-log plots of the second-order structure functions vs r/η . Upper inset: Local slopes of S_p^u and $S_p^{\delta \cos \theta}$ for $p = 2$ (lower set of curves) and $p = 6$ (upper set of curves); the pair of vertical lines denote the inertial range. Lower inset: Analogous plots as those in the upper inset but for local slopes extracted via ESS.

unambiguously (see Fig. 4.1) that $\langle u_A u_B \cos \theta_A \cos \theta_B \rangle \neq \langle u_A u_B \rangle \langle \cos \theta_A \cos \theta_B \rangle$. Furthermore, the velocity amplitude correlation $\langle u_A u_B \rangle$ decays faster with r than the correlation function of the angular part $\langle \cos \theta_A \cos \theta_B \rangle$ (Fig. 4.1, inset). This result [125] underlines the fact that the statistics of the amplitudes and geometry of the velocity vectors are just as complex and perhaps equally fundamental, which deserves a thorough investigation.

A key observation emerging from this decomposition is that the angular variables, unlike the velocity amplitudes, do not depend explicitly on the magnitude of the velocity field. In particular, quantities such as $\cos \theta$ are bounded and encode purely geometric information about the orientation of the velocity vectors relative to the separation direction. This suggests that such observables may be comparatively insensitive to the extreme amplitude fluctuations that underlie intermittency in turbulence.

This raises a natural question: is the multiscaling behaviour associated with such geometric observables a uniquely turbulent phenomenon, or can it arise more generally in correlated fields that share similar two-point statistics? In other words, to what extent is the observed scaling controlled by the nonlinear dynamics of the turbulent cascade, and to what extent is it a consequence of more generic statistical and kinematic constraints?

There is one other reason which motivates the present study. Recent measurements [126] of the scaling exponents ζ_p^T of the transverse structure functions — $S_p^T(r) = \langle (\delta u_r^T)^p \rangle$, where $\delta u_r^T = \delta \mathbf{u} \cdot \hat{\mathbf{r}}^T$ with $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}^T = 0$ — find a systematic, statistically significant departure from ζ_p^L suggestive of the role of geometry. Specifically, Buaria and Sreenivasan [126] show that ζ_p^T has stronger signatures of multiscaling, with a possible saturation of ζ_p^T for $p \gtrsim 8$.

This motivates the question of whether the classical projected increments, from which longitudinal and transverse structure functions are built, truly capture the strongest manifestations of intermittency — or whether alternative increments can reveal even richer multiscaling behaviour.

By separating velocity amplitudes from the local flow geometry via increments of $\cos \theta_A$ and $\cos \theta_B$, we uncover a family of structure functions that exhibit equally strong anomalous scaling and, strikingly, reveal persistent intermittency in the *nominally non-intermittent* inverse-cascade regime of two-dimensional turbulence when viewed solely through ζ_p^L .

In what follows, we first present detailed results from direct numerical simulations of three- and two-dimensional turbulence, which establish the multiscaling properties of these geometric observables. We then turn to synthetic Gaussian fields with similar two-point statistics to assess the extent to which these features are specific to turbulence or reflect a more general statistical mechanism.

We use the publicly available data from the Johns Hopkins Turbulence Database (JHTD) [95–97] — with Taylor-scale based Reynolds numbers $Re_\lambda = 418$ and 610 corresponding to resolutions 1024^3 and 4096^3 , respectively — which is obtained from pseudo-spectral direct numerical simulations (DNSs) of the incompressible, 3D Navier-Stokes equation with forcing at large scales. The exponents we report are, within error-bars, the same for both these Reynolds number; the data we present in the figures are from the larger Reynolds number simulation.

4.2 GEOMETRIC OBSERVABLES IN HOMOGENEOUS AND ISOTROPIC TURBULENCE

We define structure functions $S_p^u \equiv \langle |\delta u_r|^p \rangle = C_p^u \left(\frac{\ell}{r} \right)^{p/3 - \zeta_p} (\epsilon r)^{p/3}$, for the amplitude increment $\delta u_r = u_B - u_A$, and $S_p^{\delta \cos \theta} \equiv \langle |\delta \cos \theta_r|^p \rangle = C_p^{\delta \cos \theta} \left(\frac{\ell}{r} \right)^{-\chi_p}$, for the cosine-angle increment $\delta \cos \theta_r = \cos \theta_B - \cos \theta_A$. The scaling form Ansatz is in direct corollary with that assumed for the longitudinal structure functions in the inertial range. In End Matter, we show consistency of such scaling forms and find, in particular, the Kolmogorov constants $C_2^L \approx C_2^u \gg C_2^{\delta \cos \theta}$ [127].

We begin by analysing these observables in direct numerical simulations of homogeneous and isotropic turbulence in both three and two dimensions. The aim of this section is primarily phenomenological: to establish the scaling properties of the geometric observables and compare them with those associated with more conventional velocity increments. Questions concerning the degree to which these properties are uniquely turbulent, or instead reflect more general statistical constraints, will be addressed in Sec. III.

In Fig. 4.1 we show representative log-log plots of S_2^u and $S_2^{\delta \cos \theta}$ against r/η from which emerge the scaling exponents ζ_2 and χ_2 . From purely dimensional arguments it is reasonable to expect $\zeta_2 = \zeta_2^L$. In the upper inset of Fig. 4.1 (lower pair of curves) we show semilog plots of the local slopes for S_2^u and $S_2^{\delta \cos \theta}$. In the inertial range, denoted by the pair of vertical dashed lines, we find a plateau whose value yields $\zeta_2 \approx \chi_2 \approx \zeta_2^L = 2/3$ (up to intermittency corrections).

What is most revealing is the behaviour of higher-order exponents. For illustration, we consider the sixth-order structure functions. A local-slope analysis (upper inset of Fig. 4.1) gives the corresponding exponents. While the velocity-amplitude increment shows only a marginal inertial-range plateau, the plateau for $S_6^{\delta \cos \theta}$ is markedly clearer. Moreover, the resulting exponents differ substantially from the classical longitudinal value (dash-dotted line), with

$\chi_6 \neq \zeta_6 < \zeta_6^L$. Because high-order exponents in turbulence are notoriously sensitive to finite-Reynolds-number effects [124], we further validate this result using extended self-similarity (ESS) [19, 128]. The ESS local slopes, shown in the lower inset of Fig. 4.1 for $p = 2$ and $p = 6$, confirm that the exponent ratios $\tilde{\chi}_p = \chi_p/\chi_3$ and $\tilde{\zeta}_p = \zeta_p/\zeta_3$ deviate significantly from the corresponding longitudinal ratios ζ_p^L/ζ_3^L throughout the inertial range.

The contrast between the scaling behaviour of the velocity amplitudes and that of the angle-based observables is particularly striking at higher orders. Since the angular increments are bounded and insensitive to the overall magnitude of the velocity field, this already hints that the observed multiscaling may not arise from precisely the same mechanism responsible for the intermittency of conventional longitudinal increments.

A note of caution is warranted. Because our observables involve velocity amplitudes and directional cosines, there is no Kármán–Howarth–type constraint that would enforce a unit third-order exponent. Consistently, as shown in the inset of Fig. 4.2, we find $\chi_3 = \zeta_3 \neq 1$ within error bars. Therefore, our use of ESS with the third-order structure function is strictly procedural: it extends the apparent scaling range (lower inset of Fig. 4.1) and yields exponent ratios rather than bare exponents [19, 128]. Throughout this manuscript we use the term “bare exponents” to denote those obtained from direct log–log fits of the structure functions versus r , in contrast to the ESS-based exponent ratios.

The comparison of the local slopes of the second and sixth-order moments — and the fact that they deviate increasingly from the classical longitudinal exponents at higher orders — motivates a systematic calculation of the scaling exponents and the corresponding exponent ratios through ESS.

In Fig. 4.2 we show plots of the exponent ratios $\tilde{\chi}_p$ and $\tilde{\zeta}_p$ versus p [129], with both showing a pronounced departure from normal scaling. To confirm that this is not an artefact of extended self similarity, we calculate the bare exponents as a function of p as well. In the inset of Fig. 4.2 we show the corresponding bare exponents and find trends consistent with those obtained from ESS. In particular, the scaling exponents associated with the angle increments display a markedly stronger departure from the longitudinal exponents than those associated with the velocity-amplitude increments.

While our investigation is motivated by the structure of the longitudinal increment, the results reveal that multiscaling behaviour is robust across a broad family of geometric observables. In End Matter we show that several other angle-based increments, such as $\delta\theta_r = \theta_A - \theta_B$ (with exponents η_p and $-\pi \leq \delta\theta_r \leq \pi$), $\delta \sin \theta_r = \sin \theta_A - \sin \theta_B$ (with exponents β_p), and $\sin \delta\theta_r = \sin[\theta_A - \theta_B]$ (with exponents γ_p), also exhibit clear signatures of multiscaling with $\chi_p \approx \eta_p \neq \beta_p \approx \gamma_p$.

Given the lack of a Kármán–Howarth constraint, these observables may be more susceptible to finite-Reynolds-number effects and subdominant corrections. Nevertheless, the corresponding ESS exponent ratios — which compensate for infrared and ultraviolet corrections that contaminate the finite-Re inertial range [19, 128] — collapse onto a single curve. This collapse strongly suggests that the different angular increments probe a common underlying statistical structure associated with the geometry of the velocity field.

An important point, however, is that such a collapse by itself does not establish that the underlying mechanism is uniquely tied to the nonlinear dynamics of turbulence. As we shall show later, remarkably similar scaling behaviour can emerge even in synthetic correlated fields that are not solutions of the Navier–Stokes equations.

All of this throws up a tantalising question in the context of two-dimensional turbulence [130]. We know that the inverse cascade regime $L \gg r \gg L_F$, where L_F is the forcing scale and L the large scale of the system set by the drag coefficient, in forced, two-dimensional turbulence obeys the Kolmogorov normal scaling theory and the associated longitudinal exponents $\zeta_p^L = p/3$ [131–134]. It is therefore natural to investigate whether structure functions constructed from velocity magnitudes and angular increments exhibit departures from this conventional scaling behaviour.

We perform pseudo-spectral DNSs of the incompressible, 2D Navier-Stokes equation in the vorticity-streamfunction formulation in a 2π periodic domain with hyperviscosity ($\nu\Delta^8$) and hypodrag ($\alpha\Delta^{-1}$) terms. We force the system at high wavenumber $\mathbf{k}$ such that $k_f - 1 < |\mathbf{k}| < k_f + 1$ with $k_f = 997$. Hyperviscosity $\nu = 1.05 \times 10^{-47}$ and hypodrag $\alpha = 35$ [135]. From statistically steady velocity fields obtained from our DNSs, we measure, as in the 3D problem discussed above, the equal-time scaling exponents χ_p , ζ_p , β_p , γ_p , and η_p along with their counterparts $\tilde{\chi}_p$, $\tilde{\zeta}_p$, $\tilde{\beta}_p$, $\tilde{\gamma}_p$, and $\tilde{\eta}_p$ obtained from ESS [19, 128].

In the inset of Fig. 4.2 we plot the bare exponents as a function of the order p . Remarkably, the exponents associated with the angle-based structure functions show clear anomalous scaling with a suggestion of saturation at large p , more pronounced than in three-dimensional turbulence (see Fig. 4.2). In the main panel of Fig. 4.2 we show the corresponding exponent ratios. While $\tilde{\zeta}_p$ does not significantly deviate from normal scaling, pronounced multiscaling emerges for the exponents related to the angular observables. This is qualitatively similar to what is seen in three-dimensional turbulence, including the observation that while $\chi_p \approx \eta_p \neq \beta_p \approx \gamma_p$, the corresponding exponent ratios collapse once more onto a single multiscaling curve.

The appearance of strong multiscaling in angle-based observables within the inverse-cascade regime is particularly intriguing because conventional longitudinal structure functions in this regime exhibit near-normal Kolmogorov scaling. At face value, this could be interpreted as a previously hidden form of intermittency associated with the geometry of the flow. However, as we demonstrate in the next section, much of this behaviour is already reproduced by synthetic Gaussian fields with comparable two-point statistics, pointing instead toward a more general geometric and statistical origin.

4.3 GENERALITY OF GEOMETRIC MULTISCALING: COMPARISON WITH GAUSSIAN FIELDS

The results of the previous section establish that structure functions constructed from angular increments exhibit strong multiscaling behaviour in both three- and two-dimensional turbulence. A natural question therefore arises: to what extent are these observations a consequence of the nonlinear dynamics of turbulence itself, and to what extent do they reflect more general statistical properties of correlated vector fields?

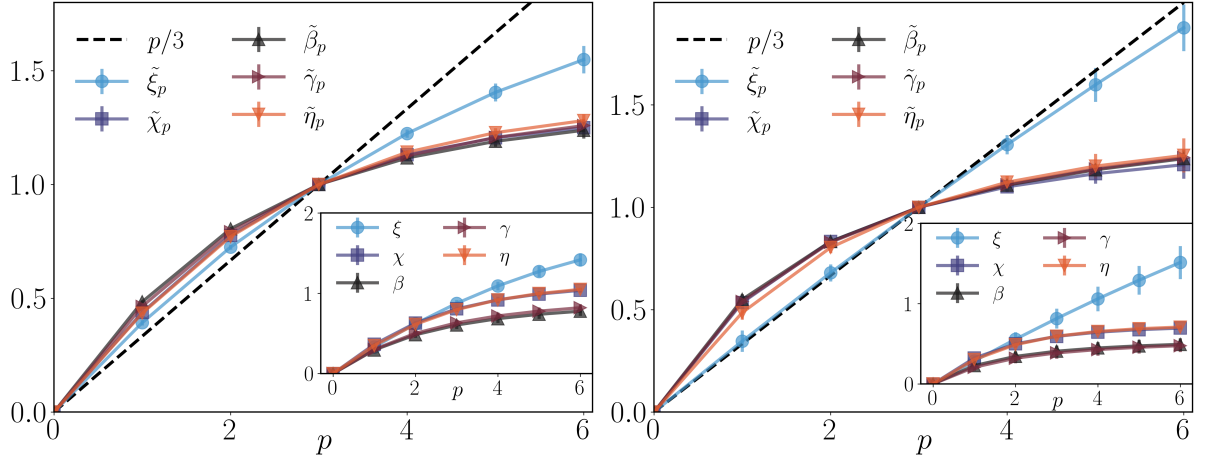


Figure 4.2. Left panel: Plots of the scaling exponent ratios $\tilde{\xi}_p$, $\tilde{\chi}_p$, $\tilde{\beta}_p$, $\tilde{\gamma}_p$ and $\tilde{\eta}_p$ (obtained via ESS) and (inset) the bare exponents ξ_p , χ_p , β_p , γ_p and η_p for three-dimensional turbulence. Right panel: Plots of the scaling exponent ratios $\tilde{\chi}_p$, $\tilde{\xi}_p$, $\tilde{\beta}_p$, $\tilde{\gamma}_p$ and $\tilde{\eta}_p$ (obtained via ESS) and (inset) the bare exponents χ_p , ξ_p , β_p , γ_p and η_p for two-dimensional turbulence.

To address this question, we construct synthetic Gaussian velocity fields with prescribed two-point statistics matching the inertial-range energy spectrum of turbulence. In particular, we generate statistically homogeneous and isotropic Gaussian fields with energy spectrum $E(k) \sim k^{-5/3}$ and analyse the same geometric observables introduced in Sec. II. By construction, these synthetic fields contain neither coherent turbulent structures nor nonlinear cascade dynamics, while retaining the same low-order correlation properties as turbulent velocity fields.

As expected, conventional longitudinal structure functions constructed from the synthetic velocity increments display essentially normal scaling, with no significant signatures of anomalous multiscaling. Remarkably, however, the angular structure functions exhibit scaling behaviour that is nearly indistinguishable from that observed in turbulent flows. In particular, the ESS exponent ratios associated with the angular increments show clear nonlinear dependence on the order p , together with indications of saturation at large p , closely paralleling the behaviour seen in the Navier–Stokes data (see Fig. 4.3).

This observation has an important implication. The multiscaling behaviour associated with the geometric observables cannot be attributed solely to the intermittent cascade dynamics of turbulence, since it already survives in Gaussian correlated fields that are not solutions of the Navier–Stokes equations. Instead, the dominant mechanism responsible for the observed scaling must arise more generally from the statistical structure of the observables themselves.

The key point is that the angular increments are fundamentally different from conventional velocity increments. The latter directly probe fluctuations in the velocity amplitude and are therefore highly sensitive to rare intense events. In contrast, the angular observables involve ratios of velocity components to the local velocity magnitude. To leading order in a scaling sense, the angular increment behaves as

$$\delta(\cos \theta_r) \sim \frac{\delta u_r}{|u|},$$

where this relation should be interpreted as a scaling estimate rather than a controlled Taylor expansion. The corresponding structure functions therefore involve moments of the form

$$S_p^{\delta \cos \theta}(r) \sim \left\langle \frac{|\delta u_r|^p}{|u|^p} \right\rangle.$$

This has several important consequences. First, the angular increments are bounded quantities, unlike conventional velocity increments. Consequently, the corresponding structure functions are far less sensitive to extreme amplitude fluctuations. Second, because the observable has a ratio structure, correlations between the increment δu_r and the local velocity magnitude $|u|$ become important. Finally, events associated with small velocity magnitude contribute disproportionately to the angular increments, enhancing the role of geometric structures associated with the near-zero set of the velocity field.

These mechanisms are already present in Gaussian correlated fields possessing the same two-point statistics. Consequently, synthetic fields reproduce much of the observed multiscaling phenomenology despite the absence of genuine turbulent intermittency in the conventional sense. The similarity between the turbulent and synthetic results therefore suggests that geometric observables probe a statistical regime that is dominated less by cascade dynamics and more by correlations, boundedness, and kinematic constraints.

At the same time, turbulence is not entirely irrelevant. Quantitative differences remain visible in the detailed statistics of the increments and in the geometry of the near-zero regions of the velocity field. For example, the fractal properties of the zero sets show measurable differences between turbulent and synthetic fields (see Fig. 4.4), even though both display similar large-scale scaling behaviour. Likewise, the probability distribution functions of the angular increments retain non-Gaussian features in the turbulent case. These differences indicate that turbulence modifies the statistical parameters of the geometric multiscaling, while leaving the underlying mechanism largely intact.

The comparison with synthetic fields therefore leads to a reinterpretation of the results of Sec. II. Rather than identifying a form of multiscaling unique to turbulence, the angular observables appear to probe a broader class of statistical and geometric effects that can already emerge in correlated Gaussian fields. Turbulence then acts to renormalise and modify these effects quantitatively through the dynamics of the cascade.

In the next section, we develop a phenomenological framework for understanding these observations in terms of ratio observables, correlation effects, and the geometry of the near-zero sets of the velocity field.

4.4 INTERPRETATION: GEOMETRY, CORRELATIONS, AND SATURATION

The comparison with synthetic Gaussian fields suggests that the multiscaling properties of the angular observables are controlled less by the detailed nonlinear dynamics of turbulence and more by the statistical and geometric structure of the observables themselves. In this section, we

develop a phenomenological interpretation of the scaling behaviour observed in Secs. II and III, focusing in particular on the role of boundedness, ratio structures, and the geometry of the near-zero sets of the velocity field.

The starting point is the observation that the angular increments are fundamentally different from conventional longitudinal velocity increments. Whereas the latter directly probe fluctuations of the velocity field itself, the former depend only on the orientation of the velocity vector. Consequently, the angular observables are dimensionless and bounded: $|\delta \cos \theta_r| \leq 2$, $|\delta \theta_r| \leq \pi$, and similarly for the related observables introduced in Sec. II.

This boundedness has an immediate consequence for the scaling exponents. Conventional structure functions can continue to acquire increasing contributions from progressively rarer and more intense fluctuations as the order p increases. In contrast, for bounded observables the dominant contributions at sufficiently large order arise not from arbitrarily large events but from increasingly localised spatial regions where the angular increments remain finite. This naturally leads to a tendency toward saturation of the scaling exponents at large p , consistent with the behaviour observed in both the turbulent and synthetic fields.

A simple scaling argument illustrates the origin of this behaviour. To leading order in a scaling sense, the angular increments behave as

$$\delta(\cos \theta_r) \sim \frac{\delta u_r}{|u|}.$$

To make this more explicit, let us consider regions where the velocity field locally vanishes with a scaling exponent h , such that $|u| \sim r^h$. In such regions, the angular increments remain finite even as the velocity magnitude becomes arbitrarily small. If the set supporting these events possesses an effective fractal dimension $D(h)$, then standard multifractal arguments suggest a contribution of the form

$$S_p^{\delta \cos \theta}(r) \sim r^{ph+d-D(h)},$$

where d is the spatial dimension. Since the observables are bounded, the minimisation over h can lead to saturation of the exponents at large order p ,

$$\chi_p \rightarrow \chi_\infty, \quad p \rightarrow \infty.$$

The observed flattening of the exponent curves in both two- and three-dimensional turbulence is qualitatively consistent with this scenario.

Importantly, however, the existence of saturation itself does not require fully developed turbulence. Correlated Gaussian fields already possess nontrivial zero sets and spatial correlations, and therefore generate similar scaling behaviour for ratio observables. In this sense, the dominant mechanism responsible for the observed multiscaling is geometric and statistical rather than exclusively dynamical. Turbulence then modifies the details of the scaling through the non-Gaussian statistics generated by the cascade.

This interpretation also sheds light on the behaviour of the inverse cascade in two-dimensional turbulence. Conventional longitudinal structure functions in the inverse cascade exhibit near-normal Kolmogorov scaling and correspondingly weak signatures of intermittency. Nevertheless, the angular observables continue to display strong multiscaling and apparent saturation.

From the perspective developed here, this does not necessarily imply the existence of strong intermittent amplitude fluctuations in the inverse cascade. Rather, it reflects the fact that geometric observables probe a distinct statistical sector dominated by correlations, boundedness, and the geometry of low-velocity regions.

The emergence of similar exponent ratios for several distinct angular observables — including $\delta \cos \theta_r$, $\delta \theta_r$, $\delta \sin \theta_r$, and $\sin(\delta \theta_r)$ — further supports this picture. Although these observables differ microscopically, they share common kinematic constraints and probe similar geometric structures of the velocity field. Their collapse onto nearly identical ESS scaling curves therefore suggests the existence of a broader class of geometric multiscaling observables.

The phenomenology developed here naturally raises a broader question concerning the interpretation of multiscaling in turbulent flows. The results of this work suggest that the existence of anomalous scaling alone may not always be sufficient to identify genuinely turbulence-specific intermittency mechanisms. Instead, the statistical content of multiscaling can depend strongly on the observable under consideration. Some observables probe extreme amplitude fluctuations generated by the nonlinear cascade, whereas others may primarily reflect more generic geometric and correlation-based effects already present in correlated Gaussian fields.

Viewed from this perspective, geometric observables provide a useful diagnostic framework for disentangling different sources of multiscaling in complex flows. Turbulence remains crucial in quantitatively modifying the scaling exponents and probability distributions, but the underlying geometric mechanism responsible for the multiscaling of bounded angular observables appears to be considerably more general.

4.5 PHENOMENOLOGICAL INTERPRETATION OF GEOMETRIC MULTISCALING

The results of Secs. II and III suggest that the multiscaling properties of the geometric observables are controlled not solely by turbulent cascade dynamics, but also by the statistical structure of the observables themselves. In particular, the strong similarity between turbulence and synthetic Gaussian fields indicates that the dominant mechanism must be sufficiently general to survive even in correlated random fields lacking nonlinear Navier–Stokes dynamics. In this section, we develop a phenomenological interpretation of the observed scaling behaviour based on three key ingredients: the ratio structure of the observables, the boundedness of the angular increments, and the enhanced role of regions where the velocity magnitude becomes small.

We consider angular structure functions of the form $S_p^\theta(r) = \langle |\delta(\cos \theta)|^p \rangle$, where $\cos \theta = (\mathbf{u} \cdot \hat{\mathbf{r}})/|\mathbf{u}|$ and

$$\delta(\cos \theta) \sim \frac{\delta u}{|u|}.$$

This relation should be interpreted as a scaling estimate rather than a controlled Taylor expansion. Numerically, these structure functions display three distinct regimes: approximately linear scaling at low orders, weak but systematic multiscaling at intermediate orders, and signatures of saturation at sufficiently large p . Remarkably, this phenomenology is observed not only in

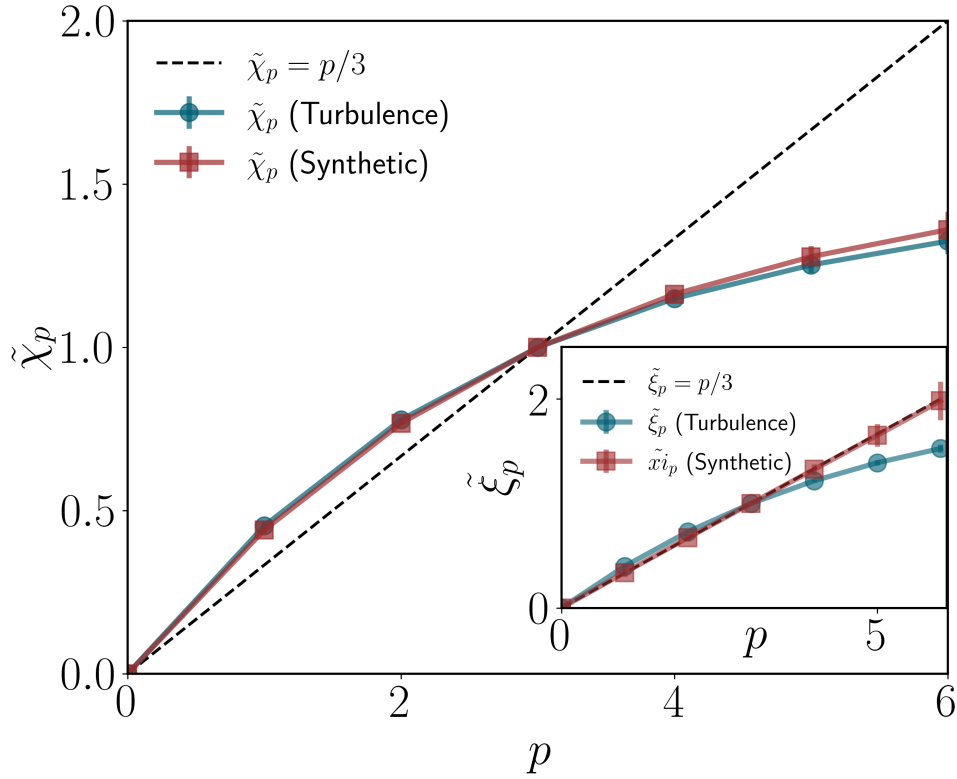


Figure 4.3. Main panel: ESS exponent ratios $\tilde{\chi}_p$ associated with the angular structure functions in homogeneous and isotropic turbulence and in synthetic Gaussian velocity fields possessing the same two-point statistics. Despite lacking nonlinear cascade dynamics and coherent turbulent structures, the Gaussian fields reproduce much of the multiscaling phenomenology observed in turbulence, including pronounced departures from normal scaling and indications of saturation at large order p . The dashed line denotes normal scaling for reference. Inset: ESS exponent ratios $\tilde{\xi}_p$ associated with conventional velocity-amplitude increments in the synthetic Gaussian field. In contrast to the angular observables, the conventional structure functions remain close to normal scaling. Together, these results suggest that the dominant mechanism underlying the multiscaling of bounded geometric observables originates not solely from turbulent intermittency, but more generally from correlations, ratio structures, and the geometry of low-velocity regions.

turbulent flows but also in synthetic Gaussian fields with the same two-point statistics.

The key point is that the angular increments behave as bounded ratio observables. The corresponding structure functions therefore involve moments of the form

$$S_p^\theta(r) \sim \left\langle \frac{|\delta u|^p}{|u|^p} \right\rangle,$$

which differ fundamentally from conventional velocity structure functions. Such moments are strongly sensitive to correlations between the increment δu and the local velocity magnitude $|u|$, while simultaneously enhancing the contribution of rare events associated with small velocity magnitude.

The persistence of multiscaling in synthetic Gaussian fields can already be understood within this framework. Let $u_1 = u(x)$ and $u_2 = u(x+r)$ be jointly Gaussian variables with covariance $\langle u_1 u_2 \rangle = \sigma^2 \rho(r)$, where $\rho(r) = 1 - \alpha r^{2h}$. Then $u_2 = \rho u_1 + \sqrt{1 - \rho^2} \eta$, with η an independent

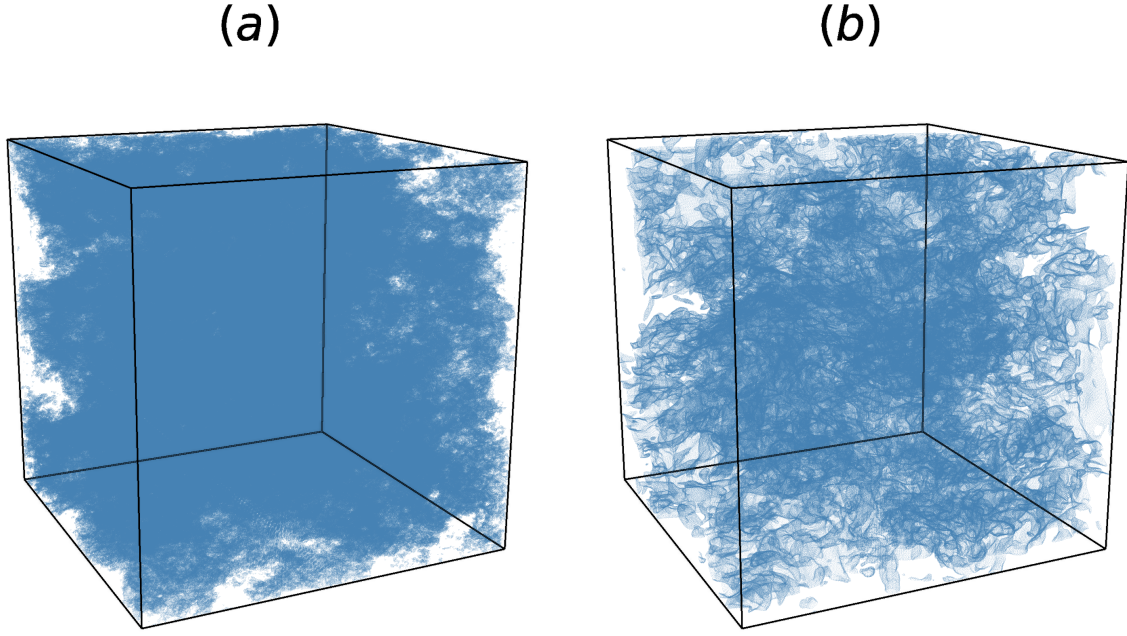


Figure 4.4. Representative iso-surfaces of small velocity magnitude in (a) correlated Gaussian velocity fields and (b) homogeneous and isotropic turbulence with matching two-point statistics. The visualisations highlight the complex spatial organisation of low-velocity regions and near-zero sets that play an enhanced role in the statistics of bounded ratio observables such as $\delta(\cos \theta)$. Although the detailed morphology differs between the turbulent and Gaussian fields, both exhibit highly structured extended low-velocity regions, consistent with the phenomenological interpretation developed in Sec. IV. Estimates of the corresponding fractal dimensions yield $d_F \approx 2.7$ for the turbulent field and $d_F \approx 2.5$ for the Gaussian field, while the phenomenological estimate based on the local Hölder exponent gives $d_F \approx 2.65$. These values are broadly consistent with the idea that the multiscaling and saturation properties of the geometric observables are influenced by the geometry of the near-zero sets of the velocity field.

Gaussian variable. The increment therefore takes the form $\delta u = (\rho - 1)u_1 + \sqrt{1 - \rho^2} \eta$. For small separations, $\rho - 1 \sim r^{2h}$ while $\sqrt{1 - \rho^2} \sim r^h$, so that the stochastic contribution dominates and $\delta u \sim r^h$, as expected. Importantly, however, the increment remains correlated with the local velocity magnitude, implying that the ratio observable inherits nontrivial scaling behaviour even in Gaussian fields.

Motivated by these observations, we phenomenologically decompose the angular structure function into two contributions,

$$S_p^\theta(r) = S_p^{\text{typ}}(r) + S_p^{\text{zero}}(r),$$

where S_p^{typ} represents typical inertial-range fluctuations away from velocity zeros, while S_p^{zero} denotes contributions associated with regions where the local velocity magnitude becomes anomalously small.

The typical contribution takes the scaling form

$$S_p^{\text{typ}}(r) \sim r^{ph - \gamma_p},$$

where the effective correction exponent $\gamma_p > 0$ phenomenologically captures the influence of correlations between δu and $|u|$, together with the enhanced weighting of low-velocity events in the ratio observable. In practice, γ_p is determined empirically from the measured scaling behaviour.

The second contribution arises from the bounded nature of the angular increments. Unlike conventional velocity increments, quantities such as $\delta(\cos \theta)$ remain finite even as $|u| \rightarrow 0$. Consequently, events associated with near-zero velocity magnitude contribute disproportionately to high-order moments. If the probability of encountering such events scales as $P(r) \sim r^{dh}$, then

$$S_p^{\text{zero}}(r) \sim r^{dh}.$$

Since this contribution is independent of p , it naturally produces a tendency toward saturation of the scaling exponents at sufficiently large order.

The geometric origin of this scaling can be understood heuristically in terms of the spatial structure of the near-zero sets of the velocity field. For a field characterised locally by Hölder exponent h , the probability of encountering zeros within a region of size r decreases algebraically with r . In d spatial dimensions, the simultaneous vanishing of all velocity components corresponds to the intersection of the zero sets associated with each component. Under generic intersection assumptions, this leads phenomenologically to an effective codimension proportional to dh , and hence to a zero-set contribution scaling as r^{dh} .

Combining the two contributions yields the phenomenological form

$$S_p^\theta(r) \sim r^{ph-\gamma_p} + \Delta^p r^{dh},$$

where $\Delta = O(1)$ reflects the bounded amplitude of the angular increments. The corresponding scaling exponent is therefore controlled by the dominant of the two contributions,

$$\zeta_p = \min(ph - \gamma_p, dh).$$

This form naturally produces near-linear scaling at small order, weak multiscaling at intermediate order through the correction exponent γ_p , and eventual saturation at sufficiently large p , qualitatively consistent with the numerical observations in both two- and three-dimensional turbulence as well as in correlated Gaussian fields.

The phenomenology developed here should not be interpreted as a complete theory of geometric multiscaling. Rather, it provides a unified framework for understanding why bounded ratio observables can display strong anomalous scaling even in the absence of fully developed turbulent cascade dynamics. From this perspective, turbulence acts primarily to renormalise the detailed statistics and geometry of the underlying correlated field, while leaving the dominant geometric mechanism largely intact.

The present results are also consistent with a broader body of work showing that several statistical properties of turbulence can be captured surprisingly well by synthetic or quasi-Gaussian descriptions when the observables are dominated primarily by low-order correlations and kinematic constraints rather than by extreme amplitude fluctuations. Examples include the successful description of energy spectra and two-point statistics within quasi-normal and

EDQNM-type closures, as well as the emergence of nontrivial scaling behaviour in synthetic random-flow models such as the Kraichnan ensemble. In a related sense, correlated Gaussian fields already possess complex nodal geometries and nontrivial zero-set statistics, which naturally influence bounded ratio observables such as those studied here. Our results suggest that the geometric multiscaling of angular observables belongs partly to this broader class of correlation-dominated statistical phenomena.

4.6 CONCLUSIONS

In this work, we have investigated a class of geometric observables in homogeneous and isotropic turbulence constructed from the amplitudes and orientations of velocity vectors. By decomposing the longitudinal velocity increment into contributions associated with the velocity magnitudes and the angular geometry of the flow, we uncovered a broad family of structure functions whose scaling properties differ significantly from those of conventional longitudinal velocity increments. In particular, the angular increments exhibit pronounced multiscaling behaviour in both three-dimensional turbulence and in the inverse-cascade regime of two-dimensional turbulence, where standard longitudinal structure functions display near-normal Kolmogorov scaling.

Our results therefore demonstrate that the manifestation of multiscaling depends strongly on the observable under consideration. While traditional velocity increments probe fluctuations of the velocity amplitudes, the geometric observables introduced here probe correlations associated with the orientation of the velocity field and the structure of low-velocity regions. This distinction becomes especially important at high orders, where the angular observables display stronger departures from normal scaling and clear indications of saturation.

The most important result of this work, however, emerges from the comparison with synthetic Gaussian fields possessing the same two-point statistics as turbulent flows. Remarkably, we find that much of the observed multiscaling phenomenology survives even in correlated Gaussian fields that are not solutions of the Navier–Stokes equations and possess neither coherent turbulent structures nor nonlinear cascade dynamics. In particular, the geometric observables in such synthetic fields exhibit nonlinear ESS exponent ratios, signatures of saturation, and scaling behaviour qualitatively similar to those observed in both two- and three-dimensional turbulence.

This observation leads naturally to a reinterpretation of the multiscaling associated with geometric observables. Rather than arising solely from intermittent cascade dynamics in the conventional sense, the dominant mechanism appears to originate more generally from the statistical structure of the observables themselves. The angular increments are bounded quantities and involve ratios of velocity increments to local velocity magnitudes. Consequently, they are comparatively insensitive to extreme amplitude fluctuations while simultaneously enhancing the statistical importance of regions where the velocity magnitude becomes small. Their scaling properties therefore depend strongly on correlations, boundedness, and the geometry of the near-zero sets of the velocity field.

Viewed from this perspective, the similarity between the turbulent and Gaussian results is not accidental. Correlated Gaussian fields already possess nontrivial spatial correlations and complex zero sets, and therefore naturally reproduce many of the geometric and statistical mechanisms that control the scaling of ratio observables. Turbulence then acts not as the sole origin of the multiscaling, but rather as a dynamical process that renormalises and quantitatively modifies an underlying geometric mechanism that is already present in correlated fields more generally.

This interpretation also clarifies why the angular observables continue to display strong multiscaling in the inverse cascade of two-dimensional turbulence despite the near-normal scaling of conventional longitudinal structure functions in that regime. Our results suggest that such behaviour need not imply the presence of strong intermittent amplitude fluctuations associated with the energy cascade itself. Instead, the geometric observables probe a distinct statistical sector dominated by boundedness, kinematic constraints, and the spatial organisation of low-velocity regions. The appearance of similar scaling behaviour in synthetic Gaussian fields supports this interpretation.

More broadly, the present work suggests that the statistical content of multiscaling in turbulence may depend crucially on the class of observables being considered. Some observables, such as conventional velocity increments, are strongly tied to the non-Gaussian amplitude fluctuations generated by turbulent cascades. Others, particularly bounded geometric or ratio observables, may derive a substantial part of their multiscaling from more generic correlation-based mechanisms that survive even in Gaussian fields. In this sense, the existence of anomalous scaling alone may not always be sufficient to identify uniquely turbulence-specific intermittency mechanisms.

Our findings also suggest a broader perspective on the use of synthetic and quasi-Gaussian descriptions in turbulence. It has long been known that several low-order statistical properties of turbulent flows, including energy spectra and certain correlation functions, are often captured surprisingly well by quasi-normal closures and synthetic Gaussian models. The present work shows that this correspondence can extend significantly further for suitably chosen observables, particularly those dominated by kinematic constraints and geometric correlations rather than by extreme amplitude fluctuations. At the same time, important differences remain: turbulence modifies the detailed statistics of the increments, the geometry of the near-zero sets, and the probability distribution functions in ways not fully reproduced by Gaussian fields. Thus, while synthetic fields can capture the dominant scaling mechanism for geometric observables, they do not provide a complete description of turbulent intermittency.

The framework developed here therefore opens several directions for future work. It would be interesting to investigate whether similar geometric multiscaling emerges in compressible turbulence, magnetohydrodynamic turbulence, rotating and stratified flows, active matter systems, or quantum fluids. Equally important is the development of a more systematic theory connecting the scaling exponents of geometric observables to the statistics and dimensions of the near-zero sets of correlated vector fields. Such investigations may help clarify more generally which aspects of multiscaling are genuinely dynamical consequences of turbulence and which

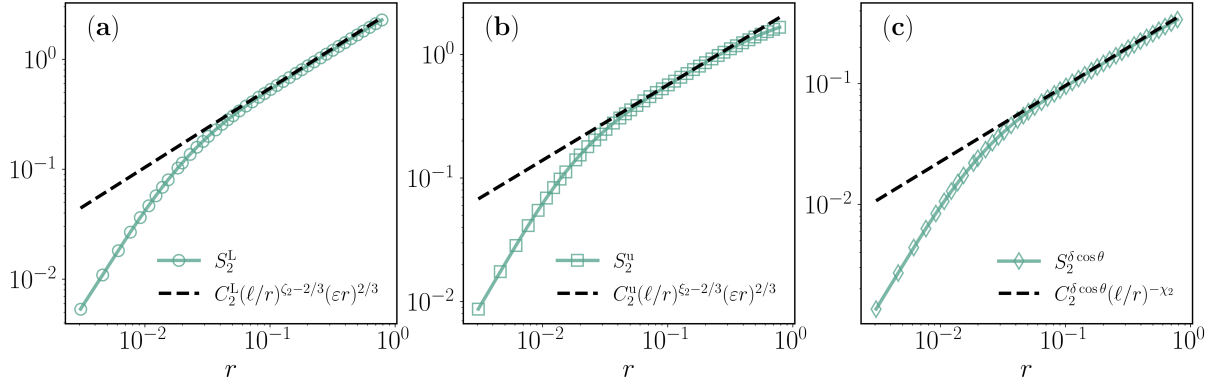


Figure 4.5. Loglog plots of the different second-order structure functions for increments of (a) the longitudinal velocity (b) velocity amplitudes and (c) the cosine-angles. Each panel is accompanied by the scaling form (valid in the inertial range) from which factors in the anomalous part of the scaling; for (c) we do not have a dimensional form for the scaling since the cosine-angle structure functions are dimensionless.

arise from broader geometric and statistical structures common to correlated random fields.

Taken together, our results suggest that geometric observables provide a complementary route for probing multiscaling in complex flows. By separating amplitude fluctuations from geometric correlations, they reveal a class of scaling phenomena that straddles the boundary between turbulence-specific intermittency and more universal statistical mechanisms present in correlated fields.

APPENDIX

4.A SCALING FORM FOR THE SECOND-ORDER STRUCTURE FUNCTIONS

We show the accuracy of the scaling form (see main text)

$$S_2^L(r) = C_2^L \left(\frac{\ell}{r} \right)^{p/3 - \zeta_p^L} (\epsilon r)^{p/3} \quad (\text{A-1})$$

$$S_2^u(r) = C_2^u \left(\frac{\ell}{r} \right)^{p/3 - \zeta_p} (\epsilon r)^{p/3} \quad (\text{A-2})$$

$$S_2^{\delta \cos \theta}(r) = C_2^{\delta \cos \theta} \left(\frac{\ell}{r} \right)^{-\chi_p} \quad (\text{A-3})$$

of the second-order structure functions. Of course, such a form for longitudinal increments is well known [124], and in Fig. 4.5(a) we fit this form to extract $C_2^L = 2.3$ [127] by using the integral length scale $\ell = 1.39$ and mean dissipation rate $\epsilon = 1.41$. We then use the same L to fit, in panels (b) and (c), respectively, the second-order structure function S_2^u for the amplitude and $S_2^{\delta \cos \theta}$ for the cosine-angle increments. In particular, our fits yield $C^u \approx 1.8$ and $C^{\delta \cos \theta} \approx 0.5$.

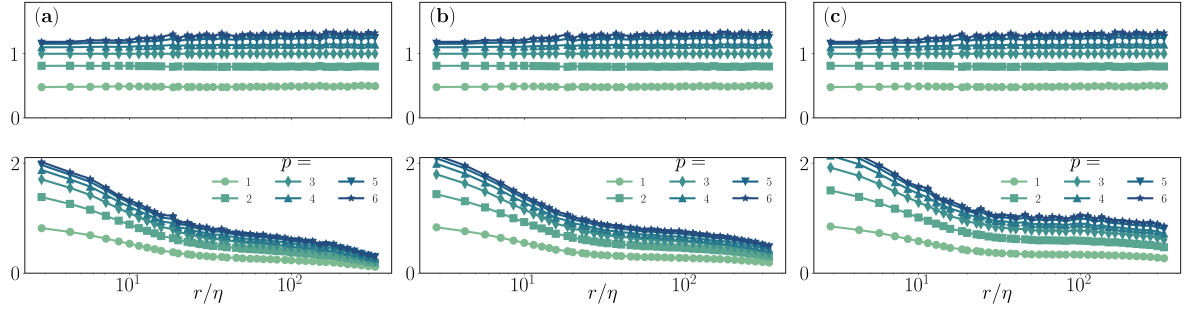


Figure 4.6. Plots of the local slopes for structure functions increments $\delta \sin \theta_r$ (b) $\delta \sin \theta_r$ and (c) $\delta \theta_r$ with (upper panel) and without (lower panel) the use of extended self-similarity in three-dimensional turbulence. A similar quality of scaling is also obtained for two-dimensional turbulence.

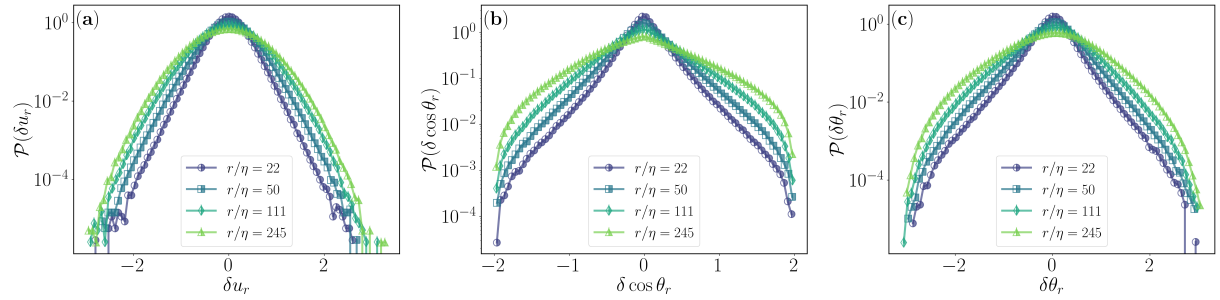


Figure 4.7. Representative plots of the probability distribution functions of (a) δu_r , (b) $\delta \cos \theta_r$, and (c) $\delta \theta_r$ for different values of r/η .

4.B APPENDIX B: MULTISCALING EXPONENTS AND THE PROBABILITY DISTRIBUTION OF INCREMENTS

In this section we analyse the multiscaling exponents for the p -order, equal-time structure function exponents, associated with the increments $\langle |\delta \sin \theta_r|^p \rangle \equiv \langle |\sin \theta_A - \sin \theta_B|^p \rangle \sim r^{\beta_p}$, $\langle |\sin \delta \theta_r|^p \rangle \equiv \langle |\sin[\theta_A - \theta_B]|^p \rangle \sim r^{\gamma_p}$, and $\langle |\delta \theta_r|^p \rangle \equiv \langle |[\theta_A - \theta_B]|^p \rangle \sim r^{\eta_p}$.

These exponents are obtained through a local slope analysis (as discussed in the main text; see Fig. 4.6). The bare exponents, as well as those obtained via ESS, are listed in Table 4.1, for three-dimensional and in Table 4.2, for two-dimensional turbulence.

Multiscaling of inertial range exponents are associated with non-Gaussian tails of the probability distribution functions of the increments themselves. In Fig. 4.7 we show representative plots of the probability distribution functions of (a) δu_r , (b) $\delta \cos \theta_r$, and (c) $\delta \theta_r$ for different values of r/η , which show clear non-Gaussian tails. In particular, for the velocity amplitude increments (Fig. 4.7(a)), large values of r/η lead to a Gaussian distribution with ever widening tails as r/η fall in the inertial range.

p	β_p	$\tilde{\beta}_p$	γ_p	$\tilde{\gamma}_p$	η_p	$\tilde{\eta}_p$
1	0.30 ± 0.03	0.462 ± 0.005	0.32 ± 0.02	0.457 ± 0.005	0.36 ± 0.01	0.424 ± 0.004
2	0.47 ± 0.04	0.790 ± 0.003	0.51 ± 0.03	0.786 ± 0.003	0.61 ± 0.02	0.769 ± 0.005
3	0.56 ± 0.06	1.00	0.63 ± 0.04	1.00	0.79 ± 0.02	1.00
4	0.63 ± 0.06	1.128 ± 0.006	0.72 ± 0.05	1.131 ± 0.005	0.92 ± 0.02	1.15 ± 0.01
5	0.68 ± 0.07	1.22 ± 0.01	0.77 ± 0.05	1.22 ± 0.01	0.98 ± 0.02	1.24 ± 0.03
6	0.72 ± 0.07	1.28 ± 0.02	0.81 ± 0.05	1.28 ± 0.02	1.03 ± 0.02	1.28 ± 0.04

Table 4.1. A summary of the equal-time scaling exponents β_p , γ_p , and η_p , with the corresponding exponent ratios obtained via ESS (denoted by a *tilde*) for three-dimensional turbulence.

p	β_p	$\tilde{\beta}_p$	γ_p	$\tilde{\gamma}_p$	η_p	$\tilde{\eta}_p$
1	0.23 ± 0.04	0.55 ± 0.03	0.21 ± 0.04	0.54 ± 0.02	0.30 ± 0.03	0.49 ± 0.03
2	0.34 ± 0.06	0.83 ± 0.01	0.32 ± 0.06	0.83 ± 0.01	0.49 ± 0.04	0.80 ± 0.03
3	0.41 ± 0.07	1.00	0.39 ± 0.07	1.00	0.59 ± 0.03	1.00
4	0.45 ± 0.07	1.11 ± 0.01	0.43 ± 0.08	1.11 ± 0.01	0.65 ± 0.01	1.10 ± 0.03
5	0.47 ± 0.08	1.18 ± 0.03	0.46 ± 0.08	1.19 ± 0.02	0.69 ± 0.01	1.16 ± 0.05
6	0.49 ± 0.08	1.24 ± 0.04	0.48 ± 0.08	1.24 ± 0.02	0.71 ± 0.02	1.21 ± 0.07

Table 4.2. A summary of the equal-time scaling exponents β_p , γ_p , and η_p , with the corresponding exponent ratios obtained via ESS (denoted by a *tilde*) for two-dimensional turbulence.

FORCING EFFECTS ON ENERGY AND ENSTROPY DISSIPATION IN TWO-DIMENSIONAL FLOWS

Mathematical estimates for the Navier–Stokes equations are traditionally expressed in terms of the Grashof number, which is a dimensionless measure of the magnitude of the forcing and hence a control parameter of the system. However, experimental measurements and statistical theories of turbulence are based on the Reynolds number. Thus, a meaningful comparison between mathematical and physical results requires a conversion of the mathematical estimates to a Reynolds-dependent form. In two dimensions, this was achieved for sufficiently smooth forcing. Nonetheless, numerical simulations have shown that the phenomenology of turbulence is sensitive to the degree of regularity of the forcing that sustains it. Therefore, we extend the available estimates for the energy and enstrophy dissipation rates as well as the attractor dimension to the full range of forcing regularity as allowed by the theory of weak solutions. The main result is the existence of three distinct regimes as a function of the regularity of the forcing.

This chapter closely follows the work [79]¹.

5.1 INTRODUCTION

The complexity and the limited predictability of turbulent flows justify the use of statistical tools to describe their dynamics. Statistical approaches have indeed been successful in capturing key aspects of turbulence, offering explanations that align well with experimental observations [44? –46]. The Navier–Stokes equations, however, are a deterministic (infinite-dimensional) dynamical system. Thus, an important goal of mathematical analysis is to prove the main facts in turbulence theories within a deterministic framework [13, 47–50]. Mathematical results generally take the form of bounds for the time average of Sobolev norms of Navier–Stokes

¹This work is done in collaboration with John D. Gibbon and Dario Vincenzi.

weak solutions; interestingly, statistical predictions often correspond to the saturation of the mathematical bounds.

The early rigorous estimates for the body-forced Navier–Stokes equations were expressed in terms of the generalized Grashof number Gr , a dimensionless measure of the magnitude of the forcing introduced by Foias et al. [54]. This is a natural choice from a mathematical point of view, since the magnitude of the forcing appears as an input parameter for the problem. However, experimental measurements and statistical theories of turbulence are typically expressed in terms of the Reynolds number Re , which is based on the velocity or the energy dissipation rate and therefore emerges as a response of the system. Thus, a meaningful comparison between mathematical and physical results requires a conversion of the mathematical bounds to a Re -dependent form. For periodic boundary conditions and square-integrable forcing, this conversion was carried out by Doering and Foias [8]. They showed that $Gr \leq c_1 Re + c_2 Re^2$, where the prefactors depend only on the functional shape of the forcing and are uniform in the other parameters of the system. Moreover, Doering and Foias [8] derived a rigorous upper bound for the time-averaged energy dissipation rate ε that captures the laminar flow behaviour at small Re and is consistent with the dissipative anomaly² in three dimensions at large Re . Under the assumption of narrow-band forcing, this estimate was later extended to Sobolev norms of the velocity of any order [137, 138].

As mentioned above, the result in [8] holds for square-integrable forcing. However, in a periodic domain, weak solutions of the Navier–Stokes equations are known to exist under more general conditions, namely the forcing must only be in the Sobolev space $\dot{H}^{-1}$ [69]. Loosely speaking, $\dot{H}^s$ is the space of functions that have zero spatial average and whose Fourier coefficients decay faster than $k^{-s-d/2}$ at large k , where k is the wavenumber and $d = 2, 3$ is the space dimension. Hence, for weak solutions to exist, the Fourier coefficients of the forcing must only decay faster than $k^{1-d/2}$ (as opposed to $k^{-d/2}$ for square-integrable forcing). Although the existence theory ensures that the global energy balance holds in the sense of an energy inequality also for these less regular forcings, it does not immediately clarify how dissipation bounds are modified. In three dimensions, this issue was addressed by Cheskidov, Doering, and Petrov [9]. Their analysis shows a qualitative transition: when the forcing is no longer square-integrable, ε is allowed to grow as a power of Re with an exponent that depends on the degree of regularity of the forcing.

Broad-band forcings have been considered in the physics literature in order to investigate a potential breakdown of the classical cascade phenomenology. In three dimensions, Kolmogorov’s theory assumes that energy injection is concentrated around a characteristic length scale which itself is much larger than the viscous scale. Forcings whose action extends over a wide range of scales, and even reach the viscous range, do not display the necessary scale separation between injection and dissipation. This overlap leads them to generate a different phenomenology. For instance, if the Fourier spectrum of the forcing scales as a power of k – a setting appropriate for turbulent flows generated by fractal grids [51, 52] – the law of finite energy

²An estimate of ε for body-forced turbulence in terms of Re was first derived by Foias [136]. That estimate captured the dissipative anomaly in the infinite- Re limit, but it included a spurious volume-dependent prefactor.

dissipation is violated and ε grows linearly with Re [53]. This is consistent with the result predicted by Cheskidov et al. [9]. Power-law forcings are also at the heart of renormalisation-group (RG) approaches. The canonical setting is white-in-time Gaussian forcing whose covariance in Fourier space scales like k^{4-d-y} , where y is an expansion parameter³. Some of the key objectives of the RG approach have been the calculation of the Kolmogorov constant and the dependence of the energy spectrum on the scaling exponent of the forcing [140?–142]. The same kind of forcing has been employed beyond the RG framework to investigate the effect of the injection mechanisms on the statistics of the velocity field [143–147]. In particular, the critical threshold $y_c = 4$ was found to separate two regimes: for $y < y_c$ the small scales are forcing-dominated and display nearly dimensional scaling, whereas for $y > y_c$ the small-scale statistics become essentially forcing-independent and exhibit the same anomalous scaling found in flows forced only at large scales [144]. This transition is again consistent with the result of Cheskidov *et al.* [9], since $y_c = 4$ is the value below which the covariance of the forcing ceases to be square-integrable. Thus, these studies demonstrate the strong impact of the forcing regularity on the dissipation of energy in three dimensions.

In two dimensions, the Navier–Stokes equations possess two quadratic invariants in the sense of strong solutions ; namely, energy and enstrophy. This fact strongly modifies the dynamics of the flow. Indeed, the Kraichnan–Leith–Batchelor (KLB) theory states that an inverse transfer of energy to large scales coexists with a direct cascade of enstrophy to small scales (see [58, 59] for a review). This phenomenology suggests that, in a forced state, ε vanishes as viscosity tends to zero, while the time-averaged enstrophy dissipation rate χ remains finite. As in the three-dimensional case, mathematical estimates of ε and χ have typically been expressed in terms of Gr [60–62], but once again the comparison with experimental and numerical results requires a conversion of the results to a Re -dependent form. Rigorous upper bounds in terms of Re were obtained by Alexakis and Doering [63] and Gibbon and Pavliotis [64] for periodic boundary conditions and under the assumption that the forcing is in $\dot{H}^2$; i.e. its second derivative is square integrable. The strategy employed was to apply the methods of Doering and Foias [8] to the vorticity equation instead of the equations for the velocity. The infinite- Re limit of the resulting bounds is consistent with the KLB prediction of a dual cascade. More stringent bounds apply when the forcing is monochromatic or injects energy in a finite set of wavenumbers at a constant rate. In these cases, both ε and χ vanish as Re tends to infinity, and the enstrophy cascade disappears [63, 148, 149]. An analogous result holds for white noise forcing [150]. Incidentally, enstrophy dissipation also vanishes in the inviscid limit when forcing is absent and turbulence decays at long times [151, 152] – see also [153] and references therein.

Returning to the forced case, we have seen that, in two dimensions, the available Re -dependent estimates of ε and χ require that the forcing possesses a relatively high degree of regularity (the forcing must be at least in $\dot{H}^2$). Contrary to the three-dimensional case, to our knowledge there has been no extension of such estimates to less regular forcing. Notwithstanding this, weak solutions are known to exist for forcing in $\dot{H}^{-1}$, and numerical simulations have

³Using a forcing term with power-law correlation is a strategy for introducing a small parameter into the Navier–Stokes equations. The *functional* RG formalism now allows for non-perturbative approaches [139].

demonstrated that even in two dimensions the regularity of the forcing has a strong impact on the turbulent flow [65, 66]. In particular, these simulations have clarified certain discrepancies between the RG predictions and the KLB theory and have shown the existence of different dynamical regimes, the KLB scenario being observed only when the forcing is sufficiently regular. Therefore, our aim here is to extend the existing Re -dependent estimates for the time-averaged dissipation rates ε and χ to forcings in $\dot{H}^s$ with $-1 \leq s < 2$. To this end, we combine the methods of [63] and [64] with those of [9]. In essence, our strategy is to apply the techniques developed for the three-dimensional Navier-Stokes equations with $\dot{H}^s$ -forcing to the vorticity equation in two dimensions.

Another crucial difference between two and three dimensions is that the two-dimensional Navier-Stokes equations on a periodic domain are known to possess a global attractor $\mathcal{A}$ if the forcing is square-integrable. $\mathcal{A}$ is a compact set on which the long-time dynamics is confined [68] (see also [69, 70]). Thus, asymptotically in time, any initial condition outside $\mathcal{A}$ approaches it, while any trajectory starting on $\mathcal{A}$ remains there for all time. The existence of such an attractor reflects the effectively finite-dimensional behaviour of an otherwise infinite-dimensional dynamical system. Bounds for the fractal dimension of the attractor, $d_f(\mathcal{A})$, indeed provide a mathematically rigorous notion of the ‘degrees of freedom’ of the system (see also [67, 71], for an informal introduction). Restricting to periodic boundary conditions – the setting considered here – Constantin, Foias, and Temam [72] were the first to obtain an estimate of $d_f(\mathcal{A})$ in terms of Gr (see also [73] for a simpler proof based on the vorticity rather than the velocity field). Later, Gibbon and Pavliotis [64] converted this estimate into a Re -dependent estimate under the more restrictive assumption of $\dot{H}^2$ -forcing. The Re -based estimate of [64] matches the heuristic prediction for the degrees of freedom of two-dimensional turbulence. Here, by using the new bound for χ , we also extend the estimate of $d_f(\mathcal{A})$ in [64] to the case of less regular forcing.

Our results for ε , χ , and $d_f(\mathcal{A})$ are summarized in table ?? in which only the large- Re bounds are shown. While the first line represents the special case of monochromatic or constant- ε forcing, the rest of the table refers to the case of $\dot{H}^s$ -forcing. Our analysis reveals the existence of three distinct regimes depending on the value of s . The case $s \geq 2$ is that studied in [63, 64]. Our estimates correspond to $0 \leq s < 2$. The last line ($-1 \leq s < 0$) is a straightforward adaptation of the three-dimensional result of [9] to two dimensions. The rest of this article is organised as follows. In §5.2 we introduce the functional setting and briefly recall the relevant results from the literature. In §5.3 we present the derivation of our estimates, and in §5.4 we offer some concluding remarks.

5.2 DEFINITIONS AND PREVIOUS TWO-DIMENSIONAL RESULTS

We consider the incompressible Navier-Stokes equations on the periodic domain $\Omega = [0, \ell]^2$:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \quad (5.1)$$

Forcing	$\chi \ell^3 / U^3$	$\varepsilon \ell / U^3$	$\ell \eta_\chi^{-1}$	$\ell \eta_\varepsilon^{-1}$	$d_f(\mathcal{A})$	Refs.
single- k or fixed- ε	Re^{-1}	Re^{-1}	$Re^{1/3}$	$Re^{1/2}$	$Re^{2/3}(1 + \ln Re)^{1/3}$	[63, 148, 149]
$s \geq 2$	Re^0	$Re^{-1/2}$	$Re^{1/2}$	$Re^{5/8}$	$Re(1 + \ln Re)^{1/3}$	[63, 64]
$0 \leq s < 2$	$Re^{\frac{2-s}{2+s}}$	$Re^{-\frac{s}{s+2}}$	$Re^{\frac{4+s}{3(2+s)}}$	$Re^{\frac{3+s}{2(2+s)}}$	$Re^{\frac{2(4+s)}{3(2+s)}} (1 + \ln Re)^{1/3}$	
$-1 \leq s < 0$	—	$Re^{-\frac{s}{s+2}}$	—	$Re^{\frac{3+s}{2(2+s)}}$	—	[9]

Table 5.1. Large- Re estimates for the two-dimensional Navier–Stokes equations with $\dot{H}^s$ constant-in-time forcing. If the forcing is time dependent, the estimates for the time-averaged dissipation rates are multiplied by a prefactor that depends on the Strouhal number, i.e. the ratio of the convective time of the flow to the characteristic time scale of the forcing (see § 5.3.3 for the details).

where $\mathbf{u}(x, t)$ is the velocity field, $p(x, t)$ pressure, ν the kinematic viscosity, and $\mathbf{f}(x)$ an externally applied, time-independent, divergence-free forcing (in § 5.4, we will comment on the extension of the results to the case of time-dependent forcing). We also assume that both the forcing and the initial velocity $\mathbf{u}_0(x)$ have zero mean, so that $\mathbf{u}(x, t)$ remains zero-mean at all times. To take advantage of the structure of the Navier–Stokes equations in two dimensions, it is convenient to study the evolution of the scalar vorticity

$$\omega = \mathbf{e}_3 \cdot (\nabla \times \mathbf{u}). \quad (5.2)$$

Taking the curl of (5.1) yields the vorticity equation

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega + \phi, \quad (5.3)$$

where $\phi = \mathbf{e}_3 \cdot (\nabla \times \mathbf{f})$.

Given a vector field $\mathbf{g}(x, t)$, we define its Fourier coefficients $\hat{\mathbf{g}}_k(t)$ through

$$\mathbf{g}(x, t) = \sum_k \hat{\mathbf{g}}_k(t) e^{ik \cdot x}, \quad \hat{\mathbf{g}}_k(t) = \ell^{-2} \int_{\Omega} \mathbf{g}(x, t) e^{-ik \cdot x} dx. \quad (5.4)$$

Since we only consider zero-mean vector fields, all Fourier series are understood to be over the wave-vectors $\mathbf{k} = 2\pi \mathbf{n} / \ell$ with $\mathbf{n} \in \mathbb{Z}^2$ such that $k = |\mathbf{k}| \neq 0$. In particular, if the field $\mathbf{g}(x, t)$ is divergenceless, then $\mathbf{k} \cdot \hat{\mathbf{g}}_k(t) = 0$ and the following identity holds

$$|(\widehat{\nabla \times \mathbf{g}})_k|^2 = |\mathbf{k} \times \hat{\mathbf{g}}_k|^2 = k^2 |\hat{\mathbf{g}}_k|^2 - (\mathbf{k} \cdot \hat{\mathbf{g}}_k)^2 = k^2 |\hat{\mathbf{g}}_k|^2. \quad (5.5)$$

We aim to study (5.1) and (5.3) as the degree of regularity of the forcing varies. To this end, we consider the homogeneous Sobolev space $\dot{H}^s(\Omega)$. A field $\mathbf{g} \in \dot{H}^s(\Omega)$ if and only if it has zero mean and its $\dot{H}^s$ -norm, defined by

$$\|\mathbf{g}\|_{\dot{H}^s}^2 = \sum_k (\ell k)^{2s} |\hat{\mathbf{g}}_k|^2, \quad (5.6)$$

is finite. Note that we follow the definition of the $\dot{H}^s$ -norm used in [9]; this definition differs from the usual one by a factor of ℓ^{s-1} but has the advantage of simplifying the calculations,

since $\|\mathbf{g}\|_{\dot{H}^s}$ has the dimension of $|\mathbf{g}|$ for all s . The parameter s characterizes the regularity of the field, higher values of s corresponding to smoother fields. In particular, $s = 0$ corresponds to the L^2 -norm (here defined with the volume normalization prefactor for consistency with the definition of the $\dot{H}^s$ -norm)

$$\|\mathbf{g}\|_{L^2}^2 = \ell^{-2} \int_{\Omega} |\mathbf{g}(\mathbf{x}, t)|^2 d\mathbf{x} = \sum_k |\hat{\mathbf{g}}_k(t)|^2 = \|\mathbf{g}\|_{\dot{H}^0}^2. \quad (5.7)$$

The forcing in (5.1) is thus taken to be in $\dot{H}^s(\Omega)$ and of the form

$$\mathbf{f}(\mathbf{x}) = F \mathbf{\Phi}(\ell^{-1}\mathbf{x}), \quad (5.8)$$

where $F > 0$ is the magnitude and $\mathbf{\Phi}$ is a dimensionless field such that

$$\int_{\Omega} \mathbf{\Phi}(\mathbf{x}) d\mathbf{x} = 0, \quad \nabla \cdot \mathbf{\Phi} = 0, \quad \|\mathbf{\Phi}\|_{\dot{H}^s} = 1. \quad (5.9)$$

We shall refer to $\mathbf{\Phi}$ as the ‘shape’ of the forcing. Recall that weak solutions of (5.1) are known to exist for $s \geq -1$ [69].

The time-averaged energy and enstrophy dissipation rates are defined as

$$\varepsilon = \nu \ell^{-2} \left\langle \int_{\Omega} |\nabla \mathbf{u}|^2 d\mathbf{x} \right\rangle, \quad \chi = \nu \ell^{-2} \left\langle \int_{\Omega} |\nabla \omega|^2 d\mathbf{x} \right\rangle, \quad (5.10)$$

respectively, where

$$\langle \cdot \rangle = \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \cdot dt. \quad (5.11)$$

In terms of the $\dot{H}^s$ -norms the same quantities are expressed as

$$\varepsilon = \nu \ell^{-2} \langle \|\mathbf{u}\|_{\dot{H}^1}^2 \rangle = \nu \langle \|\omega\|_{L^2}^2 \rangle, \quad \chi = \nu \ell^{-2} \langle \|\omega\|_{\dot{H}^1}^2 \rangle = \nu \ell^{-4} \langle \|\mathbf{u}\|_{\dot{H}^2}^2 \rangle. \quad (5.12)$$

The length scales η_{χ} and η_{ε} corresponding to energy and enstrophy dissipation are

$$\eta_{\varepsilon} = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}, \quad \eta_{\chi} = \left(\frac{\nu^3}{\chi} \right)^{1/6}. \quad (5.13)$$

These scales are also known as the Kolmogorov and Kraichnan length scales, respectively.

We now briefly recall the previous estimates of ε , χ , and the attractor dimension. For $\mathbf{f} \in \dot{H}^{-1}$, only a bound for ε is available

$$\varepsilon \leq \nu^3 \ell^{-4} Gr_*^2, \quad (5.14)$$

where $Gr_* = \nu^{-2} \ell^3 \|\mathbf{f}\|_{\dot{H}^{-1}}$ is a Grashof number based on the $\dot{H}^{-1}$ -norm [154, 155]. The absence of a bound for χ means that it is not known whether a global attractor exists in this case. The traditional estimates of ε and χ in terms of the Grashof number assume $\mathbf{f} \in L^2(\Omega)$ and define Gr as

$$Gr = \frac{\ell^3 \|\mathbf{f}\|_{L^2}}{\nu^2}. \quad (5.15)$$

The corresponding bounds on the energy and enstrophy dissipation rates read (see e.g. [47, 48])

$$\varepsilon \leq \nu^3 \ell^{-4} Gr^2, \quad \chi \leq \nu^3 \ell^{-6} Gr^2. \quad (5.16)$$

Under the same assumptions on the forcing, the bound for the attractor dimension is

$$d_f(\mathcal{A}) \leq c_0 Gr^{2/3} (1 + \ln Gr)^{1/3}, \quad (5.17)$$

where c_0 is a positive dimensionless constant [72, 73]. When the forcing is at small scales, the bound in (5.17) is improved by a bound that is linear in Gr_* [154, 155].

Converting these estimates into a Re -dependent form requires a definition of the Reynolds number based on a characteristic velocity that remains bounded for all time. To this end, take the dot product of (5.1) with $\mathbf{u}$ and average over Ω to obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|_{L^2}^2 = \nu \|\nabla \mathbf{u}\|_{L^2}^2 + \frac{1}{\ell^2} \int_{\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}, t) d\mathbf{x}. \quad (5.18)$$

Applying the interpolation inequality $\ell^{-2} \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\mathbf{x} \leq \ell \|\mathbf{f}\|_{\dot{H}^{-1}} \|\nabla \mathbf{u}\|_{L^2}$ and Young's inequality to the forcing term gives

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|_{L^2}^2 \leq -\frac{\nu}{2} \|\nabla \mathbf{u}\|_{L^2}^2 + \frac{\ell^2}{2\nu} \|\mathbf{f}\|_{\dot{H}^{-1}}^2. \quad (5.19)$$

Then, the Poincaré inequality implies

$$\frac{d}{dt} \left(\frac{1}{2} \|\mathbf{u}\|_{L^2}^2 \right) \leq -\frac{4\pi^2\nu}{\ell^2} \left(\frac{1}{2} \|\mathbf{u}\|_{L^2}^2 \right) + \frac{\ell^2}{2\nu} \|\mathbf{f}\|_{\dot{H}^{-1}}^2. \quad (5.20)$$

Finally Gronwall's lemma applied to this linear differential inequality gives

$$\frac{1}{2} \|\mathbf{u}\|_{L^2}^2 \leq \frac{1}{2} \|\mathbf{u}_0\|_{L^2}^2 \exp(-4\pi^2\nu t / \ell^2) + \frac{\ell^4}{8\pi^2\nu^2} \|\mathbf{f}\|_{\dot{H}^{-1}}^2 (1 - \exp(-4\pi^2\nu t / \ell^2)). \quad (5.21)$$

Hence $\|\mathbf{u}\|_{L^2}^2$ is uniformly bounded in time. This fact lead Doering and Foias [8] to define the Reynolds number in terms of the time-averaged L^2 -norm of the velocity as

$$Re = \frac{U\ell}{\nu}, \quad U^2 = \langle \|\mathbf{u}\|_{L^2}^2 \rangle. \quad (5.22)$$

The conversion of (5.16) and (5.17) to a form based on this definition of Re was achieved under the assumption of sufficiently regular forcing. For $\mathbf{f} \in \dot{H}^2(\Omega)$, the estimates equivalent to (5.16) and (5.17) are [63, 64]

$$\frac{\varepsilon\ell}{U^3} \leq Re^{-1/2} \left(c_1 + \frac{c_2}{Re} \right)^{1/2}, \quad \frac{\chi\ell^3}{U^3} \leq c_1 + \frac{c_2}{Re}, \quad (5.23)$$

and

$$d_f(\mathcal{A}) \leq c_0 Re (1 + \ln Re)^{1/3}, \quad (5.24)$$

with c_1 and c_2 positive dimensionless constants. The estimates of ε and χ are consistent with a residual dissipation of enstrophy and the absence of a direct cascade of energy in the infinite- Re limit. Regarding the attractor dimension, $d_f(\mathcal{A})$ can be interpreted as a mathematical definition of the number of degrees of freedom of the system [67]. Since η_χ is the smallest length scale in the system, the number of degrees of freedom can in turn be related to the number of boxes of size η_χ that fit into the domain Ω , which leads to the identification $d_f(\mathcal{A}) \sim (\ell/\eta_\chi)^2$. Comparing

this interpretation of $d_f(\mathcal{A})$ with (5.24) yields an estimate of the enstrophy dissipation scale, $\eta_\chi \sim \ell Re^{-1/2}$, consistent with the KLB theory [58]. Finally, in the special case of single- k or constant- ε forcing, the energy and enstrophy dissipation rates are both bounded by Re^{-1} , and a direct cascade of enstrophy must be ruled out in the limit of infinite Re [63, 148, 149].

In the next section, we show how to obtain Re -dependent estimates for less regular forcing, namely for $f \in \dot{H}^s(\Omega)$ with $-1 \leq s < 2$.

5.3 ESTIMATES FOR $\dot{H}^s$ FORCING

5.3.1 THE CASE $0 \leq s \leq 2$

We first consider the range $0 \leq s \leq 2$, for which (5.1) has strong solutions [69, 70]. The enstrophy balance is obtained by multiplying (5.3) by ω and integrating over space

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \nu \|\nabla \omega\|_{L^2}^2 = \ell^{-2} \int_{\Omega} \omega \phi \, dx. \quad (5.25)$$

The enstrophy $\|\omega\|_{L^2}^2$ can be shown to be bounded for all time in a similar way to $\|u\|_{L^2}^2$. First, the forcing term is estimated by using the interpolation inequality

$$\ell^{-2} \int_{\Omega} \omega \phi \, dx \leq \ell \|\nabla \omega\|_{L^2} \|\phi\|_{\dot{H}^{-1}} = \ell \|\nabla \omega\|_{L^2} \|f\|_{L^2}. \quad (5.26)$$

Then, Young's inequality is applied to the forcing term and Poincaré inequality to the dissipation term, as in (5.19) and (5.20). Finally, Gronwall's lemma gives

$$\frac{1}{2} \|\omega\|_{L^2}^2 \leq \frac{1}{2} \|\omega_0\|_{L^2}^2 \exp(-4\pi^2 \nu t / \ell^2) + \frac{\ell^2}{8\pi^2 \nu^2} \|f\|_{L^2}^2 (1 - \exp(-4\pi^2 \nu t / \ell^2)), \quad (5.27)$$

where $\omega_0 = e_3 \cdot (\nabla \times u_0)$. Therefore, the time average of the time derivative of $\|\omega\|_{L^2}^2$ vanishes. It follows that the time-averaged enstrophy dissipation rate satisfies

$$\chi = \ell^{-2} \left\langle \int_{\Omega} \omega \phi \, dx \right\rangle. \quad (5.28)$$

To estimate the right-hand side of (5.28), we use Parseval's theorem, the triangle inequality, and identity (5.5)

$$\begin{aligned} \ell^{-2} \int_{\Omega} \omega \phi \, dx &\leq \ell^{-2} \left| \int_{\Omega} (\nabla \times u) \cdot (\nabla \times f) \, dx \right| \\ &\leq \sum_k |\mathbf{k} \times \hat{u}_k| |\mathbf{k} \times \hat{f}_k| \\ &= \sum_k k^2 |\hat{u}_k| |\hat{f}_k|. \end{aligned} \quad (5.29)$$

We then rearrange the sum and apply the Cauchy–Schwarz inequality

$$\begin{aligned} \ell^{-2} \int_{\Omega} \omega \phi \, dx &\leq \ell^{-2} \sum_k (\ell k)^2 |\hat{u}_k| |\hat{f}_k| \leq \ell^{-2} \sum_k (\ell k)^{2-s} |\hat{u}_k| (\ell k)^s |\hat{f}_k| \\ &\leq \ell^{-2} \left[\sum_k (\ell k)^{2(2-s)} |\hat{u}_k|^2 \right]^{1/2} \left[\sum_k (\ell k)^{2s} |\hat{f}_k|^2 \right]^{1/2} \\ &= F \ell^{-2} \|u\|_{\dot{H}^{2-s}}. \end{aligned} \quad (5.30)$$

To relate χ to Re , we need to estimate the time average of $\|\mathbf{u}\|_{\dot{H}^{2-s}}$ in terms of χ and U . The following interpolation identity is obtained by rearranging the terms in $\|\mathbf{u}\|_{\dot{H}^{2-s}}$ and then using a Hölder inequality

$$\begin{aligned} \|\mathbf{u}\|_{\dot{H}^{2-s}}^2 &= \sum_k (\ell k)^{2(2-s)} |\hat{\mathbf{u}}_k|^2 = \sum_k [(\ell k)^4 |\hat{\mathbf{u}}_k|^2]^{\frac{2-s}{2}} (|\hat{\mathbf{u}}_k|^2)^{1-\frac{2-s}{2}} \\ &\leq \left(\sum_k (\ell k)^4 |\hat{\mathbf{u}}_k|^2 \right)^{1-\frac{s}{2}} \left(\sum_k |\hat{\mathbf{u}}_k|^2 \right)^{\frac{s}{2}} \\ &= \|\mathbf{u}\|_{\dot{H}^2}^{2-s} \|\mathbf{u}\|_{L^2}^s, \end{aligned} \quad (5.31)$$

where we have used the fact that $0 \leq s \leq 2$. Taking the square root of both sides, time averaging, and using a Hölder inequality leads to

$$\langle \|\mathbf{u}\|_{\dot{H}^{2-s}} \rangle \leq \left\langle \|\mathbf{u}\|_{\dot{H}^2}^{1-\frac{s}{2}} \|\mathbf{u}\|_{L^2}^{\frac{s}{2}} \right\rangle \leq \left\langle \|\mathbf{u}\|_{\dot{H}^2}^2 \right\rangle^{\frac{2-s}{4}} \left\langle \|\mathbf{u}\|_{L^2}^{\frac{2s}{2+s}} \right\rangle^{\frac{2+s}{4}}. \quad (5.32)$$

An application of Jensen's inequality to the last term, together with the definitions of χ and U , gives

$$\begin{aligned} \langle \|\mathbf{u}\|_{\dot{H}^{2-s}} \rangle &\leq \left\langle \|\mathbf{u}\|_{\dot{H}^2}^2 \right\rangle^{\frac{2-s}{4}} \left\langle \|\mathbf{u}\|_{L^2}^2 \right\rangle^{\frac{s}{4}} \\ &= (\nu \ell^{-4})^{\frac{s-2}{4}} \left\langle \nu \ell^{-4} \|\mathbf{u}\|_{\dot{H}^2}^2 \right\rangle^{\frac{2-s}{4}} \left\langle \|\mathbf{u}\|_{L^2}^2 \right\rangle^{\frac{s}{4}} \\ &\leq (\nu \ell^{-4})^{\frac{s-2}{4}} \chi^{\frac{2-s}{4}} U^{\frac{s}{2}}. \end{aligned} \quad (5.33)$$

Combining (5.28), (5.30) and (5.33) gives

$$\chi \leq \ell^{-2} F \left(\nu \ell^{-4} \right)^{\frac{s-2}{4}} \chi^{\frac{2-s}{4}} U^{\frac{s}{2}} \quad (5.34)$$

which leads to

$$\chi \leq \nu^{-\frac{2-s}{2+s}} \ell^{-\frac{4s}{2+s}} F^{\frac{4}{2+s}} U^{\frac{2s}{2+s}}. \quad (5.35)$$

It remains to estimate F . We take a smooth time-independent field $\boldsymbol{\psi}$ such that

$$\nabla \cdot \boldsymbol{\psi} = 0, \quad \int_{\Omega} \boldsymbol{\Phi} \cdot \boldsymbol{\psi} \, dx > 0 \quad (5.36)$$

($\boldsymbol{\psi}$ can be, for example, a suitable Galerkin truncation of $\boldsymbol{\Phi}$). We then multiply the Navier–Stokes equation (5.1) by $\boldsymbol{\psi}$ and integrate over space

$$\int_{\Omega} \boldsymbol{\psi} \cdot \partial_t \mathbf{u} \, dx + \int_{\Omega} \boldsymbol{\psi} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} \, dx = - \int_{\Omega} \boldsymbol{\psi} \cdot \nabla p \, dx + \nu \int_{\Omega} \boldsymbol{\psi} \cdot \Delta \mathbf{u} \, dx + \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\psi} \, dx. \quad (5.37)$$

By integrating by parts and using $\nabla \cdot \mathbf{u} = 0$ on the nonlinear term and $\nabla \cdot \boldsymbol{\psi} = 0$ on the pressure term, we obtain

$$\int_{\Omega} \mathbf{f} \cdot \boldsymbol{\psi} \, dx = \int_{\Omega} \boldsymbol{\psi} \cdot \partial_t \mathbf{u} \, dx + \int_{\Omega} \mathbf{u} \cdot \nabla \boldsymbol{\psi} \cdot \mathbf{u} \, dx - \nu \int_{\Omega} \mathbf{u} \cdot \Delta \boldsymbol{\psi} \, dx. \quad (5.38)$$

The long-time average of both sides eliminates the time-derivative term, since $\|\mathbf{u}\|_{L^2}^2$ is uniformly bounded in time, see (5.21). Therefore

$$\ell^{-2} \left\langle \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\psi} \, dx \right\rangle \leq \ell^{-2} \left| \left\langle \int_{\Omega} \mathbf{u} \cdot \nabla \boldsymbol{\psi} \cdot \mathbf{u} \, dx \right\rangle \right| + \nu \ell^{-2} \left| \left\langle \int_{\Omega} \mathbf{u} \cdot \Delta \boldsymbol{\psi} \, dx \right\rangle \right|. \quad (5.39)$$

Now, we apply the Cauchy–Schwarz and Jensen’s inequalities and get

$$\begin{aligned}
 \ell^{-2} \left\langle \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\psi} \, d\mathbf{x} \right\rangle &\leq \|\nabla \boldsymbol{\psi}\|_{L^\infty} \langle \|\mathbf{u}\|_{L^2}^2 \rangle + \nu \|\Delta \boldsymbol{\psi}\|_{L^2} \langle \|\mathbf{u}\|_{L^2} \rangle \\
 &\leq \|\nabla \boldsymbol{\psi}\|_{L^\infty} \langle \|\mathbf{u}\|_{L^2}^2 \rangle + \nu \|\Delta \boldsymbol{\psi}\|_{L^2} \langle \|\mathbf{u}\|_{L^2}^2 \rangle^{1/2} \\
 &= \ell^{-1} \|\tilde{\nabla} \boldsymbol{\psi}\|_{L^\infty} U^2 + \nu \ell^{-2} \|\boldsymbol{\psi}\|_{\dot{H}^2} U,
 \end{aligned} \tag{5.40}$$

where $\tilde{\nabla} = \ell \nabla$. After rewriting the left-hand side of (5.40) as

$$\ell^{-2} \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\psi} \, d\mathbf{x} = F \ell^{-2} \int_{\Omega} \boldsymbol{\Phi} \cdot \boldsymbol{\psi} \, d\mathbf{x}, \tag{5.41}$$

we find

$$F \leq c_1 \ell^{-1} U^2 + c_2 \nu \ell^{-2} U, \tag{5.42}$$

together with

$$c_1 = \left(\ell^{-2} \int_{\Omega} \boldsymbol{\Phi} \cdot \boldsymbol{\psi} \, d\mathbf{x} \right)^{-1} \|\tilde{\nabla} \boldsymbol{\psi}\|_{L^\infty}, \quad c_2 = \left(\ell^{-2} \int_{\Omega} \boldsymbol{\Phi} \cdot \boldsymbol{\psi} \, d\mathbf{x} \right)^{-1} \|\boldsymbol{\psi}\|_{\dot{H}^2}, \tag{5.43}$$

where c_1 and c_2 are not related to the same coefficients in §5.2. Finally, combining (5.35) and (5.42), we obtain the estimate for the rescaled time-averaged enstrophy dissipation rate

$$\frac{\chi \ell^3}{U^3} \leq Re^{\frac{2-s}{2+s}} \left(c_1 + c_2 Re^{-1} \right)^{\frac{4}{2+s}}. \tag{5.44}$$

Note that the prefactors only depend on the shape of the forcing. For $Re \gg 1$, (5.44) yields the following estimate for the inverse of the enstrophy dissipation length scale

$$\ell \eta_\chi^{-1} = \ell \left(\frac{\chi}{\nu^3} \right)^{1/6} \leq c'_1 Re^{\frac{4+s}{3(2+s)}}, \tag{5.45}$$

with $c'_1 = c_1^{2/3(2+s)}$. To estimate the time-averaged energy dissipation rate, we use the interpolation inequality in (5.31) for $s = 1$

$$\|\mathbf{u}\|_{\dot{H}^1} \leq \|\mathbf{u}\|_{\dot{H}^2}^{1/2} \|\mathbf{u}\|_{L^2}^{1/2}. \tag{5.46}$$

We now square both members, time average, and use the Cauchy–Schwarz inequality to obtain

$$\langle \|\mathbf{u}\|_{\dot{H}^1}^2 \rangle \leq \langle \|\mathbf{u}\|_{\dot{H}^2} \|\mathbf{u}\|_{L^2} \rangle \leq \langle \|\mathbf{u}\|_{\dot{H}^2}^2 \rangle^{1/2} \langle \|\mathbf{u}\|_{L^2}^2 \rangle^{1/2}. \tag{5.47}$$

After multiplying by $\nu \ell^{-2}$, we find

$$\varepsilon \leq \nu^{1/2} \chi^{1/2} U. \tag{5.48}$$

The estimate for χ in (5.44) can be used to bound the right-hand side

$$\frac{\varepsilon \ell}{U^3} \leq Re^{-\frac{s}{2+s}} \left(c_1 + c_2 Re^{-1} \right)^{\frac{2}{2+s}}. \tag{5.49}$$

The inverse of the energy dissipation length scale can now be estimated using (5.49) in the definition of η_ε

$$\ell \eta_\varepsilon^{-1} = \ell \left(\frac{\varepsilon}{\nu^3} \right)^{1/4} \leq c''_1 Re^{\frac{3+s}{2(2+s)}}, \quad c''_1 = c_1^{\frac{1}{2(2+s)}}. \tag{5.50}$$

We conclude this study of the case $0 \leq s \leq 2$ with the estimate of the attractor dimension $d_f(\mathcal{A})$. We recall that $d_f(\mathcal{A})$ is related to η_χ via the following bound [72]

$$d_f(\mathcal{A}) \leq c_3 (\ell \eta_\chi^{-1})^2 [1 + \ln(\ell \eta_\chi^{-1})]^{1/3}. \quad (5.51)$$

Plugging (5.45) into (5.51) gives

$$d_f(\mathcal{A}) \leq c'_3 Re^{\frac{2(4+s)}{3(2+s)}} \left(1 + \ln Re^{\frac{4+s}{3(2+s)}}\right)^{1/3} \leq c'_3 Re^{\frac{2(4+s)}{3(2+s)}} (1 + \ln Re)^{1/3}, \quad (5.52)$$

where c'_3 is a constant related to c'_1 and c_3 . The scaling exponent varies between unity for $s = 2$ and $4/3$ for $s = 0$.

5.3.2 THE CASE $-1 \leq s \leq 0$

If $-1 \leq s \leq 0$, the above derivation does not work as a bound for χ is not available. However, we can obtain a bound for ε by working with the velocity equation instead of the vorticity equation and following the same procedure used in [9] for three dimensions. For completeness, we recall the main steps below (see [9] for the details). Since $\|\mathbf{u}\|_{L^2}^2$ is uniformly bounded, the time-derivative term in (5.18) vanishes upon taking a time average. Thus, ε can be expressed as

$$\varepsilon = \ell^{-2} \left\langle \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, dx \right\rangle. \quad (5.53)$$

The integral in the right-hand side can be estimated as

$$\ell^{-2} \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, dx \leq \sum_k |\hat{f}_k| |\hat{u}_k| \leq \left[\sum_k (\ell k)^{2s} |\hat{f}_k|^2 \right]^{1/2} \left[\sum_k (\ell k)^{-2s} |\hat{u}_k|^2 \right]^{1/2} \quad (5.54)$$

$$= F \|\mathbf{u}\|_{\dot{H}^{-s}} \leq F \|\mathbf{u}\|_{\dot{H}^1}^{-s} \|\mathbf{u}\|_{L^2}^{1+s}, \quad (5.55)$$

where an interpolation inequality similar to (5.31) has been used in the last step. Taking the time average of both sides and using Hölder's and Jensen's inequalities gives

$$\varepsilon \leq F \langle \|\mathbf{u}\|_{\dot{H}^1}^{-s} \|\mathbf{u}\|_{L^2}^{1+s} \rangle \leq F \langle \|\mathbf{u}\|_{\dot{H}^1}^2 \rangle^{-\frac{s}{2}} \left\langle \|\mathbf{u}\|_{L^2}^{\frac{2(1+s)}{(2+s)}} \right\rangle^{\frac{2+s}{2}} \leq F \nu^{\frac{s}{2}} \ell^{-s} \varepsilon^{-\frac{s}{2}} U^{1+s}. \quad (5.56)$$

Using the bound on F in (5.42) and rearranging gives

$$\frac{\varepsilon \ell}{U^3} \leq Re^{-\frac{s}{2+s}} (c_1 + c_2 Re^{-1})^{\frac{2}{2+s}} \quad (5.57)$$

The final estimate has the same form as (5.49), but now $-1 \leq s < 0$.

5.3.3 TIME-DEPENDENT FORCING

The estimates derived in the previous sections can be generalised to the case of deterministic but time-dependent forcing by applying the methods outlined in section 3 of [63]. However, some variations are necessary, since here the forcing is not in $\dot{H}^2(\Omega)$. We give the main steps below.

The forcing is now taken of the form $\mathbf{f}(\mathbf{x}, t) = F(t)\Phi(\ell^{-1}\mathbf{x}, t)$, where Φ satisfies the same properties as in (5.9) for all t . It is also assumed that the magnitude of the forcing is uniformly bounded in time and has finite mean square

$$\bar{F}^2 = \langle F^2(t) \rangle < \infty. \quad (5.58)$$

Let us first consider the case $0 \leq s \leq 2$. The bound in (5.27) continues to hold provided that $\|\mathbf{f}\|_{L^2}^2$ is replaced by $\sup_t \|\mathbf{f}\|_{L^2}^2$. Hence, $\|\omega\|_{L^2}$ remains bounded for all t , and χ can again be estimated as in (5.28). An intermediate bound for χ is obtained by using the Cauchy–Schwarz inequality to split the time average of the product of $F(t)$ and the $\dot{H}^{2-s}$ -norm of the velocity and then proceeding as in § 5.3.1:

$$\begin{aligned} \chi &\leq \ell^{-2} \langle F \|\mathbf{u}\|_{\dot{H}^{2-s}} \rangle \leq \ell^{-2} \bar{F} \langle \|\mathbf{u}\|_{\dot{H}^{2-s}}^2 \rangle^{1/2} \leq \ell^{-2} \bar{F} \left\langle \|\mathbf{u}\|_{\dot{H}^2}^{2-s} \|\mathbf{u}\|_{L^2}^s \right\rangle^{1/2} \\ &\leq \ell^{-2} \bar{F} \langle \|\mathbf{u}\|_{\dot{H}^2}^2 \rangle^{\frac{2-s}{4}} \langle \|\mathbf{u}\|_{L^2}^2 \rangle^{\frac{s}{4}} = \ell^{-2} \bar{F} \left(\nu \ell^{-4} \right)^{\frac{s-2}{4}} \chi^{\frac{2-s}{4}} U^{\frac{s}{2}}. \end{aligned} \quad (5.59)$$

Rearranging yields a bound analogous to (5.35) where now F is replaced by $\bar{F}$:

$$\chi \leq \nu^{-\frac{2-s}{2+s}} \ell^{-\frac{4s}{2+s}} \bar{F}^{\frac{4}{2+s}} U^{\frac{2s}{2+s}}. \quad (5.60)$$

To estimate $\bar{F}$ in terms of U , we multiply the Navier–Stokes equations by $\boldsymbol{\psi} = (-\tilde{\Delta})^{-M} \mathbf{f}$, where $\tilde{\Delta} = \ell^2 \Delta$ and $M \geq 1 - s/2$ (the reason for this condition will be clear immediately). After integrating by parts and time averaging, we obtain the analogous of (5.39) and (5.40) (note that the term involving the time derivative of the velocity can no longer be ignored)

$$\begin{aligned} \ell^{-2} \left\langle \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\psi} \, d\mathbf{x} \right\rangle &\leq \ell^{-2} \left| \left\langle \int_{\Omega} \mathbf{u} \cdot \partial_t \boldsymbol{\psi} \, d\mathbf{x} \right\rangle \right| + \ell^{-2} \left| \left\langle \int_{\Omega} \mathbf{u} \cdot \nabla \boldsymbol{\psi} \cdot \mathbf{u} \, d\mathbf{x} \right\rangle \right| + \nu \ell^{-2} \left| \left\langle \int_{\Omega} \mathbf{u} \cdot \Delta \boldsymbol{\psi} \, d\mathbf{x} \right\rangle \right| \\ &\leq \langle \|\partial_t \boldsymbol{\psi}\|_{L^2}^2 \rangle^{1/2} \langle \|\mathbf{u}\|_{L^2}^2 \rangle^{1/2} + \left(\sup_t \|\nabla \boldsymbol{\psi}\|_{L^\infty} \right) \langle \|\mathbf{u}\|_{L^2}^2 \rangle + \nu \langle \|\Delta \boldsymbol{\psi}\|_{L^2}^2 \rangle^{1/2} \langle \|\mathbf{u}\|_{L^2}^2 \rangle^{1/2} \\ &= \langle \|\partial_t (-\tilde{\Delta})^{-M} \mathbf{f}\|_{L^2}^2 \rangle^{1/2} U + \ell^{-1} \left(\sup_t \|\tilde{\nabla} (-\tilde{\Delta})^{-M} \mathbf{f}\|_{L^\infty} \right) U^2 + \nu \ell^{-2} \langle \|\mathbf{f}\|_{\dot{H}^{2(1-M)}}^2 \rangle^{1/2} U. \end{aligned} \quad (5.61)$$

The constraint $M \geq 1 - s/2$ ensures the existence of the last term. The left-hand side can be bounded from below so as to make the dependence on $\bar{F}$ explicit:

$$\ell^{-2} \left\langle \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\psi} \, d\mathbf{x} \right\rangle = \langle F^2 \|\Phi\|_{\dot{H}^{-M}}^2 \rangle \geq a^{-1} \bar{F}^2 \quad (5.62)$$

with

$$a^{-1} = \inf_t \|\Phi\|_{\dot{H}^{-M}}^2. \quad (5.63)$$

Plugging (5.62) into (5.61) and dividing through by $a^{-1} \bar{F}$ gives

$$\bar{F} \leq a \omega_f U + b_1 \ell^{-1} U^2 + b_2 \nu \ell^{-2} U, \quad (5.64)$$

where

$$\omega_f = \bar{F}^{-1} \left\langle \|\partial_t (-\tilde{\Delta})^{-M} \mathbf{f}\|_{L^2}^2 \right\rangle^{1/2} \quad (5.65)$$

is a time frequency associated with the forcing and

$$b_1 = a\bar{F}^{-1} \sup_t \|\tilde{\nabla}(-\tilde{\Delta})^{-M} f\|_{L^\infty}, \quad b_2 = a\bar{F}^{-1} \langle \|f\|_{\dot{H}^{2(1-M)}}^2 \rangle^{1/2}. \quad (5.66)$$

We now use (5.64) in (5.60) to obtain

$$\frac{\chi \ell^3}{U^3} \leq Re^{\frac{2-s}{2+s}} \left(aSr + b_1 + b_2 Re^{-1} \right)^{\frac{4}{2+s}}, \quad (5.67)$$

where

$$Sr = \frac{\omega_f \ell}{U} \quad (5.68)$$

is the Strouhal number, i.e. the ratio of the eddy turnover time of the flow to the characteristic time scale of the forcing (e.g. [156, 157]). The prefactor b_2 can be made independent of $F(t)$ by noting that

$$b_2 \leq a \sup_t \|\Phi\|_{\dot{H}^{2(1-M)}}. \quad (5.69)$$

However, b_1 depends not only on the shape of the forcing, but also on the ratio of the maximum excursion of $F(t)$ to its root-mean-square value. It can indeed be written as

$$b_1 = a \sup_t \|\tilde{\nabla}(-\tilde{\Delta})^{-M} \Phi\|_{L^\infty} \frac{\sup_t F(t)}{\bar{F}(t)}. \quad (5.70)$$

Finally, by applying (5.48), we find the following bound for the time-averaged energy dissipation rate

$$\frac{\varepsilon \ell}{U^3} \leq Re^{-\frac{s}{2+s}} \left(aSr + b_1 + b_2 Re^{-1} \right)^{\frac{2}{2+s}}. \quad (5.71)$$

Let us now consider the case $-1 \leq s \leq 0$. When the forcing depends on time, the bound on the energy, see (5.21), holds with $\|f\|_{L^2}^2$ replaced by $\sup_t \|f\|_{L^2}^2$. Therefore, $\|u\|_{L^2}$ is uniformly bounded in time, and ε is estimated again as in (5.53). We then proceed as in § 5.3.2 with an additional application of the Cauchy-Schwarz inequality to split the time average:

$$\begin{aligned} \varepsilon &\leq \langle F \|u\|_{\dot{H}^{-s}} \rangle \leq \bar{F} \langle \|u\|_{\dot{H}^{-s}}^2 \rangle^{1/2} \leq \bar{F} \left\langle \|u\|_{\dot{H}^1}^{-2s} \|u\|_{L^2}^{2(1+s)} \right\rangle^{1/2} \\ &\leq \bar{F} \langle \|u\|_{\dot{H}^1}^2 \rangle^{-\frac{s}{2}} \langle \|u\|_{L^2}^2 \rangle^{\frac{1+s}{2}} = \bar{F} \nu^{\frac{s}{2}} \ell^{-s} \varepsilon^{-\frac{s}{2}} U^{1+s}. \end{aligned} \quad (5.72)$$

This bound is the same as (5.56) with F replaced by $\bar{F}$. Therefore, by using (5.64) and rearranging, we find

$$\frac{\varepsilon \ell}{U^3} \leq Re^{-\frac{s}{2+s}} (aSr + b_1 + b_2 Re^{-1})^{\frac{2}{2+s}}. \quad (5.73)$$

Thus, for all $-1 \leq s \leq 0$, the bounds on the time-averaged dissipation rates retain the same form as in the case of time-independent forcing with the addition of a Sr -dependent prefactor that becomes relevant at large Re . Moreover, b_1 now depends on features of the forcing other than just the shape.

5.4 CONCLUDING REMARKS

Our results for constant-in-time forcing are summarised in table ??, in which only the large- Re bounds are shown. Leaving aside the special case of single- k or constant- ε forcing, three distinct regimes can be identified as the degree of regularity of the forcing is varied. For $s \geq 2$, the estimates obtained in [63, 64] are consistent with the dual-cascade scenario. In the second regime ($0 < s < 2$), a direct cascade of energy must still be ruled out in the $Re \rightarrow \infty$ limit. However, the amount of energy injected at small scales increases as s decreases, so that ε decreases monotonically as a function of Re , but increasingly slowly. In this regime, χ is allowed to grow with Re . As energy injection at small scales increases, the bound on the number of degrees of freedom of the system (quantified by the attractor dimension) also grows from Re for $s = 2$ to $Re^{4/3}$ for $s = 0$. Finally, for $-1 < s < 0$, the time-averaged energy dissipation rate can increase with Re , similar to what is found in three dimensions for non-square-integrable forcing [9, 53]. For all $-1 \leq s \leq 2$, if the forcing is time-dependent, the large- Re estimates of the mean dissipation rates are multiplied by a Sr -dependent prefactor.

It is noteworthy that three dynamical regimes are also observed in the two-dimensional Navier–Stokes equations with power-law *stochastic* forcing [65, 66]. The covariance of the forcing in Fourier space is taken proportional to k^{2-y} with $y \geq 0$. If $y > 6$, the enstrophy flux is constant, and a direct cascade of enstrophy is observed. In the range $4 < y < 6$, there is scale-by-scale balance of the energy and enstrophy fluxes. In addition, the enstrophy flux increases with k , whereas the energy flux decays. For $0 \leq y < 4$, the system displays an inverse energy cascade. While some similarities with our results may exist for the two regimes with more regular forcing, a systematic comparison of the two systems may not be appropriate. Indeed, our forcing is deterministic whereas in [65, 66] the forcing is stochastic and the analysis explicitly uses its Gaussian and white-in-time properties. Moreover, the range $0 \leq y < 2$ is not covered by our study: for weak solutions to exist, the Fourier coefficients of the forcing must decay with k .

It is also interesting to note that the lower bound on the mean square velocity in (5.42) holds independently of the degree of regularity of the forcing. Therefore, if the forcing is constant in time and the Grashof number is defined in terms of the $\dot{H}^s$ -norm of the forcing as $Gr = \ell^3 \nu^{-2} F$, then the Doering and Foias relation between Gr and Re does not depend on s , and

$$Gr \leq c_1 Re^2 + c_2 Re \quad (5.74)$$

for all $s \geq -1$. If the forcing is time-dependent, it is appropriate to define Gr in terms of the root-mean-square of the magnitude of the forcing, i.e. $Gr = \ell^3 \nu^{-2} \bar{F}$. In this case,

$$Gr \leq (aSr + b_1) Re^2 + b_2 Re. \quad (5.75)$$

In conclusion, the main result of our study is that the scenario predicted in [63] and [64] only applies if the forcing is highly regular. By investigating the full range of forcing regularity (as allowed by the theory of weak solutions), we have shown that alternative scaling regimes are

possible and as the forcing regularity is varied, the Navier–Stokes equations undergo transitions between three different dynamical states.

EXTREME DYNAMICS AND RELAXATION OF QUANTUM GASES

The evolution of quantum gases, released from traps, is studied through hydrodynamics, both analytically and numerically, in one and two dimensions. In particular, we demonstrate the existence of long time self-similar solutions of the Euler equations, for the density and velocity fields, and derive the scaling exponents as well as the scaling functions. We find that the expanding gas develops a shock-like front and the size of the cloud grows in time as a power law. We relate the associated exponent to that appearing in the corresponding equation of state of the quantum gas. Furthermore, we study the relaxation dynamics of a trapped quantum gas and show that the resulting steady state is in excellent agreement with that derived analytically. Our hydrodynamic approach is versatile and can be used to unravel several other far-from-equilibrium collective phenomenon of extreme nature, relevant to the growing experimental interests in quantum gases.

This chapter closely follows the work [80]¹.

6.1 INTRODUCTION

The non-equilibrium dynamics of cold atomic gases has been a subject of great interest recently both experimentally and theoretically [158–167]. Fascinating experiments have been performed [168–172] to observe the free expansion and relaxation of trapped quantum gases and there has been significant progress in their theoretical understanding through the framework of hydrodynamics [75, 173–177] and generalized hydrodynamics [178]. While many of the observed experimental features can be explained by Euler hydrodynamics, the role of viscosity and its quantum aspects has been explored in experiments [179–181] and some features are now understood from theory [182–184]. However, the exact form of the viscous dissipation in the hydrodynamic equation is far from known.

¹This work is done in collaboration with Abhishek Dhar, Manas Kulkarni, and Samriddhi Sankar Ray.

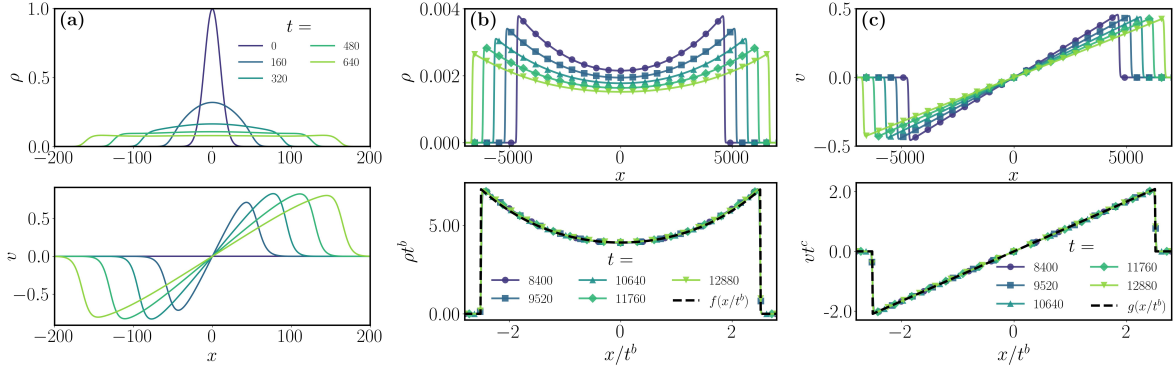


Figure 6.1. (a) Time evolution of the density and velocity profiles of a freely expanding 1D quantum gas with $\delta = 2/5$, shown at early times before the emergence of scaling behavior. (b) The upper panel shows the late-time evolution of the density profile. The lower panel displays rescaled density plots, $t^b \rho$ versus $\zeta = x/t^b$, exhibiting excellent data collapse and precise agreement with the exact scaling function (black dashed line) given by Eq. (6.12a). (c) The upper panel shows the velocity profile at late times, while the lower sub-panel presents the rescaled velocity profiles, which shows excellent data collapse and perfect agreement with the exact scaling function described by Eq. (6.12b).

In the context of dynamics in quantum gases, there has been immense activity in bosons [185], fermions [186] and various mixtures [187–189]. Such dynamics shines light on the consequences of the underlying interactions that lead to exotic collective phenomenon. There has been studies in understanding dynamics near an equilibrium background (such as sound waves [190–196]) and dynamics very far from equilibrium (for e.g., resulting in shock waves [75, 171, 197, 198] or energy dynamics [199]). However, in these systems dynamics of extreme nature is far less understood. One such specific problem is to study an evolution that mimics a blast that can arise from a sudden release of a substantial amount of particles or energy from a localized region. Some natural challenges while studying these problems include: (i) establishing rigorously the collective hydrodynamic description (ii) unavailability of analytical solutions, especially very far from equilibrium even if a phenomenological hydrodynamic theory is available.

At the level of Euler hydrodynamics, one of the most interesting results are the so-called self-similar scaling solutions [200], first discussed by Taylor, von Neumann and Sedov [201–204] in the context of atomic explosions. The main question there is to understand the time evolution of the hydrodynamic fields, namely density, momentum and energy, for an initial state where a huge amount of localized energy is released in an otherwise cold ambient gas — this is referred to as the blast problem. Following the classic work of TvNS, there has been much recent interest in the blast problem, in particular looking at direct comparisons of the Euler predictions with results from microscopic molecular dynamics simulations and understanding the effects of dissipation [205–212]. In a different context there has been immense activity devoted to the expansion of gaseous ellipsoids into a vacuum under the assumption that the gas velocity is proportional to radius. Flows satisfying this condition, known as homologous flows, preserve the geometric structure of the ellipsoid as it expands, and such and other self-similar solutions in the context of astrophysics were analyzed in several works [213–215].

So far, work on self-similar solutions has remained largely focused on classical systems. The

main goal of the present paper is to examine self similar solutions for blasts in quantum gases. The simplest set up is the release of a gas from a trap, the Euler equations would predict the formation of a sharp shock [216–218] front and a basic question is how this front evolves with time. In this work we show examples where this can be predicted precisely. Furthermore we also consider quantum gases confined in traps and study the approach to equilibrium using the framework of hydrodynamics. While the physics of blasts is rooted in the fascinating, collective dynamics of the constituent bosons or fermions, it is possible to capture many of the essential, experimentally measurable dynamics through a coarse-grained hydrodynamic description.

Here we consider the zero temperature case where it is sufficient to consider the equations for the density field, $\rho(\mathbf{x}, t)$, and velocity field, $\mathbf{v}(\mathbf{x}, t)$. The hydrodynamic Euler equations, in the presence of an external trapping potential, $U_{\text{trap}}(\mathbf{x})$, are given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (6.1a)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla w = -\nabla U_{\text{trap}}, \quad (6.1b)$$

where the specific enthalpy w is related to the pressure, P , via the relation $\partial_\rho P = \rho \partial_\rho w$. A knowledge of the zero-temperature equation of state of the quantum gas thus enables us to write the hydrodynamics equations. As is well known [216–218], for a broad class of initial conditions, Euler equations develop shock discontinuities which, in real systems, get regularized due to the presence of dissipation (Navier-Stokes terms). In our numerical study we use such a dissipative regularization scheme. Interestingly, a large number of quantum gases, both bosonic as well as fermionic, have a simple power-law form for the specific enthalpy

$$w(\rho) = D \rho^\delta. \quad (6.2)$$

As examples in one-dimensional ($d = 1$) systems, we have: $\delta = 1$ and $\delta = 2$ for the Lieb-Liniger gas of bosons at weak and strong coupling respectively [219]; $\delta = 2/5$ is the limiting case of a quasi-1D (cigar-shaped) unitary Fermi gas [75, 171, 220–222]; $\delta = 2/3$ for unitary Fermi gas in plane-wave limit. In dimensions $d > 1$ examples include: $\delta = 2/d$ for weakly interacting fermions at zero temperature. On the other hand, the unitary Fermi gas in d dimensions also shares the same dependence on δ . However, there is an overall renormalization in comparison to weakly interacting case [171, 197]. Remarkably, the unitary fermi gas, being a strongly correlated quantum fluid, shares deep resemblance to very different systems such as the quark–gluon plasma [159, 223–225] thereby enabling a hydrodynamic description. It is important to note that quantum statistics (bosonic or fermionic) and interactions is completely contained in the equation of state.

It is worth noting that, beyond cold gases, the well-known Riesz family of gases [226–230] provides an important and distinctly complementary example described by Eqs. (6.1, 6.2). This family comprises of particles with all-to-all interactions of power-law type characterized by an exponent, say s . The collective theory of the Riesz family of gases in a broad range of parameter space falls under the general structure of Eqs. (6.1a) and (6.1b). In particular, a class of these models are of finite-ranged type [228]. It was shown in Ref [228] that a finite-ranged version

of the Riesz gas in one dimension characterised by exponent s , would correspond to $\delta = s$ [Eq. (6.2)] for any $s > 0$. This immediately offers us a plethora of possibility of realization of the same collective dynamics for a wide window of δ .

Before proceeding further, it is important to emphasize that Eq. (6.1) along with Eq. (6.2) or its generalizations appears in several other contexts. One notable example is in the quantum hydrodynamics of several types of Bose Einstein Condensates [175, 177]. Self-similar solutions of Nonlinear solutions in Madelung form (containing dispersive terms) have been discussed [231].

The precise set-up where we study blasts in quantum gases is motivated by experiments in cold atoms using, e.g, time-of-flight (TOF) techniques, a standard tool in atomic physics for probing momentum and coherence in quantum gases. In TOF experiments, atoms are cooled in a (typically harmonic) trap, abruptly released, and allowed to expand for a set time, and then imaged via absorption imaging techniques [232–234]. In the present work we study, through Euler hydrodynamics (Eqs. 6.1), the dynamical evolution of such gases after their release from the trap. We show that, at long times, they evolve into self-similar forms and provide a complete analytic and numerical understanding. Secondly, we also consider the problem of a quantum gas in a trap ($U_{\text{trap}} \neq 0$) and study the relaxation dynamics to the steady state distribution. In this case, we add the Navier-Stokes corrections to Eqs. (6.1). For reasons explained later, we restrict ourselves to $0 < \delta < 2$ for the one dimensional case and $0 < \delta < 1$ for the two dimensional case.

Before proceeding further we briefly summarize our main findings:

1. For an expanding quantum gas initially held in a confining potential, we obtain self-similar solutions for both the density and velocity fields. These scaling laws exhibit striking robustness, as demonstrated by the collapse of spatial profiles obtained via direct numerical simulations onto universal curves.
2. We highlight the emergence and underlying physics of *shock-like fronts* — regions with very steep gradients in the hydrodynamic variables — that arise during such expansions. These structures correspond to inviscid shocks of the Euler equations, which appear as regularized shocks in numerical simulations due to the presence of dissipation. The growth of the shock front with time is shown to have a power-law form.
3. We derive analytical expressions for the associated scaling exponents and scaling functions and, crucially, establish their direct connection to the equation of state—and thus to the specific quantum gas under study. As a notable by-product, this provides an experimentally accessible method to infer the equation of state solely from the measured exponents.
4. We present a detailed, systematic analysis of the relaxation dynamics of trapped quantum gases toward their steady states. Where comparisons are possible, we find excellent agreement between our analytical predictions and high-resolution numerical simulations.

This paper is organized as follows. In Sec. 6.2, we discuss the one-dimensional gas without any external trap. Both evolution in zero (Sec. 6.2.1) and finite background (Sec. 6.2.2) are discussed. We then discuss the relaxation dynamics of one-dimensional gas in a trap in Sec. 6.3. The two dimensional case is discussed in Sec. 6.4. We end with conclusions along with an

outlook in Sec. 6.5. Details of direct numerical simulations are provided in Appendix. 6.A.

6.2 FREE EVOLUTION IN ONE-DIMENSIONAL CASE

We begin with the one-dimensional (1D) problem with the trap switched off and consider the Euler equations for the density and velocity fields:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0, \quad (6.3a)$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left(\frac{v^2}{2} + D\rho^\delta \right) = 0, \quad (6.3b)$$

We consider initial condition where we start with a density profile on top of a uniform density background ρ_B and zero velocity field. The cases with background density either zero and non-zero will be considered separately in the following sections. Let us assume that the density profile has a spatial spread over a characteristic length σ and a typical size ρ_0 . We can then transform to dimensionless variables

$$\rho/\rho_0 \rightarrow \rho, \quad x/\sigma \rightarrow x, \quad \frac{v}{(D\rho_0^\delta)^{1/2}} \rightarrow v, \quad \frac{t(D\rho_0^\delta)^{1/2}}{\sigma} \rightarrow t. \quad (6.4)$$

This corresponds to setting $D = 1$ in Eq. (6.3b) and we will henceforth assume this.

We will consider below the two case where an excess of particles is introduced in a localized region, either in an empty region (Sec. 6.2.1) or one with a finite density of particles (Sec. 6.2.2).

6.2.1 EVOLUTION IN A VACUUM

In this section, the initial condition is a spatially localized mass (with centre of mass at the origin) of a gas with $\rho_B = 0$ and net zero momentum. It is expected that the Euler equations should, at long times, develop a shock and the hydrodynamic fields have scaling forms. At long times the fields ρ, v will therefore take the forms

$$\rho(x, t) = t^{-b} f(\xi), \quad v(x, t) = t^{-c} g(\xi), \quad \text{where } \xi = \frac{x}{t^b}, \quad (6.5)$$

f and g are scaling functions and b and c are scaling exponents. Plugging the forms in Eq. (6.5) into Eq. (6.3b) and requiring that the resulting equations in $f(\xi)$ and $g(\xi)$ have no explicit time-dependence fixes the scaling exponents b and c as

$$b = \frac{2}{\delta + 2}, \quad c = \frac{\delta}{b + \delta}. \quad (6.6)$$

We then get the following ordinary differential equations (ODEs) for the scaling functions f and g :

$$\frac{d(fg)}{d\xi} - b\xi \frac{df}{d\xi} - bf = 0; \quad (6.7a)$$

$$\frac{df^\delta}{d\xi} + \frac{1}{2} \frac{dg^2}{d\xi} - b\xi \frac{dg}{d\xi} - cg = 0. \quad (6.7b)$$

Let us indicate the location of the shock in scaled variable as ξ_f . Therefore, it follows that the position of the shock is given by

$$R(t) = \xi_f t^b, \quad (6.8)$$

At the level of the inviscid Euler equations, the solutions described below correspond to weak solutions featuring genuine shock discontinuities. In practice, all numerical simulations necessarily involve a small regularization, implemented either through artificial viscosity or dissipative terms, to ensure numerical stability. As a result, the shock appears as a narrow but smooth front. In the limit of vanishing regularization, this front converges to the inviscid shock predicted analytically. Throughout the paper, we therefore refer to the numerically observed structures as shock-like fronts or regularized shocks, while reserving the term “shock” for the corresponding inviscid solution. Let us denote the shock velocity

$$U = \dot{R}, \quad (6.9)$$

and the fields (ρ_-, v_-) and $(\rho_+ = 0, v_+ = 0)$ just behind and in front of the shock, respectively. The Rankine–Hugoniot [210, 211] boundary conditions then give:

$$v_- = U, \quad \frac{v_-^2/2 + \rho_-^\delta}{v_-} = U, \quad (6.10)$$

and hence

$$f_- = \left(\frac{\xi_f^2 b^2}{2} \right)^{1/\delta}, \quad g_- = \xi_f b. \quad (6.11)$$

With these boundary conditions, we can solve Eq. (6.7) to get

$$f(\xi) = \left(\frac{\xi_f^2 (2 - \delta) + \delta \xi^2}{(\delta + 2)^2} \right)^{1/\delta}, \quad (6.12a)$$

$$g(\xi) = b \xi. \quad (6.12b)$$

The factor ξ_f is fixed by using the mass conservation equation

$$\int_0^{\xi_f} d\xi f(\xi) = M, \quad (6.13)$$

which, for $\delta < 2$, gives

$$\xi_f = \left(\frac{(2 - \delta)^{-1/\delta} (\delta + 2)^{2/\delta} M}{{}_2F_1\left(\frac{1}{2}, -\frac{1}{\delta}, \frac{3}{2}; \frac{\delta}{\delta - 2}\right)} \right)^{\frac{\delta}{\delta + 2}}, \quad (6.14)$$

where ${}_2F_1$ represents the Hypergeometric function. Equation. (6.14) shows that we can get an explicit form of shock location ξ_f for $\delta \leq 2$. One can immediately see that the case $\delta = 1$ gives $\xi_f = 3(M/4)^{1/3}$.

We now test the scaling solutions in Eqs. (6.12a) and (6.12b) through direct numerical solution of the hydrodynamics equations. The details of the pseudospectral numerical method employed

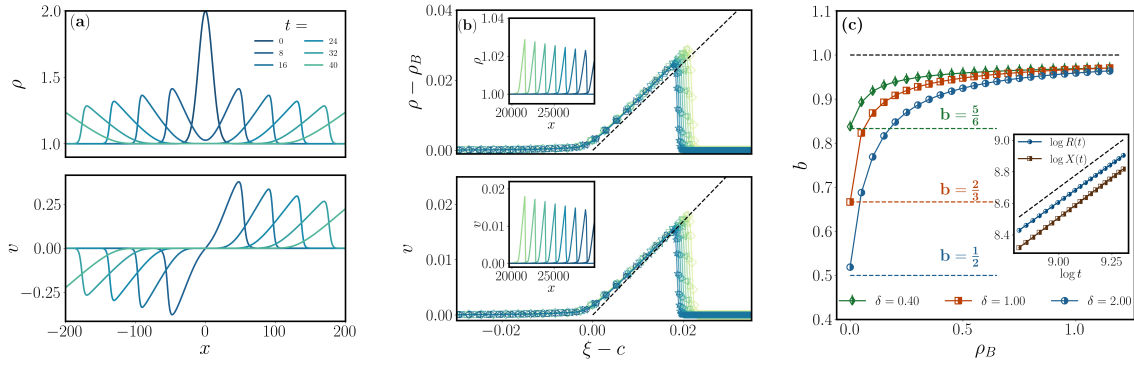


Figure 6.2. (a) Early time profiles for density and velocity for $\delta = 2/5$ and for background density $\rho_B = 1$ before the development of scaling structure. (b) Long time density and velocity profiles for $\delta = 2/5$ with the insets depicting the density profiles at different times. In the main panel the markers represent $\rho - \rho_B$ at different times with the space rescaled as $\xi = x/t$. The dashed line represents the analytical formula for the density given in Eq. (6.22). (c) The inset shows the shock location $R(t)$ (midpoint of the smoothed shock) and ramp location $X(t)$ as a function of time. Dashed line is a guide to the eye showing the ballistic behaviour. The main panel shows the scaling exponent as a function of background density ρ_B for the same excess density same. As expected $\rho_B \rightarrow 0$ the exponents goes to the vacuum exponents.

is provided in Appendix. 6.A. We chose extremely localised yet generic, initial conditions. For a system of length L ,

$$\rho(x, 0) = \frac{1}{2\sigma^2} \exp(-x^2/\sigma^2) \quad \text{with} \quad \sigma \ll L \quad (6.15a)$$

$$v(x, 0) = 0. \quad (6.15b)$$

In Fig. 6.1 we show results from direct numerical simulation for the time evolution of density and velocity for $\delta = 2/5$. In panel (a), we show relatively early time evolution and the subsequent emergence of the shock-like fronts. In panels (b) and (c) we show the late time evolution of density and velocity and their anticipated scaling and fit to the analytical predictions for the scaling functions given in Eqs. (6.12a) and (6.12b). It is interesting to note that the self-similar solution is symmetric about the origin and insensitive to the details of the initial condition. We have numerically verified that, for the class of asymmetric initial density profiles we have tested, starting from highly skewed initial conditions instead of the symmetric one in Eq. (6.15a), the system evolves toward the same long-time scaling solution. Representative examples of such asymmetric initial conditions and their subsequent evolution are shown in Appendix A (see Fig. 6.A.1), where we demonstrate convergence to the same scaling form.

6.2.2 EVOLUTION IN A FINITE DENSITY BACKGROUND

We now consider the case where we add mass to an existing finite background density ρ_B . The localized mass density will then propagate and simultaneously spread. At sufficiently large times, the excess density is a perturbation and one can then do a linearized theory, which would predict the propagation to be ballistic characterized by a speed of sound v_s . If the initial excess density is large then the initial propagation will be sub-ballistic. However, after a considerable

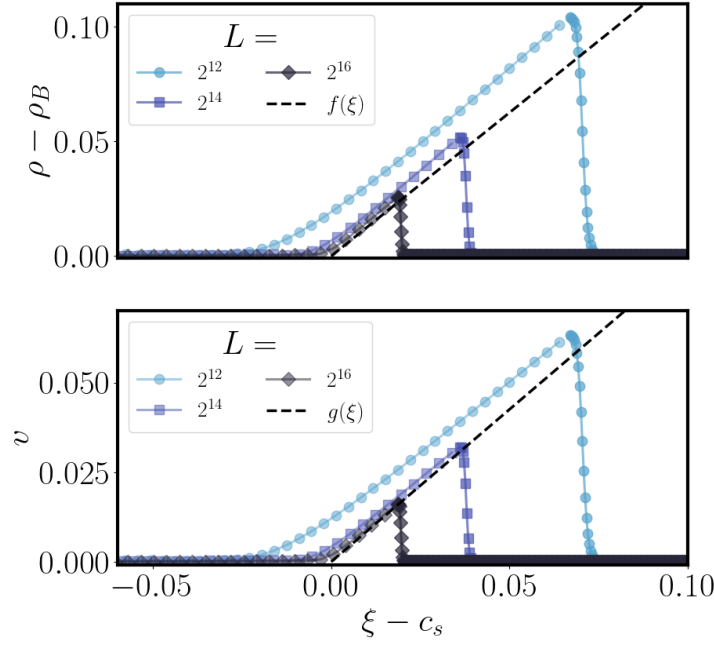


Figure 6.3. This plot shows the convergence of the rescaled profile to the analytical solution given in Eqs. (6.22a) and (6.22b) for $\delta = \frac{2}{5}$ and $\rho_B = 1$. As the system size increases, the numerical profile approaches the asymptotic limit and converges to the analytical expression.

time has passed, the excess density will become small compared to the background thereby validating a linearized theory. The linearized theory can be obtained from Eq. (6.3) as (setting $D = 1$)

$$\frac{\partial \Delta \rho}{\partial t} + \rho_B \frac{\partial \Delta v}{\partial x} = 0, \quad \frac{\partial \Delta v}{\partial t} + \delta \rho_B^{\delta-1} \frac{\partial}{\partial x} (\Delta \rho) = 0, \quad (6.16)$$

where $\Delta \rho$ and Δv denote small perturbations to justify linearization. Equation (6.16) immediately gives the sound speed [75, 222, 235]

$$c_s = \sqrt{\delta \rho_B^\delta}. \quad (6.17)$$

We now discuss the scaling solutions at long times. We expect ballistic behaviour at long times and hence we assume the form

$$\rho = \rho_B + f(x/t), \quad v = g(x/t), \quad (6.18)$$

where $f(x/t)$ and $g(x/t)$ are scaling functions that we seek to compute analytically. Plugging Eq. (6.18) into Eq. (6.3) yields

$$(g - \xi) \frac{df}{d\xi} + (f + \rho_B) \frac{dg}{d\xi} = 0; \quad (6.19a)$$

$$\delta(f + \rho_B)^{\delta-1} \frac{df}{d\xi} + (g - \xi) \frac{dg}{d\xi} = 0, \quad (6.19b)$$

where we recall that $\xi = x/t$. Equation (6.19) is a linear homogeneous equation in f' and g' and therefore a solution exists only if the determinant $(g - \xi)^2 - \delta(f + \rho_B)^\delta = 0$. This immediately gives us a relation between g and f

$$g = \xi \pm \sqrt{\delta} (f + \rho_B)^{\delta/2}. \quad (6.20)$$

Plugging Eq. (6.20) into Eq (6.19a) one gets

$$\pm(1 + \delta/2)\sqrt{\delta}(f + \rho_B)^{\delta/2}f' + (f + \rho_B) = 0. \quad (6.21)$$

This can be integrated to yield (the physical solution)

$$f(\xi) = \rho_B - \Phi \quad (6.22a)$$

$$g(\xi) = \xi - \sqrt{\delta}(2\rho_B - \Phi)^{\delta/2} \quad (6.22b)$$

where,

$$\Phi = \left(\frac{2(\delta + 1)\rho_B^{\delta/2} - \delta^{1/2}\xi}{\delta + 2} \right)^{2/\delta}. \quad (6.23)$$

Note that, as required, the following conditions are satisfied

$$f(\xi = c_s) = 0, \quad g(\xi = c_s) = 0. \quad (6.24)$$

In Fig. 6.2(a) we show the density (upper panel) and velocity (lower panel) profiles for different times with the background density $\rho_B = 1$ with $\delta = 2/5$. Clearly, both fields evolve ballistically. After some time, the shock front $R(t)$ moves with the sound speed, i.e., $R(t) = c_s t$ where we recall that c_s is given in Eq. (6.17). Given the symmetric nature of the evolution, we track the shock moving towards the right. In Fig. 6.2(b) we show the plot in scaled and shifted variables $(\xi - c_s)$ where $\xi = x/t$. In Fig. 6.2(c) we plot the scaling exponent b as a function of background density ρ_B . In the large system size limit, we observe exponent $b = 1$ independent of δ . Clearly, as $\rho_B \rightarrow 0$, we recover the freely-evolving gas limit where, as seen earlier $b = \frac{2}{2+\delta}$. In the inset of Fig. 2(c) we plot the position of shock and the ramp as a function of time. The thick black line, as a guide to the eye, confirms the ballistic motion of $R(t)$. We further analyze the shock front in Fig. 6.3. We demonstrate that as one increases the system size, the numerical profile approaches the asymptotic limit and converges to the analytical expression given in Eq. (6.22).

We next discuss some analytical estimates for the location of the shock and its scaling with time. For this, we revisit the expression of $f(\xi)$ in Eq. (6.22a). Note that for relatively small $\xi - c_s$, we have

$$f(\xi - c_s) = A(\xi - c_s) + O((\xi - c_s)^2), \quad (6.25)$$

where

$$A = \frac{2}{2 + \delta} \frac{\rho_B^{1-\delta/2}}{\sqrt{\delta}}. \quad (6.26)$$

Next we require by normalization,

$$t \int_{c_s}^{(c_s + x_{\max}/t)} f(\xi) d\xi = \frac{M}{2}, \quad (6.27)$$

where we recall that M is the excess mass and the factor of 2 is because we are considering one side of the profile. Simplifying Eq. (6.27) by using Eq. (6.25) gives

$$x_{\max} = \sqrt{\frac{M t}{A}}. \quad (6.28)$$

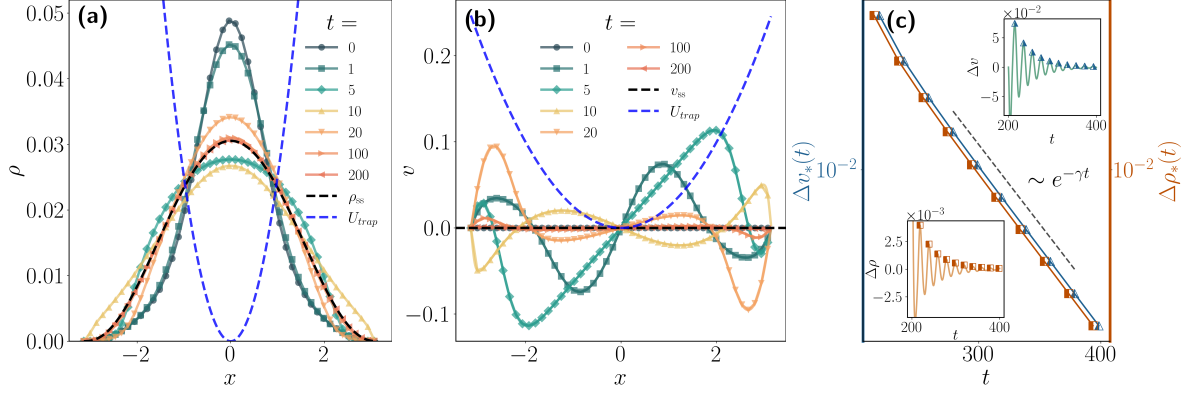


Figure 6.4. Spatial profiles, at different times, of the (a) density and (b) velocity fields of a gas confined in a trap (dashed blue line) with $\omega^2 = 0.1$ for the fully nonlinear set of equations in Eq. (6.30). (c) Time series of the perturbation in the density $\Delta\rho$ (lower inset) and velocity Δv (upper inset) at a fixed chosen location showing a decaying, oscillatory relaxation to the steady state. The successive peaks $\Delta\rho_*$ and Δv_* are indicated by red and blue markers. In the main panel we show a semi-log plot of $\Delta\rho_*$ and Δv_* suggesting exponentially damped solutions of the evolution equations of the perturbed field. The decay time scale $1/\gamma$ is in excellent agreement with the time-scale obtained from the linearized analysis (Sec. 6.3) as shown by the dashed line.

We next evaluate the height of the shock at $\xi = c_s + x_{\max}/t$. To do so, we plug the value of $x_{\max}/t$ in Eq. (6.25) to give

$$f(\xi - c_s = x_{\max}/t) = \sqrt{\frac{MA}{t}}. \quad (6.29)$$

We have numerically verified that $f(\xi - c_s = x_{\max}/t) \sim 1/\sqrt{t}$. In fact, our numerical results are also consistent with the precise coefficient $\sqrt{MA}$ given in Eq. (6.29).

6.3 RELAXATION OF A ONE-DIMENSIONAL TRAPPED GAS

So far, we have discussed cases when there is no external confinement. In this section, we discuss the case when we have a confining potential $U_{\text{trap}} \neq 0$. Starting from some nonequilibrium initial condition we study how the gas evolves and eventually reaches a steady state. The hydrodynamic equations are given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0, \quad (6.30a)$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left(\frac{v^2}{2} + \rho^\delta + \frac{\omega^2}{2} x^2 \right) = \nu \frac{\partial^2 v}{\partial x^2}, \quad (6.30b)$$

where we are using dimensionless variables as in Eq. (6.4) and ω and ν are also made dimensionless via

$$\frac{\omega \sigma}{(D\rho_0^\delta)^{1/2}} \rightarrow \omega, \quad \frac{\nu}{(D\rho_0^\delta)^{1/2}\sigma} \rightarrow \nu. \quad (6.31)$$

In contrast to the free expansion case, the presence of viscosity in Eq. (6.30b) prevents the formation of shock discontinuities; here dissipation plays a physical role and governs the

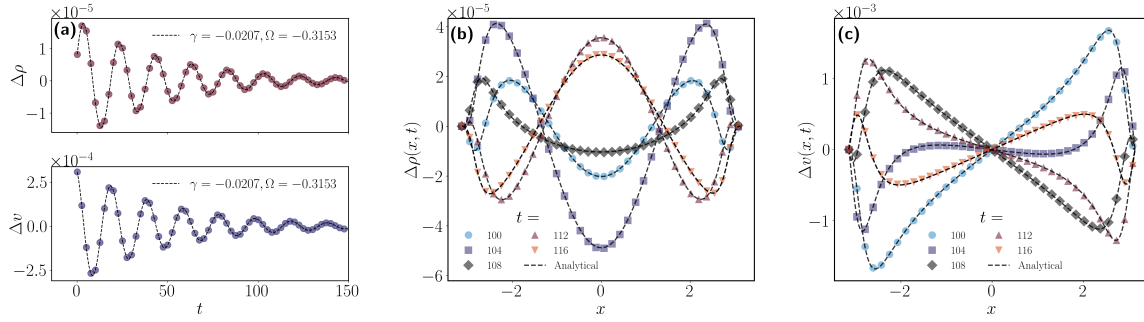


Figure 6.5. Upper and lower half of panel (a) shows the time series of $\Delta\rho$ and Δv respectively at a chosen spatial location. The markers shows the data obtained from direct numerical simulation of Eq. (6.35) and the dashed lines show the fit with a damped sinusoidal function of the form $C \sin(\Omega t + D)e^{-\gamma t}$. The legend shows the extracted fitting parameters namely γ, Ω and this agrees perfectly (up-to four decimal places) with the values obtained by using the boundary conditions given in Eq. (6.42). Panels (b) and (c) shows the perturbed density and velocity profiles for different time snapshots computed by solving the linearized Eq. (6.35) represented by markers. The dashed line shows the analytical expressions given in Eqs. (6.50) and (6.51) which perfectly agrees with the numerical simulations of linearised equations.

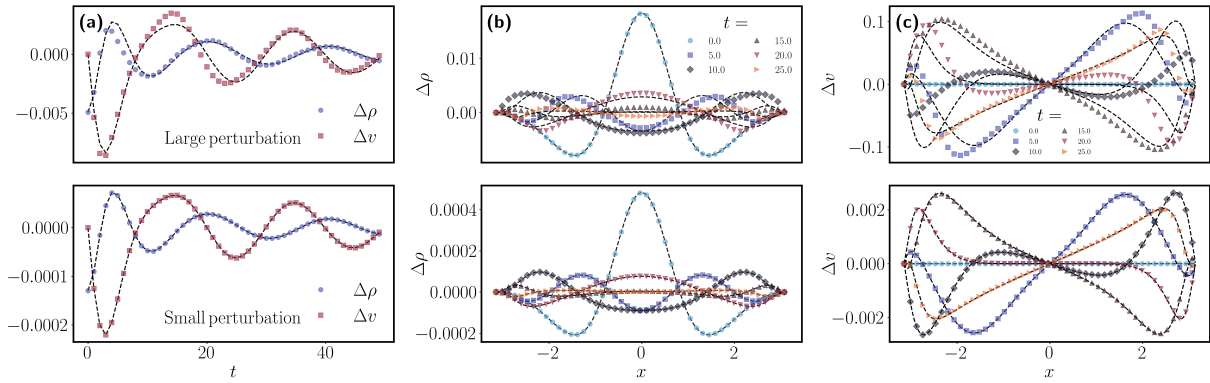


Figure 6.6. (a) Time series of the density $\Delta\rho$ and velocity perturbations Δv at a fixed spatial location for large (upper panel) and small (lower panel) initial perturbations. Markers represent the nonlinear evolution [Eq. (6.30)], while the black dashed line indicates the corresponding linear evolution [Eq. (6.35)]. Since Δv differs significantly in magnitude from $\Delta\rho$, it has been scaled by a constant factor for visualization on the same plot. Panels (b) and (c) show the comparison of the full nonlinear and linearized dynamics for the time evolution of $\Delta\rho$ and Δv for large (upper panel) and small (lower panel) perturbations. Markers denote the nonlinear evolution, while dashed lines represent the solutions of the linearized equations. Upper panels of (b) and (c) clearly demonstrates that for large perturbations, the linearized and nonlinear evolutions do not agree, whereas the lower panel of (b) and (c) show that for smaller perturbations the linear approximation is excellent agrees with the nonlinear evolution.

relaxation toward the steady state. In Eq. (6.30b), the trap we consider is of quadratic type. This is because, this is most ubiquitous in cold atom experimental setups [222, 236]. Our formalism can also be adapted and generalized for other trapping potentials such as of quartic type [237] and box potential [238–240]. Note that a more realistic form for the dissipative term in Eq. (6.31) would be $\partial_x(\rho\partial_x v)/\rho$. For simplicity, we replace it with the form $\nu\partial_x^2 v$ which is expected to be a good approximation for shallow traps and avoids numerical issues related to divergences associated with the $1/\rho$ factor.

It is easy to see that the steady state solutions of Eq. (6.30) takes the form

$$\rho_{ss} = \left(\frac{\omega^2 a^2}{2}\right)^{1/\delta} \left(1 - \frac{x^2}{a^2}\right)^{1/\delta}, \quad v_{ss} = 0 \quad (6.32)$$

where the constant a is fixed by the constraint

$$\int_{-a}^a dx \rho_{ss}(x) = M. \quad (6.33)$$

In Fig. 6.4, we show results from direct numerical simulations of Eq. (6.30) for $\delta = 2/5$. Since we are using a pseudospectral method with Fourier basis which is suited for periodic boundaries we smoothen the quadratic potential near the boundary and make it periodic to avoid numerical errors (see Appendix. 6.A). In other words, without the smoothening, $\frac{\partial U_{trap}}{\partial x}$ is discontinuous at the boundary making the pseudospectral method ill-suited. We show in Fig. 6.4(a) density and (b) velocity profiles at several different times. Clearly at late time both the density and velocity fields converge to analytical expressions given in Eq. (6.32). Figure 6.4(c) displays an exponential decay in ρ and v at a fixed spatial location. We analytically derive the corresponding decay timescale later in this section.

As can be noticed in Fig. 6.4 (a) and (b), both the density and velocity profiles show damped oscillatory relaxation to the steady state. We now provide an in-depth understanding of this transient behaviour and, in particular, compute the associated damping rate and frequency. A full solution of the time-dependent hydrodynamic equations is challenging. However, for initial conditions that are close to the steady state one can consider a linearized dynamics. As we shall now show, such equations are amenable to analytic treatment and in fact also provide insights into the late time dynamics of even far-from equilibrium initial conditions.

To proceed further, we make the following substitution into Eq. (6.30)

$$\rho = \rho_{ss} + \Delta\rho, \quad v = \Delta v, \quad (6.34)$$

where recall that ρ_{ss} is given in Eq. (6.32). Retaining terms linear in the perturbation we obtain

$$\frac{\partial \Delta\rho}{\partial t} + \frac{\partial}{\partial x} \rho_{ss} \Delta v = 0, \quad (6.35a)$$

$$\frac{\partial \Delta v}{\partial t} + \delta \frac{\partial}{\partial x} \rho_{ss}^{\delta-1} \Delta\rho = \nu \frac{\partial^2 \Delta v}{\partial x^2}. \quad (6.35b)$$

Note that Eq. (6.35) is linear in $\Delta\rho$ and Δv but with space dependent coefficients making them still challenging to solve. Nevertheless we can in fact obtain the spectrum completely. Let us look for solutions of Eq. (6.35) of the form

$$(\Delta\rho(x, t), \Delta v(x, t)) = (\phi(x), \psi(x)) e^{-\lambda t}. \quad (6.36)$$

Plugging this into Eq. (6.35) gives us the following equations for ψ and ϕ :

$$\nu \frac{d^2\psi}{dx^2} - \frac{\delta}{\lambda} \frac{d}{dx} \rho_{ss}^{\delta-1} \frac{d(\rho_{ss}\psi)}{dx} + \lambda\psi = 0, \quad (6.37)$$

$$\phi(x) = \frac{1}{\lambda} \frac{d(\rho_{ss}\psi)}{dx}. \quad (6.38)$$

Plugging in the explicit form of ρ_{ss} in Eq. (6.32) into Eq (6.37) leads to the form:

$$(c_1 + c_2 x^2) \frac{d^2\psi}{dx^2} + c_3 x \frac{d\psi}{dx} + c_4 \psi = 0, \quad (6.39)$$

where,

$$c_1 = -\nu + \frac{\omega^2 a^2 \delta}{\lambda}, \quad c_2 = -\frac{\omega^2 \delta}{2\lambda}, \quad (6.40)$$

$$c_3 = -\frac{\omega^2}{\lambda} (\delta + 1), \quad c_4 = -\left(\lambda + \frac{\omega^2}{\lambda}\right). \quad (6.41)$$

These equations have to be solved with the boundary conditions

$$\psi(a) = \psi(-a) = 0. \quad (6.42)$$

This will give us to a set of eigenfunctions $\{\psi_n, \phi_n\}$ and corresponding allowed eigenvalues $\{\lambda_n\}$ for our linear equations. From the structure of the eigenvalue equations [Eq. (6.38)], it is clear that the eigenvalues and eigenfunctions come in complex conjugate pairs. We now note that Eq. (6.39) is in the form of the Legendre equation whose general solution can be expressed in terms of the associated Legendre functions of the first and second kinds:

$$\psi(x) = R(x) [A_1 P(a_1, a_2, a_3 x) + A_2 Q(a_1, a_2, a_3 x)], \quad (6.43)$$

where

$$R(x) = (c_1 + c_2 x^2)^{-\frac{2c_2+c_3}{4c_2}} \quad (6.44)$$

$$a_1 = \frac{\sqrt{c_2^2 - 2c_2c_3 - 4c_2c_4 + c_3^2 - c_2}}{2c_2} \quad (6.45)$$

$$a_2 = -\frac{2c_2 + c_3}{2c_2}, \quad a_3 = i\sqrt{\frac{c_2}{c_1}}, \quad (6.46)$$

and A_1, A_2 are arbitrary constants. Without loss of generality we can set $A_1 = 1$ since this can be fixed by normalization. We then have two complex unknown parameters, namely A_2 and λ , These are determined by imposing the two boundary conditions $\psi_n(\pm a) = 0$. We solve for these unknown parameters numerically using Mathematica.

The general solution of Eq. (6.35) is given by

$$(\Delta\rho(x, t), \Delta v(x, t)) = \sum_{n=0}^{\infty} p_n [\phi_n(x), \psi_n(x)] e^{-\lambda_n t}, \quad (6.47)$$

where the constants p_n are determined from the initial conditions. At large times, the eigenvalue with the smallest value for the real part will dominate. There are two such eigenvalues which we denote as

$$\lambda_m = \gamma \pm i\Omega, \quad (6.48)$$

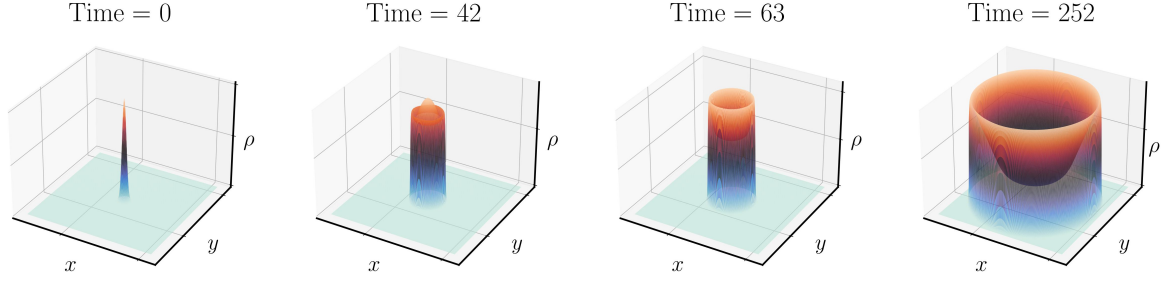


Figure 6.7. The time evolution of the two-dimensional density for $\delta = \frac{2}{5}$ starting from a localized density profile and zero momentum, we clearly observe the emergence of a shock-like structure and a scaling form. Animation of the time evolution is available in Ref [241].

where $\gamma > 0$ and Ω are real. These provide us with the time scale for decay and the frequency of the oscillations, namely γ^{-1} and Ω respectively. The expected forms for the density and velocity fields at late times can also be determined. Writing the dominant eigenfunctions [Eq. 6.47] in the form

$$\phi_m(x) = F_1(x)e^{\pm i\alpha_1(x)}, \quad \psi_m(x) = F_2(x)e^{\pm i\alpha_2(x)}, \quad (6.49)$$

we have the following late time forms:

$$\Delta\rho(x, t) = C_1 F_1(x) \sin(\alpha_1(x) - \beta_1 - \Omega t) e^{-\gamma t}, \quad (6.50)$$

$$\Delta v(x, t) = C_2 F_2(x) \sin(\alpha_2(x) - \beta_2 - \Omega t) e^{-\gamma t}, \quad (6.51)$$

where $C_1, C_2, \beta_1, \beta_2$ are constants that are fixed by the initial conditions. In the numerics for the nonlinear case these constants can be fixed by using the value of $\Delta\rho$ and Δv at some late time snapshot because in the early time there might be contribution from other eigen-functions which may lead to an incorrect estimation of the constants. In Fig. (6.5), we compare the analytical predictions [Eqs. (6.50), (6.51)] with direct numerical solutions of the linearized equations [Eq. (6.35)] and find excellent agreement. To further validate the calculations with the linearized equations, in Fig. 6.6 we explicitly compare numerical simulations of the full nonlinear system [Eq. (6.30)] with those of the perturbation equations [Eq. (6.35)]. For small perturbations, the linearized equations accurately approximate the full nonlinear dynamics. However, as the perturbation strength increases, deviations become apparent, indicating the breakdown of the linear approximation.

6.4 TWO-DIMENSIONAL CASE

In this section, we discuss the evolution, in two dimensions, of an initially localized mass of a quantum gas with zero total momentum. We only consider expansion into a vacuum. The initial mass distribution is assumed to have radial symmetry and so we look for solutions of the hydrodynamic equations which have this symmetry. In particular, we need to only consider the density field ρ , and the radial component of the velocity vector field v . Clearly, the space-time dependence of the fields will have the form $\rho = \rho(r, t), v = v(r, t)$ and the corresponding Euler

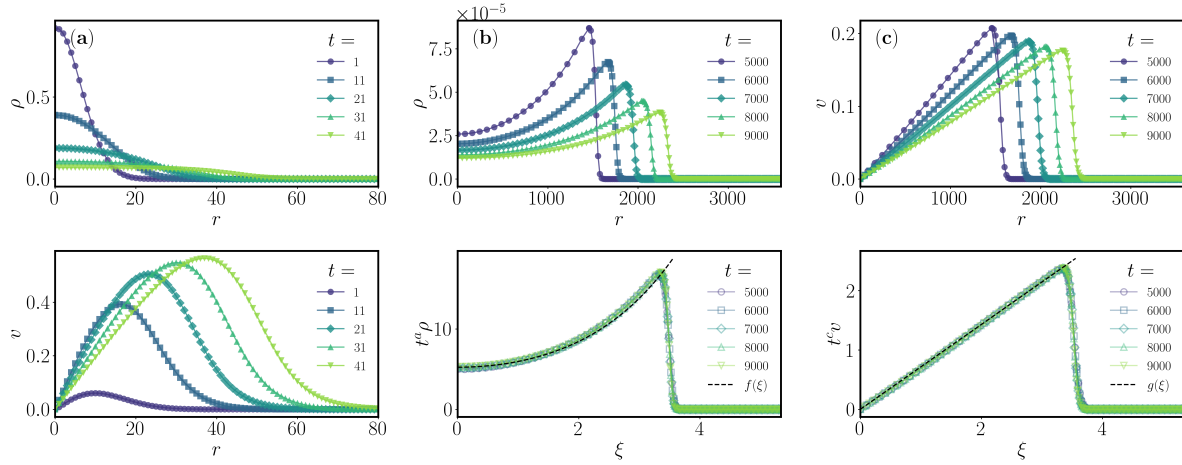


Figure 6.8. (a) The early time evolution of the radial density and the velocity before developing the scaling form as described in Eq. (6.52) for $d = 2$ and $\delta = 2/5$. Upper panel of (b) and (c) show the radial density and velocity profiles for late times after development of the scaling form whereas the lower half of panel (b) and (c) show the rescaled density and velocity profiles and the analytically computed scaling functions f and g described in Eq. (6.69). Note that the shock is smoothened; hence the location of the shock front used in Eq. (6.69) is the mean of this smoothened shock front which leads to a difference of 1% from the analytical obtained shock position in Eq. (6.71).

equations in radial coordinates are

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + \frac{d-1}{r} \rho v_r = 0 \quad (6.52a)$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial r} \left(\frac{v^2}{2} + \rho^\delta \right) = 0. \quad (6.52b)$$

As in Sec. 6.2.1 we expect that asymptotically the dynamics develops a scaling form as

$$\rho(r, t) = t^{-2b} f(\xi), \quad v(r, t) = t^{-c} g(\xi), \quad \text{with } \xi = \frac{r}{t^b}, \quad (6.53)$$

where the factor, $2b$, for the density scaling appears since $\int_0^\infty dr r \rho(r, t)$ has to be conserved. Plugging Eq. (6.53) into Eq. (6.52) and requiring that any explicit time dependence goes away, we get

$$b = \frac{1}{1+\delta}, \quad c = \delta b \quad (6.54)$$

and the corresponding ordinary differential equations for the scaling functions

$$(g - b\xi) \frac{df}{d\xi} + f \frac{dg}{d\xi} + \frac{fg}{\xi} - 2bf = 0; \quad (6.55a)$$

$$\frac{df^\delta}{d\xi} + (g - b\xi) \frac{dg}{d\xi} - cg = 0. \quad (6.55b)$$

We now discuss the Rankine-Hugoniot conditions which turn out to be identical to the 1D case given by Eqs. (6.10) or (6.11). To derive the first Rankine-Hugoniot condition, we integrate the first conserved quantity, $\int r \rho dr$, across the shock at position R , i.e., over the interval $[R - \Delta, R + \Delta]$ and then take the limit as $\Delta \rightarrow 0$. This yields

$$\frac{d}{dt} \int_{R-\Delta}^{R+\Delta} r \rho dr = 0 \quad (6.56)$$

which implies

$$(\rho_- - \rho_+)RU - (\rho_-v_- - \rho_+v_+)R = 0, \quad (6.57)$$

where ρ_- and v_- denote the radial density and velocity before the shock, and ρ_+ and v_+ denote the density and velocity after the shock. Noting that $\rho_+ = v_+ = 0$ gives the first Rankine–Hugoniot condition

$$v_- = U. \quad (6.58)$$

To derive the second Rankine–Hugoniot condition, we use the second conserved quantity, $\int rv \, dr$, and integrate it around the shock

$$\frac{d}{dt} \int_{R-\Delta}^{R+\Delta} rv \, dr = 0. \quad (6.59)$$

which implies

$$(r_-v_- - r_+v_+)U + \int_{R-\Delta}^{R+\Delta} r \partial_t v \, dr = 0. \quad (6.60)$$

We now simplify the second term

$$\begin{aligned} \int_{R-\Delta}^{R+\Delta} r \partial_t v \, dx &= - \int_{R-\Delta}^{R+\Delta} r \partial_r \left(\frac{1}{2} v^2 + \rho^\delta \right) dr \\ &= \int_{R-\Delta}^{R+\Delta} \left[\left(\frac{1}{2} v^2 + \rho^\delta \right) \right. \\ &\quad \left. - \partial_r \left(\frac{1}{2} r v^2 + r \rho^\delta \right) \right] dr. \end{aligned} \quad (6.61)$$

Plugging Eq. (6.61) back in Eq. (6.60) and taking the limit $\Delta \rightarrow 0$ gives us the second Rankine–Hugoniot condition

$$v_- U = \frac{1}{2} v_-^2 + \rho_-^\delta. \quad (6.62)$$

By using Eq. (6.58) in Eq. (6.62), we get

$$\rho_-^\delta = \frac{1}{2} U^2. \quad (6.63)$$

In terms of scaled variables given in Eq. (6.53), we get

$$g_- = b \zeta_f, \quad f_- = \left(\frac{1}{2} b^2 \zeta_f^2 \right)^{1/\delta}, \quad (6.64)$$

which takes into account that the shock position is given by $R = \zeta_f t^b$ and shock velocity $U = \dot{R}$.

We now want to find the solutions to Eq. (6.54). We take an *Ansatz* $g(\zeta) = k \zeta$ and substitute it in Eq. (6.55a) yielding

$$(-b \zeta + k \zeta) \frac{df}{d\zeta} + kf - af + kf = 0. \quad (6.65)$$

To ensure a non-zero solution for Eq. (6.65), we need $k = b$. So, $g(\xi) = b \xi$. We verify via direct numerical simulations that this is indeed the case. Now we substitute $g(\xi) = b \xi$ in Eq. (6.55b) and get

$$\delta f^{\delta-1} \frac{df}{d\xi} - c b \xi = 0, \quad (6.66)$$

which simplifies to

$$\frac{df^\delta}{d\xi} = c b \xi. \quad (6.67)$$

Integrating Eq. (6.67) from ξ_f to ξ we get,

$$f^\delta(\xi) = \left(f^\delta(\xi_f) - \frac{c b}{2} (\xi_f^2 - \xi^2) \right)^{1/\delta}. \quad (6.68)$$

Finally, the analytical solution for f and g can be summarized as

$$f^\delta(\xi) = \left(f^\delta(\xi_f) - \frac{c b}{2} (\xi_f^2 - \xi^2) \right)^{1/\delta}, \quad g = b \xi \quad (6.69)$$

where we recall that b is given in Eq. (6.54) and $f(\xi_f)$ can be obtained from the Rankine-Hugoniot condition [Eq. (6.64)]. What remains is to find ξ_f from mass conservation

$$\int_0^{\xi_f} d\xi \xi f(\xi) = M, \quad (6.70)$$

which yields (for $\delta < 1$)

$$\xi_f = \left(\frac{\left(1 - (1 - \delta)^{\frac{1}{\delta}+1} \right) (\delta + 1)^{-\frac{\delta+2}{\delta}}}{2^{\frac{\delta+1}{\delta}} M} \right)^{-\frac{\delta}{2\delta+2}}. \quad (6.71)$$

Next we test our analytical prediction via direct numerical simulations. First we solve the full two-dimensional equations i.e. Eqs (6.1a) and (6.1b) for $\delta = 2/5$ and $U_{trap} = 0$ starting from a localised density profile and zero velocity. Figure. 6.7 shows the density evolution for early and late times demonstrating blast-like phenomenon and the emergence of a scaling form. In Fig. 6.8, we show the numerical results obtained from solving Eq. (6.52) numerically for $d = 2$ and $\delta = 2/5$. (a) Figure 6.8(a) shows the early time behavior and upper half of panels (b) and (c) show the radial density and velocity for relatively late times. In the lower panels pf Fig. 6.8 (b) and (c) we show excellent agreement between rescaled numerical profiles and analytical scaling functions given in Eq. (6.69).

6.5 CONCLUSIONS AND OUTLOOK

To summarize, we have investigated quantum dynamics in highly out-of-equilibrium situations, both in one and two dimension, via hydrodynamics and using both analytical calculations and direct numerical simulations. We demonstrated that at long times, the hydrodynamic fields

evolve to self-similar forms, that are independent of the details of the localized initial conditions. We derived analytically the scaling exponents and scaling functions and thus characterized completely the features of this highly nonequilibrium process. In particular, for a gas with the equation of state $P \sim \rho^{\delta+1}$ and evolving in a vacuum, the front grows with time as $R(t) \sim t^b$, where $b = 2/(2 + \delta)$ and $1/(1 + \delta)$ in one and two dimensions respectively. We also investigated the relaxation dynamics of a confined quantum gas. At late times we see a damped oscillatory relaxation, where the damping coefficient and frequency can be computed analytically from a linearized theory. We validated our analytical findings by extensive numerical simulations and found excellent agreement between the two.

Our work targets a distinct and largely unexplored regime: the extreme dynamics of expanding or very strongly confined quantum gases and the existence of exact self-similar solutions. This focus has led to several genuinely new results. First, we provide an explicit analytical derivation of the scaling exponents and scaling functions governing the expansion—including the formation of shock-like fronts—and establish their direct link to the equation of state and vice versa. Second, while numerical simulations in one and two dimensions are well established, the level of quantitative agreement we demonstrate between analytical predictions and high-resolution simulations has not previously been achieved. Moreover, the rapid expansions and shock-driven phenomena at the heart of our study stand in sharp contrast to the field’s traditional emphasis on equilibrium or close-to-equilibrium, which has left such far-from-equilibrium behavior largely uncharted. Consequently, our work establishes the robustness of the hydrodynamic framework for these experimentally relevant problems and opens the door to systematic exploration of even more complex nonequilibrium dynamics in quantum gases.

An important and interesting question concerns the extension of our results to finite, but low, temperatures. A proper treatment would require incorporating energy conservation and an additional hydrodynamic equation for the temperature field, leading to a coupled set of equations for density, velocity, and energy. While temperature-dependent corrections to the equation of state are expected to modify the scaling behavior, a systematic analysis of these effects lies beyond the scope of the present work and is left for future study. A second point is the choice of initial conditions. Based on the numerical simulations performed, we find that in one dimension the long-time scaling solution is robust with respect to the symmetry of the initial condition for the class of initial profiles we have examined. In two dimensions, more general initial conditions could lead to solutions with different symmetries, for example, elliptic shock profiles, that have been discussed earlier [214, 215] and it will be interesting to investigate in the quantum context.

Our methodology is highly adaptive and expected to be of relevance in understanding far from equilibrium dynamics in a variety of interacting quantum systems. It will be interesting to explore quantum gases in higher spatial dimensions, relaxing the assumptions of symmetry [242–244], which would make the effective one dimensional description ill-suited. It would also be important and interesting to extend our techniques to open quantum systems such as dissipative quantum fluids [245, 246]. Beyond cold quantum gases, our collective formalism and subsequent analysis of self-similar solutions is also relevant in several other contexts

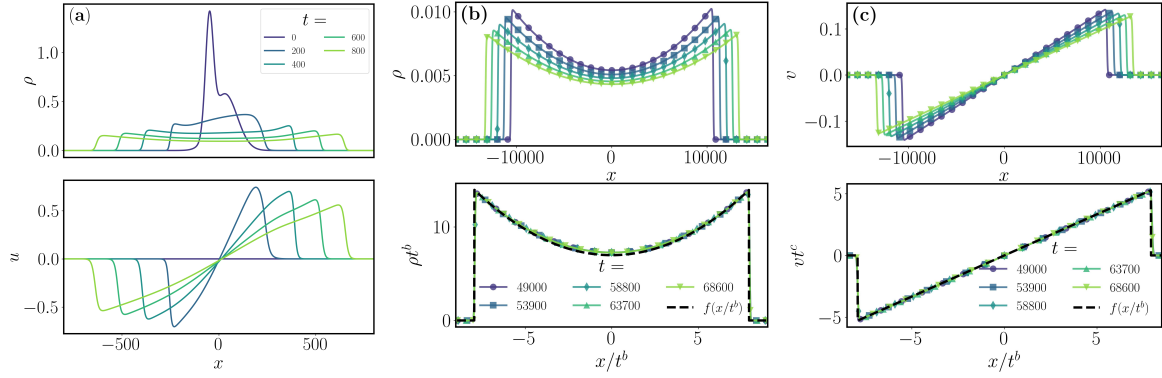


Figure 6.A.1. (a) Early-time evolution of density and velocity profiles, showing the effects of initial asymmetry reducing gradually. (b) The upper panel shows the late-time evolution of the density profile. The lower panel displays rescaled density plots, $t^b \rho$ versus $\xi = x/t^b$, exhibiting precise agreement with the exact scaling function (black dashed line) given by Eq. (6.12a). (c) The upper panel shows the velocity profile at late times, while the lower sub-panel presents the rescaled velocity profiles, which show data collapse and perfect agreement with the exact scaling function described by Eq. (6.12b). This provides an example where the initial asymmetry tends to vanish over time, and the solution asymptotically converges to the scaling form given in Eqs. (6.12a) and (6.12b).

such as finite-ranged Riesz gas [228] and hydrodynamics of a plasma of strongly interacting quarks and gluons produced in relativistic heavy ion collisions [159, 223, 224, 247]. As noted earlier, our protocol is well-suited to TOF measurements, where atoms are cooled, released, and imaged via absorption [232–234]. Testing these predictions appears to be within reach of current experimental capabilities, and the potential observation of self-similar scaling solutions warrants further investigation.

6.A DIRECT NUMERICAL SIMULATIONS AND ROBUSTNESS OF INITIAL CONDITIONS

We perform direct numerical simulations using a fully-dealiased pseudospectral method with periodic boundary conditions in both one and two dimensions. Spatial discretization is carried out using a Fourier basis, and time integration is performed via either the classical fourth-order Runge–Kutta (RK4) method or the integrating factor RK4 (IFRK4) scheme, depending on the stiffness of the equations [248].

To simulate the inviscid equations, we introduce a small artificial viscosity in both the density and velocity equations to regularize sharp gradients and suppress numerical instabilities. More precisely, in the simulations we replace the Euler equations in Eqs. (6.1a)-(6.1b) and work with the following dissipative equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \eta \nabla^2 \rho, \quad (6.72a)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla w + \nabla U_{\text{trap}} = \nu \nabla^2 \mathbf{v}. \quad (6.72b)$$

For the range of values that we consider in this paper, i.e., $0 < \delta < 2$ in 1D and $0 < \delta < 1$ in 2D, one can easily check that, for the case of free evolution, the dissipation terms become irrelevant in the long time scaling regime. The viscosity is chosen to ensure numerical stability while preserving the key features of the inviscid dynamics. Convergence is verified by simultaneously increasing the resolution and decreasing the viscosity [249]; in the asymptotic regime, we confirm that the features of the solution remain unaffected for the case of free evolution. Note however, that for the 1D trapped gas discussed in Sec. 6.3, the viscosity plays a significant role in the long time relaxation to the steady state.

For one-dimensional simulations, we use $2^{12} \leq N \leq 2^{16}$ collocation points. In two dimensions, we use $N \times N$ grids with $2^9 \leq N \leq 2^{11}$. The time step, dt , is chosen adaptively based on the system size, subject to the Courant–Friedrichs–Lewy (CFL) stability condition.

For freely evolving systems (i.e., without background density), we set the system size to $L = N$ ($dx = 1$), and find convergence in the scaling regime for $L \sim 2^{14}$. For cases with a finite background density, convergence is slower, and we obtain reliable results for $L \sim 2^{16}$.

In Fig. 6.A.1, we examine the influence of initial conditions on the scaling function. We analyzed localized initial conditions that are skewed but have their center of mass at the origin. We observe that while the system initially retains this asymmetry, it tends to lose memory of the initial condition over time, eventually converging to the asymptotic scaling solution. To quantify this relaxation, we compared symmetric and asymmetric simulations using the deviation metric $\sqrt{\langle (\rho_{\text{sym}} - \rho_{\text{asym}})^2 \rangle}$, where ρ_{sym} and ρ_{asym} correspond to the solutions for symmetric and asymmetric initial conditions, respectively and $\langle \rangle$ mean an integral over space. We found that the deviation decreases from an initial value of 0.08 at $t = 0$ to 0.0002 at $t = 70000$, corresponding to a reduction to 0.25% of the initial value. Furthermore, we verified these results across various system sizes, confirming that the deviation metric decreases consistently as the system size increases.

In the presence of a harmonic trapping potential, we solve Eq. (6.30) without adding artificial viscosity to the continuity equation. While the Fourier pseudospectral method remains applicable, the non-periodic nature of the potential is incompatible with strict periodic boundary conditions. To suppress boundary artifacts, we apply a mild smoothing of the potential near the edges of the domain. All simulations are performed with a fixed computational domain of length $L = 2\pi$. This issue does not arise in the linearized perturbation equations [Eq. (6.35)], where the quadratic trap does not appear explicitly and its effects are encoded through the steady-state background.

To solve the two-dimensional equations in polar coordinates [Eq. (6.52)], we impose non-periodic boundary conditions appropriate for the radial geometry. Specifically, we set $v = 0$ and $\partial_r \rho = 0$ at $r = 0$, following Ref [250]. In contrast to the Cartesian case, periodic boundary conditions hence pseudospectral methods are no longer applicable. We used a second-order finite difference scheme to approximate the spatial derivatives, while time integration was carried out using the classical fourth-order Runge–Kutta (RK4) method.

Most of the codes used in the numerical results reported here can be found in Ref [241].

HYDRODYNAMICS OF A HARD-CORE ACTIVE LATTICE GAS

In this chapter we present a fluctuating hydrodynamic description of an active lattice gas model with excluded volume interactions that exhibits motility-induced phase separation under appropriate conditions. For quasi-one dimension and higher, stability analysis of the noiseless hydrodynamics gives quantitative bounds on the phase boundary of the motility-induced phase separation in terms of spinodal and binodal. Inclusion of the multiplicative noise in the fluctuating hydrodynamics describes the exponentially decaying two-point correlations in the stationary-state homogeneous phase. Our hydrodynamic description and theoretical predictions based on it are in excellent agreement with our Monte Carlo simulations and pseudospectral iteration of the hydrodynamics equations. Our construction of hydrodynamics for this model is not suitable in strictly one-dimension with single-file constraints, and we argue that this breakdown is associated with micro-phase separation.

This chapter closely follows the work [81]¹.

7.1 INTRODUCTION

Active matter commonly refers to nonequilibrium systems with energy injections at the microscopic scale that independently drives individual constituents and often leads to large scale self-organized structures [251–254]. Although they are inspired by living matter, active systems have been realized in synthetic objects such as Janus particles [255], vibrated rods [256], driven granular systems [257], motor driven robots [258, 259], and even in quantum matter [260]. The prominence of the field comes from the novel emergent collective behaviors that often comes as a surprise from an equilibrium viewpoint. Beside their potential applicability [261, 262], these emergent many-body structures pose an intriguing challenge for nonequilibrium statistical

¹This work is done in collaboration with Soumyabrata Saha, Abhishek Dhar, Tridib Sadhu, and Sanjib Sabhapandit; Soumyabrata is a graduate student at TIFR.

mechanics. In recent decades, major efforts have been devoted to developing a theoretical understanding based on effective field theories [263] and large scale computer simulations [264–266]. However, even for simple characteristic many-body phases of active matter, namely the Motility-Induced Phase Separation (MIPS) [267, 268], there is no clear bottom-up theoretical understanding, analogous to equilibrium statistical mechanics for liquid-gas coexistence, that could relate the large scale phases to the underlying microscopic dynamics.

A promising bottom-up approach for many-body dynamics is in terms of fluctuating hydrodynamics, that has been phenomenally successful in capturing nonequilibrium fluctuations in diffusive transport models [269], Hamiltonian dynamical systems [270, 271], and even in quantum integrable models [272–274]. Hydrodynamics description has also proven tremendously effective in active matter [275–279], although a rigorous derivation relating to microscopic dynamics is available only for a handful of cases. One such example is an active lattice gas [280, 281], whose hydrodynamics was proven in mathematical terms and its fluctuation extension incorporating a stochasticity has been recently obtained [282, 283] using intriguing mapping to a well-known ABC exclusion model. However, the model involves an exchange dynamics between particles that seems unphysical, presumably incorporated to sustain MIPS. Most well-known off-lattice active dynamics [267, 284–286] are for impassable particles with finite volume hard core which provides the much-needed activity-induced caging effect to sustain MIPS. In fact, volume exclusion is expected in natural examples of living or synthetic active matter.

In this article, we propose a lattice model, that is inspired by [280], but without the exchange dynamics. The dynamics of our model is closer to realistic off-lattice dynamics of active matter that are known to exhibit MIPS [267]. For our model in the quasi-one-dimension of a periodic two-lane ladder lattice, there is a stable MIPS state at certain range of high activity and density. For this model, we derive the fluctuating hydrodynamics using a field-theoretical approach that is generalizable and independent of the previous method in [282, 283]. Within the hydrodynamics we show how dynamical instabilities of the coupled equations of hydrodynamic fields quantitatively bound the MIPS boundaries. Moreover, we find that correlations from the fluctuating hydrodynamics have excellent agreement with their Monte Carlo results and capture the qualitative features seen [287] in well-known off-lattice active matter models. Our construction of the hydrodynamics straightforwardly extends for the higher dimensional generalization of the model and in particular, they capture the existence of MIPS in two dimensions.

The rest of the article is organized as follows. In Section 7.2 we precisely define the model in the quasi-one-dimensional (quasi-1D) ladder lattice and present its fluctuating hydrodynamic description in Section 7.3. The spinodal and binodal bounds on the MIPS phase boundary obtained using stability analysis of the noiseless hydrodynamics is discussed in Section 7.4. In Section 7.5 we compute the spatial correlations of density and magnetization by incorporating the noise into the hydrodynamic description. Further generalizations to higher dimensions are presented in Section 7.6. The limitations of our hydrodynamic construction in strictly one-dimensional lattice are discussed in Section 7.7 and conclude the article in Section 7.8. Additional details about the construction of the fluctuating hydrodynamics and numerical analysis are

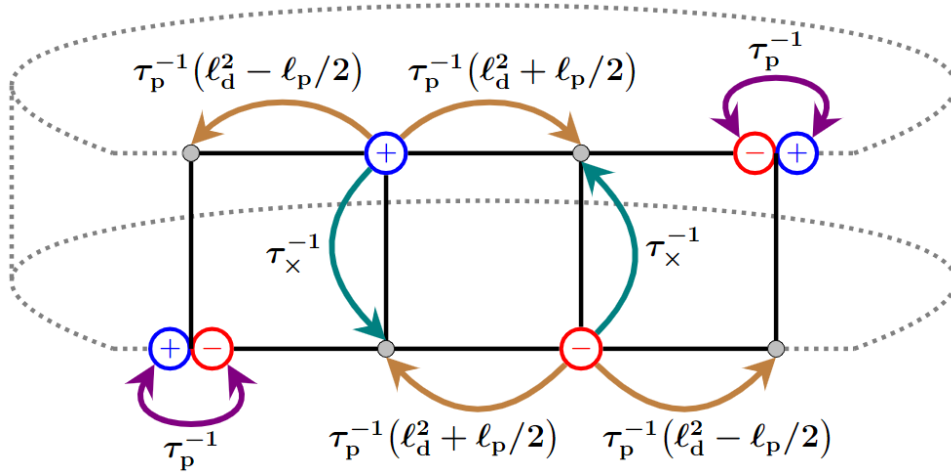


Figure 7.1.1. Quasi-1D model: A schematic of the model defined in items (1)–(3) on a two-lane ladder lattice with periodic boundary condition.

relegated to the Appendixes.

7.2 A QUASI-1D MODEL

We consider a two-species lattice gas with average density $\rho_0 \in [0, 1]$, on a periodic two-lane ladder-lattice, with each lane $j = \{1, 2\}$ containing L sites indexed as $i = \{1, 2, \dots, L\}$. To incorporate hard-core repulsion among particles, we enforce a simple exclusion condition that each site can contain at most one particle at a given instance. The two species denoted by $(+)$ and $(-)$, representing internal orientations of an active particle, follow a persistent dynamics on the lattice with intermittent switching of the orientations, as defined below. (See Fig. 7.1.1 for a schematic of the same.)

1. Biased diffusion: a $(+)$ particle at site (i, j) hops to site $(i \pm 1, j)$ at rate $\tau_p^{-1} [(\ell_d/a)^2 \pm (\ell_p/a)/2]$ while a $(-)$ particle at site (i, j) hops to $(i \pm 1, j)$ site at rate $\tau_p^{-1} [(\ell_d/a)^2 \mp (\ell_p/a)/2]$, provided the target site is empty.
2. Tumbling: a $(+)$ particle at any site converts to a $(-)$ particle at rate τ_p^{-1} and vice versa.
3. Lane crossing: a particle at a site $(i, 1)$ on lane 1 hops to the site $(i, 2)$ on lane 2 at rate τ_x^{-1} and vice versa, provided the target site is empty.

The above dynamics mimics self-propelled particles with persistence time τ_p , self-propulsion speed ℓ_p/τ_p , and thermal diffusivity $D = \ell_d^2/\tau_p$. The internal state $(\pm)$ represents the particle's preferred direction of motion, towards right and left, respectively. The ratio of the persistent length ℓ_p and the diffusive length ℓ_d , known as the Péclet number $Pe = \ell_p/\ell_d$, is a standard measure of activity. The parameter a denotes lattice spacing, which we set to unity throughout this article.

Multilane lattice models are natural descriptions of protein transport along microtubule [288] and for breaking integrability in quantum transport [289]. The quasi-one-dimensional geometry

with the lane crossing effectively describes motion inside narrow channels, where particle size is comparable to the diameter of the channel, allowing occasional crossing [290–293].

A microscopic configuration of the system at a given time τ is specified in terms of the binary occupation variables $n_{i,j}^\pm(\tau)$ for each species $\sigma = (\pm)$ respectively, where $n_{i,j}^\sigma(\tau) = 1$ or 0 depending on whether the site (i, j) is occupied by the species σ or not. A more convenient choice to describe the dynamics is in terms of the total occupation variable, $n_{i,j} = n_{i,j}^+ + n_{i,j}^-$ and the polarization variable, $M_{i,j} = n_{i,j}^+ - n_{i,j}^-$. Due to the exclusion condition, $n_{i,j} = \{0, 1\}$ depending on the occupancy of the site, while $M_{i,j} = \{0, \pm 1\}$ depending on the species of the occupant.

7.3 A HYDRODYNAMIC DESCRIPTION

We now present a hydrodynamic description by coarse-graining these variables such that fast fluctuations are locally equilibrated over a hydrodynamic scale and slow modes smoothly evolve. For our dynamics the relevant length and time scales are ℓ_d and τ_p . The hydrodynamics is defined in the rescaled coordinates $(x, t) \equiv (i/\ell_d, \tau/\tau_p)$, by taking ℓ_d, ℓ_p and τ_p large, while keeping the diffusivity $D = \ell_d^2/\tau_p$ and the Péclet number $Pe = \ell_p/\ell_d$ finite. For finite $\tau_x \ll \tau_p$, the adjacent sites of the two lanes effectively equilibrate at the hydrodynamic scale, implying their evolution is described by the same coarse-grained variables $n_{i,j}(\tau) \simeq \rho(x, t)$ and $M_{i,j}(\tau) \simeq m(x, t)$, independent of the lane index j .

We show in Appendix 7.A that the evolution of $\rho(x, t)$ and $m(x, t)$ follows the coupled fluctuating hydrodynamics

$$\partial_t \rho = \partial_x^2 \rho - Pe \partial_x [m(1 - \rho)] + \ell_d^{-1/2} \partial_x \eta_\rho, \quad (7.1a)$$

$$\begin{aligned} \partial_t m = (1 - \rho) \partial_x^2 m + m \partial_x^2 \rho - Pe \partial_x [\rho(1 - \rho)] - 2m \\ + \ell_d^{-1/2} (\partial_x \eta_m + 2\eta_f), \end{aligned} \quad (7.1b)$$

where η_ρ, η_m and η_f are Gaussian noises each of zero mean with covariance $\langle \eta_p(x, t) \eta_q(x', t') \rangle = S_{p,q} \delta(x - x') \delta(t - t')$, with the mobility matrix $S_{\rho,\rho} = S_{m,m} = 2\rho(1 - \rho)$, $S_{\rho,m} = S_{m,\rho} = 2m(1 - \rho)$, $S_{f,f} = \rho$, and zero for the rest. The nonconservative noise η_f is due to tumbling events in the dynamics. The same hydrodynamics extends for multilane generalizations with the number of lanes $\ll \ell_d$. The effective particle exchange rate along the quasi-one dimension is much reduced compared to the exchange rate in [280], resulting in a different hydrodynamics.

Our detailed derivation for (7.1), presented in Appendix 7.A, is based on an evaluation of the Martin-Siggia-Rose-Janssen-de Dominicis action [294–297] of the stochastic differential equation (7.1) following a coarse-graining of the microscopic generator [298]. This method is generalizable for variations of the dynamics and dimensionality. It is fundamentally different from a derivation [282] for a related model [280], which rests on mapping to stochastic diffusive models whose hydrodynamics is well tested. Such mapping relies on a very specific choice of the parameters, and not extendable for our model. Compared to the hydrodynamics in [280, 282], the diffusion matrix and the mobility matrix are significantly different.

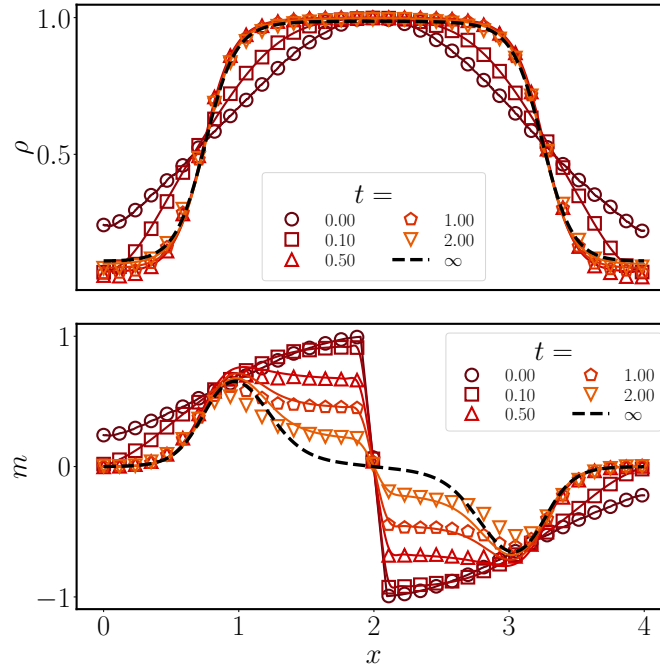


Figure 7.3.1. Test of hydrodynamics: Time evolution of the hydrodynamic density $\rho(x, t)$ and polarization $m(x, t)$, starting from a Gaussian profile $\rho(x, 0)$ centered around the middle of the system, with (+) particles placed at the left half and (−) particles at the right half. The solid lines represent noiseless hydrodynamic evolution (7.1), while the points represent Monte Carlo simulations of the microscopic dynamics in (1)–(3). The dashed lines indicate the steady-state profile from hydrodynamics. The plots are for bulk density $\rho_0 = 0.66$ and microscopic parameter values $\ell_d = 1024$, $\tau_p = 2.5\ell_d^2$, $\ell_p = 10\ell_d$, and $\tau_x = 0.1$, and system size $L = 4\ell_d$, which corresponds to $Pe = 10$.

In Fig. 7.3.1, we compare the noiseless limit of hydrodynamics (7.1) with direct Monte Carlo simulation of the microscopic dynamics (1)–(3). In the initial state, the left half of the system is occupied with (+) particles and the right half with (−) particles, each drawn from a Bernoulli distribution with a site-dependent average density $\langle n_i \rangle = \rho(i/\ell_d, 0)$, shown in Fig. 7.3.1. Hydrodynamic density and polarization profiles are obtained by coarse-graining over boxes containing 200 sites. Only profiles of lane-one, averaged over 64 ensembles, are shown in Fig. 7.3.1. For iterating the hydrodynamics we numerically integrate (7.1) with zero noise using a Fourier-pseudospectral method [299]. An excellent agreement seen in the evolution of the profiles, asymptotically converging towards an inhomogeneous state, indicates a phase separation between high density and low-density regions.

7.4 SPINODAL AND BINODAL ANALYSIS

It is readily seen from (7.1) that uniform density $\rho(x) = \rho_0$ and $m(x) = 0$ is a stationary solution of the noiseless hydrodynamics. An inhomogeneous phase-separated state, widely known as the MIPS for active matter systems, is an indication of the instability of this globally homogeneous solution. The spinodal curve on the (Pe, ρ_0) plane describes the onset of instability against

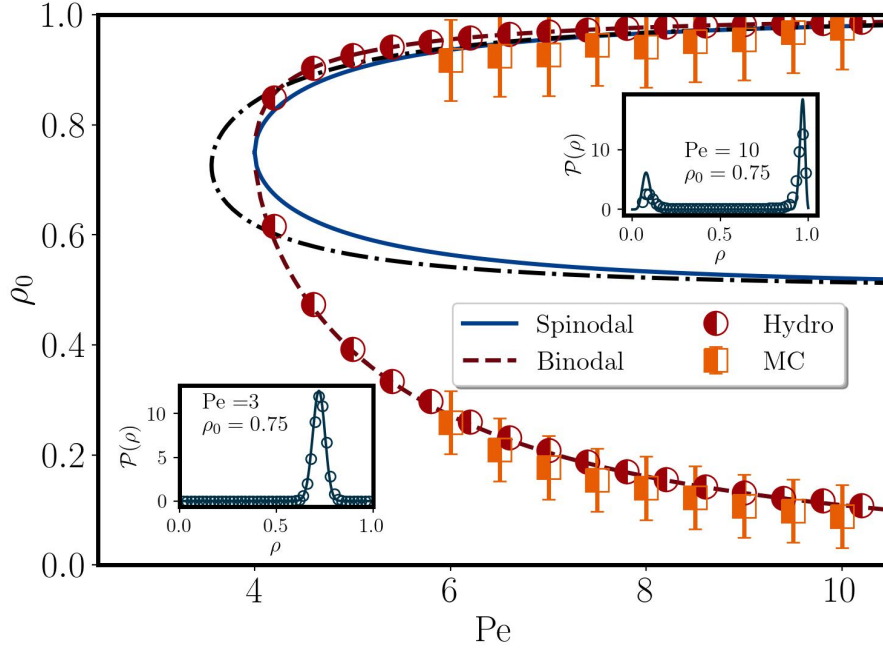


Figure 7.3.2. Phase diagram: Spinodal and binodal, given by (7.2) and (7.3) respectively, plotted with the solid and dashed lines. Along the dash-dotted line, the prefactor of the exponential in (7.4) diverges. The points represent the “liquid” and “gas” densities obtained from Monte Carlo simulations and numerical iteration of hydrodynamics. The error bars indicate the uncertainty due to the width of the density distribution around the peaks shown in the insets. The bimodal density distribution inside the spinodal region is an indication of MIPS, whereas the unimodal density distribution outside the binodal region is an indication of the homogeneous phase. In the inset, the points are from Monte Carlo simulations and the solid lines are from an analytical calculation in Appendix 7.D. We use $L = 4096$ in Monte Carlo simulations and hydrodynamics equation solved numerically using Fourier spectral method with 1024 modes.

linear perturbations. A standard linear stability analysis of the noiseless hydrodynamics (7.1) (shown in Appendix 7.E) predicts $Pe^2(1 - \rho_0)(2\rho_0 - 1) > 2$ as the unstable regimes. Thus, for $0 < Pe < 4$, a homogeneous state with any density $0 < \rho_0 < 1$ is always stable under linear perturbations. On the other hand, for $Pe > 4$, the homogeneous state is unstable if

$$\frac{3}{4} - \frac{1}{4}\sqrt{1 - \left(\frac{4}{Pe}\right)^2} < \rho_0 < \frac{3}{4} + \frac{1}{4}\sqrt{1 - \left(\frac{4}{Pe}\right)^2}. \quad (7.2)$$

The spinodal curve is shown in Fig. 7.3.2. Indeed, we see from Fig. 7.3.1 that if the parameters (Pe, ρ_0) are chosen from within the spinodal region, the density profile $\rho(x, t)$ evolves to an inhomogeneous state. The spinodal region given by Eq. (7.2) incidentally coincides with that for the model in [280].

The binodal curve gives the two densities in a phase-separated state where a “liquidlike” high-density ρ_l phase coexists with a “gaslike” low-density ρ_g phase. Following [300, 301], we find that for our noiseless hydrodynamics (7.1) (details in Appendix 7.F), the two coexistence

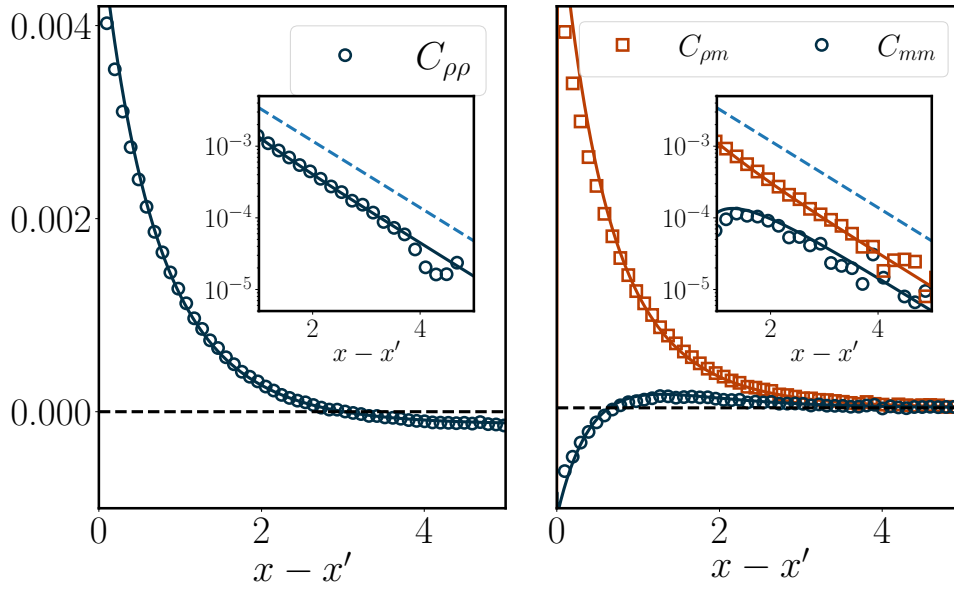


Figure 7.4.1. Correlations: Solid lines plot the analytical results of the correlation functions whereas the points are from numerical simulations. The insets show the asymptotic decay of the correlation functions (with a constant shift in order to show up in the log-linear plot), with the dashed guiding lines plot (7.4) with a shift for better visibility. The parameters used in the simulations are $\ell_d = 10.24$, $\tau_p = 2\ell_d^2$, $\ell_p = 2\ell_d$, $\tau_x = 0.1$, $L = 100\ell_d$ and $\rho_0 = 0.25$, and we average over 10^5 statistically independent time steps to generate a clean curve for comparing with hydrodynamic predictions.

densities are determined by a pair of relations

$$g_0(\rho_g) = g_0(\rho_l) \quad \text{and} \quad h_0(\rho_g) = h_0(\rho_l), \quad (7.3)$$

where $g_0(\rho) = \text{Pe} \rho (1 - \rho) - (2/\text{Pe}) \ln(1 - \rho)$ and $h_0(\rho) = [2/(9\text{Pe})] (1 - \rho)^{-3} + (\text{Pe}/6) (3 - 4\rho) (1 - \rho)^{-2}$. We numerically solve the above pair of relations (7.3) and obtain the binodal curve, which we plot in Fig. 7.3.2 along with the spinodal. Evidently, the volume fraction v of the “liquid” region(s) is given by $\rho_l v + \rho_g (1 - v) = \rho_0$, i.e., $v = (\rho_0 - \rho_g)/(\rho_l - \rho_g)$. Therefore, for $\text{Pe} > 4$, for any global density $\rho_g < \rho_0 < \rho_l$, it is possible to sustain a nonzero fraction $0 < v < 1$ of a liquid region within the analysis of the noiseless hydrodynamics. In particular, the parameters between the spinodal and the binodal curves correspond to a metastable region where, depending on the initial condition, the noiseless hydrodynamics (7.1) leads to either a homogeneous state or a phase-separated state (see Fig. 7.F.2).

We also numerically iterate the noiseless hydrodynamics (7.1) for a long time so that it reaches a stationary state, and determine the steady-state “liquid” and “gas” densities, by choosing the minimum and maximum densities in the density profile, respectively. However, in Monte Carlo simulations, identifying the lowest and highest densities is less straightforward due to the fluctuating nature of the density profile. To address this, we extract density distributions and average them over steady-state profiles (depicted in the inset). The resulting bimodal density distribution indicates the presence of two distinct densities, characteristic of MIPS.

7.5 TWO-POINT CORRELATIONS

We now compute the correlation functions in the stationary-state homogeneous phase, using the fluctuating hydrodynamics of (7.1). For example, the density-density correlation for far-separated points in the stationary-state homogeneous phase has the asymptote (see Appendix 7.G)

$$C_{\rho\rho}(x, x') \simeq \frac{A}{B\ell_d} \exp\left\{\left(-\frac{|x - x'|}{\xi_1}\right)\right\}, \quad (7.4)$$

where $A = \text{Pe}^2 \rho_0^2 (1 - \rho_0)^2 / \sqrt{1 - \rho_0/2}$, $B = 2 - \text{Pe}^2 (1 - \rho_0)(2 - \rho_0)(2\rho_0 - 1)$, and the correlation length $\xi_1 = \sqrt{1 - \rho_0/2}$. The line along which the denominator B vanishes, indicating breakdown of linearized hydrodynamics, is shown in Fig. 7.3.2. Its full form and the other correlation functions are given in Appendix 7.G. We compare our analytical correlation predictions with Monte Carlo simulations in Fig. 7.4.1 and find excellent agreements. An indirect verification of the multiplicative noise terms in (7.1) comes from this comparison of correlations with Monte Carlo results.

7.6 GENERALIZATION TO d DIMENSIONS

There are several ways one can generalize our quasi-1D model to higher dimensions. The straightforward generalization takes the number of lanes to be $O(\ell_d)$ with $\tau_\times = \ell_d^2 / \tau_p$. In this case, the noiseless hydrodynamic description along the *active* direction remains the same as in the quasi-1D case (see Appendix 7.B). On the other hand, in an isotropic generalization, we define the d -dimensional version of the model on a periodic hypercubic lattice of L^d sites and average density $\rho_0 \in [0, 1]$. The lattice sites indexed as $(i_1, i_2, \dots, i_d)$, where $i_k = 1, 2, \dots, L$ with $k = 1, 2, \dots, d$. The system is composed of $2d$ distinct species denoted by $\sigma = 1, 2, \dots, 2d$ corresponding to the $2d$ possible directions of self-propulsion denoted by $\hat{r}_\sigma = \pm \hat{x}_1, \pm \hat{x}_2, \dots, \pm \hat{x}_d$. In this model, a particle of σ species hops at rate $\tau_p^{-1}(\ell_d^2 + \ell_p/2)$ in the $\hat{r}_\sigma$ direction and at rate $\tau_p^{-1}(\ell_d^2 - \ell_p/2)$ in the other $2d - 1$ directions, provided that the target site is empty. In addition, a particle changes its species, i.e. tumbles, at rate τ_p^{-1} . The fluctuating hydrodynamics for this microscopic dynamics turns out to be (detailed derivation in Appendix 7.B)

$$\begin{aligned} \partial_t \rho_\sigma = & (1 - \rho) \nabla^2 \rho_\sigma + \rho_\sigma \nabla^2 \rho - \text{Pe} \vec{\nabla}[\rho_\sigma (1 - \rho)] \cdot \hat{r}_\sigma \\ & - \sum_{\sigma' \neq \sigma} (\rho_\sigma - \rho_{\sigma'}) + \ell_d^{-d/2} \left[\vec{\nabla} \cdot \vec{\eta}_\sigma + \sum_{\sigma' \neq \sigma} \eta_{\sigma \rightarrow \sigma'} \right]. \end{aligned} \quad (7.5)$$

Here, the conservative zero-mean Gaussian noise, $\vec{\eta}_\sigma = (\eta_\sigma^{(1)}, \eta_\sigma^{(2)}, \dots, \eta_\sigma^{(d)})$ arising from the biased-hopping dynamics, has the covariance $\langle \eta_\sigma^{(k)}(\vec{x}, t) \eta_{\sigma'}^{(k')}(\vec{x}', t') \rangle = 2\rho_\sigma(1 - \rho) \delta_{\sigma, \sigma'} \delta_{k, k'} \delta(\vec{x} - \vec{x}') \delta(t - t')$ while the nonconservative zero-mean Gaussian noise, $\eta_{\sigma \rightarrow \sigma'} = -\eta_{\sigma' \rightarrow \sigma}$, arising from the tumbling dynamics, has the covariance $\langle \eta_{\sigma \rightarrow \sigma'}(\vec{x}, t) \eta_{\psi \rightarrow \psi'}(\vec{x}', t') \rangle = (\rho_\sigma + \rho_{\sigma'}) \delta_{\sigma, \psi} \delta_{\sigma', \psi'} \delta(\vec{x} - \vec{x}') \delta(t - t')$.

Since the analysis of the above hydrodynamics equation (7.5) is similar to the quasi-1D case, we do not repeat it here. Instead, we perform Monte Carlo simulations for the two-dimensional

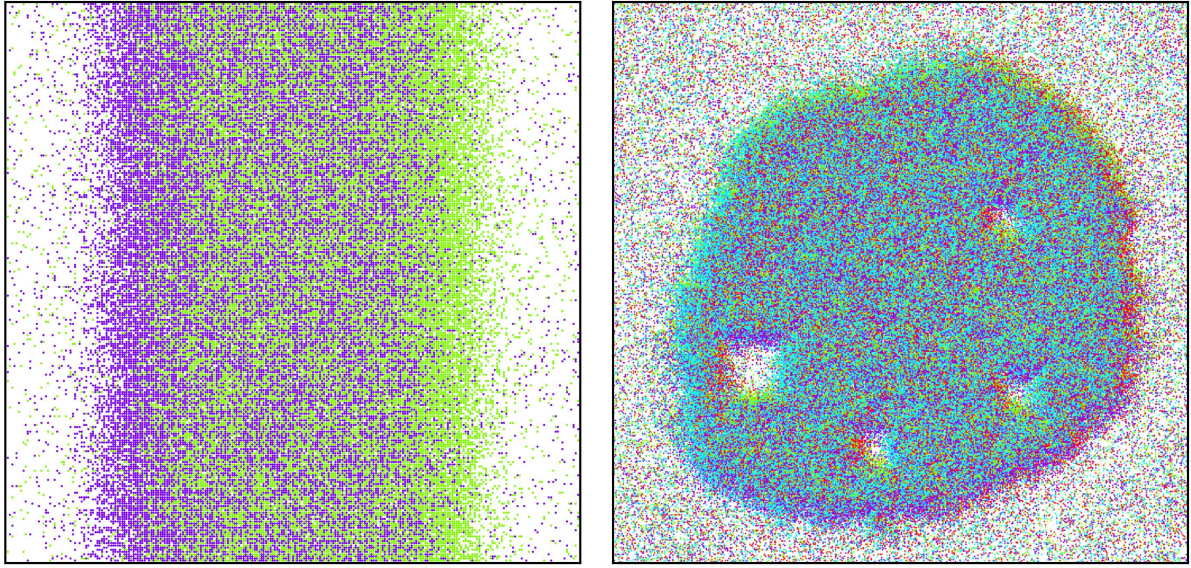


Figure 7.6.1. MIPS: Monte Carlo simulation for the two-dimensional models on a 512×512 square lattice with $\rho_0 = 0.62$. Left figure corresponds to the two-species ($\rightarrow$ and $\leftarrow$) generalization and right to the four-species ($\rightarrow$, $\uparrow$, $\leftarrow$ and $\downarrow$) generalization. The colors indicate species (magenta $\equiv \rightarrow$, cyan $\equiv \uparrow$, green $\equiv \leftarrow$, red $\equiv \downarrow$). For the two-species model $\ell_d = 128$ and for the four-species model $\ell_d = 32$, with $Pe = 12$ and $\tau_p = \ell_d^2$ in both cases. The holes in the high-density phase were seen earlier in active systems [302, 303].

microscopic model ($d = 2$) with both generalizations. We observe MIPS in our simulations for some parameters, as shown in Fig. 7.6.1, indicating that MIPS can emerge in two-dimensional systems, even without the exchange dynamics of [280]. Therefore, in dimensions higher than one, neither achieving a strong mixing effect nor observing the MIPS requires an exchange dynamics between the particles making our choice of dynamics naturally conducive for the desired results.

7.7 A STRICTLY 1D MODEL

For the version of our model on a periodic one-lane lattice with the dynamics (1) and (2), exclusion interaction introduces single-file constraint, which preserves positional order of the particles. Our hydrodynamics construction, relying on an assumption of local equilibrium measure, predicts the same fluctuating hydrodynamics (7.1) as in the quasi-1D case, which predicts the same phase boundary as in Fig. 7.3.2. However, our Monte Carlo simulation of this 1D model does not support MIPS [304] for the similar parameter ranges as in Fig. 7.3.2; neither do the correlations match with their hydrodynamics prediction (see Fig. 7.G.2). Evidently, the predicted hydrodynamics does not work for this strictly 1D geometry, resonating similar findings [305, 306]. This is further confirmed by a direct comparison of the noiseless evolution (7.1) of average fields with Monte Carlo simulation (see Fig. 7.D.4).

Hydrodynamics is by construction about effective dynamics of slow modes rising from local equilibrium of fast modes. Compared to higher dimensions, the single-file constraint in 1D

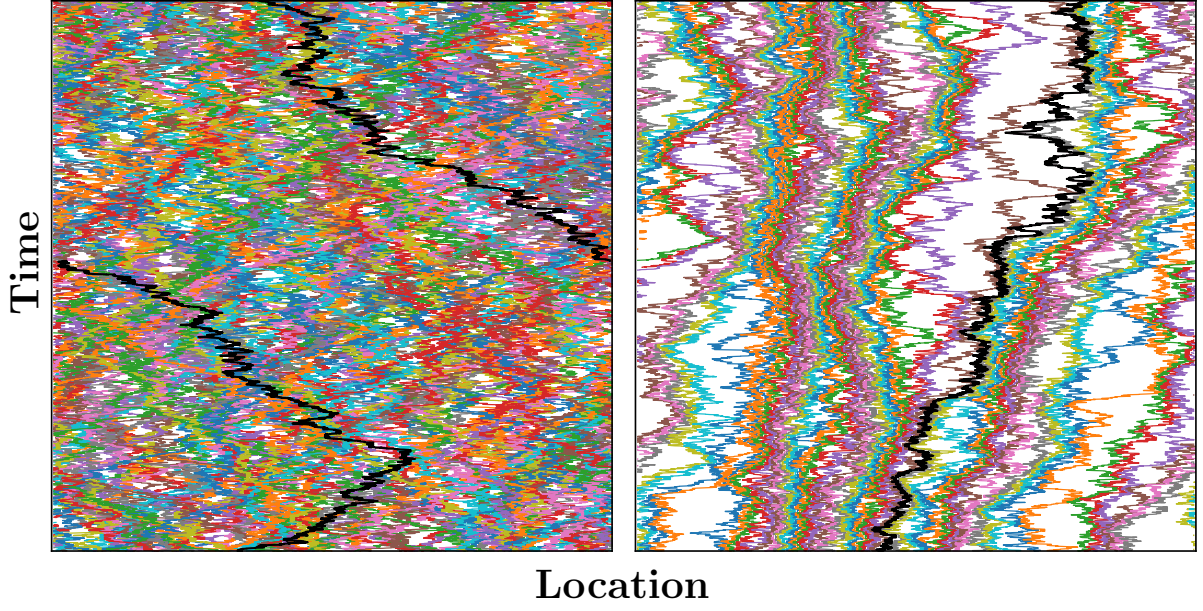


Figure 7.7.1. Kymographs: Comparison of space-time trajectories of particles at long hydrodynamic times for two-lane (left) and one-lane (right) models. Black indicates a tracer. The figures correspond to parameter values ($Pe = 6$, and $\rho_0 = 0.2$) outside the binodal curve in Fig. 7.3.2. Time evolution of the trajectories for different Pe are available in [307].

preserves the local order of species, and therefore, local polarization at microscopic time scales $\ll \tau_p$. This fight between activity and local conservation, results in the formation of a large number of microclusters, that does not coarsen as seen in the kymographs of particles in Fig. 7.7.1. Compared to the fluidlike evolution in the two-lane model in its homogeneous phase, the particle movements are severely restricted in the one-lane model at the same parameter values. This reduced effective diffusivity and drift of particles break our specific assumption about local equilibration (details in Appendix 7.D), invalidating the hydrodynamics construction.

7.8 CONCLUSION

In the spirit of statistical mechanics, we have proposed a minimal model of interacting active particles where the simplest kind of steric interaction leads to macroscopic phase separation even in quasi-one dimension. It would be of interest to verify our prediction in experimental realizations of active transport inside narrow channels [308] or in optical lattices [289]. The simplicity of our model allows us to theoretically analyze the macroscopic properties using a systematic bottom-up hydrodynamics theory. This hydrodynamic formulation potentially could serve as a minimal macroscopic description for MIPS, that is shared between different microscopic models, in the spirit of the Landau theory of phase transitions. There is a plethora of active dynamics [265, 284, 309, 310] belonging to this class for which a detailed fluctuating hydrodynamics is not available. It would be interesting to see how the fluctuating hydrodynamics results for the present work compare with other models, including a full characterization of macroscopic fluctuations in terms of large deviations [311–314]. Our current work provides a

crucial step in this promising direction.

7.A AN ACTION FORMULATION FOR THE QUASI-1D MODEL

Here, we present a derivation of the fluctuating hydrodynamic description for the quasi-1D model introduced in the main text, following a similar route that was used earlier in [298, 315] for the symmetric simple exclusion process (SSEP) and related stochastic lattice gases. The dynamics of our model is defined in items (1)–(3) of the main text and schematically shown in Fig. 7.1.1.

The configuration of the system at any time τ is described by $\mathbf{n}(\tau) \equiv \{n_{i,j}^+(\tau), n_{i,j}^-(\tau)\}$ where the occupation variables take the values

$$(n_{i,j}^+(\tau), n_{i,j}^-(\tau)) = \{(1, 0), (0, 1), (0, 0)\} \quad (7.6)$$

depending on whether the i th site of the j lane is occupied by a particle of species (+) or (−), or unoccupied, respectively. The simple exclusion interactions ensure that at any given time, no more than one particle of either species occupies a site. The system configuration evolves in time due to same-lane biased hopping, tumbling and lane crossing. The changes in occupation variables of a site relate to counters associated to the dynamical events in the infinitesimal time interval between τ and $\tau + d\tau$:

$$n_{i,1}^\pm(\tau + d\tau) - n_{i,1}^\pm(\tau) = \mathcal{B}_{i-1,1}^\pm(\tau) - \mathcal{B}_{i,1}^\pm(\tau) \mp \mathcal{T}_{i,1}(\tau) - \mathcal{L}_i^\pm(\tau), \quad (7.7a)$$

$$n_{i,2}^\pm(\tau + d\tau) - n_{i,2}^\pm(\tau) = \mathcal{B}_{i-1,2}^\pm(\tau) - \mathcal{B}_{i,2}^\pm(\tau) \mp \mathcal{T}_{i,2}(\tau) + \mathcal{L}_i^\pm(\tau), \quad (7.7b)$$

where $\mathcal{B}_{i,j}^\sigma(\tau)$ is the net number of same-lane hopping of the σ -species particles between the i th and $(i + 1)$ th sites of the j th lane; $\mathcal{L}_i^\sigma$ is the net number of lane crossings of the σ -species particles from the $(i, 1)$ th site to the $(i, 2)$ th site; $\mathcal{T}_{i,j}(\tau)$ corresponds the net number of (+)-species to (−)-species tumblings at the i th site of the j lane. For an infinitesimal $d\tau$, the various possible values that these variables can take are listed in Table 7.1. The rates ensure the exclusion condition and preserve the total number of particles.

Our interest lies in determining the probability that the system starting from an initial configuration $\mathbf{n}(0)$ evolves into a final configuration $\mathbf{n}(T)$. This transition probability can be written as a path integral over all possible evolution of the trajectories $\{\mathbf{n}(\tau), \mathcal{B}(\tau), \mathcal{T}(\tau), \mathcal{L}(\tau)\}$ as

$$\begin{aligned} \Pr[\mathbf{n}(T) | \mathbf{n}(0)] = \int_{\mathbf{n}(0)}^{\mathbf{n}(T)} [\mathcal{D}\mathbf{n}] \left\langle \prod_{\tau=0}^{T-d\tau} \prod_{i=1}^L \left(\delta_{n_{i,1}^+(\tau+d\tau)-n_{i,1}^+(\tau), \mathcal{B}_{i-1,1}^+(\tau)-\mathcal{B}_{i,1}^+(\tau) \mp \mathcal{T}_{i,1}(\tau) - \mathcal{L}_i^+(\tau)} \right. \right. \\ \left. \left. \times \delta_{n_{i,2}^+(\tau+d\tau)-n_{i,2}^+(\tau), \mathcal{B}_{i-1,2}^+(\tau)-\mathcal{B}_{i,2}^+(\tau) \mp \mathcal{T}_{i,2}(\tau) + \mathcal{L}_i^+(\tau)} \right) \right\rangle_{\{\mathcal{B}(\tau), \mathcal{T}(\tau), \mathcal{L}(\tau)\}}, \end{aligned} \quad (7.8)$$

$(\mathcal{B}_{i,1}^+, \mathcal{B}_{i,1}^-, \mathcal{B}_{i,2}^+, \mathcal{B}_{i,2}^-, \mathcal{T}_{i,1}, \mathcal{T}_{i,2}, \mathcal{L}_i^+, \mathcal{L}_i^-)$	Rate
$(1, 0, 0, 0, 0, 0, 0, 0)$	$\tau_p^{-1} (\ell_d^2 + \ell_p/2) n_{i,1}^+ (1 - n_{i+1,1}^+ - n_{i+1,1}^-)$
$(-1, 0, 0, 0, 0, 0, 0, 0)$	$\tau_p^{-1} (\ell_d^2 - \ell_p/2) n_{i+1,1}^+ (1 - n_{i,1}^+ - n_{i,1}^-)$
$(0, 1, 0, 0, 0, 0, 0, 0)$	$\tau_p^{-1} (\ell_d^2 - \ell_p/2) n_{i,1}^- (1 - n_{i+1,1}^+ - n_{i+1,1}^-)$
$(0, -1, 0, 0, 0, 0, 0, 0)$	$\tau_p^{-1} (\ell_d^2 + \ell_p/2) n_{i+1,1}^- (1 - n_{i,1}^+ - n_{i,1}^-)$
$(0, 0, 1, 0, 0, 0, 0, 0)$	$\tau_p^{-1} (\ell_d^2 + \ell_p/2) n_{i,2}^+ (1 - n_{i+1,2}^+ - n_{i+1,2}^-)$
$(0, 0, -1, 0, 0, 0, 0, 0)$	$\tau_p^{-1} (\ell_d^2 - \ell_p/2) n_{i+1,2}^+ (1 - n_{i,2}^+ - n_{i,2}^-)$
$(0, 0, 0, 1, 0, 0, 0, 0)$	$\tau_p^{-1} (\ell_d^2 - \ell_p/2) n_{i,2}^- (1 - n_{i+1,2}^+ - n_{i+1,2}^-)$
$(0, 0, 0, -1, 0, 0, 0, 0)$	$\tau_p^{-1} (\ell_d^2 + \ell_p/2) n_{i+1,2}^- (1 - n_{i,2}^+ - n_{i,2}^-)$
$(0, 0, 0, 0, \pm 1, 0, 0, 0)$	$\tau_p^{-1} n_{i,1}^\pm$
$(0, 0, 0, 0, 0, \pm 1, 0, 0)$	$\tau_p^{-1} n_{i,2}^\pm$
$(0, 0, 0, 0, 0, 0, 1, 0)$	$\tau_\times^{-1} n_{i,1}^+ (1 - n_{i,2}^+ - n_{i,2}^-)$
$(0, 0, 0, 0, 0, 0, -1, 0)$	$\tau_\times^{-1} n_{i,2}^+ (1 - n_{i,1}^+ - n_{i,1}^-)$
$(0, 0, 0, 0, 0, 0, 0, 1)$	$\tau_\times^{-1} n_{i,1}^- (1 - n_{i,2}^+ - n_{i,2}^-)$
$(0, 0, 0, 0, 0, 0, 0, -1)$	$\tau_\times^{-1} n_{i,2}^- (1 - n_{i,1}^+ - n_{i,1}^-)$

Table 7.1. Possible values of the various dynamical events for the quantities in (7.7) and corresponding rates.

where the Kronecker-delta function $\delta_{a,b}$ incorporates the particle conservation in (7.7), and the angular brackets denote the average over histories of $\{\mathcal{B}(\tau), \mathcal{T}(\tau), \mathcal{L}(\tau)\}$. The path integral measure on the occupation variables is defined as

$$\int [\mathcal{D}\mathbf{n}] := \prod_{\tau=0}^T \prod_{i=1}^L \prod_{j=1}^2 \prod_{\sigma=(\pm)} \sum_{n_{i,j}^{\sigma}(\tau)=0}^1. \quad (7.9)$$

We use an integral representation of the Kronecker-delta function $\delta_{a,b} = (2\pi i)^{-1} \int_{-i\pi}^{i\pi} d\hat{n} e^{-\hat{n}(a-b)}$ and introduce a response variable $\hat{n}_{i,j}^{\sigma}(\tau)$ at every site for both species. Subsequently, performing the averages over the variables $\mathcal{B}_{i,j}^{\sigma}$, $\mathcal{T}_{i,j}$ and $\mathcal{L}^{\sigma}$ using Table 7.1, we write the transition probability in an action formulation,

$$\Pr[\mathbf{n}(T) | \mathbf{n}(0)] = \int_{\mathbf{n}(0)}^{\mathbf{n}(T)} [\mathcal{D}\mathbf{n}] [\mathcal{D}\hat{\mathbf{n}}] e^{\mathcal{K}[\mathbf{n}, \hat{\mathbf{n}}] + \mathcal{H}[\mathbf{n}, \hat{\mathbf{n}}]} \quad (7.10)$$

with the “kinetic” term

$$\mathcal{K}[\mathbf{n}, \hat{\mathbf{n}}] = \sum_{i=1}^L \sum_{j=1}^2 \sum_{\sigma=(\pm)} \left[n_{i,j}^{\sigma}(0) \hat{n}_{i,j}^{\sigma}(0) - n_{i,j}^{\sigma}(T) \hat{n}_{i,j}^{\sigma}(T) + \int_0^T d\tau \left(n_{i,j}^{\sigma}(\tau) \frac{d\hat{n}_{i,j}^{\sigma}(\tau)}{d\tau} \right) \right] \quad (7.11)$$

and the “Hamiltonian” term

$$\begin{aligned} \mathcal{H}[\mathbf{n}, \hat{\mathbf{n}}] = & \frac{\ell_d^2}{\tau_p} \sum_{i=1}^L \sum_{j=1}^2 \sum_{\sigma=(\pm)} \int_0^T d\tau \left[(e^{\hat{n}_{i+1,j}^{\sigma} - \hat{n}_{i,j}^{\sigma}} - 1) n_{i,j}^{\sigma} (1 - n_{i+1,j}) + (e^{\hat{n}_{i,j}^{\sigma} - \hat{n}_{i+1,j}^{\sigma}} - 1) n_{i+1,j}^{\sigma} (1 - n_{i,j}) \right] \\ & + \frac{\ell_p}{2\tau_p} \sum_{i=1}^L \sum_{j=1}^2 \int_0^T d\tau \left[(e^{\hat{n}_{i+1,j}^{+} - \hat{n}_{i,j}^{+}} - 1) n_{i,j}^{+} (1 - n_{i+1,j}) - (e^{\hat{n}_{i,j}^{+} - \hat{n}_{i+1,j}^{+}} - 1) n_{i+1,j}^{+} (1 - n_{i,j}) \right. \\ & \left. - (e^{\hat{n}_{i+1,j}^{-} - \hat{n}_{i,j}^{-}} - 1) n_{i,j}^{-} (1 - n_{i+1,j}) + (e^{\hat{n}_{i,j}^{-} - \hat{n}_{i+1,j}^{-}} - 1) n_{i+1,j}^{-} (1 - n_{i,j}) \right] \\ & + \frac{1}{\tau_p} \sum_{i=1}^L \sum_{j=1}^2 \int_0^T d\tau \left[(e^{\hat{n}_{i,j}^{+} - \hat{n}_{i,j}^{-}} - 1) n_{i,j}^{+} + (e^{\hat{n}_{i,j}^{-} - \hat{n}_{i,j}^{+}} - 1) n_{i,j}^{-} \right] \\ & + \frac{1}{\tau_{\times}} \sum_{i=1}^L \sum_{\sigma=(\pm)} \int_0^T d\tau \left[(e^{\hat{n}_{i,2}^{\sigma} - \hat{n}_{i,1}^{\sigma}} - 1) n_{i,1}^{\sigma} (1 - n_{i,2}) + (e^{\hat{n}_{i,1}^{\sigma} - \hat{n}_{i,2}^{\sigma}} - 1) n_{i,2}^{\sigma} (1 - n_{i,1}) \right], \end{aligned} \quad (7.12)$$

where $n_{i,j}(\tau) = \sum_{\sigma} n_{i,j}^{\sigma}(\tau)$ denotes the total occupation number of the i th site of j lane at time τ . The path integral measure on the conjugate response variable is defined as

$$\int [\mathcal{D}\hat{\mathbf{n}}] := \prod_{\tau=0}^T \prod_{i=1}^L \prod_{j=1}^2 \prod_{\sigma=(\pm)} \int_{-i\pi}^{i\pi} \frac{d\hat{n}_{i,j}^{\sigma}(\tau)}{2\pi i}. \quad (7.13)$$

7.A.1 THE HYDRODYNAMIC LIMIT

Given the diffusive nature of the dynamics for our quasi-1D model, the local statistics of occupation variables in a region of length scale ℓ_d reach a local equilibrium over a period of time scale $\tau_p = \ell_d^2/D$ with respect to a slowly varying density profile. Considering that, in these

length and time scales, the diffusion dominates over drift or tumbling, this local equilibrium measure is best approximated by the product measure,

$$(n_{i,j}^+(\tau), n_{i,j}^-(\tau)) = \begin{cases} (1, 0) & \text{with prob. } \rho_i^+(\tau) \\ (0, 1) & \text{with prob. } \rho_i^-(\tau) \\ (0, 0) & \text{with prob. } 1 - \rho_i(\tau) \end{cases} \quad \begin{matrix} (7.14a) \\ (7.14b) \\ (7.14c) \end{matrix}$$

similar to the SSEP on two lanes. Here, $\rho_i^\pm(\tau) = \rho_\pm(i/\ell_d, \tau/\tau_p)$ are slowly varying local average densities with $\rho_i(\tau) = \rho_i^+(\tau) + \rho_i^-(\tau)$.

An important assumption is that $\rho_i^\pm(\tau)$ in the two lanes are the same. This is justified considering that in the time scale $\tau_p \gg \tau_\times$, particles frequently switch lanes before diffusing on the lane. This results in fast equilibration of adjacent sites on the two lanes.

To be consistent with this local equilibrium assumption, the response variables are also considered as slowly varying and the same on the two lanes. More precisely,

$$\hat{n}_{i,j}^\pm(\tau) = \hat{\rho}_\pm\left(\frac{i}{\ell_d}, \frac{\tau}{\tau_p}\right) \equiv \hat{\rho}_i^\pm(\tau). \quad (7.15)$$

In the hydrodynamic scale, defined by the rescaled coordinates $(x, t) \equiv (i/\ell_d, \tau/\tau_p)$, the hydrodynamics description of the model is in terms of the evolution of these slowly varying fields $\rho_\pm(x, t)$ and $\hat{\rho}_\pm(x, t)$ gives.

In obtaining the transition probability between the initial and final hydrodynamic densities, $\rho(0)$ and $\rho(T)$, from the action formulation in (7.10)–(7.12), the leading contribution comes from typical microscopic configurations associated to the hydrodynamic fields, such that

$$\begin{aligned} \sum_i n_{i,j}^\sigma(0) \hat{n}_{i,j}^\sigma(0) &\simeq \sum_i \rho_i^\sigma(0) \hat{\rho}_i^\sigma(0), \text{ and} \\ \sum_i n_{i,j}^\sigma(T) \hat{n}_{i,j}^\sigma(T) &\simeq \sum_i \rho_i^\sigma(T) \hat{\rho}_i^\sigma(T) \end{aligned} \quad (7.16)$$

for large ℓ_d and τ_p . Taking average of $\Pr[\mathbf{n}(T) | \mathbf{n}(0)]$ in (7.10)–(7.12) over occupation variables using the local equilibrium measure (7.14) give the transition probability in hydrodynamic scale,

$$\Pr[\rho(T) | \rho(0)] = \int_{\rho(0)}^{\rho(T)} [\mathcal{D}\rho] [\mathcal{D}\hat{\rho}] \left\langle e^{\mathcal{K}[n, \hat{\rho}] + \mathcal{H}[n, \hat{\rho}]} \right\rangle_{\{\mathbf{n}(\tau)\}} \quad (7.17)$$

with $\mathcal{K}$ and $\mathcal{H}$ in (7.11) and (7.12), respectively.

In performing the average, in the large ℓ_d and τ_p limit, the local equilibrium probability measure at different times are assumed to be independent of one another, except correlations through their slowly varying density fields. This way, the leading-order behavior of the transition probability,

$$\Pr[\rho(T) | \rho(0)] \simeq \int_{\rho(0)}^{\rho(T)} [\mathcal{D}\rho] [\mathcal{D}\hat{\rho}] \prod_{\tau=0}^T \left\langle e^{\mathrm{d}\tau \mathcal{S}(\tau)} \right\rangle_{\mathbf{n}(\tau)} e^{2 \sum_{i=1}^L \sum_{\sigma=\pm} (\rho_i^\sigma(0) \hat{\rho}_i^\sigma(0) - \rho_i^\sigma(T) \hat{\rho}_i^\sigma(T))} \quad (7.18)$$

with the action

$$\begin{aligned} S(\tau) = & \sum_{i=1}^L \sum_{j=1}^2 \left\{ \sum_{\sigma=(\pm)} \left(n_{i,j}^{\sigma} \frac{d\hat{\rho}_i^{\sigma}}{d\tau} \right) + \frac{\ell_d^2}{\tau_p} \sum_{\sigma=(\pm)} \left[(e^{\hat{\rho}_{i+1}^{\sigma} - \hat{\rho}_i^{\sigma}} - 1) n_{i,j}^{\sigma} (1 - n_{i+1,j}) + (e^{\hat{\rho}_i^{\sigma} - \hat{\rho}_{i+1}^{\sigma}} - 1) n_{i+1,j}^{\sigma} (1 - n_{i,j}) \right] \right. \\ & + \frac{\ell_p}{2\tau_p} \left[(e^{\hat{\rho}_{i+1}^{+} - \hat{\rho}_i^{+}} - 1) n_{i,j}^{+} (1 - n_{i+1,j}) - (e^{\hat{\rho}_i^{+} - \hat{\rho}_{i+1}^{+}} - 1) n_{i+1,j}^{+} (1 - n_{i,j}) - (e^{\hat{\rho}_{i+1}^{-} - \hat{\rho}_i^{-}} - 1) n_{i,j}^{-} (1 - n_{i+1,j}) \right. \\ & \left. \left. + (e^{\hat{\rho}_i^{-} - \hat{\rho}_{i+1}^{-}} - 1) n_{i+1,j}^{-} (1 - n_{i,j}) \right] + \frac{1}{\tau_p} \left[(e^{\hat{\rho}_i^{+} - \hat{\rho}_i^{-}} - 1) n_{i,j}^{+} + (e^{\hat{\rho}_i^{-} - \hat{\rho}_i^{+}} - 1) n_{i,j}^{-} \right] \right\}, \end{aligned} \quad (7.19a)$$

where we used (7.16). Notably, the contribution in the action from lane-crossing (involving $\tau_{\times}$) vanishes due to the fast equilibration assumption between the two lanes, equating their local densities and conjugate fields in (7.15).

Further simplification comes from the $d\tau \rightarrow 0$ limit, where

$$\prod_{\tau} \langle e^{d\tau S(\tau)} \rangle_{n(\tau)} \simeq \prod_{\tau} e^{d\tau \langle S(\tau) \rangle_{n(\tau)}}. \quad (7.20)$$

The averaging of the action essentially reduces to replacing $n_{i,1}^{\pm}(\tau)$ and $n_{i,2}^{\pm}(\tau)$ by $\rho_i^{\pm}(\tau)$ in (7.19a). This leads to

$$\Pr[\rho(T) | \rho(0)] = \int_{\rho(0)}^{\rho(T)} [\mathcal{D}\rho] [\mathcal{D}\hat{\rho}] e^{2 \int_0^T d\tau \sum_{i=1}^L \mathcal{S}_i}, \quad (7.21)$$

where the action is

$$\begin{aligned} \mathcal{S}_i = & -\hat{\rho}_i^{+} \frac{d\rho_i^{+}}{d\tau} - \hat{\rho}_i^{-} \frac{d\rho_i^{-}}{d\tau} + \frac{\ell_d^2}{\tau_p} \left[(e^{\hat{\rho}_{i+1}^{+} - \hat{\rho}_i^{+}} - 1) \rho_i^{+} (1 - \rho_{i+1}) + (e^{\hat{\rho}_{i+1}^{-} - \hat{\rho}_i^{-}} - 1) \rho_i^{-} (1 - \rho_{i+1}) \right. \\ & + (e^{\hat{\rho}_i^{+} - \hat{\rho}_{i+1}^{+}} - 1) \rho_{i+1}^{+} (1 - \rho_i) + (e^{\hat{\rho}_i^{-} - \hat{\rho}_{i+1}^{-}} - 1) \rho_{i+1}^{-} (1 - \rho_i) \left. \right] \\ & + \frac{\ell_p}{2\tau_p} \left[(e^{\hat{\rho}_{i+1}^{+} - \hat{\rho}_i^{+}} - 1) \rho_i^{+} (1 - \rho_{i+1}) - (e^{\hat{\rho}_i^{+} - \hat{\rho}_{i+1}^{+}} - 1) \rho_{i+1}^{+} (1 - \rho_i) \right. \\ & - (e^{\hat{\rho}_{i+1}^{-} - \hat{\rho}_i^{-}} - 1) \rho_i^{-} (1 - \rho_{i+1}) + (e^{\hat{\rho}_i^{-} - \hat{\rho}_{i+1}^{-}} - 1) \rho_{i+1}^{-} (1 - \rho_i) \left. \right] \\ & + \frac{1}{\tau_p} \left[(e^{\hat{\rho}_{i,j}^{+} - \hat{\rho}_{i,j}^{-}} - 1) \rho_{i,j}^{+} + (e^{\hat{\rho}_{i,j}^{-} - \hat{\rho}_{i,j}^{+}} - 1) \rho_{i,j}^{-} \right]. \end{aligned} \quad (7.22)$$

In writing (7.22), we used an integration by parts in the τ variable to cancel the exponential term outside the averaging in (7.18).

For writing the path integral in terms of the hydrodynamic fields, we use the definition of $\rho_{\pm}(x, t)$ and $\hat{\rho}_{\pm}(x, t)$ in (7.14) and (7.15) and perform gradient expansions, for large ℓ_d and τ_p . Keeping the leading-order terms

$$\Pr[\rho_{\pm}(x, \mathbb{T}) | \rho_{\pm}(x, 0)] = \int_{\rho_{\pm}(x, 0)}^{\rho_{\pm}(x, \mathbb{T})} [\mathcal{D}\rho_{\pm}] [\mathcal{D}\hat{\rho}_{\pm}] e^{-2\ell_d \int_0^{\mathbb{T}} dt \left[\int_0^{\mathbb{L}} dx (\hat{\rho}_{+} \partial_t \rho_{+} + \hat{\rho}_{-} \partial_t \rho_{-}) - H[\hat{\rho}_{\pm}, \rho_{\pm}] \right]} \quad (7.23)$$

with the effective Hamiltonian

$$\begin{aligned} H = & \int_0^{\mathbb{L}} dx \left\{ \rho_{+} (1 - \rho) (\partial_x \hat{\rho}_{+})^2 + \rho_{-} (1 - \rho) (\partial_x \hat{\rho}_{-})^2 \right. \\ & - [(1 - \rho_{-}) \partial_x \rho_{+} + \rho_{+} \partial_x \rho_{-} - \text{Pe} \rho_{+} (1 - \rho)] \partial_x \hat{\rho}_{+} \\ & - [(1 - \rho_{+}) \partial_x \rho_{-} + \rho_{-} \partial_x \rho_{+} + \text{Pe} \rho_{-} (1 - \rho)] \partial_x \hat{\rho}_{-} \\ & \left. + (e^{-\hat{\rho}_{+} + \hat{\rho}_{-}} - 1) \rho_{+} + (e^{\hat{\rho}_{+} - \hat{\rho}_{-}} - 1) \rho_{-} \right\} \end{aligned} \quad (7.24)$$

where we denote $T/\tau_p = \mathbb{T}$ and $L/\ell_d = \mathbb{L}$. Note, that in our choice of rates, the thermal diffusivity, $D = \ell_d^2/\tau_p$ and Péclet number, $Pe = \ell_p/\ell_d$ are kept finite.

Action in terms of density and polarization fields

The action in (7.23) describes the stochastic hydrodynamics of the model. More convenient hydrodynamic fields for active matter are the total density ($\rho = \rho_+ + \rho_-$) and the polarization ($m = \rho_+ - \rho_-$) fields. By defining the corresponding response fields $\hat{\rho} = (\hat{\rho}_+ + \hat{\rho}_-)/2$ and $\hat{m} = (\hat{\rho}_+ - \hat{\rho}_-)/2$ in (7.23) we get the transition probability

$$\begin{aligned} & \Pr [\rho(x, \mathbb{T}), m(x, \mathbb{T}) \mid \rho(x, 0), m(x, 0)] \\ & \simeq \int [\mathcal{D}\rho] [\mathcal{D}m] [\mathcal{D}\hat{\rho}] [\mathcal{D}\hat{m}] e^{-2\ell_d \int_0^{\mathbb{T}} dt \left[\int_0^{\mathbb{L}} dx (\hat{\rho} \partial_t \rho + \hat{m} \partial_t m) - H[\hat{\rho}, \hat{m}, \rho, m] \right]} \end{aligned} \quad (7.25)$$

with the effective Hamiltonian

$$\begin{aligned} H = \int_0^{\mathbb{L}} dx & \left\{ \rho (1 - \rho) (\partial_x \hat{\rho})^2 + \rho (1 - \rho) (\partial_x \hat{m})^2 + 2m (1 - \rho) \partial_x \hat{\rho} \partial_x \hat{m} \right. \\ & - [\partial_x \rho - Pe m (1 - \rho)] \partial_x \hat{\rho} [(1 - \rho) \partial_x m + m \partial_x \rho - Pe \rho (1 - \rho)] \partial_x \hat{m} \\ & \left. + 2\rho \sinh^2 \hat{m} - m \sinh 2\hat{m} \right\}. \end{aligned} \quad (7.26)$$

Gaussian approximation

The effective Hamiltonian in (7.26) is non-Gaussian in $\hat{m}$. For most states away from transition lines, fluctuations are small and it is reasonable to assume $\hat{m}$ to be small. A similar quadratic approximation was done in [282] for a related dynamics. Keeping up to the quadratic order terms in small $\hat{m}$, the Hamiltonian is given by

$$H = \int_0^{\mathbb{L}} dx \left[\frac{S_{\rho,\rho}}{2} (\partial_x \hat{\rho})^2 + \frac{S_{m,m}}{2} (\partial_x \hat{m})^2 + 2S_{f,f} \hat{m}^2 + S_{\rho,m} \partial_x \hat{\rho} \partial_x \hat{m} + \bar{J}_\rho \partial_x \hat{\rho} + \bar{J}_m \partial_x \hat{m} - 2\bar{J}_f \hat{m} \right] \quad (7.27)$$

with $S_{\rho,\rho} = S_{m,m} = 2\rho(1-\rho)$, $S_{f,f} = \rho$, $S_{\rho,m} = 2m(1-\rho)$ and the average conserved currents due to diffusion drift

$$\bar{J}_\rho = -\partial_x \rho + Pe m (1 - \rho) \quad (7.28)$$

$$\bar{J}_m = -(1 - \rho) \partial_x m - m \partial_x \rho + Pe \rho (1 - \rho) \quad (7.29)$$

and the nonconserved tumbling current

$$\bar{J}_f = m \quad (7.30)$$

7.A.2 THE FLUCTUATING HYDRODYNAMICS EQUATIONS

The hydrodynamic action in (7.25) with the Gaussian effective Hamiltonian (7.27) is the Martin-Siggia-Rose-Janssen-De Dominicis action [294–297] of a fluctuating hydrodynamics equation,

$$\partial_t \rho = -\partial_x \left(\bar{J}_\rho - \frac{1}{\sqrt{\ell_d}} \eta_\rho \right), \quad (7.31a)$$

$$\partial_t m = -\partial_x \left(\bar{J}_m - \frac{1}{\sqrt{\ell_d}} \eta_m \right) - 2 \left(\bar{J}_f - \frac{1}{\sqrt{\ell_d}} \eta_f \right), \quad (7.31b)$$

where $\boldsymbol{\eta} \equiv (\eta_\rho \ \eta_m \ \eta_f)^\top$ is a multivariate Gaussian noise vector, whose probability distribution is

$$\text{Pr}(\boldsymbol{\eta}) \sim e^{-2^{-1} \int dt dt' dx dx' \boldsymbol{\eta}(x,t)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\eta}(x',t')} \quad (7.32)$$

with the covariance matrix

$$\boldsymbol{\Sigma} = \delta(x - x') \delta(t - t') \begin{pmatrix} S_{\rho,\rho} & S_{\rho,m} & 0 \\ S_{\rho,m} & S_{m,m} & 0 \\ 0 & 0 & S_{f,f} \end{pmatrix}. \quad (7.33)$$

7.B GENERALIZATION TO d DIMENSIONS

Here, we generalize the action formalism and thus, the fluctuating hydrodynamic description for our model on a d -dimensional hyper-cubic lattice with L^d sites with periodic boundaries. The lattice sites are indexed by their Cartesian coordinates $(i_1, i_2, \dots, i_d)$ with $i_k = \{1, 2, \dots, L\}$ for $k = 1, 2, \dots, d$. We discuss two possible generalizations of the active dynamics, as described in the main text.

7.B.1 CASE I: TWO-SPECIES GENERALIZATION

This case corresponds to the system composed of particles of two-species [denoted as $(\pm)$] similar to the quasi-1D model. The $(\pm)$ particles move persistently in the $\pm \hat{x}_1$ direction (which we refer to as *active* direction) with the motion being symmetric along all other directions $\{\pm \hat{x}_2, \dots, \pm \hat{x}_d\}$ (which we refer to as *passive* directions). The microscopic dynamics is defined as

- Biased hopping: A $(+)$ particle at site $(i_1, i_2, \dots, i_d)$ hops to the site $(i_1 \pm 1, i_2, \dots, i_d)$ at rate $(\ell_d^2 \pm \ell_p/2)/\tau_p$ and to the sites $(i_1, i_2 \pm 1, \dots, i_d), \dots, (i_1, i_2, \dots, i_d \pm 1)$ at rate ℓ_d^2/τ_p , while a $(-)$ particle at site $(i_1, i_2, \dots, i_d)$ hops to the site $(i_1 \pm 1, i_2, \dots, i_d)$ at rate $(\ell_d^2 \mp \ell_p/2)/\tau_p$ and to the sites $(i_1, i_2 \pm 1, \dots, i_d), \dots, (i_1, i_2, \dots, i_d \pm 1)$ at rate ℓ_d^2/τ_p , provided that the target site is unoccupied
- Tumbling: A $(+)$ particle converts to a $(-)$ particle at rate $1/\tau_p$ and vice versa

Starting from an initial configuration $\rho_\pm(x, t_{\text{ini}})$, the probability that the system eventually reaches a final configuration $\rho_\pm(x, t_{\text{fin}})$ has a hydrodynamic path integral description

$$\text{Pr}[\rho_\pm(\vec{x}, t_{\text{fin}}) \mid \rho_\pm(\vec{x}, t_{\text{ini}})] = \int [\mathcal{D}\rho_\pm] [\mathcal{D}\hat{\rho}_\pm] e^{-\ell_d^d \int dt [\int d\vec{x} (\hat{\rho}_+ \partial_t \rho_+ + \hat{\rho}_- \partial_t \rho_-) - H[\hat{\rho}_\pm, \rho_\pm]]} \quad (7.34)$$

with the Hamiltonian given by

$$\begin{aligned}
H = \int d\vec{x} \Big\{ & \rho_+ (1 - \rho) |\vec{\nabla} \hat{\rho}_+|^2 + \rho_- (1 - \rho) |\vec{\nabla} \hat{\rho}_-|^2 \\
& - [(1 - \rho_-) \vec{\nabla} \rho_+ + \rho_+ \vec{\nabla} \rho_- - \text{Pe} \rho_+ (1 - \rho) \hat{x}_1] \cdot \vec{\nabla} \hat{\rho}_+ \\
& - [(1 - \rho_+) \vec{\nabla} \rho_- + \rho_- \vec{\nabla} \rho_+ + \text{Pe} \rho_- (1 - \rho) \hat{x}_1] \cdot \vec{\nabla} \hat{\rho}_- \\
& + (e^{-\hat{\rho}_+ + \hat{\rho}_-} - 1) \rho_+ + (e^{\hat{\rho}_+ - \hat{\rho}_-} - 1) \rho_- \Big\}
\end{aligned} \tag{7.35}$$

and where we have denoted $\rho = \rho_+ + \rho_-$. The derivation is following similar analysis as presented in Appendix 7.A.

The fluctuating hydrodynamic equation corresponding to this action formalism is given by

$$\begin{aligned}
\partial_t \rho_{\pm} = & (1 - \rho) \nabla^2 \rho_{\pm} + \rho_{\pm} \nabla^2 \rho \mp \text{Pe} \partial_{x_1} [\rho_{\pm} (1 - \rho)] \\
& \mp (\rho_+ - \rho_-) + \frac{1}{\ell_d^{d/2}} (\vec{\nabla} \cdot \vec{\eta}_{\pm} \pm \eta_f)
\end{aligned} \tag{7.36}$$

with the Gaussian noises having zero means and covariances $\langle (\vec{\eta}_{\pm}(\vec{x}, t) \cdot \hat{x}_k) (\vec{\eta}_{\pm}(\vec{x}', t') \cdot \hat{x}_{k'}) \rangle = 2\rho_{\pm} (1 - \rho) \delta_{k,k'} \delta(\vec{x} - \vec{x}') \delta(t - t')$ and $\langle \eta_f(\vec{x}, t) \eta_f(\vec{x}', t') \rangle = \rho \delta(\vec{x} - \vec{x}') \delta(t - t')$.

7.B.2 CASE II: $2d$ -SPECIES GENERALIZATION

In this generalization, the system is composed of particles belonging to $2d$ number of species (denoted as $\sigma = \{(\pm)_k\}$ with $k = 1, 2, \dots, d$), corresponding to the possible internal orientations in the spatial directions $\hat{r}_k = \{\pm \hat{x}_k\}$ with $k = 1, 2, \dots, d$. This corresponds to the isotropic generalization of our quasi-1D model in higher dimensions. The microscopic dynamics is defined as

- **Biased hopping:** A particle of $(+)_k$ species at site $(i_1, i_2, \dots, i_d)$ hops to the site $(i_1, i_2, \dots, i_k + 1, \dots, i_d)$ at rate $(\ell_d^2 + \ell_p/2)/\tau_p$ and to the other adjacent sites at rate $(\ell_d^2 - \ell_p/2)/\tau_p$, while a particle of $(-)_k$ species at site $(i_1, i_2, \dots, i_d)$ hops to the site $(i_1, i_2, \dots, i_k - 1, \dots, i_d)$ at rate $(\ell_d^2 + \ell_p/2)/\tau_p$ and to the other adjacent sites at rate $(\ell_d^2 - \ell_p/2)/\tau_p$, provided that the target site is unoccupied.
- **Tumbling:** A particle of σ species converts to any other species at rate $1/\tau_p$.

Starting from an initial configuration $\rho_{\sigma}(x, t_{\text{ini}})$, the probability that the system eventually reaches a final configuration $\rho_{\sigma}(x, t_{\text{fin}})$ has a hydrodynamic path integral description

$$\text{Pr} [\rho_{\sigma}(\vec{x}, t_{\text{fin}}) | \rho_{\sigma}(\vec{x}, t_{\text{ini}})] \simeq \int [\mathcal{D}\rho] [\mathcal{D}\hat{\rho}] e^{-\ell_d^d \int dt \int d\vec{x} \sum_{\sigma} (\hat{\rho}_{\sigma} \partial_t \rho_{\sigma}) - H[\rho_{\sigma}, \hat{\rho}_{\sigma}]}, \tag{7.37}$$

where the Hamiltonian is given by

$$\begin{aligned}
H = \int d\vec{x} \sum_{\sigma} \Big\{ & \rho_{\sigma} (1 - \rho) |\vec{\nabla} \hat{\rho}_{\sigma}|^2 - [(1 - \rho) \vec{\nabla} \rho_{\sigma} \\
& + \rho_{\sigma} \vec{\nabla} \rho - \text{Pe} \rho_{\sigma} (1 - \rho) \hat{r}_{\sigma}] \cdot \vec{\nabla} \hat{\rho}_{\sigma} \\
& + \sum_{\sigma' \neq \sigma} \rho_{\sigma'} (e^{\hat{\rho}_{\sigma'} - \hat{\rho}_{\sigma}} - 1) \Big\}
\end{aligned} \tag{7.38}$$

and we have denoted $\rho = \sum_{\sigma} \rho_{\sigma}$.

Corresponding to this action formalism, the fluctuating hydrodynamics for the σ -species particle is then given by

$$\begin{aligned} \partial_t \rho_{\sigma} = & (1 - \rho) \nabla^2 \rho_{\sigma} + \rho_{\sigma} \nabla^2 \rho - \text{Pe} \vec{\nabla} [\rho_{\sigma} (1 - \rho)] \cdot \hat{r}_{\sigma} \\ & - \sum_{\sigma' \neq \sigma} (\rho_{\sigma} - \rho_{\sigma'}) + \frac{1}{\ell_d^{d/2}} \left(\vec{\nabla} \cdot \vec{\eta}_{\sigma} + \sum_{\sigma' \neq \sigma} \eta_{\sigma \rightarrow \sigma'} \right) \end{aligned} \quad (7.39)$$

with the Gaussian noises having zero means and covariances $\langle (\vec{\eta}_{\sigma}(\vec{x}, t) \cdot \hat{x}_k) (\vec{\eta}_{\sigma'}(\vec{x}', t') \cdot \hat{x}_{k'}) \rangle = 2 \rho_{\sigma} (1 - \rho) \delta_{\sigma, \sigma'} \delta_{k, k'} \delta(\vec{x} - \vec{x}') \delta(t - t')$ and $\langle \eta_{\sigma \rightarrow \sigma'}(\vec{x}, t) \eta_{\psi \rightarrow \psi'}(\vec{x}', t') \rangle = (\rho_{\sigma} + \rho_{\sigma'}) \delta_{\sigma, \psi} \delta_{\sigma', \psi'} \delta(\vec{x} - \vec{x}') \delta(t - t')$.

7.C THE STRICTLY 1D MODEL WITH EXCHANGE DYNAMICS

Until now, we have strictly imposed the exclusion constraint which forbids two particles in adjacent sites to cross each other on the 1D lattice. However, in certain physical examples, this single-file constraint may be partially relaxed. For example, the dynamics of active particles inside a channel whose width is of the order of the diameter of the particles [291]. Motivated by this scenario, we discuss the case where two adjacent particles on the 1D lattice are allowed to exchange their positions at a constant rate, in addition to their usual biased-hopping and tumbling dynamics described in items (1) and (2) of the main text.

Precisely, we consider the model on a one-dimensional lattice with the following dynamics:

- Biased hopping: A (+) particle at site i hops to site $i \pm 1$ at rate $(\ell_d^2 \pm \ell_p/2)/\tau_p$ while a (−) particle at site i hops to site $i \pm 1$ at rate $(\ell_d^2 \mp \ell_p/2)/\tau_p$, provided that the target site is empty.
- Tumbling: A particle changes its internal orientation (i.e. (+) to (−) and vice versa) at rate $1/\tau_p$.
- Particle-position exchanging: A pair of ($\pm$) particles at sites i and $i + 1$, exchanges their positions at rate ℓ_e^2/τ_p .

Following the same procedure outlined in Appendix 7.A, we write the transition probability of the system starting from an initial configuration $\rho_{\pm}(x, t_{\text{ini}})$ and reaching a final configuration $\rho_{\pm}(x, t_{\text{fin}})$ in terms of a path integral

$$\begin{aligned} \text{Pr} [\rho(x, t_{\text{fin}}), m(x, t_{\text{fin}}) \mid \rho(x, t_{\text{ini}}), m(x, t_{\text{ini}})] \\ \simeq \int [\mathcal{D}\rho] [\mathcal{D}m] [\mathcal{D}\hat{\rho}] [\mathcal{D}\hat{m}] e^{-\ell_d \int dt [\int dx (\hat{\rho} \partial_t \rho + \hat{m} \partial_t m) - H[\rho, m, \hat{\rho}, \hat{m}]}], \end{aligned} \quad (7.40)$$

The corresponding fluctuating hydrodynamic equation for the density field is identical to the one in (7.31a) with the same parameters, while that for the polarization field is similar to the one in (7.31b) but with the modified values of $\bar{J}_m$ and $\langle \eta_m(x, t) \eta_m(x', t') \rangle$.

The $\mathcal{E} = 1$ limit corresponds to the lattice-gas model in [280] for which the fluctuating hydrodynamic description was derived [282] using an ingenious mapping to known integrable models. The mapping does not straightforwardly extend for the dynamics with arbitrary $\mathcal{E}$.

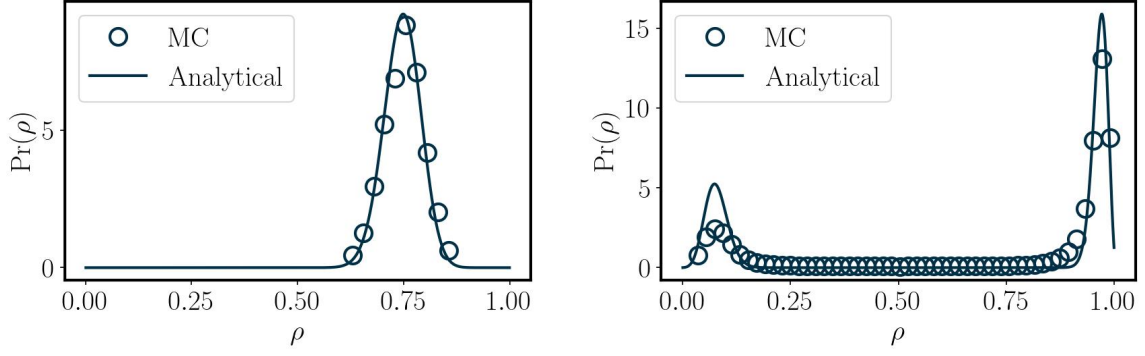
(a) Low Péclet number: $Pe = 2$ (b) High Péclet number: $Pe = 10$

Figure 7.D.1. For the quasi-1D model: The number density distribution from Monte Carlo simulation compared with the distributions (7.43) and (7.44) for appropriate phases in the two-lane ladder model defined in the main text. The bulk-average density is $\rho_0 = 0.75$. For (a), the parameter values correspond to the homogeneous phase and for (b), they are for the phase-separated state.

Remark. In the $\mathcal{E} = 0$ limit, the hydrodynamic description breaks down, as discussed in the main text and also in the next Appendix.

7.D NUMERICAL EVIDENCE FOR LOCAL EQUILIBRIUM

Our construction of the fluctuating hydrodynamics crucially relies on an assumption of local equilibrium. For our choice of rates in items (1)–(2) in the main text, diffusion dominates in length scales of order ℓ_d , and consequently, the nearby sites within this diffusive length are expected to reach a local equilibrium in times $\tau_p \sim \ell_d^2$ around a quasistatic average density defined in (7.14). A consequence of this particular local equilibrium assumption is that the total number N of particles, irrespective of their polarity, in a region of $l \sim \ell_d$ sites centered around $i \simeq x\ell_d$ at time $\tau \simeq t\tau_p$ is given by the binomial distribution [316]

$$\Pr_l(N | \rho(x, t)) = \binom{l}{N} \rho(x, t)^N (1 - \rho(x, t))^{l-N}, \quad (7.41)$$

where $\rho(x, t)$ is the slowly varying local average density. Taking N and l to be large, while keeping $n = N/l$ finite, a Stirling's approximation gives

$$\Pr_l(n | \rho) \simeq \exp \left\{ \left\{ -l \left[n \ln \frac{n}{\rho} + (1 - n) \ln \frac{1 - n}{1 - \rho} \right] \right\} \right\}. \quad (7.42)$$

In the homogeneous stationary state i.e., outside the binodal region, the most probable local average density profile is uniform $\rho(x) = \rho_0$ everywhere. The probability $\Pr(n)$ of the number-density of particles sampled over *random* boxes of fixed length l in the bulk comes from (7.42) corresponding to this typical average density ρ_0 , leading to

$$\Pr(n) = \Pr_l(n | \rho_0). \quad (7.43)$$

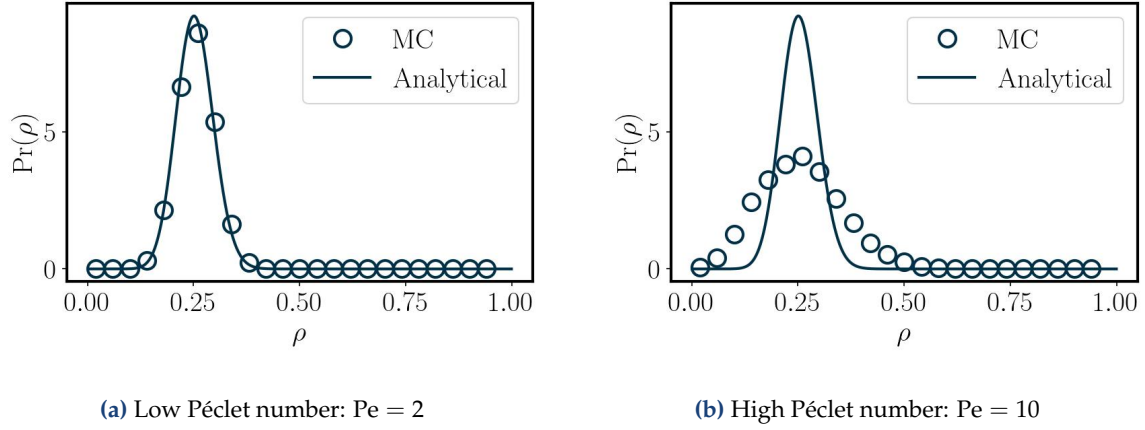


Figure 7.D.2. For the strictly-1D model: Comparison of Monte Carlo results of number density distribution and (7.43). Panel (a) show good agreement, supporting local equilibrium for bulk average density $\rho_0 = 0.25$ and low Péclet number. However, as shown in panel (b) at higher Péclet but at same ρ_0 , even though the system is in homogeneous phase, the (7.43) significantly differs from numerically observed distribution, indicating breakdown of our local equilibrium assumption.

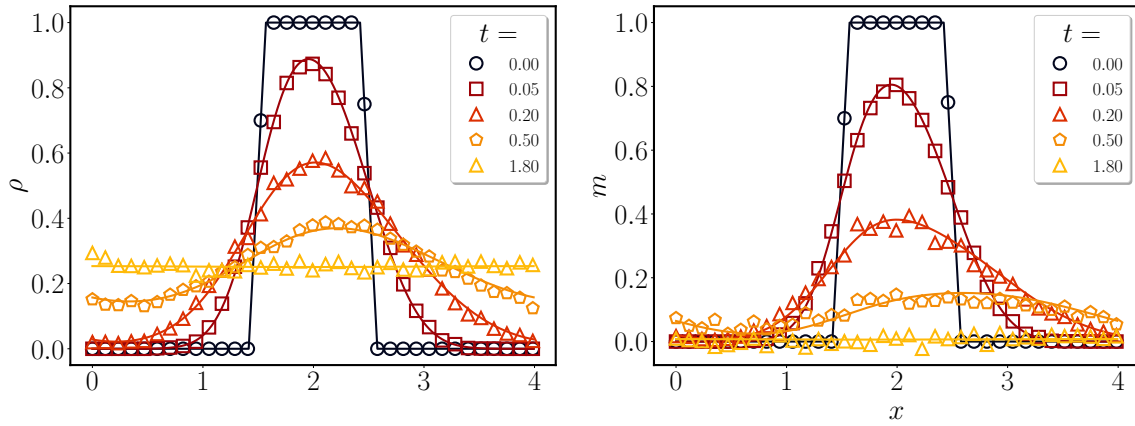


Figure 7.D.3. Validity of noiseless hydrodynamics in the quasi-1D model: A comparison of the evolution of density $\rho(x, t)$ and polarization $m(x, t)$ from Monte Carlo simulation of the microscopic dynamics and from the noiseless hydrodynamics in (7.31). The evolution starts from a step initial profile for both $\rho(x, 0)$ and $m(x, 0)$, centered around the middle of the system. The solid lines represent noiseless hydrodynamics evolution (7.31), while the markers represent Monte Carlo simulations of the microscopic dynamics. To compare with the noiseless hydrodynamics, the Monte Carlo results are binned over 20 sites and averaged over 64 realizations. The hydrodynamic simulations are performed in the domain $[0, L/\ell_d = 4]$ using a pseudospectral method with 2^9 modes, described later in Appendix 7.H. The plots are for $\rho_0 = 0.25$ and the system size $L = 4\ell_d$ with the dynamical parameters $\ell_d = 128$, $\tau_p = \ell_d^2$, $\ell_p = 2\ell_d$, and $\tau_x = 0.05$, which corresponds to $Pe = 2$. These parameters correspond to a homogeneous stationary state.

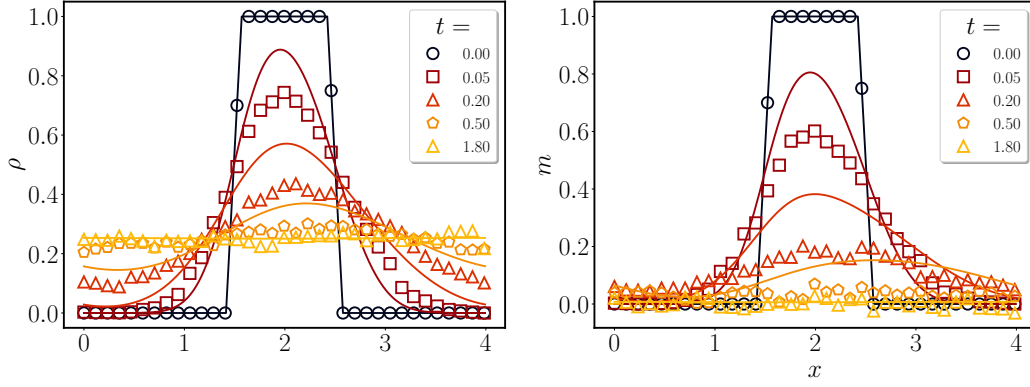


Figure 7.D.4. Breakdown of noiseless hydrodynamics in the strictly-1D model: A similar comparison between hydrodynamic evolution and Monte Carlo simulations of microscopic dynamics for the single-lane model, for identical parameter values as in Fig. 7.D.3. The noticeable deviations between the microscopic and the hydrodynamics indicate unsuitability of the latter for the 1D model.

In the phase-separated stationary state, i.e. inside the spinodal region, there are two typical average densities, ρ_l and ρ_g , corresponding to the liquid and the gas phases, respectively. In random sampling of the box position, probability for the box to be inside the liquid or the gas region is proportional to their volume fraction. This results in the probability of number density inside randomly sampled boxes,

$$\Pr(n) = v \Pr_l(n | \rho_l) + (1 - v) \Pr_l(n | \rho_g), \quad (7.44)$$

where $v = (\rho_0 - \rho_g) / (\rho_l - \rho_g)$ denotes the volume fraction of the liquid phase.

The number-density distributions in (7.43) and (7.44) are an indirect test of the local equilibrium assumption. To verify this crucial assumption, we numerically measure $\Pr(n)$ from samples of stationary-state configurations generated using Monte Carlo simulation of the quasi-one-dimensional dynamics. Comparison of the numerical results against the distributions (7.43) and (7.44) are shown in Fig. 7.D.1. The left panel, corresponding to a parameter regime outside the binodal (high density, low activity), thus in the homogeneous phase, shows excellent agreement with (7.43). The right panel depicts a high-density, high-activity regime where MIPS forms, as evidenced by the bimodal distribution. There is good agreement with (7.44), except slight deviations originating from contributions near the liquid-gas interface, which are not included in (7.44).

A similar test of local equilibrium for the strictly 1D case, shows a different scenario than in the quasi-one-dimensional case. At very low Péclet value and low density, the local equilibrium assumption is consistent as shown in the left panel of Fig. 7.D.2. However, at higher Péclet, but still inside the homogeneous phase, the number distribution differs from the theoretical prediction (7.43), suggesting a breakdown of our specific assumption of local equilibrium.

The consequence of local equilibrium for quasi-one-dimension and its breakdown for strict one dimension is reflected in the validity of hydrodynamics in describing the Monte Carlo evolution, shown in Figs. 7.D.3 and 7.D.4.

7.E THE SPINODAL ANALYSIS

The noiseless hydrodynamics is obtained from (7.31) by setting the noise terms equal to zero

$$\partial_t \rho = -\partial_x \bar{J}_\rho, \quad (7.45a)$$

$$\partial_t m = -\partial_x \bar{J}_m - 2\bar{J}_f, \quad (7.45b)$$

where the conservative currents $\bar{J}_\rho$ and $\bar{J}_m$ arising due to biased-diffusion dynamics are given in (7.28) and (7.29) respectively while the nonconservative current $\bar{J}_f$ arising due to tumbling events is given in (7.30).

A particular solution for the stationary state of (7.45) is given by the homogeneous profiles $\rho(x, t) = \rho_0$ and $m(x, t) = 0$. A linear stability analysis of the homogeneous solution give the spinodal curves for MIPS. Our analysis is similar to [280, 300] used in a related problem. We perturb the stationary homogeneous profiles as $\rho(x, t) = \rho_0 + \delta\rho(x, t)$ and $m(x, t) = \delta m(x, t)$ with $\delta\rho$ and δm being small fluctuations in the hydrodynamic fields. This allows us to write the hydrodynamics for the small perturbations up to the leading order as

$$\partial_t(\delta\rho) = \partial_x^2(\delta\rho) - \text{Pe}(1 - \rho_0)\partial_x(\delta m), \quad (7.46a)$$

$$\partial_t(\delta m) = (1 - \rho_0)\partial_x^2(\delta m) - \text{Pe}(1 - 2\rho_0)\partial_x(\delta\rho) - 2\delta m. \quad (7.46b)$$

Defining the continuous Fourier transformation

$$\begin{pmatrix} \tilde{\delta\rho}(k, t) \\ \tilde{\delta m}(k, t) \end{pmatrix} = \frac{1}{L/\ell_d} \int_0^{L/\ell_d} dx e^{-ikx} \begin{pmatrix} \delta\rho(x, t) \\ \delta m(x, t) \end{pmatrix} \quad (7.47)$$

for a large L/ℓ_d , we can rewrite the Fourier-transformed noiseless hydrodynamic equations for the small fluctuating fields as

$$\begin{aligned} \partial_t \begin{pmatrix} \tilde{\delta\rho}(k, t) \\ \tilde{\delta m}(k, t) \end{pmatrix} &= \begin{pmatrix} -k^2 & -i\text{Pe}(1 - \rho_0)k \\ i\text{Pe}(2\rho_0 - 1)k & -(1 - \rho_0)k^2 - 2 \end{pmatrix} \\ &\quad \times \begin{pmatrix} \tilde{\delta\rho}(k, t) \\ \tilde{\delta m}(k, t) \end{pmatrix}, \end{aligned} \quad (7.48)$$

where $k = 2\pi n\ell_d/L$ with $n = 0, \pm 1, \pm 2, \dots$. The eigenvalues of the above matrix are given by

$$\begin{aligned} \lambda_{\pm}(k) &= -1 - \left(1 - \frac{\rho_0}{2}\right)k^2 \\ &\quad \pm \sqrt{1 + [\text{Pe}^2(1 - \rho_0)(2\rho_0 - 1) - \rho_0]k^2 + \frac{\rho_0^2 k^4}{4}}, \end{aligned} \quad (7.49)$$

which obey the symmetry $\lambda_{\pm}(-k) = \lambda_{\pm}(k)$.

The two eigenvalues $\lambda_{+}(0) = 0$ and $\lambda_{-}(0) = -2$ corresponding to the $k = 0$ mode respectively describe the invariance of the spatially integrated total density $\int dx \rho(x, t)$ and the exponential decay of the spatially integrated polarization $M(t) = \int dx m(x, t) = M(0)e^{-2t}$, which also directly follows from (7.45).

The homogeneous solutions become linearly unstable at large wavelengths when one of the eigenvalues corresponding to the first mode [i.e., $k = \pm 2\pi/(L/\ell_d)$] becomes positive, $\lambda_{+}(2\pi/(L/\ell_d)) > 0$. This, consequently, puts a condition on the macroscopic size of the system as

$$L/\ell_d > \frac{2\pi\sqrt{1-\rho_0}}{\sqrt{\text{Pe}^2(1-\rho_0)(2\rho_0-1)-2}} \quad (7.50)$$

Note that the term within the square root in the denominator must be positive, i.e.,

$$\text{Pe}^2(1-\rho_0)(2\rho_0-1) > 2. \quad (7.51)$$

Interestingly, this is the same instability condition found earlier [280], where the single-file constraint is not respected and a $\pm$ pair of particles at neighboring sites is allowed to exchange their positions at a rate ℓ_d^2/τ_p . The above condition is equivalent to

$$(\rho_0 - \rho_l^s)(\rho_0 - \rho_h^s) < 0, \quad (7.52)$$

where

$$\rho_l^s = \frac{3}{4} - \frac{1}{4}\sqrt{1 - \left(\frac{4}{\text{Pe}}\right)^2}, \text{ and } \rho_h^s = \frac{3}{4} + \frac{1}{4}\sqrt{1 - \left(\frac{4}{\text{Pe}}\right)^2}. \quad (7.53)$$

Therefore, for $\text{Pe} > 4$, the system is linearly unstable in the spinodal region given by $\rho_0 \in [\rho_l^s, \rho_h^s]$.

7.F THE BINODAL ANALYSIS

For a system in a phase-separated state, a high-density liquidlike phase and a low-density gaslike phase coexist. The binodal or the coexistence curve gives the two densities, denoted by ρ_l and ρ_g respectively, as a function of a control parameter, which in our case is the Péclet number Pe . To compute the binodal, we use the procedure followed in [280, 300], which we outline below for the convenience of the readers.

The stationary state is given by $\partial_t \rho = 0$, which has a zero flux due to symmetry, which gives us $\bar{J}_\rho = 0$. Using (7.28), we obtain

$$m = \frac{1}{\text{Pe}} \frac{\partial_x \rho}{1-\rho} = -\frac{1}{\text{Pe}} \partial_x [\ln(1-\rho)]. \quad (7.54)$$

The stationary state for the polarization field i.e., $\partial_t m = 0$ gives us $\partial_x \bar{J}_m - 2\bar{J}_f = 0$, which, using (7.29), (7.30) and the above expression of m , can be rewritten as $\partial_x g = 0$, where

$$g(\rho) = g_0(\rho) + \Lambda(\rho) (\partial_x \rho)^2 - \kappa \partial_x^2 \rho = \text{const}, \text{ with} \quad (7.55)$$

$$g_0(\rho) = \text{Pe} \rho(1-\rho) - \frac{2}{\text{Pe}} \ln(1-\rho), \quad (7.56)$$

$$\Lambda(\rho) = -\frac{2}{\text{Pe}} \frac{1}{1-\rho}, \text{ and } \kappa = \frac{1}{\text{Pe}}. \quad (7.57)$$

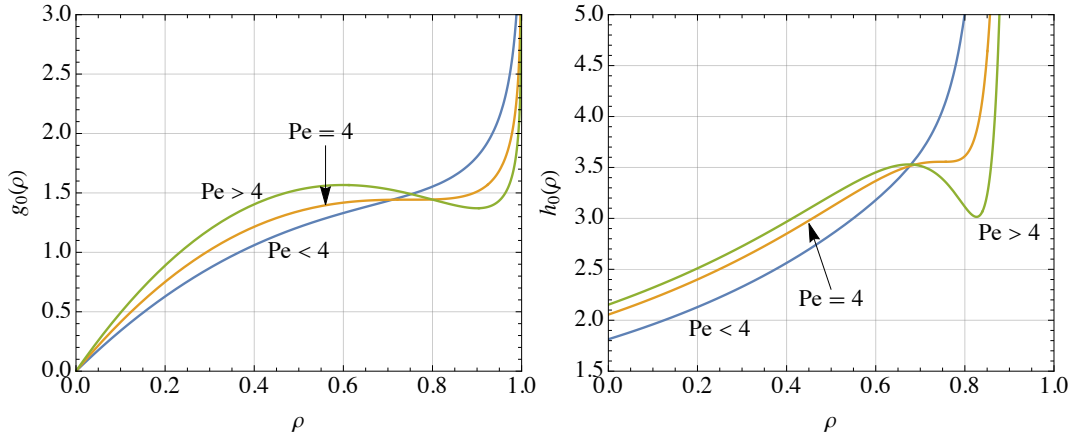


Figure 7.F.1. The binodal curves are given by the pair of conditions $g_0(\rho_g) = g_0(\rho_l)$ and $h_0(\rho_g) = h_0(\rho_l)$, where the expressions of $g_0(\rho)$ and $h_0(\rho)$ are given by (7.56) and (7.69) respectively. Here, $g_0(\rho)$ and $h_0(\rho)$ are plotted as a function of ρ on the left and right figures, respectively. For $Pe < 4$, both are monotonic functions of ρ . On the other hand, for Pe , both $g_0(\rho)$ and $h_0(\rho)$ have a minimum and a maximum. For $Pe = 4$, there is an inflection point at $\rho = 3/4$, where the first and the second derivatives of both g_0 and h_0 vanish.

The spinodal curve is equivalently defined by $g'_0(\rho) = 0$, whose two solutions are given by ρ_l^s and ρ_h^s respectively [see (7.53)].

7.F.1 FIRST RELATION

In the coexisting phase, density is homogeneous in the liquid (ρ_l) and gas (ρ_g) phases: $\partial_x \rho = \partial_x^2 \rho = 0$ in the two phases, giving

$$g_0(\rho_g) = g_0(\rho_l) \equiv \bar{g}(Pe), \quad (7.58)$$

For $Pe < 4$, $g_0(\rho)$ is a monotonic function of ρ . On the other hand, it has a maximum and minimum at ρ_l^s and ρ_h^s respectively for $Pe > 4$ [see Fig. 7.F.1]. For $Pe = 4$, there is an inflection point at $\rho = 3/4$, where both $g'_0(\rho)$ and $g''_0(\rho)$ are zero. For $Pe > 4$, any line $g_0(\rho) = \text{const} \in (g_0(\rho_l^s), g_0(\rho_h^s))$, intersects the $g_0(\rho)$ vs. ρ curve at three points, and therefore, there exists an infinite number of solutions for (7.58). Hence, we need another relation for fixing the coexistence densities, which we discuss below.

7.F.2 SECOND RELATION

Consider the following integral across the liquid-gas interface in a phase-separated state

$$I = \int_{x_g}^{x_l} dx g(\rho) \partial_x R(\rho), \quad (7.59)$$

where $R(\rho)$ is some function of ρ , and x_g and x_l are any two points inside the gas and liquid phases respectively. Since $g(\rho) = \bar{g}$ is constant, the above integral yields

$$I = \bar{g} [R(\rho_l) - R(\rho_g)]. \quad (7.60)$$

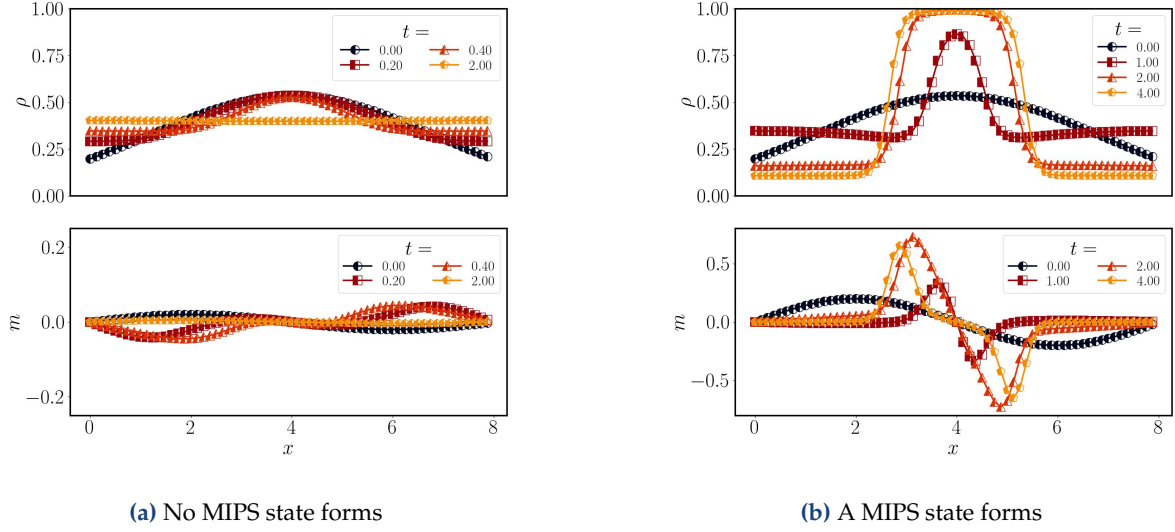


Figure 7.F.2. Metastability: Numerical evolution of the noiseless hydrodynamic equation (7.31) for regions outside the spinodal curve but inside the binodal curve demonstrates that a homogeneous solution is linearly stable, but nonlinear instabilities could drive the system to an inhomogeneous MIPS state. The two panels correspond to two initial states with same initial Gaussian total density profile $\rho(x, 0)$ but different initial polarization fields $m(x, 0) = A \sin 2\pi x/L$ with amplitude $A = 0.02$ for panel (a) and $A = 0.2$ for panel (b). The parameters $Pe = 10$ and $\rho_0 = 0.4$ are for a region between the spinodal and binodal curves. The small inhomogeneity of $m(x, 0)$ in panel (a) drives the system towards a uniform final state, whereas larger inhomogeneity in panel (b) leads to an inhomogeneous MIPS state.

On the other hand, substituting the expression of $g(\rho)$ from (7.55) in the integral gives

$$I = \int_{x_g}^{x_l} dx g_0(\rho) \partial_x R(\rho) + \int_{x_g}^{x_l} dx [\Lambda(\rho) R'(\rho) (\partial_x \rho)^2 - \kappa R'(\rho) \partial_x^2 \rho] \partial_x \rho. \quad (7.61)$$

Now we choose $R(\rho)$ such that the second integral is zero. Consider the relation

$$\partial_x [A(\rho) (\partial_x \rho)^2] = [A'(\rho) (\partial_x \rho)^2 + 2A(\rho) \partial_x^2 \rho] \partial_x \rho. \quad (7.62)$$

Choosing $2A(\rho) = -\kappa R'(\rho)$ and $\kappa R''(\rho) = -2\Lambda(\rho) R'(\rho)$, the above relation gives

$$-\partial_x \left[\frac{\kappa}{2} R'(\rho) (\partial_x \rho)^2 \right] = [\Lambda(\rho) R'(\rho) (\partial_x \rho)^2 - \kappa R'(\rho) \partial_x^2 \rho] \partial_x \rho. \quad (7.63)$$

Since $\partial_x \rho = 0$ in the liquid and the gas phases, the integral over the above expression across the interface is zero. Hence, defining $g_0(\rho) = d\phi(R)/dR$, from (7.61) we get

$$I = \phi(R_l) - \phi(R_g) \quad (7.64)$$

where $R_l = R(\rho_l)$ and $R_g = R(\rho_g)$. Combining the two expressions of I , (7.60) and (7.64), we arrive at the relation

$$h_0(R_g) = h_0(R_l) \text{ where } h_0(R) = R\phi'(R) - \phi(R). \quad (7.65)$$

For the particular form of $\Lambda(\rho)$ given above, the above differential equation of $R(\rho)$ can be solved exactly to give (up to an additive and a multiplicative constant)

$$R(\rho) = \frac{1}{3} \frac{1}{(1-\rho)^3}. \quad (7.66)$$

Inverting the above equation, we can express $\rho(R) = 1 - (3R)^{-1/3}$, which in turn gives

$$\phi'(R) = \text{Pe} [1 - (3R)^{-1/3}] (3R)^{-1/3} + \frac{2}{3\text{Pe}} \ln(3R). \quad (7.67)$$

Now, integrating with respect to R gives (up to an additive constant, which does not show up in the relation)

$$\phi(R) = \text{Pe} \left[\frac{(3R)^{1/3}}{2} - 1 \right] (3R)^{1/3} + \frac{2R}{3\text{Pe}} [\ln(3R) - 1]. \quad (7.68)$$

Therefore, $h_0(R(\rho))$ can be expressed in terms of ρ explicitly as

$$h_0(\rho) = \frac{2}{9\text{Pe}} \frac{1}{(1-\rho)^3} + \frac{\text{Pe}}{6} \frac{3-4\rho}{(1-\rho)^2} \quad (7.69)$$

and the second coexistence relation can now be written as

$$h_0(\rho_g) = h_0(\rho_l). \quad (7.70)$$

The qualitative behavior of h_0 is similar to that of $g_0(\rho)$ [see Fig. 7.F.1]. For $\text{Pe} < 4$, $h_0(\rho)$ is a monotonic function of ρ , whereas for $\text{Pe} > 4$, it has a maximum and a minimum at ρ_l^s and ρ_h^s respectively. For $\text{Pe} = 4$, there is an inflection point at $\rho = 3/4$, where both $h'_0(\rho)$ and $h''_0(\rho)$ are zero.

Interestingly, the two solutions of $h'_0(\rho) = 0$ are same as that of $g'_0(\rho) = 0$ and are given by ρ_l^s and ρ_h^s respectively [see (7.53)]. This follows from using $\phi'(R) = g_0(\rho)$, writing $h_0(\rho) = R(\rho)g_0(\rho) - \phi(R(\rho))$ and taking a derivative with respect to ρ

$$\begin{aligned} h'_0(\rho) &= R(\rho) g'_0(\rho) + R'(\rho) g_0(\rho) - R'(\rho) \phi'(R) \\ &= R(\rho) g'_0(\rho). \end{aligned} \quad (7.71)$$

The binodal curves are obtained by numerically solving the relations $g_0(\rho_g) = g_0(\rho_l)$ and $h_0(\rho_g) = h_0(\rho_l)$, where the expressions of $g_0(\rho)$ and $h_0(\rho)$ are given by (7.56) and (7.69) respectively.

Remark. In the intermediate region between the spinodal and binodal curves, the uniform homogeneous solution of the noiseless hydrodynamics is linearly stable, although there are two additional solutions corresponding to the two binodal densities ρ_l and ρ_g . In practice, these solutions are selected depending on the initial state, as shown in Fig. 7.F.2.

7.G THE TWO-POINT CORRELATIONS IN THE STATIONARY-STATE HOMOGENEOUS PHASE

In this appendix, we present the derivation of the two-point correlations of the hydrodynamic fields in the stationary state of the homogeneous phase of our quasi-1D model. This serves as an

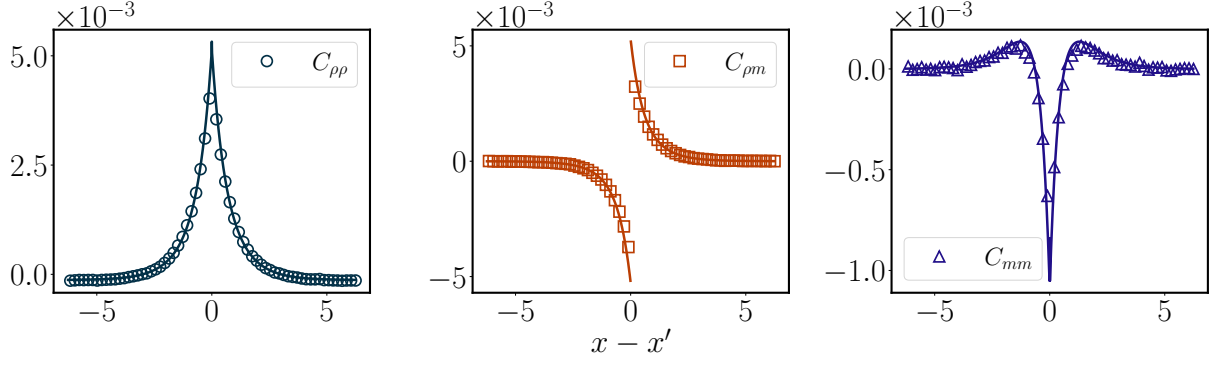


Figure 7.G.1. Correlations in the homogeneous phase in quasi-1D model: An excellent agreement between the hydrodynamic predictions in (7.83), indicated by solid lines, and their results from a Monte Carlo simulation, indicated by plot markers. The results are for the homogeneous phase with $\rho_0 = 0.25$, $Pe = 2$. For the simulation, the microscopic correlations $C_{\rho\rho}(i, i') = \langle n_i n_{i'} \rangle - \langle n_i \rangle \langle n_{i'} \rangle$, $C_{\rho m}(i, i') = \langle n_i M_{i'} \rangle - \langle n_i \rangle \langle M_{i'} \rangle$, and $C_{mm}(i, i') = \langle M_i M_{i'} \rangle - \langle M_i \rangle \langle M_{i'} \rangle$, where, i and i' denotes the lattice sites, $n_i = \{0, 1\}$ and $M_i = \{0, \pm 1\}$ denotes the total occupation and polarization of the i^{th} site, and $\langle \cdot \rangle$ denotes the averaging over time after reaching the stationary state. Using the translational symmetry, due to periodic boundary conditions, spatial averaging is conducted to further refine the averaging. For comparing with the fluctuating hydrodynamics results, we use the re-scaling as $x - x' = (i - i')/\ell_d$. Evidently, the $C_{\rho\rho}(x)$ and $C_{mm}(x)$ are symmetric in x , while the $C_{\rho m}(x)$ is anti-symmetric in x .

indirect check of our fluctuating hydrodynamic description, reported in eq. (1) of the main text. Our result for the correlations is valid up to the leading order in $1/\ell_d$ and for this purpose, we use the Gaussian noise approximation, ignoring any subleading terms.

The analysis follows a standard approach [282, 317, 318] for correlations using fluctuating hydrodynamics. We begin our derivation by adding small fluctuations, $\delta\rho(x, t)$ and $\delta m(x, t)$ which are of at most $O(1/\sqrt{\ell_d})$ around the stationary-state homogeneous fields, $\rho(x, t) = \rho_0$ and $m(x, t) = 0$ respectively. The definition of the two-point correlations of the fluctuating fields then readily follows

$$C_{\rho\rho}(x, x') = \langle \delta\rho(x, t) \delta\rho(x', t) \rangle, \quad (7.72a)$$

$$C_{\rho m}(x, x') = \langle \delta\rho(x, t) \delta m(x', t) \rangle, \text{ and} \quad (7.72b)$$

$$C_{mm}(x, x') = \langle \delta m(x, t) \delta m(x', t) \rangle. \quad (7.72c)$$

Putting these perturbed total density and polarization fields in eq. (1) of the main text and keeping only the terms which are at most linear in $\delta\rho$ and δm , we obtain

$$\partial_t \begin{pmatrix} \delta\rho(x, t) \\ \delta m(x, t) \end{pmatrix} = D_x \begin{pmatrix} \delta\rho(x, t) \\ \delta m(x, t) \end{pmatrix} + F(x, t) \quad (7.73a)$$

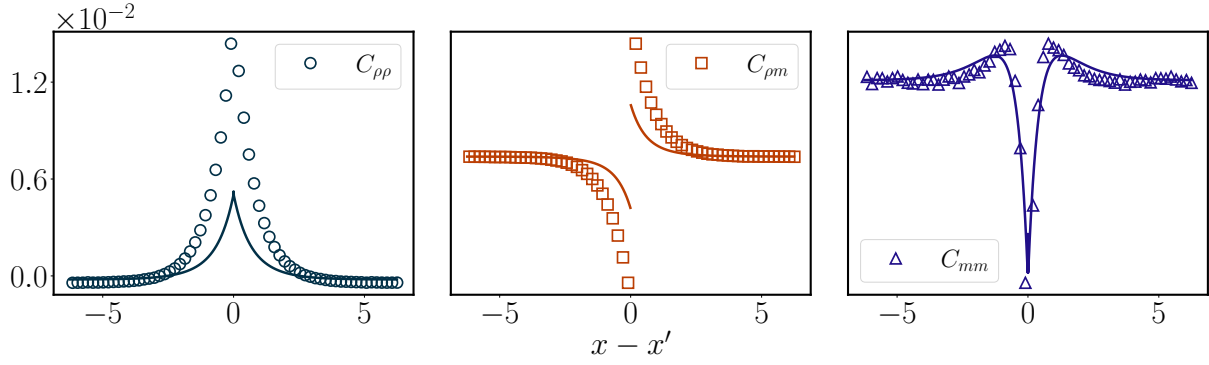


Figure 7.G.2. Correlations in the homogeneous phase in strictly-1D model: Similar comparison of correlations in the single-lane model for parameters identical to Fig. 7.G.1. The significant deviation between the hydrodynamic and the Monte Carlo results indicate breakdown of the former in the strict 1D geometry.

where the deterministic, D_x and fluctuating, $F(x, t)$ components are respectively given as

$$D_x = \begin{pmatrix} \partial_x^2 & -\text{Pe}(1 - \rho_0) \partial_x \\ -\text{Pe}(1 - 2\rho_0) \partial_x & (1 - \rho_0) \partial_x^2 - 2 \end{pmatrix}, \quad (7.73b)$$

$$F(x, t) = \sqrt{\frac{2\rho_0}{\ell_d}} \begin{pmatrix} \sqrt{1 - \rho_0} \partial_x \eta_\rho \\ \sqrt{1 - \rho_0} \partial_x \eta_m + \sqrt{2} \eta_f \end{pmatrix} \quad (7.73c)$$

where $\eta_\rho(x, t)$, $\eta_m(x, t)$ and $\eta_f(x, t)$ are delta-correlated Gaussian white noises with unit self-covariance and zero cross covariance.

In the large hydrodynamic system size limit $L/\ell_d \rightarrow \infty$, we define the continuous-Fourier-transformed fluctuating hydrodynamic fields as

$$\begin{pmatrix} \delta\tilde{\rho}(k, t) \\ \delta\tilde{m}(k, t) \end{pmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \begin{pmatrix} \delta\rho(x, t) \\ \delta m(x, t) \end{pmatrix} e^{-ikx} \quad (7.74)$$

and subsequently rewrite the fluctuating hydrodynamic equations in the Fourier space

$$\partial_t \begin{pmatrix} \delta\tilde{\rho}(k, t) \\ \delta\tilde{m}(k, t) \end{pmatrix} = \tilde{D}_k \begin{pmatrix} \delta\tilde{\rho}(k, t) \\ \delta\tilde{m}(k, t) \end{pmatrix} + \tilde{F}(k, t) \quad (7.75a)$$

with the deterministic and fluctuating components in the Fourier space respectively given by

$$\tilde{D}_k = \begin{pmatrix} -k^2 & -i \text{Pe}(1 - \rho_0) k \\ -i \text{Pe}(1 - 2\rho_0) k & -(1 - \rho_0)k^2 - 2 \end{pmatrix}, \quad (7.75b)$$

$$\tilde{F}(k, t) = \sqrt{\frac{2\rho_0}{\ell_d}} \begin{pmatrix} i \sqrt{1 - \rho_0} k \tilde{\eta}_1(k, t) \\ \sqrt{(1 - \rho_0)k^2 + 2} \tilde{\eta}_2(k, t) \end{pmatrix}, \quad (7.75c)$$

where $\tilde{\eta}_1(k, t)$ and $\tilde{\eta}_2(k, t)$ are Gaussian noises with mean and covariance

$$\langle \eta_\alpha(x, t) \rangle = 0 \text{ and} \quad (7.75d)$$

$$\langle \eta_\alpha(x, t) \eta_{\alpha'}(x', t') \rangle = \delta_{\alpha, \alpha'} \delta(x - x') \delta(t - t'), \quad (7.75e)$$

where $\alpha, \alpha' = \{1, 2\}$.

The differential equations in (7.75a) can now be readily solved which leads to the solutions of the fluctuation fields in Fourier space in terms of $\tilde{D}_k$ and $\tilde{F}(k, t)$ as

$$\begin{pmatrix} \delta\tilde{\rho}(k, t) \\ \delta\tilde{m}(k, t) \end{pmatrix} = \int_{-\infty}^t dt' e^{D_k(t-t')} F(k, t') \quad (7.76)$$

$$\tilde{C}_{\rho\rho}(k, k') = \frac{\rho_0 A \{ ABk^4 + [4A + \text{Pe}^2 A^2 + \chi^2]k^2 + 2[2 + \text{Pe}^2 A] \}}{\{ ABk^4 + [2A + B\chi^2]k^2 + 2\chi^2 \} \ell_d} \delta(k + k'), \quad (7.77)$$

where we have defined:

$$A = 1 - \rho_0, \quad B = 2 - \rho_0, \quad (7.78)$$

and the parameter

$$\chi = \sqrt{2 - \text{Pe}^2 (1 - \rho_0) (2\rho_0 - 1)} \quad (7.79)$$

determines the onset of linear instability of the stationary-state homogeneous fields under small perturbations i.e., the spinodal region.

The $k = (k' =) 0$ mode relates to the spatial integral of the fluctuations of the total density field, $\int dx \delta\rho(x, t)$. Although the total number of particles in the two-lane system as a whole is conserved, the number of particles in each of the lanes is allowed to fluctuate due to the lane-crossing dynamics. This particle number fluctuation in the two lanes of the system can be assumed to be independent of each other in the long time limit which implies that the $2 \times \tilde{C}_{\rho\rho}(k, k')$ must vanish when $k = (k' =) 0$. We take this into account by including an additional $k = (k' =) 0$ term proportional to the infinitesimal intermode width ℓ_d/L , such that the correlation in (7.77) is now given as

$$\tilde{C}_{\rho\rho}(k, k') = \left[\frac{\rho_0 A \left\{ ABk^4 + [4A + \text{Pe}^2 A^2 + \chi^2]k^2 + 2[2 + \text{Pe}^2 A] \right\}}{\{ ABk^4 + [2A + B\chi^2]k^2 + 2\chi^2 \} \ell_d} - \frac{\rho_0 A [2 + \text{Pe}^2 A]}{2\chi^2 L} \delta(k) \right] \delta(k + k') \quad (7.80a)$$

Similarly, the correlations of total density with polarization and polarization with polarization are respectively given as

$$\tilde{C}_{\rho m}(k, k') = \frac{-2i \text{Pe} \rho_0^2 A (A k^2 + 2) k}{\{ ABk^4 + [2A + B\chi^2]k^2 + 2\chi^2 \} \ell_d} \delta(k + k'), \quad (7.81)$$

$$\tilde{C}_{mm}(k, k') = \frac{\rho_0 \{ ABk^4 + [2B + \text{Pe}^2 A(1 - 2\rho_0)^2 + A\chi^2]k^2 + 2\chi^2 \}}{\{ ABk^4 + [2A + B\chi^2]k^2 + 2\chi^2 \} \ell_d} \delta(k + k'). \quad (7.82)$$

In the above two expressions of the correlations involving the polarization fields, we do not have any additional $k = (k' =) 0$ term, as the spatially-integrated fluctuating polarization fields $\int dx \delta m(x, t)$ do not vanish.

Finally, we take the inverse Fourier transformation of the expressions in (7.80a-7.81), which gives the two-point correlations for the fluctuating hydrodynamic fields in the stationary-state homogeneous phase. These are respectively given by

$$C_{\rho\rho}(x, x') = \frac{\text{Pe}^2 \rho_0^2 A^2}{\mathcal{D} \ell_d} \left[\sqrt{\frac{2}{B}} e^{-\frac{|x-x'|}{\xi_1}} + \frac{\chi^2 - 2}{\chi} \sqrt{A} e^{-\frac{|x-x'|}{\xi_2}} \right] x + \frac{\rho_0 A}{\ell_d} \delta(x - x') - \frac{\rho_0 A [2 + \text{Pe}^2 A]}{2\chi^2 L}, \quad (7.83a)$$

$$C_{\rho m}(x, x') = \text{sgn}(x - x') \frac{\text{Pe} \rho_0^2 A}{\mathcal{D} \ell_d} \left[\frac{2}{B} e^{-\frac{|x-x'|}{\xi_1}} + (\chi^2 - 2) e^{-\frac{|x-x'|}{\xi_2}} \right], \quad (7.83b)$$

$$C_{mm}(x, x') = \frac{\text{Pe}^2 \rho_0^2 A (1 - 2\rho_0)}{\mathcal{D} \ell_d} \left[\sqrt{\frac{2}{B}} e^{-\frac{|x-x'|}{\xi_1}} - \frac{\chi}{\sqrt{A}} e^{-\frac{|x-x'|}{\xi_2}} \right] + \frac{\rho_0}{\ell_d} \delta(x - x'), \quad (7.83c)$$

where the two "correlation lengths" corresponding to the length scales of the exponentially decaying long-range correlations are

$$\xi_1 = \sqrt{1 - \frac{\rho_0}{2}} \quad \text{and} \quad \xi_2 = \frac{\sqrt{1 - \rho_0}}{\chi}. \quad (7.84)$$

Two important observations are in order at this point. First, all three correlations show a divergence when $2 - \text{Pe}^2 (1 - \rho_0) (2 - \rho_0) (2\rho_0 - 1) \rightarrow 0$. In fact, our results for the correlations are not valid in the region $2 - \text{Pe}^2 (1 - \rho_0) (2 - \rho_0) (2\rho_0 - 1) \leq 0$. Second, $\xi_1 > \xi_2$ in the region $2 - \text{Pe}^2 (1 - \rho_0) (2 - \rho_0) (2\rho_0 - 1) > 0$. Consequently, the first exponential term in each of the correlations dominates over the second exponential term when the two points in space are far-separated i.e., $x - x' \gg 0$.

There is an excellent agreement, shown in Fig. 7.G.1, between the hydrodynamic results for the correlations in (7.83) and corresponding results from a direct Monte Carlo simulation.

Remark. For the strictly 1D model, one would arrive at the same two-point correlations in the stationary-state homogeneous phase owing to the similar fluctuating hydrodynamic equations

in the two geometries. Only the additional shift corresponding to the zero momentum mode in the strictly 1D model would be twice that in the quasi-1D model. However, due to a breakdown of hydrodynamics, the exact correlation functions obtained from the Monte Carlo simulations show (see Fig. 7.G.2) a clear mismatch with the analytical expressions.

7.H DETAILS OF THE NUMERICAL SIMULATIONS

For the microscopic dynamics we use the usual Monte Carlo method. We consider a periodic lattice with two lanes with L sites in each lane. We choose time step $dt = 0.01$ to convert the rates to probability for the Monte Carlo simulations. The lane-crossing rate $1/\tau_\times$ is taken as 10, 20 or 40 depending on the simulation.

To solve the hydrodynamic equations we employed the standard Fourier pseudospectral method [299], with nonlinear terms computed in real space using the $\frac{2}{3}$ dealiasing rule [299]. Time integration was performed in Fourier space utilizing the standard Runge-Kutta fourth order (RK4) method and the Integrating factor Runge-Kutta fourth order (IFRK4) method, with a time step (dt) of 10^{-3} or 10^{-4} , depending on the system's resolution. Some relevant codes for this project can be found in [319].

CONCLUSION AND OUTLOOK

This thesis spans three complementary areas: (i) statistical physics of turbulence, (ii) mathematical analysis of the Navier–Stokes equations, and (iii) hydrodynamics of active lattice gases and semiclassical description quantum gases. Much of the focus is to understand the emergence of scaling laws, two-point correlations, asymptotic scaling solutions, and rigorous bounds on certain observables. The governing equations in each case are nonlinear partial differential equations, with almost no analytical handle. Variety of tools from statistical mechanics, mathematical analysis and numerics are used to understand these hydrodynamic equations. As always, this thesis opens up more problems than it answers. Below we summarize the conclusions and the open questions.

The first three chapters addressed incompressible Navier–Stokes turbulence in three dimensions, where the Kolmogorov 1941 theory provides the natural starting point. Despite its success, K41 rests on the assumption of uniform, space-filling energy dissipation — an assumption that real turbulent flows violate. The departures from K41, collectively described as intermittency, were the central subject of these chapters.

Chapter 2 examined dissipation statistics from both Eulerian and Lagrangian perspectives using high-resolution direct numerical simulations with large scale separation. We show that, despite strong intermittency, Eulerian dissipation does not organize into a classical scale-invariant multiplicative cascade: its spatial correlations, coarse-grained statistics, and multiplier distributions are incompatible with multifractal or log-divisible descriptions. In sharp contrast, dissipation sampled along Lagrangian trajectories exhibits clear signatures of a temporal multiplicative cascade, including logarithmic time correlations, linear scaling of cumulants with the logarithm of the time scale, scale-invariant multipliers, and a well-defined large-deviation principle. Finally we propose an alternative interpretation of the classical bridge relation in which Eulerian dissipation statistics emerge from temporal cascade dynamics along Lagrangian paths, with Eulerian multipliers encoding the scale-to-scale inheritance of fluctuations.

This chapter raises several further directions. First, while we have strong indirect evidence

supporting the modified bridge relation we propose, direct verification would be valuable. Second, a natural extension of this work would be to apply the coarse-grained dissipation statistics framework to the coarse-grained flux, which may prove to be a more appropriate quantity to examine. Third, a detailed comparative study of between dissipation and pseudo-dissipation.

In Chapter 3, we studied chaos in turbulent flows through the framework of decorrelators. The largest Lyapunov exponent λ , which governs the degree of chaos at small scales, is predicted by Ruelle's classical argument to scale as $\lambda \sim Re^{1/2}$. Starting from the Navier–Stokes equations, we have analytically shown that intermittent fluctuations of the strain-rate field modify the Ruelle scaling. We further established that the degree of chaos in turbulence is controlled by fluctuations of the compressional eigenvalue of the strain-rate tensor. This provides a concrete and previously unquantified link between intermittency and predictability through the Navier–Stokes equations.

We derived scaling laws for the largest Lyapunov exponent; however, a given flow possesses a full distribution of Lyapunov exponents, and a natural direction for future work is the characterization of the complete Lyapunov spectrum.

In Chapter 4, we revisited the structure of velocity increment statistics through a geometric decomposition. Conventional structure functions measure longitudinal velocity increments, which conflate contributions from the amplitude and the orientation of the velocity difference vector. By separating these two contributions, we identified geometric multiscaling laws that are invisible to standard longitudinal measurements. These results show that intermittency leaves its signature not only in the magnitudes of velocity fluctuations but also in the geometry of the flow field, and that a complete picture of anomalous scaling requires attending to both.

Chapter 5 turned to two-dimensional Navier–Stokes turbulence, where the physics is qualitatively different and tools from mathematical analysis are used to find bounds on scaling laws. Under power-law forcing at all scales, the conventional cascade picture breaks down, and the phenomenological picture breaks down. Using mathematical analysis tools we derived strict upper bounds on the energy and enstrophy dissipation rates as functions of the Reynolds number. These bounds, obtained without any modeling assumptions, identify the scaling of dissipation with Re in a rigorous way and provide a complement to the statistical viewpoint of the earlier chapters. We also derived bounds on the attractor dimension, quantifying the degrees of freedom.

The tools that we have used are for deterministic forcing, however there are numerical simulations available for stochastic power-law forcing. It would be very interesting to see how to modify our methods so that they are suitable for two-dimensional Navier–Stokes with stochastic power-law forcing.

The final two chapters examined one-dimensional hydrodynamic models from two complementary directions. In Chapter 6, we studied the evolution of quantum gases, released from traps, is studied through hydrodynamics, both analytically and numerically, in one and two dimensions. In particular, we demonstrate the existence of long time self-similar solutions of the Euler equations, for the density and velocity fields, and derive the scaling exponents as well as

the scaling functions. We find that the expanding gas develops a shock-like front and the size of the cloud grows in time as a power law. We relate the associated exponent to that appearing in the corresponding equation of state of the quantum gas. Furthermore, we study the relaxation dynamics of a trapped quantum gas and show that the resulting steady state is in excellent agreement with that derived analytically. Our hydrodynamic approach is versatile and can be used to unravel several other far-from-equilibrium collective phenomenon of extreme nature, relevant to the growing experimental interests in quantum gases.

In this study, the extension to two-dimension is rather limited, since we assume spherical symmetry. It would be interesting to understand the asymptotic dynamics without the spherical symmetry. Relaxation dynamics in a trap is only studied for the quadratic trap

In Chapter 7, we presented a bottom-up approach to derive hydrodynamic equations from microscopic models. We present a fluctuating hydrodynamic description of an active lattice gas model with excluded volume interactions that exhibits motility-induced phase separation under appropriate conditions. For quasi-one dimension and higher, stability analysis of the noiseless hydrodynamics gives quantitative bounds on the phase boundary of the motility-induced phase separation in terms of spinodal and binodal. Inclusion of the multiplicative noise in the fluctuating hydrodynamics describes the exponentially decaying two-point correlations in the stationary-state homogeneous phase. Our hydrodynamic description and theoretical predictions based on it are in excellent agreement with our Monte Carlo simulations and pseudospectral iteration of the hydrodynamics equations. Our construction of hydrodynamics for this model is not suitable in strictly one-dimension with single-file constraints, and we argue that this breakdown is associated with micro-phase separation.

Although the model is described in two dimensions but the full analysis is left for a future project. We have explicitly shown the breakdown of hydrodynamics in strictly one dimension due to breakdown of local equilibrium but exactly what causes this breakdown and how to write down the hydrodynamics correctly is still an open question. Also, it would be interesting to see how the fluctuating hydrodynamics results for this work compare with other models, including a full characterization of macroscopic fluctuations in terms of large deviations.

BIBLIOGRAPHY

- [1] G. I. Barenblatt, *Scaling*, Cambridge Texts in Applied Mathematics (Cambridge University Press, 2003).
- [2] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, 1995).
- [3] A. Arneodo, C. Baudet, F. Belin, R. Benzi, B. Castaing, B. Chabaud, R. Chavarria, S. Ciliberto, R. Camussi, F. Chillà, B. Dubrulle, Y. Gagne, B. Hebral, J. Herweijer, M. Marchand, J. Maurer, J. F. Muzy, A. Naert, A. Noullez, J. Peinke, F. Roux, P. Tabeling, W. van de Water, and H. Willaime, *Europhysics Letters* **34**, 411 (1996).
- [4] M. Lesieur, *Turbulence in fluids*, Vol. 84 (Springer Dordrecht, 2008).
- [5] A. N. Kolmogorov, *Dokl. Akad. Nauk SSSR* **30**, 301 (1941).
- [6] A. N. Kolmogorov, *Dokl. Akad. Nauk SSSR* **32**, 16 (1941).
- [7] J. D. Gibbon and D. Vincenzi, *Journal of Nonlinear Science* **32**, 87 (2022).
- [8] C. R. Doering and C. Foias, *J. Fluid Mech.* **467**, 289 (2002).
- [9] A. Cheskidov, N. Petrov, and C. R. Doering, *J. Math. Phys.* **48**, 065208 (2007).
- [10] J. Bec and K. Khanin, *Phys. Rep.* **447**, 1 (2007).
- [11] B. Derrida, *Physics Reports* **301**, 65 (1998).
- [12] P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A kinetic view of statistical physics* (Cambridge University Press, 2010).
- [13] C. R. Doering, *Annu. Rev. Fluid Mech.* **41**, 109 (2009).
- [14] P. A. Davidson, *Turbulence: An Introduction for Scientists and Engineers* (Oxford University Press, Oxford, 2004).
- [15] L. Onsager, *Il Nuovo Cimento* (1943-1954) **6**, 279 (1949).
- [16] B. Dubrulle, *J. Fluid Mech.* **867**, P1 (2019).
- [17] G. Eyink, *Journal of Fluid Mechanics* **988**, 10.1017/jfm.2024.415 (2024).

- [18] T. Ishihara, T. Gotoh, and Y. Kaneda, *Annual review of fluid mechanics* **41**, 165 (2009).
- [19] R. Benzi, S. Ciliberto, R. Tripiccion, C. Baudet, F. Massaioli, and S. Succi, *Phys. Rev. E* **48**, R29 (1993).
- [20] Z.-S. She and E. Leveque, *Physical review letters* **72**, 336 (1994).
- [21] A. N. Kolmogorov, *J. Fluid Mech.* **13**, 82 (1962).
- [22] Z.-S. She and E. C. Waymire, *Phys. Rev. Lett.* **74**, 262 (1995).
- [23] B. Dubrulle, *Phys. Rev. Lett.* **73**, 959 (1994).
- [24] G. Parisi and U. Frisch, *Proceedings of the International School of Physics Enrico Fermi, Course LXXXVIII, Varenna, 1985* (1985).
- [25] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, *Journal of Physics A: Mathematical and General* **17**, 3521 (1984).
- [26] G. Falkovich, K. Gawędzki, and M. Vergassola, *Rev. Mod. Phys.* **73**, 913 (2001).
- [27] S. Sankar Ray, D. Mitra, and R. Pandit, *New Journal of Physics* **10**, 033003 (2008).
- [28] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, *Phys. Rev. A* **33**, 1141 (1986).
- [29] G. Stolovitzky and K. R. Sreenivasan, *Rev. Mod. Phys.* **66**, 229 (1994).
- [30] H. Touchette, *Physics Reports* **478**, 1 (2009).
- [31] C. Meneveau and K. R. Sreenivasan, *Nucl. Phys. B* **2**, 49 (1987).
- [32] A. Arnéodo, R. Benzi, J. Berg, L. Biferale, E. Bodenschatz, A. Busse, E. Calzavarini, B. Castaing, M. Cencini, L. Chevillard, *et al.*, *Phys. Rev. Lett.* **100**, 254504 (2008).
- [33] D. Mitra and R. Pandit, *Phys. Rev. Lett.* **93**, 024501 (2004).
- [34] S. Mukherjee, J. Schalkwijk, and H. J. J. Jonker, *Journal of the Atmospheric Sciences* **73**, 2715 (2016).
- [35] M. J. Powdel and S. S. Ray, arXiv preprint arXiv:2512.05656 <https://doi.org/10.48550/arXiv.2512.05656> (2025).
- [36] E. Lorenz, *J. Atmos. Sci.* **20**, 130 (1963).
- [37] D. R. Durran and M. Gingrich, *Journal of the Atmospheric Sciences* **71**, 2476 (2014).
- [38] D. Ruelle, *Physics Letters A* **72**, 81 (1979).
- [39] G. Paladin and A. Vulpiani, *Physics Reports* **156**, 147 (1987).
- [40] A. Crisanti, M. H. Jensen, A. Vulpiani, and G. Paladin, *Phys. Rev. Lett.* **70**, 166 (1993).

- [41] See Supplemental Material at [URL will be inserted by publisher] for (a) details on the direct numerical simulations and the GOY shell model, (b) the derivation of the decorrelator from the Navier-Stokes equation and its limits, and (c) the decorrelator in the GOY shell model for turbulence.
- [42] S. D. Murugan, D. Kumar, S. Bhattacharjee, and S. S. Ray, *Phys. Rev. Lett.* **127**, 124501 (2021).
- [43] S. Mukherjee, R. K. Singh, M. James, and S. S. Ray, *Nature Physics* **19**, 891 (2023).
- [44] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, edited by J. L. Lumley (MIT Press, Cambridge, MA, 1975).
- [45] M. Verma, *Energy Transfers in Fluid Flows* (Cambridge University Press, Cambridge, 2019).
- [46] R. Benzi and F. Toschi, *Phys. Rep.* **1021**, 1 (2023).
- [47] C. R. Doering and J. D. Gibbon, *Applied Analysis of the Navier–Stokes Equations* (Cambridge University Press, Cambridge, 1995).
- [48] C. Foias, O. Manley, R. Rosa, and R. Temam, *Navier–Stokes Equations and Turbulence* (Cambridge University Press, Cambridge, 2001).
- [49] P. Constantin, in *Mathematical Foundation of Turbulent Viscous Flows*, Lecture Notes in Mathematics, Vol. 1871, edited by M. Cannone and T. Miyakawa (Springer, Berlin Heidelberg, 2006) pp. 1–43.
- [50] C. W. Bardos and E. S. Titi, *J. Turbul.* **14**, 42 (2013).
- [51] D. Queiros-Conde and J. C. Vassilicos, in *Intermittency in Turbulent Flows*, Vol. 12, edited by J. C. Vassilicos (Cambridge university press Cambridge, UK, 2001) pp. 136–167.
- [52] A. Staicu, B. Mazzi, J. C. Vassilicos, and W. van de Water, *Phys. Rev. E* **67**, 066306 (2003).
- [53] B. Mazzi and J. C. Vassilicos, *J. Fluid Mech.* **502**, 65 (2004).
- [54] C. Foias, O. Manley, R. Temam, and Y. Treve, *Physica D* **9**, 157 (1983).
- [55] G. L. Eyink and K. R. Sreenivasan, *Reviews of modern physics* **78**, 87 (2006).
- [56] J. Duchon and R. Robert, *Nonlinearity* **13**, 249 (2000).
- [57] B. Dubrulle, *Journal of Fluid Mechanics* **867**, P1 (2019).
- [58] G. Boffetta and R. E. Ecke, *Annu. Rev. Fluid Mech.* **44**, 427 (2012).
- [59] A. Alexakis and L. Biferale, *Phys. Rep.* **767–769**, 1 (2018).
- [60] C. Foias, O. Manley, and R. Temam, *Phys. Lett. A* **174**, 210 (1993).
- [61] C. Foias, M. S. Jolly, O. P. Manley, and R. Rosa, *J. Stat. Phys.* **108**, 591 (2002).

- [62] R. Dascaliuc, C. Foias, and M. Jolly, *Physica D* **237**, 3020 (2008).
- [63] A. Alexakis and C. R. Doering, *Phys. Lett. A* **359**, 652 (2006).
- [64] J. D. Gibbon and G. A. Pavliotis, *J. Math. Phys.* **48**, 065202 (2007).
- [65] A. Mazzino, P. Muratore-Ginanneschi, and S. Musacchio, *Phys. Rev. Lett.* **99**, 144502 (2007).
- [66] A. Mazzino, P. Muratore-Ginanneschi, and S. Musacchio, *J. Stat. Mech.* **2009**, P10012 (2009).
- [67] J. C. Robinson, *J. Fluid Mech.* **578**, 495–507 (2007).
- [68] O. A. Ladyzhenskaya, *J. Sov. Math.* **3**, 458–479 (1975).
- [69] J. C. Robinson, *Infinite-Dimensional Dynamical Systems* (Cambridge University Press, Cambridge, 2001).
- [70] J. C. Robinson, *Int. Sch. Res. Notices* **2013**, 291823 (2013).
- [71] J. C. Robinson, *Chaos* **5**, 330 (1995).
- [72] P. Constantin, C. Foias, and R. Temam, *Physica D* **30**, 284 (1988).
- [73] C. R. Doering and J. D. Gibbon, *Physica D* **48**, 471 (1991).
- [74] P. J. Morrison, *Rev. Mod. Phys.* **70**, 467 (1998).
- [75] M. Kulkarni and A. G. Abanov, *Phys. Rev. A* **86**, 033614 (2012).
- [76] H. Spohn, *Large scale dynamics of interacting particles* (Springer Science & Business Media, 2012).
- [77] A. Banerjee, R. Mukherjee, S. D. Murugan, S. Bhattacharjee, and S. S. Ray, *arXiv preprint arXiv:2505.09538* (2025).
- [78] R. Mukherjee, S. Mukherjee, I. Kolokolov, V. Lebedev, T. Matsumoto, and S. S. Ray, *arXiv preprint arXiv:2511.06439* (2025).
- [79] R. Mukherjee, D. Vincenzi, and J. Gibbon, *arXiv preprint arXiv:2512.15188* (2025).
- [80] R. Mukherjee, A. Dhar, M. Kulkarni, and S. S. Ray, *Phys. Rev. A* **113**, 043305 (2026).
- [81] R. Mukherjee, S. Saha, T. Sadhu, A. Dhar, and S. Sabhapandit, *Phys. Rev. E* **111**, 024128 (2025).
- [82] P. K. Yeung, X. M. Zhai, and K. R. Sreenivasan, *Proceedings of the National Academy of Sciences* **112**, 12633 (2015).
- [83] D. Buaria, A. Pumir, E. Bodenschatz, and P.-K. Yeung, *New Journal of Physics* **21**, 043004 (2019).

- [84] L. Biferale, G. Boffetta, A. Celani, B. J. Devenish, A. Lanotte, and F. Toschi, *Phys. Fluids* **17**, 115101 (2005).
- [85] S. Thalabard, J. Bec, and A. A. Mailybaev, *Communications Physics* **3**, 122 (2020).
- [86] D. Bandak, A. A. Mailybaev, G. L. Eyink, and N. Goldenfeld, *Phys. Rev. Lett.* **132**, 104002 (2024).
- [87] S. Mukherjee, S. D. Murugan, R. Mukherjee, and S. S. Ray, *Phys. Rev. Lett.* **132**, 184002 (2024).
- [88] J. C. Vassilicos, *Atmosphere* **14**, 736 (2023).
- [89] C. Meneveau and K. R. Sreenivasan, *J. Fluid Mech.* **224**, 429 (1991).
- [90] A. M. Obukhov, *J. Geophys. Res.* **67**, 3011 (1962).
- [91] A. Vulpiani, F. Cecconi, M. Cencini, A. Puglisi, and D. Vergni, *The Legacy of the Law of Large Numbers* (Berlin: Springer) **10**, 978 (2014).
- [92] M. Borgas, *Phil. Trans. Roy. Soc. A* **342**, 379 (1993).
- [93] L. Chevillard, B. Castaing, A. Arneodo, E. L  v  que, J.-F. Pinton, and S. G. Roux, *C.R. Physique* **13**, 899 (2012).
- [94] C. C. Lalescu and M. Wilczek, *New J. Phys.* **20**, 013001 (2018).
- [95] P. K. Yeung, D. Donzis, and K. R. Sreenivasan, *J. Fluid Mech.* **700**, 5 (2012).
- [96] Y. Li, E. Perlman, M. Wan, Y. Yang, C. Meneveau, R. Burns, A. Chen, Shiyi and Szalay, and G. Eyink, *Journal of Turbulence* , N31 (2008).
- [97] E. Perlman, R. Burns, Y. Li, and C. Meneveau, in *Proceedings of the 2007 ACM/IEEE Conference on Supercomputing* (2007) pp. 1–11.
- [98] B. L. Sawford and P. K. Yeung, *Phys. Fluids* **27**, 065109 (2015).
- [99] J. M. Lawson, E. Bodenschatz, A. N. Knutsen, J. R. Dawson, and N. A. Worth, *Phys. Rev. Fluids* **4**, 022601 (2019).
- [100] J. O’Neil and C. Meneveau, *Physics of Fluids A: Fluid Dynamics* **5**, 158 (1993).
- [101] B. Castaing and B. Dubrulle, *J. Phys. II (France)* **5**, 895 (1995).
- [102] A. Arn  odo, E. Bacry, S. Manneville, and J. Muzy, *Phys. Rev. Lett.* **80**, 708 (1998).
- [103] R. Pandit, P. Perlekar, and S. S. Ray, *Pramana* **73**, 157 (2009).
- [104] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, *Journal of Physics A: Mathematical and General* **17**, 3521 (1984).

- [105] A. Berera and R. D. J. G. Ho, *Phys. Rev. Lett.* **120**, 024101 (2018).
- [106] M. Yamada and K. Ohkitani, *Journal of the Physical Society of Japan* **56**, 4210 (1987).
- [107] G. Boffetta and S. Musacchio, *Phys. Rev. Lett.* **119**, 054102 (2017).
- [108] P. Mohan, N. Fitzsimmons, and R. D. Moser, *Phys. Rev. Fluids* **2**, 114606 (2017).
- [109] J. Ge, J. Rolland, and J. C. Vassilicos, *Journal of Fluid Mechanics* **977**, A17 (2023).
- [110] A. Das, S. Chakrabarty, A. Dhar, A. Kundu, D. A. Huse, R. Moessner, S. S. Ray, and S. Bhattacharjee, *Phys. Rev. Lett.* **121**, 024101 (2018).
- [111] T. Bilitewski, S. Bhattacharjee, and R. Moessner, *Phys. Rev. B* **103**, 174302 (2021).
- [112] T. Bilitewski, S. Bhattacharjee, and R. Moessner, *Phys. Rev. Lett.* **121**, 250602 (2018).
- [113] A. G. Lamorgese, D. A. Caughey, and S. B. Pope, *Physics of Fluids* **17**, 015106 (2004).
- [114] [Supplemental video](#) (2025), spatio-temporal evolution of an initially localised perturbation.
- [115] L. Biferale, *Annual Review of Fluid Mechanics* **35**, 441 (2003).
- [116] P. D. Ditlevsen, *Turbulence and Shell Models* (Cambridge University Press, Cambridge, UK, 2010).
- [117] E. B. Gledzer, *Soviet Physics Doklady* **18**, 216 (1973).
- [118] K. Ohkitani and M. Yamada, *Progress of Theoretical Physics* **81**, 329 (1989).
- [119] S. H. Shenker and D. Stanford, *Journal of High Energy Physics* **2014** (2014).
- [120] J. Bec, L. Biferale, G. Boffetta, M. Cencini, S. Musacchio, and F. Toschi, *Physics of Fluids* **18**, 091702 (2006).
- [121] S. S. Ray, *Phys. Rev. Fluids* **3**, 072601 (2018).
- [122] S. S. Ray, D. Mitra, and R. Pandit, *New Journal of Physics* **10**, 033003 (2008).
- [123] Y. Chigirinskaya, D. Schertzer, and S. Lovejoy, in *Advances in Turbulence VII*, edited by U. Frisch (Springer Netherlands, Dordrecht, 1998) pp. 263–266.
- [124] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, 1996).
- [125] We have confirmed that a reconstruction of the second-order structure function by using the various terms in its expansion reproduces identically the longitudinal structure functions constructed from the velocity increments. ().
- [126] D. Buaria and K. R. Sreenivasan, *Phys. Rev. Lett.* **131**, 204001 (2023).
- [127] R. Ni and K.-Q. Xia, *Phys. Rev. E* **87**, 023002 (2013).

- [128] S. Chakraborty, U. Frisch, and S. S. Ray, *Journal of Fluid Mechanics* **649**, 275–285 (2010).
- [129] The exponents are calculated as the mean height of the plateaus from a local slope analysis (Fig. 4.1, insets) and the standard deviation of these are our error-bars. ()
- [130] G. Boffetta and R. E. Ecke, *Annual review of fluid mechanics* **44**, 427 (2012).
- [131] Y.-K. Tsang, E. Ott, T. M. Antonsen Jr, and P. N. Guzdar, *Phys. Rev. E* **71**, 066313 (2005).
- [132] P. Perlekar and R. Pandit, *New Journal of Physics* **11**, 073003 (2009).
- [133] S. S. Ray, D. Mitra, P. Perlekar, and R. Pandit, *Physical Review Letters* **107**, 184503 (2011).
- [134] R. Pandit, D. Banerjee, A. Bhatnagar, M. Brachet, A. Gupta, D. Mitra, N. Pal, P. Perlekar, S. S. Ray, V. Shukla, and D. Vincenzi, *Physics of Fluids* **29**, 111112 (2017).
- [135] T. Kishi, T. Matsumoto, and S. Toh, *Phys. Rev. Fluids* **7**, 064604 (2022).
- [136] C. Foias, *Contemp. Math.* **208**, 151 (1997).
- [137] J. D. Gibbon, *J. Nonlin. Sci.* **29**, 215 (2019).
- [138] J. D. Gibbon, *Europhys. Lett.* **131**, 64001 (2020).
- [139] L. Canet, *J. Fluid Mech.* **950**, P1 (2022).
- [140] L. M. Smith and S. L. Woodruff, *Annu. Rev. Fluid Mech.* **30**, 275 (1998).
- [141] L. T. Adzhemyan, N. V. Antonov, and A. N. Vasiliev, *Field Theoretic Renormalization Group in Fully Developed Turbulence* (CRC Press, 1999).
- [142] Y. Zhou, *Phys. Rep.* **488**, 1 (2010).
- [143] A. Sain, Manu, and R. Pandit, *Phys. Rev. Lett.* **81**, 4377 (1998).
- [144] L. Biferale, A. S. Lanotte, and F. Toschi, *Phys. Rev. Lett.* **92**, 094503 (2004).
- [145] L. Biferale, M. Cencini, A. S. Lanotte, M. Sbragaglia, and F. Toschi, *New J. Phys.* **6**, 37 (2004).
- [146] D. Mitra, J. Bec, R. Pandit, and U. Frisch, *Phys. Rev. Lett.* **94**, 194501 (2005).
- [147] S. De, D. Mitra, and R. Pandit, *Sci. Rep.* **13**, 7151 (2023).
- [148] P. Constantin, C. Foias, and O. P. Manley, *Phys. Fluids* **6**, 427 (1994).
- [149] C. V. Tran and T. G. Shepherd, *Physica D* **165**, 199 (2002).
- [150] G. L. Eyink, *Physica D* **91**, 97 (1996).
- [151] C. V. Tran and D. G. Dritschel, *J. Fluid Mech.* **559**, 107–116 (2006).

- [152] M. Lopes Filho, A. Mazzucato, and H. Nussenzveig Lopes, *Arch. Ration. Mech. Anal.* **179**, 353–387 (2006).
- [153] P. Matharu, B. Protas, and T. Yoneda, *Physica D* **441**, 133517 (2022).
- [154] J. C. Robinson, *Physica D* **181**, 39 (2003).
- [155] C. V. Tran, T. G. Shepherd, and H.-R. Cho, *Physica D* **192**, 187 (2004).
- [156] M. Horner, G. Metcalfe, S. Wiggins, and J. M. Ottino, *J. Fluid Mech.* **452**, 199 (2002).
- [157] G. A. Voth, T. C. Saint, G. Dobler, and J. P. Gollub, *Phys. Fluids* **15**, 2560 (2003).
- [158] I. Bloch, J. Dalibard, and S. Nascimbène, *Nature Physics* **8**, 267 (2012).
- [159] A. Adams, L. D. Carr, T. Schäfer, P. Steinberg, and J. E. Thomas, *New Journal of Physics* **14**, 115009 (2012).
- [160] A. Bulgac, M. M. Forbes, and P. Magierski, The unitary fermi gas: From monte carlo to density functionals, in *The BCS-BEC Crossover and the Unitary Fermi Gas*, edited by W. Zwerger (Springer Berlin Heidelberg, Berlin, Heidelberg, 2012) pp. 305–373.
- [161] T. Langen, R. Geiger, and J. Schmiedmayer, *Annual Review of Condensed Matter Physics* **6**, 201 (2015).
- [162] N. P. Proukakis, S. A. Gardiner, M. Davis, and M. Szymanska, *Quantum gases: finite temperature and non-equilibrium dynamics*, Vol. 1 (World Scientific, 2013).
- [163] T. Langen, *Non-equilibrium dynamics of one-dimensional Bose gases* (Springer, 2015).
- [164] M. Randeria and E. Taylor, *Annual Review of Condensed Matter Physics* **5**, 209 (2014).
- [165] N. Defenu, A. Lerose, and S. Pappalardi, *Physics Reports* **1074**, 1 (2024), out-of-equilibrium dynamics of quantum many-body systems with long-range interactions.
- [166] T. Langen, G. Valtolina, D. Wang, and J. Ye, *Nature Physics* **20**, 702 (2024).
- [167] L. Chomaz, I. Ferrier-Barbut, F. Ferlaino, B. Laburthe-Tolra, B. L. Lev, and T. Pfau, *Reports on Progress in Physics* **86**, 026401 (2022).
- [168] K. O'hara, S. Hemmer, M. Gehm, S. Granade, and J. Thomas, *Science* **298**, 2179 (2002).
- [169] C. Menotti, P. Pedri, and S. Stringari, *Phys. Rev. Lett.* **89**, 250402 (2002).
- [170] T. Kinoshita, T. Wenger, and D. S. Weiss, *Science* **305**, 1125 (2004).
- [171] J. A. Joseph, J. E. Thomas, M. Kulkarni, and A. G. Abanov, *Phys. Rev. Lett.* **106**, 150401 (2011).
- [172] U. Ernst, J. Schuster, F. Schreck, A. Marte, A. Kuhn, and G. Rempe, *Applied Physics B* **67**, 719 (1998).

- [173] B. N. Narozhny, *Annals of Physics* **411**, 167979 (2019).
- [174] L. Salasnich, *Europhysics Letters* **96**, 40007 (2011).
- [175] M. Tsubota, M. Kobayashi, and H. Takeuchi, *Physics Reports* **522**, 191 (2013), quantum hydrodynamics.
- [176] S. A. Khan and M. Bonitz, in *Complex plasmas: Scientific challenges and technological opportunities* (Springer, 2014) pp. 103–152.
- [177] P. A. Andreev, I. N. Mosaki, and M. I. Trukhanova, *Physics of Fluids* **33**, 067108 (2021).
- [178] P. Ruggiero, P. Calabrese, B. Doyon, and J. Dubail, *Physical review letters* **124**, 140603 (2020).
- [179] C. Cao, E. Elliott, J. Joseph, H. Wu, J. Petricka, T. Schäfer, and J. E. Thomas, *Science* **331**, 58 (2011).
- [180] X. Wang, X. Li, I. Arakelyan, and J. E. Thomas, *Phys. Rev. Lett.* **128**, 090402 (2022).
- [181] X. Li, J. Huang, and J. E. Thomas, *Phys. Rev. Res.* **6**, L042021 (2024).
- [182] G. M. Bruun and H. Smith, *Phys. Rev. A* **72**, 043605 (2005).
- [183] G. Wlazłowski, P. Magierski, and J. E. Drut, *Phys. Rev. Lett.* **109**, 020406 (2012).
- [184] R. Samanta, R. Sharma, and S. P. Trivedi, *Phys. Rev. A* **96**, 053601 (2017).
- [185] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, *Rev. Mod. Phys.* **83**, 1405 (2011).
- [186] M. Inguscio, W. Ketterle, and C. Salomon, *Ultra-cold Fermi gases*, Vol. 164 (IOS press, 2008).
- [187] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney, and L. Tarruell, *Science* **359**, 301 (2018).
- [188] G. Semeghini, G. Ferioli, L. Masi, C. Mazzinghi, L. Wolswijk, F. Minardi, M. Modugno, G. Modugno, M. Inguscio, and M. Fattori, *Phys. Rev. Lett.* **120**, 235301 (2018).
- [189] T. Sowiński and M. Ángel García-March, *Reports on Progress in Physics* **82**, 104401 (2019).
- [190] P. Capuzzi, P. Vignolo, F. Federici, and M. P. Tosi, *Phys. Rev. A* **73**, 021603 (2006).
- [191] G. Bertaina, L. Pitaevskii, and S. Stringari, *Phys. Rev. Lett.* **105**, 150402 (2010).
- [192] J. Joseph, B. Clancy, L. Luo, J. Kinast, A. Turlapov, and J. E. Thomas, *Phys. Rev. Lett.* **98**, 170401 (2007).
- [193] C. C. N. Kuhn, S. Hoinka, I. Herrera, P. Dyke, J. J. Kinnunen, G. M. Bruun, and C. J. Vale, *Phys. Rev. Lett.* **124**, 150401 (2020).

- [194] R. R. Joseph, L. E. C. Rosales-Zárate, and P. D. Drummond, *Phys. Rev. A* **98**, 013638 (2018).
- [195] X. Li, X. Luo, S. Wang, K. Xie, X.-P. Liu, H. Hu, Y.-A. Chen, X.-C. Yao, and J.-W. Pan, *Science* **375**, 528 (2022).
- [196] Z. Yan, P. B. Patel, B. Mukherjee, C. J. Vale, R. J. Fletcher, and M. W. Zwierlein, *Science* **383**, 629 (2024).
- [197] L. Salasnich, *Few-Body Systems* **54**, 697 (2013).
- [198] A. Bulgac, Y.-L. Luo, and K. J. Roche, *Phys. Rev. Lett.* **108**, 150401 (2012).
- [199] X. Yan, J. Min, D. Sun, S.-G. Peng, X. Xie, X. Wu, and K. Jiang, *arXiv:2502.03524* (2025).
- [200] L. I. Sedov, *Similarity and dimensional methods in mechanics* (CRC press, 2018).
- [201] L. I. Sedov, *Journal of Applied Mathematics and Mechanics* **10**, 241 (1946).
- [202] G. I. Taylor, *Proc. Roy. Soc. A* **201**, 159 (1950).
- [203] G. I. Taylor, *Proc. Roy. Soc. London. Series A* **201**, 175 (1950).
- [204] J. von Neumann, *Collected Works* (Pergamon Press, Oxford, 1963).
- [205] T. Antal, P. L. Krapivsky, and S. Redner, *Phys. Rev. E* **78**, 030301 (2008).
- [206] M. Barbier, D. Villamaina, and E. Trizac, *Phys. Rev. Lett.* **115**, 214301 (2015).
- [207] M. Barbier, *J. Stat. Mech.* **2015**, P11019 (2015).
- [208] J. P. Joy, S. N. Pathak, and R. Rajesh, *J. Stat. Phys.* **182**, 34 (2021).
- [209] J. P. Joy and R. Rajesh, *J. Stat. Phys.* **184**, 3 (2021).
- [210] S. Chakraborti, S. Ganapa, P. L. Krapivsky, and A. Dhar, *Phys. Rev. Lett.* **126**, 244503 (2021).
- [211] S. Ganapa, S. Chakraborti, P. L. Krapivsky, and A. Dhar, *Physics of Fluids* **33** (2021).
- [212] S. K. Singh, S. Chakraborti, A. Dhar, and P. L. Krapivsky, *J. Stat. Phys.* **190**, 118 (2023).
- [213] F. J. Dyson, *Journal of Mathematics and Mechanics* **18**, 91 (1968).
- [214] O. I. Bogoyavlensky, *Methods in the qualitative theory of dynamical systems in astrophysics and gas dynamics* (Springer, 1985).
- [215] H. Takabe, Self-similar solutions of compressible fluids, in *The Physics of Laser Plasmas and Applications - Volume 2: Fluid Models and Atomic Physics of Plasmas* (Springer International Publishing, Cham, 2024) pp. 149–196.
- [216] G. B. Whitham, *Linear and nonlinear waves* (John Wiley & Sons, 2011).

- [217] E. F. Toro, *Riemann solvers and numerical methods for fluid dynamics: a practical introduction* (Springer Science & Business Media, 2013).
- [218] L. Ambrosio, A. Bressan, D. Helbing, A. Klar, and E. Zuazua, *Modelling and Optimisation of Flows on Networks: Cetraro, Italy 2009*, Editors: Benedetto Piccoli, Michel Rascle, Vol. 2062 (Springer, 2012).
- [219] M. L. Kerr, G. D. Rosi, and K. V. Kheruntsyan, *SciPost Phys. Core* **7**, 047 (2024).
- [220] T.-L. Ho, *Physical review letters* **92**, 090402 (2004).
- [221] J. Thomas, J. Kinast, and A. Turlapov, *Physical review letters* **95**, 120402 (2005).
- [222] S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **80**, 1215 (2008).
- [223] E. Shuryak, *Progress in Particle and Nuclear Physics* **62**, 48 (2009).
- [224] T. Schäfer and D. Teaney, *Reports on Progress in Physics* **72**, 126001 (2009).
- [225] J. Berges, M. P. Heller, A. Mazeliauskas, and R. Venugopalan, *Rev. Mod. Phys.* **93**, 035003 (2021).
- [226] M. Lewin, *Journal of Mathematical Physics* **63**, 061101 (2022).
- [227] S. Agarwal, A. Dhar, M. Kulkarni, A. Kundu, S. N. Majumdar, D. Mukamel, and G. Schehr, *Phys. Rev. Lett.* **123**, 100603 (2019).
- [228] A. Kumar, M. Kulkarni, and A. Kundu, *Phys. Rev. E* **102**, 032128 (2020).
- [229] S. Santra, J. Kethepalli, S. Agarwal, A. Dhar, M. Kulkarni, and A. Kundu, *Phys. Rev. Lett.* **128**, 170603 (2022).
- [230] B. K. S., D. A. Huse, and M. Kulkarni, *Phys. Rev. E* **104**, 044117 (2021).
- [231] I. F. Barna, M. A. Pocsai, and L. Mátyás, *Advances in Mathematical Physics* **2018**, 7087295 (2018).
- [232] W. Ketterle, *Phys. Today* **52**, 30 (1999).
- [233] D. A. Smith, S. Aigner, S. Hofferberth, M. Gring, M. Andersson, S. Wildermuth, P. Krüger, S. Schneider, T. Schumm, and J. Schmiedmayer, *Opt. Express* **19**, 8471 (2011).
- [234] K. Hueck, N. Luick, L. Sobirey, J. Siegl, T. Lompe, H. Moritz, L. W. Clark, and C. Chin, *Opt. Express* **25**, 8670 (2017).
- [235] L. D. Landau and E. M. Lifshitz, *Fluid mechanics: Landau And Lifshitz: course of theoretical physics*, Vol. 6 (Elsevier, 2013).
- [236] I. Bloch, J. Dalibard, and W. Zwerger, *Rev. Mod. Phys.* **80**, 885 (2008).
- [237] V. Bretin, S. Stock, Y. Seurin, and J. Dalibard, *Phys. Rev. Lett.* **92**, 050403 (2004).

- [238] L. Baird, X. Wang, S. Roof, and J. E. Thomas, *Phys. Rev. Lett.* **123**, 160402 (2019).
- [239] X. Li, J. Huang, and J. E. Thomas, *Phys. Rev. Res.* **6**, L042021 (2024).
- [240] P. B. Patel, Z. Yan, B. Mukherjee, R. J. Fletcher, J. Struck, and M. W. Zwierlein, *Science* **370**, 1222 (2020).
- [241] R. Mukherjee, Relevant codes for quantum gases, <https://github.com/rikmukherjee/CompressibleEuler> (2025).
- [242] W. Ketterle and M. W. Zwierlein, *La Rivista del Nuovo Cimento* **31**, 247 (2008).
- [243] J. Hou and T. Schäfer, *Phys. Rev. A* **104**, 023313 (2021).
- [244] L. Salasnich, P. Comaron, M. Zambon, and F. Toigo, *Phys. Rev. A* **88**, 033610 (2013).
- [245] A. Amo, S. Pigeon, D. Sanvitto, V. G. Sala, R. Hivet, I. Carusotto, F. Pisanello, G. Leménager, R. Houdré, E. Giacobino, C. Ciuti, and A. Bramati, *Science* **332**, 1167 (2011).
- [246] I. Carusotto and C. Ciuti, *Rev. Mod. Phys.* **85**, 299 (2013).
- [247] D. A. Teaney, Viscous hydrodynamics and the quark gluon plasma, in *Quark-Gluon Plasma 4*, pp. 207–266.
- [248] S. Cox and P. Matthews, *Journal of Computational Physics* **176**, 430 (2002).
- [249] S. D. Murugan, U. Frisch, S. Nazarenko, N. Besse, and S. S. Ray, *Phys. Rev. Res.* **2**, 033202 (2020).
- [250] A. Kumar and R. Rajesh, *J. Stat. Phys.* **188**, 12 (2022).
- [251] M. J. Bowick, N. Fakhri, M. C. Marchetti, and S. Ramaswamy, *Phys. Rev. X* **12**, 010501 (2022).
- [252] S. Ramaswamy, *J. Stat. Mech.* **2017**, 054002 (2017).
- [253] S. Ramaswamy, *Annu. Rev. Condens. Matter Phys.* **1**, 323 (2010).
- [254] M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao, and R. A. Simha, *Rev. Mod. Phys.* **85**, 1143 (2013).
- [255] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, *Phys. Rev. Lett.* **75**, 1226 (1995).
- [256] J. Deseigne, S. Léonard, O. Dauchot, and H. Chaté, *Soft Matter* **8**, 5629 (2012).
- [257] J. Deseigne, O. Dauchot, and H. Chaté, *Phys. Rev. Lett.* **105**, 098001 (2010).
- [258] G. Wang, T. V. Phan, S. Li, M. Wombacher, J. Qu, Y. Peng, G. Chen, D. I. Goldman, S. A. Levin, R. H. Austin, and L. Liu, *Phys. Rev. Lett.* **126**, 108002 (2021).

- [259] V. A. Webster-Wood, M. Guix, N. W. Xu, B. Behkam, H. Sato, D. Sarkar, S. Sanchez, M. Shimizu, and K. K. Parker, *Bioinspir. Biomim.* **18**, 015001 (2022).
- [260] R. Khasseh, S. Wald, R. Moessner, C. A. Weber, and M. Heyl, Active quantum flocks (2023), [arXiv:2308.01603v2 \[quant-ph\]](https://arxiv.org/abs/2308.01603v2).
- [261] G. Frangipane, D. Dell’Arciprete, S. Petracchini, C. Maggi, F. Saglimbeni, S. Bianchi, G. Vizsnyiczai, M. L. Bernardini, and R. Di Leonardo, *eLife* **7**, e36608 (2018).
- [262] É. Fodor and M. E. Cates, *EPL* **134**, 10003 (2021).
- [263] M. E. Cates, in *Active Matter and Nonequilibrium Statistical Physics: Lecture Notes of the Les Houches Summer School: Volume 112, September 2018* (Oxford University Press, 2022) Chap. 6, p. 180.
- [264] M. R. Shaebani, A. Wysocki, R. G. Winkler, G. Gompper, and H. Rieger, *Nat. Rev. Phys.* **2**, 181 (2020).
- [265] J. U. Klamser, O. Dauchot, and J. Tailleur, *Phys. Rev. Lett.* **127**, 150602 (2021).
- [266] B. Sabass, R. G. Winkler, T. Auth, J. Elgeti, D. A. Fedosov, M. Ripoll, G. A. Vliegenthart, and G. Gompper, in *Out-of-equilibrium Soft Matter* (The Royal Society of Chemistry, 2023) Chap. 10, p. 354.
- [267] M. E. Cates and J. Tailleur, *Annu. Rev. Condens. Matter Phys.* **6**, 219 (2015).
- [268] J. O’Byrne, A. Solon, J. Tailleur, and Y. Zhao, An introduction to motility-induced phase separation, in *Out-of-equilibrium Soft Matter* (The Royal Society of Chemistry, 2023) Chap. 4, p. 107–150.
- [269] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, *Rev. Mod. Phys.* **87**, 593 (2015).
- [270] B. Doyon, *SciPost Phys.* **5**, 054 (2018).
- [271] H. Spohn, *J. Stat. Phys.* **154**, 1191 (2014).
- [272] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, *Phys. Rev. Lett.* **117**, 207201 (2016).
- [273] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, *Phys. Rev. X* **6**, 041065 (2016).
- [274] J. De Nardis, D. Bernard, and B. Doyon, *Phys. Rev. Lett.* **121**, 160603 (2018).
- [275] R. A. Simha and S. Ramaswamy, *Phys. Rev. Lett.* **89**, 058101 (2002).
- [276] J. Toner, Y. Tu, and S. Ramaswamy, *Ann. Phys.* **318**, 170 (2005).
- [277] F. Jülicher, S. W. Grill, and G. Salbreux, *Rep. Prog. Phys.* **81**, 076601 (2018).
- [278] B. Partridge and C. F. Lee, *Phys. Rev. Lett.* **123**, 068002 (2019).

- [279] L. Langford and A. K. Omar, *Phys. Rev. E* **110**, 054604 (2024).
- [280] M. Kourbane-Houssene, C. Erignoux, T. Bodineau, and J. Tailleur, *Phys. Rev. Lett.* **120**, 268003 (2018).
- [281] C. Erignoux, *Markov Process. Related Fields* **30**, 57 (2024).
- [282] T. Agranov, S. Ro, Y. Kafri, and V. Lecomte, *J. Stat. Mech.* **2021**, 083208 (2021).
- [283] T. Agranov, S. Ro, Y. Kafri, and V. Lecomte, *SciPost Phys.* **14**, 045 (2023).
- [284] P. Digregorio, D. Levis, A. Suma, L. F. Cugliandolo, G. Gonnella, and I. Pagonabarraga, *Phys. Rev. Lett.* **121**, 098003 (2018).
- [285] J. U. Klamser, S. C. Kapfer, and W. W. Krauth, *Nat. Commun.* **9**, 5045 (2018).
- [286] F. Caballero, C. Nardini, and M. E. Cates, *J. Stat. Mech.* **2018**, 123208 (2018).
- [287] G. Szamel and E. Flenner, *EPL* **133**, 60002 (2021).
- [288] L. S. Ferro, S. Can, M. A. Turner, M. M. ElShenawy, and A. Yildiz, *eLife* **1**, 031080 (2019).
- [289] J. F. Wienand, S. Karch, A. Impertro, C. Schweizer, E. McCulloch, R. Vasseur, S. Gopalakrishnan, M. Aidelsburger, and I. Bloch, *Nat. Phys.* **20**, 1732 (2024).
- [290] U. Siems, C. Kreuter, A. Erbe, N. Schwierz, S. Sengupta, P. Leiderer, and P. Nielaba, *Sci. Rep.* **2**, 1015 (2012).
- [291] A. Miron, D. Mukamel, and H. A. Posch, *J. Stat. Mech.* **2020**, 063216 (2020).
- [292] P. Wilke, E. Reithmann, and E. Frey, *Phys Rev X* **8**, 031063 (2018).
- [293] R. Nandi, U. C. Täuber, and Priyanka, *Entropy* **23**, 1343 (2021).
- [294] P. C. Martin, E. D. Siggia, and H. A. Rose, *Phys. Rev. A* **8**, 423 (1973).
- [295] H. K. Janssen, *Z. Physik B* **23**, 377 (1976).
- [296] C. De Dominicis, *Phys. Rev. B* **18**, 4913 (1978).
- [297] C. De Dominicis and L. Peliti, *Phys. Rev. B* **18**, 353 (1978).
- [298] S. Saha and T. Sadhu, *SciPost Phys.* **17**, 033 (2024).
- [299] L. N. Trefethen, *Spectral methods in MATLAB* (SIAM Philadelphia, 2000).
- [300] A. P. Solon, J. Stenhammar, M. E. Cates, Y. Kafri, and J. Tailleur, *Phys. Rev. E* **97**, 020602(R) (2018).
- [301] A. P. Solon, J. Stenhammar, M. E. Cates, Y. Kafri, and J. Tailleur, *New J. Phys.* **20**, 075001 (2018).

- [302] E. Tjhung, C. Nardini, and M. E. Cates, *Phys. Rev. X* **8**, 031080 (2018).
- [303] M. R. Nejad and J. M. Yeomans, *PRX Life* **1**, 023008 (2023).
- [304] I. Mukherjee, A. Raghu, and P. K. Mohanty, *SciPost Phys.* **14**, 165 (2023).
- [305] O. Blondel, C. Erignoux, and M. Simon, *Probab. Math. Phys.* **2**, 127 (2021).
- [306] L. Touzo, P. Le Doussal, and G. Schehr, *EPL* **142**, 61004 (2023).
- [307] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevE.111.024128> for videos.
- [308] S. Das, S. Ghosh, T. Sadhu, and J. U. Klamser, Role of kinematic constraints in the time reversal symmetry breaking of a model active matter (2024), [arXiv:2409.10425](https://arxiv.org/abs/2409.10425) [cond-mat.soft] .
- [309] Y-E Keta, R. L. Jack, and L. Berthier, *Phys. Rev. Lett.* **129**, 048002 (2022).
- [310] J. Mason, C. Erignoux, R. L. Jack, and M. Bruna, *Proc. R. Soc. A* **479**, 20230524 (2023).
- [311] T. Bodineau and B. Derrida, *Phys. Rev. E* **72**, 066110 (2005).
- [312] G. Bunin, Y. Kafri, and D. Podolsky, *J. Stat. Mech.* **2012**, L10001 (2012).
- [313] P. Tsobgni Nyawo and H. Touchette, *Phys. Rev. E* **94**, 032101 (2016).
- [314] Y. Baek, Y. Kafri, and V. Lecomte, *J. Phys. A: Math. Theor.* **51**, 105001 (2018).
- [315] A. Lefèvre and G. Biroli, *J. Stat. Mech.* **2007**, P07024 (2007).
- [316] B. Derrida, *J. Stat. Mech.* **2007**, P07023 (2007).
- [317] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, *J. Stat. Phys.* **135**, 857 (2009).
- [318] T. Sadhu and B. Derrida, *J. Stat. Mech.* **2016**, 113202 (2016).
- [319] R. Mukherjee, Active lattice gas, <https://github.com/rikmukherjee/Active-Lattice-Gas.git> (2024).