

# Holography, Cosmology and Tachyons

A Thesis

Submitted to the  
Tata Institute of Fundamental Research, Mumbai  
Subject Board of Physics  
for the degree of Doctor of Philosophy

by

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September, 2026

# DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgment of collaborative research and discussions.

The work was done under the guidance of Professor Suvrat Raju at the Tata Institute of Fundamental Research, Mumbai.



Ashik H

In my capacity as the formal supervisor of record of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.



Suvrat Raju

Date: 27 September 2026

## **Acknowledgements**

I don't think I can put into words the love and gratitude I have for people who have been supportive during this journey.

However, to put it concisely, I would like to thank my supervisor, collaborators, colleagues, friends, and family for their support and encouragement throughout this work.

# Abstract

We study the ‘unusual localisation’ of information in theories of Quantum Gravity. For this, we mainly consider a low-energy effective field theory description of gravity in asymptotically flat and de Sitter spacetimes. In addition to this, we also consider a scenario in string theory, related to unstable D-branes.

As for the effective theory of gravity in flat and de Sitter spacetimes, we see that even in the  $G_N \rightarrow 0$  limit, imposing the gravitational constraints leads to information being holographic in nature. An explicit example of this is discussed by considering scalar field theory in flat spacetime and arguing that the gravitational Gauss’ law leads to holography of information, even for finite regions.

We then turn to a cosmological setting, where we study implications of gravitational constraints for scalar field theory in de Sitter. We see that the constraints are much stronger in de Sitter, owing to the fact that Cauchy slices are compact. We compute cosmological correlators in the gravitationally constrained states and see signatures of holography of information for perturbative states with a small number of excitations. We then propose a heavy observer model which reproduces QFT-like correlations, thereby helping to reconcile holography with the notion of locality.

Finally, we study tachyon dynamics on an unstable D-brane in string theory. Earlier works on this subject suggest, in the classical ( $g_s \rightarrow 0$ ) limit, an equality between open and closed string descriptions of the physics of a decaying D-brane. We advance this map further and show that by quantising the tachyon effective theory, using collective field theory methods, one can map it to a sub-sector of Klein-Gordon theory. And the Hilbert space we obtain consists of coherent states and excitations on top of them. This is in agreement with the fact that a decaying D-brane produces a coherent state of closed strings. We then discuss how this result is amenable to an interpretation of having an ‘unusual localisation’ of information in the sense that the open string degrees of freedom know about the closed string states into which the D-brane is expected to decay.

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## Works included in the thesis

1. N. Gaddam and A. H, *Holography of information in a ball of finite radius*, [JHEP 06 \(2025\) 034](#) [[2410.17316](#)]
2. T. Chakraborty, A. H and S. Raju, *Cosmological correlators in gravitationally constrained de Sitter states*, [JHEP 01 \(2026\) 004](#) [[2507.15926](#)]
3. P.V. Athira, A. H and P. Paul, *Mapping Tachyon effective field theory to a subsector of Klein-Gordon theory*, [2604.18679](#)

# Chapter 1

## Introduction

Understanding Quantum Gravity has been one of the primary goals of theoretical high-energy physics. Not only does it give a single framework to address all four fundamental forces of nature, but it also allows us to answer questions regarding black holes, the physics of the early universe, and so on. But I think it is equally important to acknowledge the hope that understanding Quantum Gravity will widen the horizon of our knowledge and will lead to more questions regarding our universe that we are not even equipped to comprehend, given our current theoretical framework. This hope stems from a historical standpoint. Let us take electricity and magnetism, phenomena which were initially understood to be completely unrelated. As we made progress, it was understood that a single theory could explain them. This led to further questions and a better understanding. A similar story holds for the Standard Model as well - from the developments in quantum mechanics to finally arriving at a reasonably deep understanding of gauge theories - we saw that all the fundamental forces except gravity could be consistently addressed within the framework of quantum field theory. Furthermore, the development of the theory forced us to ask questions related to the Higgs particle and so on. It is in this spirit that I would like to motivate the need for studies on Quantum Gravity. They will answer some questions that we already have, but more importantly, they will also enable us to ask new questions.

There are many approaches to understanding Quantum Gravity. The most straightforward one is to develop a theory that could consistently address gravity and other quantum field theories. String theory is one such framework. Developments in String theory led to a better understanding of gauge theories and their relationship with theories of gravity. One of its most important achievements is the AdS/CFT correspondence, which provides us with a toy model of quantum gravity for a universe with a negative cosmological constant. There is still a long way to go before we can understand quantum gravity in universes with a positive or zero cosmological constant.

Alternatively, one could try to understand properties expected of a theory of quantum gravity by working within a low-energy effective field theory approximation. Here, one looks for peculiar properties of quantum gravity that could be extracted at this energy scale. In my opinion, a historical analogy would be the development of the least action principle. We had Newtonian mechanics, which could explain particle dynamics. However, the principle of least action explains the same theory in a rather different way: the particle takes the path which minimises the action. And we know that this notion naturally carries over to quantum mechanics, *with necessary modifications*. In the case of an effective field theory analysis of quantum gravity, our hope is the same. We would like to find properties and principles that would then naturally have an extension in a full-fledged Quantum theory of gravity.

In this thesis, we will mostly follow the low-energy effective field theory approach to quantum gravity. We treat gravity as a gauge theory with diffeomorphisms as the gauge redundancies. The principal objective will then be to understand the differences between gravity and other well-known gauge theories like electromagnetism. From physical intuition alone, one can immediately see that gauge theories like electromagnetism have particles that can attract or repel each other, whereas in gravity, particles can only attract. This means that electromagnetism has positive as well as negative charge whereas gravity has only one kind of charge and it only attracts. We will study implications stemming from this physical distinction. In particular, we will study how, in gravity, information is stored in a radically different way compared to other gauge theories.

On a more technical note, our approach to understanding distinctions between the effective theory of gravity and other gauge theories is to carefully study the constraint equations in both cases. As we know, a gauge theory has redundant degrees of freedom. Thus, not all equations of motion will be evolution equations<sup>1</sup>. Rather, some will be equations that relate the initial data of the degrees of freedom to each other. These equations are the constraint equations, as they constrain the initial data, like Gauss' law in electromagnetism. The Hamiltonian formalism is well-suited to understanding the implications of constraint equations. For example, consider a system with phase space variables  $q_i, p_i; i = 1, 2, \dots, N$ , out of which some are redundant. Let the constraint equation be denoted as  $C[q_i, p_i] = 0$ , which will be obeyed by all classical solutions<sup>2</sup>. Furthermore, these constraints generate gauge transformations, under which all physically measurable quantities remain invariant, as gauge symmetry is an

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<sup>1</sup>Evolution equations relate the second time derivative of a degree of freedom to its first and zeroth derivatives. For example,  $\frac{1}{m}Fx(t), \dot{x}(t) = \ddot{x}(t)$

<sup>2</sup>Recall that the set of classical solutions is in one-to-one correspondence with the set of all allowed initial conditions.

artifact of the redundancy in the description [1].

$$\delta F_{obs} = \{F, C\} = 0, \quad C[\{q_i^{(cl)}\}, \{p_i^{(cl)}\}] = 0 \quad (1.1)$$

Here  $F_{obs}$  is a quantity that can be measured, and it is a function of the phase space variables. The Poisson bracket is denoted by  $\{\dots\}$  and  $\{q_i^{(cl)}\}, \{p_i^{(cl)}\}$  denote a classical solution. In quantum theory, these conditions become statements regarding the set of all allowed states and operators in the theory.

$$\delta \hat{F}_{obs} = -i[\hat{F}, \hat{C}] = 0, \quad \hat{C}[\{\hat{q}_i\}, \{\hat{p}_i\}]|\psi\rangle = 0 \quad (1.2)$$

Here, we used the hat symbol to emphasize that we are dealing with quantum operators<sup>3</sup>. Equations (1.2) are much stronger than (1.1) because, roughly speaking, the classical constraints are like conditions imposed on the expectation values, whereas quantum constraints are imposed directly on the states and operators.

In the case of gravity, the constraint equations are known as Wheeler-DeWitt equations, and they are more powerful compared to other gauge constraints. As we explained earlier, this is because gravity couples to everything, and it is always attractive. In electromagnetism, you can hide a region of positive charge by covering it with a region of negative charge such that, when measuring from the outside, one sees a neutral object. In the case of gravity, you cannot hide anything, as there is no compensating negative energy one can use. These differences between gravity and other gauge theories can be made mathematically precise by analysing (1.2). Furthermore, one can establish that, in the case of gravity, information available in a region is also available in its complement. This property is called the Holography of information.

We start with a simple scenario in Chapter 2 where the holographic nature of information in theories of gravity could be explicitly seen. We consider scalar field theory in 3+1-dimensional asymptotically Minkowski spacetime and show - within the linearised gravity approximation- that information is holographic in nature, even for finite regions. The setup of the problem involves a sphere of finite radius  $R$ , in the  $t = 0$  slice, and two observers, one stationed inside the ball and another at the boundary of the ball (fig. 1.1).

Both observers are given access to a state  $|\psi\rangle$ , by which we mean that they could make measurements on this state. Then, by computing correlation functions, they are asked to uniquely identify the state  $|\psi\rangle$ <sup>4</sup>. But the domain in which they are allowed to make measurements is restricted. The red observer is bound to stay inside the ball, whereas

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<sup>3</sup>Also, we have put  $\hbar = 1$ .

<sup>4</sup>Or, in other words, given two states  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , they are asked to distinguish between them.

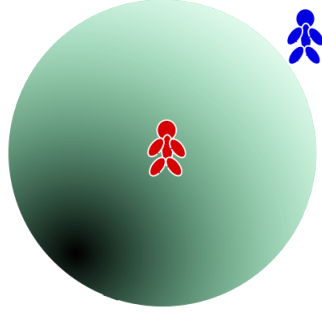


Figure 1.1: A ball of finite radius, with an observer inside the ball and one at the boundary.

the blue observer is allowed to make measurements only in a radial band outside the ball. One can then show that, depending on the state, the observers may either be able to uniquely identify the state or obtain partial information about the state. However, in both cases, *whatever information is accessible to the inside observer is also available to the outside observer*. We will show this explicitly by providing an algorithm with which the observers can go about extracting information from the state, by means of measuring correlation functions. As we will elaborate further in Chapter 2, this ‘unusual localisation of information’ happens only in theories of gravity, and it is a direct consequence of the gravitational constraints.

The implications of gravitational constraints become much more interesting when we consider gravity in asymptotically de-Sitter spacetime. This is discussed in Chapter 3, where we will again consider scalar field theory coupled to gravity and work in the  $G_N \rightarrow 0$  limit. Here, the constraints are much stronger owing to the fact that de-Sitter has compact spatial slices. A simple analogy from electromagnetism can explain the reason for this. The electromagnetic Gauss’ law tells you that the total charge is equal to the flux at the boundary. But if there is no boundary, then the total charge should be zero.

$$Q_\Sigma = \int_\Sigma \rho = \int_{\partial\Sigma} \vec{E} \cdot \vec{n} = 0; \text{ as } \partial\Sigma = 0 \quad (1.3)$$

In quantum theory, the corresponding statement is that the allowed states in the theory would be annihilated by  $Q_\Sigma$ , i.e.,  $Q_\Sigma|\psi\rangle = 0$ . Similarly, for gravitational constraints in de Sitter, one obtains the result that the allowed states in the theory will be annihilated by charges generating de Sitter isometries, i.e., states have  $SO(1,d+1)$  invariance. This will amount to the fact that information is holographic in nature. We will explicitly demonstrate perturbative signatures of holography by computing lower-point functions in microscopically simple states. One important aspect that will be of great interest is the fact that correlation functions in simple states will not look like the vacuum<sup>5</sup> correlations in a non-gravitational QFT. Such correlation functions are of utmost

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<sup>5</sup>Also called the Hartle-Hawking state or the Bunch-Davis vacuum

importance in the study of Cosmology, and one would naturally expect some limit in which the correlation functions computed in our gravitational theory reduce to local QFT correlators. We show that this indeed is possible by introducing a ‘heavy’ observer and coarse-graining the set of observables. More precisely, correlation functions defined relationally with respect to the state reduce to vacuum-like correlations in an ordinary QFT. This construction further makes holography of information consistent with our mundane notion of locality, as it is possible to obtain local QFT correlations within the framework of gravitationally constrained states and operators.

Given these examples where one studies the holographic nature of information using the effective theory of gravity in flat and de Sitter space-times, it is natural to ask whether such an ‘unusual localisation of information’ is seen in String theory. Of course, for gravitational theories in AdS spacetimes, we have the AdS/CFT correspondence, which is a statement of holographic duality and it is much stronger than holography of information. In the case of flat and de Sitter spacetimes, we don’t have anything as concrete as the AdS/CFT correspondence yet. However, it is still possible to look for instances where there is some notion of an ‘unusual localisation of information’ in String theory. It is in this context that we study tachyon dynamics on an unstable D-brane in bosonic/super string theory. An unstable D-brane has a tachyonic mode in the open string spectrum owing to the tachyonic field ( $T$ ) sitting at the maximum of the potential  $V(T)$ . If one displaces the tachyon away from the maximum, it rolls towards the minimum in a peculiar way, and such a scenario is described by time-dependent solutions in string theory. Physically, the results of this tree-level open-string theory analysis can be summarised as follows: the D-brane decays when the tachyon rolls down the potential, and in the  $x^0 \rightarrow \infty$  limit, one is left with a system of non-rotating, non-interacting dust. Interestingly, one can turn on  $g_s$  and study the closed string radiation produced by the decaying D-brane. Then, taking the  $g_s \rightarrow 0$  limit, one observes that the final closed string radiation is localised in the plane of the original D-brane, and it has properties similar to the system of dust obtained from open string analysis. This equivalence in description, in the classical limit, suggests that the open string degrees of freedom already know about the closed string states into which the D-brane is expected to decay. This could be considered as an instance of ‘unusual localisation of information’. We study this further and show that it is possible to map the tachyon effective theory, which describes the dynamics of tachyon in an unstable D-brane, to a subsector of Klein-Gordon theory. In particular, the Hilbert space consists of coherent states and excitations on top. This is in agreement with the coherent state of closed string modes that results from the decay of the D-brane, as the D-brane acts as a time-dependent source for closed strings. Although one needs a string theory computation to prove the equivalence between the open string de-

scription and the closed string description at the quantum level, our results from the tachyon effective field theory seem to suggest that this might be possible. Furthermore, one might be able to make the notion ‘open string degrees of freedom already know about the closed string states into which the D-brane is expected to decay’ more precise.

The thesis consists of the following:

- Holography of information in a ball of finite radius is discussed in Chapter 2. Here, emphasis is given to how the gravitational Gauss’ law gives the observer at the boundary access to information that the inside observer already has. It is a slight rewriting of the paper [2], and I thank my collaborator Nava Gaddam for his contributions to this project.
- Cosmological correlators in gravitationally constrained de Sitter states are discussed in Chapter 3. Here, we focus more on the implications of gravitational constraints rather than describing how to solve the constraints, i.e., we start with gravitationally constrained states, citing previous works on this subject, and compute correlation functions in these states and analyse the result. Except for the introduction and part of the discussion section, this is a verbatim reproduction of the work [3], and I would like to thank my collaborators, Tuneer Chakraborty and Suvrat Raju, for their contributions to this work.
- Tachyon effective field theory and its mapping to a subsector of Klein-Gordon theory are discussed in Chapter 4. This chapter is heavily based on [4], and I thank my collaborators Athira P V and Priyadarshi Paul for their contributions. I would also like to thank Ashoke Sen for his guidance in this work.

# Chapter 2

## Holography

In theories of gravity, information is localised in an unusual way compared to other gauge theories. Working in the low-energy effective field theory, one can observe the holographic nature of information by a careful analysis of the gravitational constraints. This was performed in asymptotically AdS, flat, and de Sitter spacetimes in [5–7] respectively. The general conclusion one arrives at after these studies is termed the ‘principle of Holography of information’, and it states [7,8]:

*“Information that is available in a region of the Cauchy slice is also available in its complement.”*

In this chapter we will describe a simple treatment of scalar field theory in flat space-time and show how the notion of holography of information arises from the gravitational Gauss’ law [2].

In AdS and flat spacetimes, the way to prove holography of information goes by putting an observer at a small radial band at the boundary of the Cauchy slice (fig. 2.1) and arguing that this observer can distinguish between any two states in the theory [5,6]. In this Chapter, we will consider a simpler question, “What happens when the observer is stationed at a finite radius? What amount of information is available to this observer?”. In particular, we will be interested in the case of asymptotically flat spacetimes and ask whether correlation functions on the boundary of the ball capture all the information of all correlators in the bulk of the ball. We will follow the analysis presented in [2].

One can already guess the answer by stating the results of [5] and [6] in a different way. Here, the statement is that an observer at the boundary of the Cauchy slice has access to all the information in the bulk. Equivalently, information available to an observer

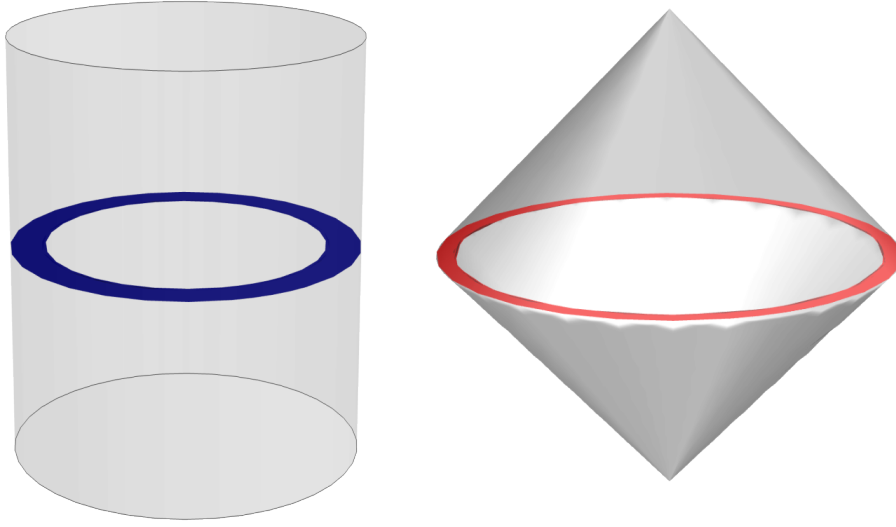


Figure 2.1: An observer at the boundary of the Cauchy slice of AdS (left) and flat (right) spacetimes can distinguish between any two states in the bulk.

having access to the full Cauchy slice<sup>1</sup> is also available to an observer in a small radial band at the boundary of the Cauchy slice. An analogous statement for finite regions will be that information available to an observer *inside a ball of finite radius* will also be available to an observer *in a small radial band at the boundary of the ball*. In this chapter, we will show that this statement is true within the confines of linearised quantum gravity for generic low-energy (free Klein-Gordon) states in flat space created by the action of unitaries on the vacuum to leading order in a certain perturbative parameter. For an infinite sub-class of these states, our results are perturbatively exact and valid to all orders in this same parameter. Unlike at infinity, massive and massless fields can be captured within the same framework.

It should be mentioned right away that our result for a finite region [2] is weaker than the corresponding result at infinity [5, 6]. This is due to the fact that there are a lot of advantages to making a statement at a small band at the boundary of the Cauchy slice. Most importantly, one can formulate the result at infinity in a gauge-invariant way by mapping this region to a small band at the boundary of the spacetime<sup>2</sup>, such that the results are given in terms of boundary variables that are gauge invariant. We will discuss this issue and related limitations of the finite region result in Section 2.6.

<sup>1</sup>Recall that the Cauchy slice has full information of the theory

<sup>2</sup>For AdS, this is a small time band  $t \in [0, \epsilon]$  at the boundary of AdS [5], whereas for flat spacetime, it is a small band  $u \in [-\infty, -\frac{1}{\epsilon}]$  at null infinity (for a massless field) [6]

## 2.1 Setup

We consider a scalar field coupled to gravity in asymptotically flat spacetime. The action is given by,

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{g} R - \frac{1}{2} \int d^4x \left( g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + m^2 \phi(x)^2 \right) \quad (2.1)$$

Although we have chosen a mass parameter  $m$  for the scalar field, our analysis will go through for the massless case ( $m = 0$ ) also. In perturbation theory, one can write the metric as <sup>3</sup>,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (2.2)$$

Although it is common usage to pull out a factor of  $\sqrt{G_N}$  from  $h_{\mu\nu}$ , we will not do it as we will only work up to the first non-trivial order in  $G_N$ .

One can now consider a spherical region of radius  $R$  in the  $t = 0$  Cauchy slice. Fix a coordinate system  $\{t, \vec{x}\}$ , such that integrating the local Gauss' law constraint <sup>4</sup> over the ball, we get the familiar statement of the Gauss' law [9].

$$\int_{|\vec{x}| \leq R} \left( T_{00}^{(\phi)} + T_{00}^{(h)} \right) = \frac{1}{16\pi G_N} \int_{|\vec{x}|=R} d^2x n_i (\partial_j h_{ij} - \partial_i h_{jj}) \quad (2.3)$$

Here  $n_i$  is a normal vector to the surface of the ball, and  $i, j$  run over the spatial components. We have kept only the leading order term in  $G_N$ , such that the LHS is  $O(G_N^0)$ . At this order, the Gauss' law could be read as follows: the energy density of the scalar field  $(T_{00}^{(\phi)})$  and free gravitons  $(T_{00}^{(h)})$  integrated over the bulk of the ball is equal to a boundary integral of the gravitational fluctuations.

Let us omit the energy of the gravitons for simplicity. They can be easily taken into account at this order in perturbation theory as they are just free propagating gravitons, decoupled from the scalar field. Upon canonical quantisation of the theory, (2.3) becomes an operator statement.

$$\begin{aligned} H_R[\phi] &= H_{\partial R}[h] \\ \text{Where } H_R[\phi] &\equiv \frac{1}{2} \int_{|\vec{x}| \leq R} d^3x : \left( \pi(\vec{x})^2 + (\vec{\nabla} \phi(\vec{x}))^2 + m^2 \phi(\vec{x})^2 \right) : \\ \text{and } H_{\partial R}[h] &\equiv \frac{1}{16\pi G_N} \int_{|\vec{x}|=R} d^2x n_i (\partial_j h_{ij} - \partial_i h_{jj}) \end{aligned} \quad (2.4)$$

Here  $\pi$  is the conjugate momentum of  $\phi$  and  $: :$  denotes normal ordering. The fact that energy density of a scalar field integrated over a finite volume ( $H_R$ ) becomes equal to a

<sup>3</sup>We work in the  $-+++$  signature for the Minkowski metric

<sup>4</sup>which is just the time-time component of Einstein's equations

boundary operator ( $H_{\partial R}$ ) given in terms of  $h_{ij}$ , is a key implication of the gravitational constraint that we will use throughout. As a technical aside, one should keep in mind that the finite region Hamiltonian  $H_R$  has quadratic fluctuations that scale with  $R$ . One has to smear this operator appropriately such that it has a well-defined  $R \rightarrow \infty$  limit. This point is discussed in Appendix A.1. We will not worry about this, as we will only work with a finite value of  $R$ .

## Observers and observables

We can now introduce the observers and observables so as to set up the problem. Consider an observer stationed at the boundary of this ball (See fig. 1.1), who is only allowed to make measurements in a small radial band at the boundary. We would like to understand the amount of information accessible to this observer. To make the question precise one has to first define what is meant by information in a region.

*Definition: Given a state  $|\psi\rangle$ , the information regarding the state  $|\psi\rangle$  available in a region  $\mathcal{R}$  is defined as the set of all correlation functions involving the field variables and their conjugate momenta one can measure in the region  $\mathcal{R}$ .*

For example, consider a function  $O^{(n)}(\vec{y}_1, \vec{y}_2, \dots, \vec{y}_n)$  which is made using the field variables and their conjugate momenta inserted at the points  $\vec{y}_i$ ;  $i = 1, 2, \dots, n$ <sup>5</sup>. The correlation function  $\langle \psi | O^{(n)}(\vec{y}_1, \vec{y}_2, \dots, \vec{y}_n) | \psi \rangle$ ;  $\forall \vec{y}_i \in \mathcal{R}$  is, according to our definition, part of the information about the state  $|\psi\rangle$  available in the region  $\mathcal{R}$ . It should be obvious why this defines information about the state  $|\psi\rangle$  available in the region  $\mathcal{R}$ . An observer restricted to the region  $\mathcal{R}$  has access to only operators of the form,

$$\int_{\mathcal{R}} d^3x_1 d^3x_2 \dots d^3x_n f(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) O^{(n)}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$$

Here  $f$  is a smearing function and the integral is restricted to the region  $\mathcal{R}$ <sup>6</sup>. So, given the expectation value,  $\langle \psi | O(\vec{y}_1, \vec{y}_2, \dots, \vec{y}_n) | \psi \rangle$ , for all values of  $\vec{y}_i$  in the region  $\mathcal{R}$ , one can compute the expectation value of any operator having local support in  $\mathcal{R}$ .

Having defined what we mean by information in a region, we would like to show that for theories of gravity, information is holographic even for finite regions. The key idea is to consider two observers, one inside the ball and the other at the boundary. Then, by analysing the measurements that can be performed by both observers, we show that the information available to the observer at the boundary is the same as

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<sup>5</sup> $n$  can be any positive integer.

<sup>6</sup>We have considered  $\mathcal{R}$  as a spatial region, as the observers we will consider in this chapter will be making measurements about a ball of finite radius in the  $t = 0$  slice.

that available to the observer inside the ball.

Our strategy will be the following:

1. An observer is placed inside the ball, allowed to make measurements of the form,  $\langle \psi | O(\vec{y}_i) | \psi \rangle$ ;  $|\vec{y}_i| \leq R$ , where  $\vec{y}_i = \vec{y}_1, \vec{y}_2, \dots$ . Note that  $O(\vec{y}_i)$  would consist of monomials of the field  $\phi$  and momenta  $\pi$  inserted at the points  $\vec{y}_1, \vec{y}_2, \dots$
2. An observer is placed at a small radial band at the boundary of the ball. The set of all measurements that the boundary observer can make on a state  $|\psi\rangle$  is of the form  $\langle \psi | O(\vec{z}_i) | \psi \rangle$ ;  $R \leq |\vec{z}_i| \leq R + \epsilon$ . *Here, in addition to monomials of the field  $\phi$  and momenta  $\pi$  inserted at the points  $\vec{z}_1, \vec{z}_2, \dots$ , the observer also has access to the operator  $H_{\partial R}[h]$ , owing to Gauss' law (2.4).*
3. One has to now consider a state  $|\psi\rangle$ , and show that whatever information can be extracted by the inside observer regarding this state, by measuring correlation functions, can also be extracted by the observer at the boundary.

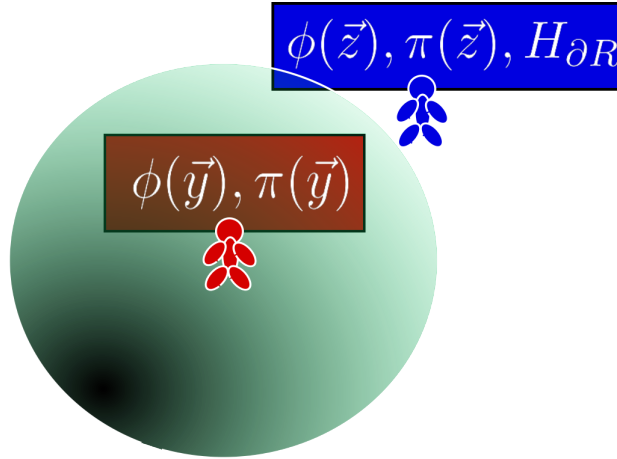


Figure 2.2: Observers inside the ball and outside the ball, along with the operators they could choose to use.

This completes the setup of our problem. One has to now define an ‘interesting’ class of states  $|\psi\rangle$  to work with. This is because some classes of states are more interesting in the sense that, in a non-gravitational theory, the boundary observer might not be able to detect this state at all. We will next describe this interesting set of states, which are of the form of localised unitary excitations on top of the vacuum.

#### Notation:

- States are defined, and measurements are being made in the  $t = 0$  Cauchy slice.
- Points that may take values anywhere in the  $t = 0$  slice will be labelled by  $\vec{x}_i$ .

- Points that may take values exclusively inside a ball of radius  $R$  will be labelled by  $\vec{y}_i$ . So,  $\vec{y}_i$  will label the measurements made by the observer inside the ball. This observer will often be called ‘inside/bulk observer’.
- Points that may take values exclusively in a small radial band near the boundary of the ball of radius  $R$  will be labelled by  $\vec{z}_i$ . So,  $\vec{z}_i$  will label the measurements made by the observer at the boundary of the ball. This observer will often be called ‘outside/boundary observer’.
- $n$ -point field insertions of the form  $\phi(\vec{x}_1)\phi(\vec{x}_2)\dots\phi(\vec{x}_n)$  will be compactly denoted as  $\phi^n(\{\vec{x}\})$ .
- Vacuum two-point functions of fields are denoted using the usual Wick contraction symbol,  $\langle 0|\phi\phi|0\rangle = \overline{\phi\phi}$

## 2.2 Class of states

We motivate an interesting class of states that are of relevance to the holography of information by a simple example. Consider a localised unitary excitation on top of the vacuum, parametrised by the function  $f(\vec{x})$ .

$$|\psi\rangle = e^{i \int d^3x f(\vec{x})\phi(\vec{x})}|0\rangle \equiv U|0\rangle \quad (2.5)$$

Take  $f(\vec{x})$  to have support only in the compact region  $\mathcal{R}$  in the Cauchy slice. Put an observer in the complement  $\bar{\mathcal{R}}$ , such that  $\mathcal{R}$  and  $\bar{\mathcal{R}}$  are space-like separated by definition. In a non-gravitational theory, the observer will not be able to ‘see’ the state  $|\psi\rangle$ . This can be shown as follows. Note that in a non-gravitational theory, the observer can measure a correlation function of operators of the form  $O(\{\vec{z}_i\}) \equiv O(\vec{z}_1, \vec{z}_2, \dots)$ , which are built from  $\phi$  and  $\pi$  inserted at the points  $z_i$ . Then, owing to the fact that  $\vec{z}_i \in \bar{\mathcal{R}}$  is space-like separated from  $\mathcal{R}$ , we have  $[U, O(\{\vec{z}_i\})] = 0$ . Using this, we get,

$$\begin{aligned} \langle \psi | O(\{\vec{z}_i\}) | \psi \rangle &= \langle 0 | U^\dagger O(\{\vec{z}_i\}) U | 0 \rangle \\ &= \langle 0 | O(\{\vec{z}_i\}) U^\dagger U | 0 \rangle \\ &= \langle 0 | O(\{\vec{z}_i\}) | 0 \rangle \end{aligned} \quad (2.6)$$

Thus, in a non-gravitational theory, the localised unitary excitation is inaccessible to an observer in the complement of the unitary.

The above argument can be generalised as follows: any two states  $|\psi_1\rangle$  and  $|\psi_2\rangle$  that differ by the action of a localised unitary,  $|\psi_2\rangle = U|\psi_1\rangle$ , will be indistinguishable in the region spacelike separated from the region of support of the unitary (for a non-

gravitational theory).

Thus, the interesting class of states that we would like to consider is of the form of unitary excitations on top of the vacuum,

$$|\psi\rangle = \exp \left\{ i\gamma \sum_{n=1}^{\infty} \int \prod_{i=1}^n d^3x_i f_n(x_1, x_2, \dots, x_n) : O^{(n)}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) : \right\} |0\rangle \quad (2.7)$$

Here,  $O^{(n)}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$  is a function given in terms of monomials in  $\phi$  and  $\pi$  evaluated at  $n$  points. The  $::$  sign denotes normal ordering, which, as we are in the free theory limit, corresponds to subtracting out all the lower-point Wick contractions from the expression for  $O^{(n)}$ . This could be considered as a suitable choice of basis, which will provide technical simplicity as we will see later. The  $f_n$  are real-valued functions such that the operator acting on the vacuum is a unitary<sup>7</sup>.  $\gamma$  is a small parameter we introduced so as to keep the energy of the state controlled, to avoid backreaction<sup>8</sup>.

We will work with states given in (2.7), and show that, in theories of gravity, one can use Gauss' law to access localised unitaries that are space-like separated. We will be able to prove holography of information to all orders in  $\gamma$ , when the  $O^{(n)}$  in (2.7) consists only of either  $\phi$  or  $\pi$ . In the general case where  $O^{(n)}$  has both  $\phi$  and  $\pi$  insertions, we work in perturbation theory in  $\gamma$  and show that the results hold to linear order in  $\gamma$ .

## 2.3 Simple example

Having defined observers, observables, and the interesting class of states, one should now show that information is holographic. This proceeds by working with the set of states in (2.7) and showing that the bulk and boundary observers can extract the same amount of information regarding the functions  $\{f_n\}$  by measuring correlation functions.

Before getting into long calculations, we would like to discuss the simplest scenario, which will make the basic idea clear. From (2.7), consider the case where only  $f_1 \neq 0$ .

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<sup>7</sup>Contrast this class of states with the often used basis of states in QFT,

$$|\psi\rangle = \sum_{n=0}^{n_{\max}} \int \prod_{i=1}^n d^3x_i g_n(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) O^{(n)}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) |0\rangle$$

If  $n_{\max}$  is finite, then two states of the form (7) do not differ by a localised unitary, and hence there exists no region in which they look the same. So, it is trivial to distinguish between two such states.

<sup>8</sup>One can easily see that  $\langle\psi|H|\psi\rangle \propto \gamma^2$

The state is given by<sup>9</sup>,

$$|\psi\rangle = e^{i\gamma \int d^3x f_1(\vec{x})\phi(\vec{x})} |0\rangle \quad (2.8)$$

We have a set of states parametrised by the function  $\{f_1(\vec{x})\}$ . We do not assume anything about the region of support of  $f_1(\vec{x})$ .

*Given such a state to an observer, if they could make some measurements and extract the profile of the function  $f_1(\vec{x})$ , then we say that they have uniquely identified the state.*

We will now show how observers inside and outside the ball will go about it. The observer inside the ball will measure  $\pi(\vec{y})$  correlation function to get,

$$\begin{aligned} \langle\psi|\pi(\vec{y})|\psi\rangle &= \langle 0|e^{-i\gamma \int d^3x f_1(\vec{x})\phi(\vec{x})}\pi(\vec{y})e^{i\gamma \int d^3x f_1(\vec{x})\phi(\vec{x})}|0\rangle \\ &= \langle 0|\left(\pi(\vec{y}) - i\gamma \int d^3x f_1(\vec{x})[\phi(\vec{x}), \pi(\vec{y})]\right)|0\rangle \\ &= \gamma \int d^3x f_1(\vec{x}) \delta(\vec{x} - \vec{y}) \\ &= \gamma f_1(\vec{y}) \end{aligned} \quad (2.9)$$

Here, in the intermediate steps, we used the BCH formula,

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \dots$$

and the canonical commutation relation,

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x} - \vec{y}).$$

Equation (2.9) tells you that the inside observer can measure  $\pi(\vec{y})$  correlation function to get the profile of the function  $f_1(\vec{x})$  inside the ball (Recall that, by definition, the inside observer can only make measurements inside the ball). The next natural question is whether the outside observer can access the same amount of information.

We already saw that in a non-gravitational theory, the outside observer will never be able to see a unitary excitation localised inside. However, in gravity, you can make use of the gravitational Gauss' law (2.4) to see the excitation localised inside. To this end, the outside observer will measure the  $H_{\partial R}\pi(\vec{z})$  correlation function.

$$\begin{aligned} \langle\psi|H_{\partial R}\pi(\vec{z})|\psi\rangle &= \langle\psi|H_R\pi(\vec{z})|\psi\rangle \\ &= \langle 0|\left(H_R\pi(\vec{z}) - i\gamma \int d^3x f_1(\vec{x})[\phi(\vec{x}), H_R\pi(\vec{z})] + \dots\right)|0\rangle \\ &= \gamma \int d^3x \Theta(R - |\vec{x}|) f_1(\vec{x}) \langle 0|\pi(\vec{x})\pi(\vec{z})|0\rangle \end{aligned} \quad (2.10)$$

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<sup>9</sup>We consider  $O^{(1)} = \phi$ , and the other case  $O^{(1)} = \pi$  can be treated in the same manner.

In the first step, we used Gauss' law (2.4) to replace  $H_{\partial R}$  with  $H_R$ . The commutator of  $\phi(\vec{x})$  with  $H_R$  could be easily computed from the definition of  $H_R$  in (2.4) by use of the canonical commutation relations. The  $\Theta$  function is the usual Heaviside theta, which follows from the fact that Hamiltonian density is integrated only up to radius  $R$ .

$$[\phi(\vec{x}), H_R] = i\Theta(R - |\vec{x}|) \pi(\vec{x}) \quad (2.11)$$

Note that there should have been a contribution to (2.10) from the following term,

$$\langle 0 | (-i\gamma) \int d^3x f_1(\vec{x}) H_R [\phi(\vec{x}), \pi(\vec{z})] | 0 \rangle = \gamma f_1(\vec{z}) \langle 0 | H_R | 0 \rangle$$

Here, this is zero,  $\langle 0 | H_R | 0 \rangle = 0$ , as the Hamiltonian is normal ordered. However, when we generalise this procedure to more general states, we would also assume that the function  $f_n$  doesn't have support in the radial band accessible to the boundary observer. This is just for the sake of technical simplicity, because our main aim is to show the usage of Gauss' law to extract information. Terms which arise due to  $[\phi(\vec{x}), \pi(\vec{z})] = i\delta(\vec{x} - \vec{z})$  could be locally measured by the boundary observer and separately taken care of.

Equation (2.10) tells you that the boundary observer can make measurements at different points  $\vec{z}$ , so as to effectively invert the integral equation to get  $f_1(\vec{x})$ . But, owing to the presence of the theta function,  $\Theta(R - |\vec{x}|)$ , they will only be able to invert for the profile of the function  $f_1(\vec{x})$  inside the ball. And this precisely matches the information accessible to the inside observer.

*Thus, we can conclude, at least for the simple state,  $|\psi\rangle = e^{i\gamma \int d^3x f_1(\vec{x}) \phi(\vec{x})} |0\rangle$ , that whatever information about the state is accessible to the inside observer is also available to the outside observer.*

This simple example presents the key idea involved in showing that information is holographic. Since we want this simple example to be as self-contained as possible, we will also sketch a procedure by which the outside observer can invert the integral equation (2.10) to extract the function  $f_1$ .

Let us assume for simplicity that the function  $f_1$  is spherically symmetric (Methods to take care of the angular dependence are discussed in Appendix A.3). For massive scalar fields, the two-point function of  $\pi$  is given by [2],

$$\langle 0 | \pi(\vec{x}) \pi(\vec{z}) | 0 \rangle = -\frac{m^2 K_2(|\vec{x} - \vec{z}|)}{4\pi |\vec{x} - \vec{z}|^2} \quad (2.12)$$

Where  $K_n(z)$  is the modified Bessel function of the second kind. Substituting this in

(2.10), series expanding in  $\frac{x}{z}$  (we use the notation  $|\vec{x}| \equiv x$ ), and performing the angular integrals, we get,

$$\begin{aligned}
\langle \psi | H_{\partial R} \pi(\vec{z}) | \psi \rangle &= \gamma \int d^3x \Theta(R - |\vec{x}|) f_1(\vec{x}) \left( -\frac{m^2 K_2(|\vec{x} - \vec{z}|)}{4\pi |\vec{x} - \vec{z}|^2} \right) \\
&= -2\pi\gamma \int dx d\theta x^2 \sin\theta f_1(x) \frac{m^2 K_2\left(z\sqrt{1 - 2\frac{x}{z}\cos\theta + \frac{x^2}{z^2}}\right)}{4\pi z^2 \left(1 - 2\frac{x}{z}\cos\theta + \frac{x^2}{z^2}\right)^2} \quad (2.13) \\
&= -\gamma \int dx f_1(x) \left\{ x^2 c_1(z) + x^4 c_2(z) + x^6 c_3(z) + \dots \right\}
\end{aligned}$$

where,

$$\begin{aligned}
c_1(z) &= \frac{m^2 K_2(mz)}{z^2} \\
c_2(z) &= \frac{m^2 K_2(mz)}{24z^2} - \frac{m^3 K_3(mz)}{4z^3} \\
c_3(z) &= \frac{m^6 K_2(mz)}{1920z^2} - \frac{m^5 K_3(mz)}{80z^3} \\
&\dots
\end{aligned} \quad (2.14)$$

Thus, we get a linear combination of the moments of the function,  $\int dx x^n f_1(x)$ , with known coefficients ( $c_n(z)$ ). One can take the  $m \rightarrow 0$  limit of the  $c_n(z)$ s given above to get the corresponding coefficients in the case of a massless scalar field. As for inverting for the function  $f_1$ , note that, if the function  $f_1(x)$  is determined by  $N$  moments, then we have to make measurements at  $N$  points, get a system of linear equations, and solve for the moments. For most of the functions that are of interest, you can uniquely identify the function from all of its moments <sup>10</sup>.

This concludes our analysis of the simplest scenario, where the state is parametrised by a single function  $f_1$ . We saw the importance of Gauss' law, which allows the observer at the boundary to access localised excitations inside by use of  $H_{\partial R} = H_R$ . We will next generalise this procedure.

## 2.4 A protocol for extracting information

The basic idea involved in showing the equivalence of information available for the inside and outside observers was outlined in the last section by using a simple state. We will now consider a general state of the form (2.7) and devise a protocol by which

<sup>10</sup>More on the technical details of this inversion procedure is explained in Appendix A.3.

the inside and outside observers can access information on the functions  $\{f_n\}$  that define the state. We will first consider the case where the unitary has only  $\phi$  excitations, which could be denoted as,

$$\begin{aligned} |\psi\rangle &= \exp\left\{i\gamma \sum_{n=1}^{\infty} \int \prod_{i=1}^n d^3x_i f_n(x_1, x_2, \dots, x_n) : \phi(\vec{x}_1)\phi(\vec{x}_2)\dots\phi(\vec{x}_n) : \right\} |0\rangle \\ &\equiv \exp\left\{i\gamma \sum_{n=1}^{\infty} \int f_n \phi^n\right\} \end{aligned} \quad (2.15)$$

Like in the example of the simple  $f_1$  case, our protocol to show holography of information will be the following:

- We start with states of the form (2.15), parametrised by a set of functions  $\{f_n\}$ .
- We provide a step-by-step measurement procedure for the inside observer to extract information about the functions  $\{f_n\}$ , one by one.
- An analogous measurement procedure, involving the use of Gauss' law, is provided for the outside observer to extract information about the functions  $\{f_n\}$ .
- We show that the information extracted by both observers is the same, and hence, information is holographic.

### 2.4.1 Extracting $f_1$

The bulk observer will measure the  $\pi$  correlator so as to get  $f_1$ .

$$\langle \psi | \pi(\vec{y}) | \psi \rangle = \gamma f_1(\vec{y}) \quad (2.16)$$

It is important to note that the result is unchanged from (2.9), although we have a more general state, with  $f_n \neq 0$ . This is due to the fact that the  $\phi$  insertions in the unitary (2.15) are normal ordered. So, while applying the BCH formula on a general term  $\int f_n \phi^n$ , one of the  $\phi$ s will commute with the  $\pi$  to produce a Dirac delta function. But the vacuum expectation value of the remaining  $n - 1$  fields is zero, since this is subtracted off at the beginning, by normal ordering.

For example,

$$\begin{aligned} \langle 0 | \left[ \int f_3 \phi^3, \pi(\vec{y}) \right] | 0 \rangle &\sim \int d^3x_1 d^3x_2 d^3x_3 f_3(\vec{x}_1, \vec{x}_2, \vec{x}_3) \langle 0 | : \phi(\vec{x}_1)\phi(\vec{x}_2)\delta(\vec{x}_3 - \vec{y}) : | 0 \rangle \sim 0 \\ \text{Since } : \phi(\vec{x}_1)\phi(\vec{x}_2)\phi(\vec{x}_3) &:= \phi(\vec{x}_1)\phi(\vec{x}_2)\phi(\vec{x}_3) - \overline{\phi(\vec{x}_1)\phi(\vec{x}_2)}\phi(\vec{x}_3) - \dots \end{aligned} \quad (2.17)$$

Similarly, one can show that, for the boundary observer,  $H_{\partial R}\pi(\vec{z})$  isolates  $f_1$ .

$$\langle \psi | H_{\partial R} \pi(\vec{z}) | \psi \rangle = \gamma \int d^3x \Theta(R - |\vec{x}|) f_1(\vec{x}) \langle 0 | \pi(\vec{x}) \pi(\vec{z}) | 0 \rangle \quad (2.18)$$

This follows from normal ordering as well as the fact that we've chosen the functions  $f_n$  to have no local support in the radial band where the boundary observer is making measurements. We emphasize that the requirement is just for technical simplicity; in the general case, one can always find this contribution by local measurements and subtract it off [2].

As we explained in the previous section, these measurements,  $\pi$  for the insider and  $H_{\partial R}\pi$  for the outsider, help them to extract the profile of the function  $f_1$  inside the ball. One can next go about extracting  $f_2$ .

### 2.4.2 Extracting $f_2$

The inside observer will measure  $\pi(\vec{y}_1)\phi(\vec{y}_2)$  and the outside observer will measure  $H_{\partial R}\pi(\vec{z}_1)\phi(\vec{z}_2)$  to isolate  $f_2$ . Just like in the case of  $f_1$ , one can show that, owing to normal ordering and the fact that the unitary is solely composed of  $\phi$ , one can isolate  $f_2$  to get,

$$\begin{aligned} &: \langle \psi | \pi(\vec{y}_1)\phi(\vec{y}_2) | \psi \rangle := \gamma \int d^3x_2 \tilde{f}_2(\vec{y}_1, \vec{x}_2) \overline{\phi(\vec{x}_2)} \phi(\vec{y}_2) \\ &: \langle \psi | H_{\partial R} \pi(\vec{z}_1)\phi(\vec{z}_2) | \psi \rangle := \gamma \int d^3x_1 d^3x_2 \tilde{f}_2(\vec{x}_1, \vec{x}_2) \Theta(R - |\vec{x}_1|) \overline{\pi(\vec{x}_1)} \pi(\vec{z}_1) \overline{\phi(\vec{x}_2)} \phi(\vec{z}_2) \end{aligned} \quad (2.19)$$

We have introduced some notation here. The measurement is defined by subtracting the state-independent vacuum contribution,

$$: \langle \psi | \pi(\vec{y}_1)\phi(\vec{y}_2) | \psi \rangle := \langle \psi | \pi(\vec{y}_1)\phi(\vec{y}_2) | \psi \rangle - \langle 0 | \pi(\vec{y}_1)\phi(\vec{y}_2) | 0 \rangle \quad (2.20)$$

And,  $\tilde{f}_2$  is defined as,

$$\tilde{f}_2(\vec{x}_1, \vec{x}_2) = f_2(\vec{x}_1, \vec{x}_2) + f_2(\vec{x}_2, \vec{x}_1) \quad (2.21)$$

The symmetrization of the function  $f_2$  is a result of the symmetry in interchanging the  $\phi$  fields in the definition of the state,  $\int d^3x_1 d^3x_2 f_2(\vec{x}_1, \vec{x}_2) \phi(\vec{x}_1)\phi(\vec{x}_2)$ , given in (2.15).

It is clear from (2.19) that the bulk and boundary observers could extract the same amount of information about the function  $f_2$ . One could explicitly find the dependence of  $\tilde{f}_2$  on one of the arguments inside the ball. As for the other argument, the integration kernel is the same for both bulk and boundary observers, so whatever information could be extracted out will be the same. Technical details involved in inverting for  $f_2$

are given in Appendix A.3.

It is important at this point to distinguish between formal reconstructibility and practical reconstructibility. Take the example of  $f_2$  discussed above. Here the second argument of  $\tilde{f}_2$  is integrated against the same kernel for bulk and boundary correlators. So whatever practical difficulty is there for the bulk observer to invert for the function will also be there for the boundary observer. As for the first argument, measurement proceeds exactly as in the  $f_1$  case: the bulk observer makes local measurements to extract the profile of the function and the boundary observer has to make the same number of measurements to get the function. However, practically, things are not as easy as they look. Realistically, both the observers will make a finite number of measurements and then try to invert for the function. One easy way to think about it is to consider expanding the function  $\tilde{f}_2$  as a linear combination of a finite set of basis functions. The coefficients of this expansion will define the data associated with the function  $\tilde{f}_2$ . So, the number of measurements made by both the bulk and boundary observers has to be equal to the number of unknown coefficients. But this finiteness in data means that we are approximating the function  $\tilde{f}_2$  to be a linear combination of the basis functions. This procedure, in general, has a finite amount of error associated with it. Increasing the number of elements in the appropriately chosen set of basis functions will reduce the error. This argument makes it clear that practical reconstructibility is different from formal reconstructibility. But for our purpose, which is to show equivalence of information available to the bulk and boundary observers, the formal reconstructibility will suffice.

### 2.4.3 Extracting $f_n$

The strategy for extracting information about the functions  $f_n$  might be clear by now. One has to successively extract out  $f_1, f_2, \dots$  by means of  $\pi, \pi\phi, \dots$  correlators for the bulk observer, and  $H_{\partial R}\pi, H_{\partial R}\pi\phi, \dots$  for the boundary observer. We saw how it works for  $f_1$  and  $f_2$  in the previous sections. One would next compute  $\pi(\vec{y}_1)\phi(\vec{y}_2)\phi(\vec{y}_3)$  for the bulk and  $H_{\partial R}\pi(\vec{z}_1)\phi(\vec{z}_2)\phi(\vec{z}_3)$  for the boundary so as to isolate  $f_3$ . However, as one can easily check, the corresponding correlators will have, in addition to  $f_3$ , terms involving  $f_1$ . You can then use the data you already have about  $f_1$  to effectively isolate  $f_3$ . This is the reason for extracting out information in a particular order. A general formula for extracting out the function could be given as follows,

**Inside observer.** The inside observer measures  $\pi(\vec{y}_1)\phi(\vec{y}_2)\dots\phi(\vec{y}_m)$ , denoted compactly as  $\pi(\vec{y}_1)\phi^{m-1}(\{\vec{y}\})$ . As we saw in the computations for  $f_1$  and  $f_2$ , inserting exactly one  $\pi$  means that the BCH formula truncates at  $O(\gamma)$ , and we get a linear integral equation for  $f_n$ .

$$\begin{aligned}
& : \langle \psi | \pi(\vec{y}_1) \phi^{m-1}(\{\vec{y}\}) | \psi \rangle \equiv \langle \psi | \pi(\vec{y}_1) \phi^{m-1}(\{\vec{y}\}) | \psi \rangle - \langle 0 | \pi(\vec{y}) \phi^{m-1}(\{\vec{y}\}) | 0 \rangle \\
& = -i\gamma \sum_n \int \left( \prod_{i=1}^n d^3 x_i \right) f_n(\{\vec{x}\}) \sum_{k=1}^n [\phi(\vec{x}_k), \pi(\vec{y}_1)] \langle 0 | : \phi^{n-1}(\{\vec{x}\}) : \phi^{m-1}(\{\vec{y}\}) | 0 \rangle \\
& \quad \text{where } \phi^{n-1}(\{\vec{x}\}) \text{ does not have an insertion at } \vec{x}_k \\
& = \gamma \sum_n \int \left( \prod_{i=1}^n d^3 x_i \right) f_n(\{\vec{x}\}) \sum_{k=1}^n \delta(\vec{x}_k - \vec{y}_1) \langle 0 | : \phi^{n-1}(\{\vec{x}\}) : \phi^{m-1}(\{\vec{y}\}) | 0 \rangle \quad (2.22) \\
& \quad \text{where } \phi^{n-1}(\{\vec{x}\}) \text{ does not have an insertion at } \vec{x}_k
\end{aligned}$$

One can see by use of wick contractions that the RHS is non-zero only when  $m + n$  is even. Furthermore, if  $n > m$ , then the term vanishes because the  $n - 1$  field insertions,  $: \phi^{n-1}(\{\vec{x}\}) :$ , coming from the state, are normal ordered such that any self contraction is zero, by definition. So, we get a non-zero correlator only when  $n \leq m$ . Thus, the correlator  $\pi(\vec{y}_1) \phi^{m-1}(\{\vec{y}\})$  gets contributions from  $f_m$  and other lower point functions evenly separated,  $f_{m-2}, f_{m-4}, \dots$ . So, one has to extract the functions starting from the lowest ones ( $f_1, f_2, \dots$ ), and then use this data to obtain the higher point functions. A similar protocol could also be followed by the boundary observer.

**Outside observer.** The outside observer measures,  $H_{\partial R} \pi(\vec{y}_1) \phi^{m-1}(\{\vec{y}\})$  to isolate the function  $f_m$ , starting from  $m = 1$ .

$$\begin{aligned}
& : \langle \psi | H_{\partial R} \pi(\vec{z}_1) \phi^{m-1}(\{\vec{z}\}) | \psi \rangle \equiv \langle \psi | H_{\partial R} \pi(\vec{z}_1) \phi^{m-1}(\{\vec{z}\}) | \psi \rangle - \langle 0 | H_{\partial R} \pi(\vec{z}_1) \phi^{m-1}(\{\vec{z}\}) | 0 \rangle \\
& = -i\gamma \sum_n \int \left( \prod_{i=1}^n d^3 x_i \right) f_n(\{\vec{x}\}) \sum_{k=1}^n [\phi(\vec{x}_k), H_{\partial R}] \langle 0 | : \phi^{n-1}(\{\vec{x}\}) : \pi(\vec{z}_1) \phi^{m-1}(\{\vec{z}\}) | 0 \rangle \\
& \quad \text{where } \phi^{n-1}(\{\vec{x}\}) \text{ does not have an insertion at } \vec{x}_k \\
& = \gamma \sum_n \int \left( \prod_{i=1}^n d^3 x_i \right) f_n(\{\vec{x}\}) \sum_{k=1}^n \Theta(R - |\vec{x}_k|) \overline{\pi(\vec{x}_k)} \pi(\vec{z}_1) \langle 0 | : \phi^{n-1}(\{\vec{x}\}) : \phi^{m-1}(\{\vec{z}\}) | 0 \rangle \quad (2.23) \\
& \quad \text{where } \phi^{n-1}(\{\vec{x}\}) \text{ does not have an insertion at } \vec{x}_k
\end{aligned}$$

Just like we explicitly computed for  $f_1$  and  $f_2$ , except for the point  $\vec{x}_k$ , integration kernels are the same for bulk and boundary observables. As for the point  $\vec{x}_k$ , the kernel is  $\delta(\vec{x}_k - \vec{y}_1)$  for the bulk observable, and  $\Theta(R - |\vec{x}_k|) \overline{\pi(\vec{x}_k)} \pi(\vec{z}_1)$  for the boundary observable. This makes sure that the information accessible to the inside observer is the same as that for the outside observer.

The following table summarises the protocol we have outlined so far, which allows the bulk and boundary observers of the ball to access the same amount of information

from a unitary excitation of  $\phi$  fields on top of the vacuum.

| $ \psi\rangle = \exp\{\sum_{n=1}^{\infty} \int f_n \phi^n\}$ |                                                                       |             |
|--------------------------------------------------------------|-----------------------------------------------------------------------|-------------|
| Bulk observer                                                | Boundary Observer                                                     | Information |
| $\pi(\vec{y})$                                               | $H_{\partial R} \pi(\vec{z})$                                         | $f_1$       |
| $\pi(\vec{y}_1) \phi(\vec{y}_2)$                             | $H_{\partial R} \pi(\vec{z}_1) \phi(\vec{z}_2)$                       | $f_2$       |
| $\dots$                                                      | $\dots$                                                               | $\dots$     |
| $\pi(\vec{y}_1) \phi(\vec{y}_2) \dots \phi(\vec{y}_n)$       | $H_{\partial R} \pi(\vec{z}_1) \phi(\vec{z}_2) \dots \phi(\vec{z}_n)$ | $f_n$       |

## 2.5 Most general unitary excitations on the vacuum

In all of the calculations so far, we have restricted ourselves to a sub-space of states of the form (2.15). While this is an infinite class of states parameterised by an infinite set of functions  $f_n$ , we know that the most general states contain insertions of the conjugate momentum too, and the class is (at least morally speaking) doubly infinite, parameterised by functions  $f_{n,m}$  as shown below,

$$|\psi\rangle = \exp\left\{\sum_{n,m} \int \left(\prod_{i=1}^{n+m} d^3 x_i\right) f_{n,m}(\{\vec{x}\}) \phi^{(n)}(\{\vec{x}\}) \pi^{(m)}(\{\vec{x}\})\right\} \quad (2.24)$$

Owing to the extra insertions of the conjugate momentum in the states, none of the correlation functions terminate at any finite order in  $\gamma$ . Therefore, even identifying what information the bulk correlators contain about the functions  $f_{n,m}$  that define the state is difficult to quantify.

However, in this section, we show that at linear order in  $\gamma$ , the bulk correlators can indeed be reproduced by certain boundary correlators, lending credibility to the generality of the principle. In order to do so, we will consider an  $\mathcal{O}(\gamma)$  expansion of the general states up to two particles (this can be done to incorporate any finite  $n$ -particle scenario, but with more tedious work)

$$|\psi\rangle = \exp\left(i\gamma \int \left(f_{1,0}\phi + f_{0,1}\pi + f_{2,0}:\phi^2: + f_{1,1}:\phi\pi: + f_{0,2}:\pi^2:\right)\right) |0\rangle, \quad (2.25)$$

where the arguments of the functions and fields are left implicit.

## One-point functions

The bulk one-point functions in this state can easily be computed to give

$$: \langle \psi | \pi(\vec{y}) | \psi \rangle : = \gamma f_{1,0}(\vec{y}) + \mathcal{O}(\gamma^2) \quad \text{and} \quad (2.26)$$

$$: \langle \psi | \phi(\vec{y}) | \psi \rangle : = -\gamma f_{0,1}(\vec{y}) + \mathcal{O}(\gamma^2). \quad (2.27)$$

The corresponding boundary one-point functions are then

$$: \langle \psi | H_R \pi(\vec{z}) | \psi \rangle : = \gamma \int f_{1,0}(\vec{x}) \overline{\pi(\vec{x})} \pi(\vec{z}) \Theta(R - |\vec{x}|) + \mathcal{O}(\gamma^2) \quad \text{and} \quad (2.28)$$

$$: \langle \psi | H_R \phi(\vec{z}) | \psi \rangle : = \gamma \int f_{0,1}(\vec{x}) \overline{\partial_t \pi(\vec{x})} \phi(\vec{z}) \Theta(R - |\vec{x}|) + \mathcal{O}(\gamma^2). \quad (2.29)$$

Here the kernel,  $\overline{\partial_t \pi(\vec{x})} \phi(\vec{y})$  is given by [2],

$$\overline{\partial_t \pi(\vec{x})} \phi(\vec{y}) = \frac{m^2}{4\pi^2 |\vec{x} - \vec{y}|^2} K_2(m|\vec{x} - \vec{y}|). \quad (2.30)$$

Applying the inversion algorithm that we have already used in the case of  $f_1$  in Section 2.3, we see that these boundary correlators allow us to find exactly the same information as the bulk correlators <sup>11</sup>.

## Two-point functions

There are three inequivalent bulk two-point functions at hand, all of which are non-trivial in the general state (2.25). These are given by

$$\langle \psi | \phi(\vec{y}_1) \phi(\vec{y}_2) | \psi \rangle = -\gamma \int f_{1,1}(\vec{x}_1, \vec{y}_1) \overline{\phi(\vec{x}_1)} \phi(\vec{y}_2) + (\vec{y}_1 \leftrightarrow \vec{y}_2) + \mathcal{O}(\gamma^2) \quad (2.31)$$

$$\langle \psi | \pi(\vec{y}_1) \pi(\vec{y}_2) | \psi \rangle = \gamma \int f_{1,1}(\vec{y}_1, \vec{x}_2) \overline{\pi(\vec{x}_2)} \pi(\vec{y}_2) + (\vec{y}_1 \leftrightarrow \vec{y}_2) + \mathcal{O}(\gamma^2) \quad (2.32)$$

$$\begin{aligned} \langle \psi | \pi(\vec{y}_1) \phi(\vec{y}_2) | \psi \rangle &= \gamma \int \tilde{f}_{2,0}(\vec{y}_1, \vec{x}_2) \overline{\phi(\vec{x}_2)} \phi(\vec{y}_2) - \gamma \int \tilde{f}_{0,2}(\vec{y}_2, \vec{x}_2) \overline{\pi(\vec{x}_2)} \pi(\vec{y}_1) \\ &\quad + \mathcal{O}(\gamma^2), \end{aligned} \quad (2.33)$$

where the tilde on the functions refers to a symmetrisation in  $\vec{x}_1$  and  $\vec{x}_2$ . Of course, a fourth correlator  $\langle \psi | \phi(\vec{z}_1) \pi(\vec{z}_2) | \psi \rangle$  can be obtained from the third one above. The corresponding boundary two-point functions are then given, respectively, by the stan-

<sup>11</sup> $\vec{y}$  is restricted to be inside the ball by definition of a bulk measurement whereas the boundary kernel is multiplied by the theta function,  $\Theta(R - |\vec{x}|)$

standard prescription with an insertion of the finite-radius Hamiltonian

$$\begin{aligned} \langle \psi | H_R \phi(\vec{z}_1) \phi(\vec{z}_2) | \psi \rangle &= -\gamma \int f_{1,1}(\vec{x}_1, \vec{x}_2) \overline{\phi(\vec{x}_1)} \phi(\vec{z}_1) \overline{\partial_t \pi(\vec{x}_2)} \phi(\vec{z}_2) \Theta(R - |\vec{x}_2|) \\ &\quad + (\vec{z}_1 \leftrightarrow \vec{z}_2) + \mathcal{O}(\gamma^2) \end{aligned} \quad (2.34)$$

$$\begin{aligned} \langle \psi | H_R \pi(\vec{z}_1) \pi(\vec{z}_2) | \psi \rangle &= \gamma \int f_{1,1}(\vec{x}_1, \vec{x}_2) \overline{\pi(\vec{x}_1)} \pi(\vec{z}_1) \overline{\pi(\vec{x}_2)} \pi(\vec{z}_2) \Theta(R - |\vec{x}_1|) \\ &\quad + (\vec{z}_1 \leftrightarrow \vec{z}_2) + \mathcal{O}(\gamma^2) \end{aligned} \quad (2.35)$$

$$\begin{aligned} \langle \psi | H_R \pi(\vec{z}_1) \phi(\vec{z}_2) | \psi \rangle &= \gamma \int \tilde{f}_{2,0}(\vec{x}_1, \vec{x}_2) \overline{\pi(\vec{x}_1)} \pi(\vec{z}_1) \overline{\phi(\vec{x}_2)} \phi(\vec{z}_2) \Theta(R - |\vec{x}_1|) \\ &\quad - \gamma \int \tilde{f}_{0,2}(\vec{x}_1, \vec{x}_2) \overline{\partial_t \pi(\vec{x}_1)} \phi(\vec{z}_2) \overline{\pi(\vec{x}_2)} \pi(\vec{z}_1) \Theta(R - |\vec{x}_1|) \\ &\quad + \mathcal{O}(\gamma^2). \end{aligned} \quad (2.36)$$

Up to  $\mathcal{O}(\gamma)$ , it is easy to see that both bulk and boundary correlators contain the same information about the state <sup>12</sup>. As an example, let us consider (2.33) and (2.36). First, it is evident that the kernels conjugating the functions  $f_{2,0}$  and  $f_{0,2}$  in both the bulk and boundary integrals are identical for the argument  $x_2$ . These are the field and conjugate momentum two-point functions, respectively. Moreover, the two functions appear in the same linear combination in both equations. Therefore, both the bulk and boundary equations yield the same information about the functions in terms of one variable. The bulk equation gives us a linear combination of the functions in terms of the second variable, but restricted to be inside the ball. From the boundary equation, owing to the step function, we may extract exactly the same information.

Therefore, even without having to invert these expressions for the functions defining the state, it is nevertheless evident that the boundary correlators allow for exactly the same information about the state to be extracted.

## 2.6 Discussion

In this chapter, we considered linearised gravity coupled minimally to a free Klein-Gordon scalar field theory. We defined a certain basis of states that are excitations built out of unitary operators acting on the vacuum in flat space (2.7). These states were motivated by an interest in understanding a notion of holography of information in finite regions of space. We chose a ball of radius  $R$  for this end. We put two

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<sup>12</sup>See Appendix A.3 for a discussion on the possibility of expanding the functions in a basis and extracting the coefficients by solving linear equations associated with the  $\mathcal{O}(\gamma)$  term

observers, one inside the ball and one at the boundary of the ball. We could show, for states of the form of unitaries made of  $\phi$  acting on the vacuum (2.15), that there exists a protocol by which the bulk and boundary observers can make measurements to identify a state. However, as the bulk and boundary observers are placed in finite regions, unlike observers at infinity, they could not uniquely identify any arbitrary state. We saw this, for example, in the case of the state  $|\psi\rangle = e^{i\int f_1\phi}|0\rangle$ , that the profile of the function  $f_1$  outside the ball is inaccessible to both the bulk and boundary observers. Thus, we say that these finite region observers have only partial information regarding states defined in general over the entire Cauchy slice. But even this information, which is available in finite regions, is holographic. This is because whatever information regarding the states (2.15) is available to the observer inside the ball is also available to the observer at the boundary of the ball. We could show this for states of the form (2.15) to all orders in the parameter  $\gamma$ , whereas for the most general unitaries of the form (2.24), we could make a statement at  $O(\gamma)$ . These results lend credibility to a notion of holography of information in finite regions of spacetime, without having to go to infinity, at least within the confines of our setup.

However, there are several obvious shortcomings of our work that warrant mention. As we already mentioned, our results are valid to linear order in  $\gamma$  for the most general unitaries in (2.24). Our technology fails to give us conclusive answers either way at higher orders. It would be a significant improvement to find a method to determine whether the principle of holography of information in a ball of finite radius holds to all orders in perturbation theory. Also, we have restricted our attention to linearised gravity. Therefore, all states we considered are only built out of scalar operators. However, it is rather straightforward (even if slightly more cumbersome) to include free gravitons in the theory. This requires us to include the gravitational stress-tensor in (2.4), which we ignored. Since our construction for finite region boundary observables that contain the bulk information involves a special choice of basis of states and corresponding correlators, this may easily be extended to the case of graviton field insertions and the corresponding conjugate momentum insertions. However, going beyond free gravitons is a significant challenge. It comes with further challenges that entail the positivity (or potential lack thereof) of the finite-radius Hamiltonian.

Although we have not incorporated these effects in the present chapter, a self-interacting Klein-Gordon field should pose no conceptual problems. The kernels that are involved in inverting for the information of the states from the (bulk/boundary) correlators will now be significantly more involved, depending on the nature of the interactions. Nevertheless, we expect the conceptual conclusions to remain unchanged.

Another comment is that we have fixed a finite radius in the bulk to define our lo-

cal region of interest. This is, of course, an ill-defined quantity because it is entirely gauge dependent. Of course, we expect that physical quantities measured in any gauge should all be consistent with each other. Strictly speaking, there are no local gauge-invariant observables in quantum gravity. At the leading order in interactions that we are concerned with, this is not an issue. However, when non-linearities discussed above are taken into consideration, observables can be defined in a gauge-invariant manner, for example, by defining them relationally to points on the boundary of spacetime. It would be good if the finite region notion of holography could be formulated in a gauge-invariant way by defining quantities with suitable gravitational dressing [10, 11].

Despite the possibility for further improvements, even the simplistic scenario we have considered here is enough to demonstrate the powerful implications of the gravitational constraints in relation to quantum information. Taking Gauss' law of gravity into account for a ball of finite radius in flat spacetime, we saw that the gravitational fluctuations at the boundary of the ball store information holographically. An observer at the boundary of the ball can use these gravitational fluctuations to access the same amount of information as is available to an observer inside the ball. As we discussed at the beginning of this section, constraints are much stronger in gravity compared to other gauge theories. For example, consider scalar electrodynamics. Here,  $Q_R = i \int_R (\phi^* \partial_t \phi - \phi \partial_t \phi^*)$  will be a boundary term given in terms of the electric field at the boundary. However, if you have a unitary excitation inside the ball of the form  $e^{i\gamma \int d^3x f(\vec{x}) \phi^*(\vec{x}) \phi(\vec{x})} |0\rangle$ , then the  $Q_R$  operator will not see this excitation,

$$\langle 0 | e^{-i\gamma \int f \phi^* \phi} Q_R e^{-i\gamma \int f \phi^* \phi} | 0 \rangle = \langle 0 | Q_R | 0 \rangle; \text{ as } [Q_R, \phi^* \phi] = 0$$

Thus, we can find localised unitaries that commute with  $Q_R$ , such that the boundary observer cannot access them using Gauss' law. In the case of gravity,  $Q_R$  is replaced by  $H_R$ , and we don't have a negative charge analogue of energy so as to hide information. This makes gravity more powerful, and information gets stored holographically in theories of gravity.

In the next chapter, we will pursue this further and analyse the implications of gravitational constraints in de Sitter spacetime. Here, Gauss' law becomes much more powerful, as the Cauchy slices are compact.

# Chapter 3

## Cosmology

In the previous chapter, we studied the implications of gravitational constraints in asymptotically flat space-time by considering scalar field theory minimally coupled to gravity. We saw that, upon canonical quantisation, the Gauss' law constraint results in quantum information being holographic in nature. One can use the same methods to study scalar field theory in de Sitter space-time. While quantum field theory in de Sitter space has been studied extensively, and already provides a rich theoretical playground, the study of gravity [12] has received less attention. New physical effects in gravity arise from the gravitational constraints. Since our intuition is commonly based on classical solutions of the equations of motion, the significance of the constraints in the quantum theory is often overlooked. But as we saw in the previous chapter, a careful study of gravitational constraints will reveal interesting aspects of gravity, like holography of information.

*Furthermore, as the spatial slices are compact for de Sitter, the gravitational constraints will become more powerful than in flat spacetime, such that the set of allowed states and operators will be considerably restricted. One can study correlation functions in such gravitationally constrained states so as to understand the novel features contributed by gravity.*

From a different perspective, the early Universe is believed to have undergone a period of accelerated expansion when it was well-approximated by de Sitter space. Therefore, a study of de Sitter quantum gravity might be relevant for inflationary physics.

As this chapter is a bit long compared to the previous one, we will quickly summarise the setup and main results in the following section.

### 3.1 Brief summary

The papers [7,13] studied the effects of the quantum gravitational constraints in asymptotically de Sitter space by classifying all solutions of the Wheeler-DeWitt equation. These papers also proposed a norm on this space to obtain a Hilbert space. Finally, they introduced a class of gauge-fixed observables, termed “cosmological correlators”, that are labelled by points on the asymptotically late-time slice, although their gauge-invariant description involves an integral on the entire late-time slice. These observables are the natural microscopically simple observables in a theory of quantum gravity in asymptotically de Sitter space and are amenable to straightforward perturbative computations. In this chapter, we will continue to use the phrase “cosmological correlators” to describe them, although we comment more on the terminology below.

These observables obey a surprising property: cosmological correlators labelled by points in any open set of the late-time slice form a complete set of observables. This result can be viewed as an extension of the principle of holography of information [5,6,8,14–17] to de Sitter space. This result, which is related to similar results in anti-de Sitter space and flat space, tells us that quantum information about the entire space-time is available in a restricted region but does not provide a full-fledged holographic dual that captures the dynamics of the theory. The natural hope for a full holographic dual would be to show, by some extension of these arguments, that dynamics on all of de Sitter space can be captured by degrees of freedom on a codimension-1 region of the late-time slice.

In this chapter, we study the norm and cosmological correlators in the limit where  $G_N \rightarrow 0$  but the fields continue interacting via perturbative nongravitational interactions. One might naively expect that, in the limit  $G_N \rightarrow 0$ , all gravitational effects vanish. However, this is incorrect; in [7,13], it was shown that the  $G_N \rightarrow 0$  limit of the gravitational Hilbert space is not the QFT Hilbert space. Physically, this is because as one takes  $G_N \rightarrow 0$ , it is necessary to still impose the Gauss law [18–23]. The Gauss law restricts the Hilbert space by imposing de Sitter invariance on all states and observables.

We find the following results:

1. As explained in [7,13], the Euclidean vacuum (also known as the Hartle-Hawking state [24]), which we denote by  $|0\rangle$ , is not perturbatively normalizable. A necessary condition for excited states to be normalizable is that they should be orthogonal to  $|0\rangle$ . We formulate Feynman rules for computing the norm of such states in perturbation theory and provide some sample computations.
2. We formulate Feynman rules for computing cosmological correlators and pro-

vide sample computations of these observables. We formulate necessary conditions for such computations to give finite values that are free of group-volume divergences.

3. These cosmological correlators always differ from QFT correlators in the vacuum. For instance, they can display interesting non-Gaussian features even when the vacuum corresponds to a Gaussian wavefunction. We present several explicit examples in this chapter. There is nothing mysterious about this because (a) every normalizable state has excitations on top of the vacuum, (b) the gauge-fixed operators that we study are labelled by points but are secretly smeared over the entire late-time slice. Therefore, their correlators pick up the presence of the excitation. One noteworthy difference from QFT correlators in excited states is that these correlators are always conformally invariant.
4. Nevertheless, in the presence of an appropriate background state, it is possible to define “relational observables” whose expectation values resemble QFT correlators in the vacuum. This construction generalizes the AdS construction of relational observables presented in [25–29]. Here, the microscopic description of both the state and the observables is complicated, but the final expectation values correspond to simple QFT observables.

The background state can be thought of as an “observer” as discussed in the recent literature [30]. However, the “observer” is not introduced by hand as an auxiliary tool to solve the constraints. Instead, we study an excitation about the vacuum and set up relational observables while manifestly respecting all the original constraints.

These results buttress the broad narrative that has developed over a number of studies. The gravitational constraints allow one to deduce a version of the holographic principle; this can be explicitly verified for simple perturbative states and observables; nevertheless, it is possible to construct states and observables where our usual notions of locality in quantum field theory are restored to a good approximation.

We believe that our perturbative computations are of theoretical interest in understanding de Sitter quantum gravity. However, we do not suggest that they have any immediate observational consequence. It is more natural that the correlations we observe in the cosmic microwave background or the large-scale structure of the Universe correspond to correlators of the “relational observables” described above, which are close to standard QFT observables. To avoid confusion, we do not use the phrase “cosmological correlators” for correlators of these relational observables in this chapter.

These relational observables themselves deserve further study — both from a techni-

cal and a fundamental perspective. An important technical problem is to develop a systematic perturbative treatment of these relational observables, going beyond the leading-order treatment that we provide here. A more conceptual issue is to understand their “state dependence”, which is similar to the state dependence of observables [31,32] in the black-hole interior and raises deeper questions about observations in a cosmological setting, where the observer is part of the Universe they are studying.

A brief overview of this chapter is as follows. In Section 3.2, we review our setup, including key results from [7,13]. In Section 3.3, we set up Feynman rules to compute perturbative observables in the nongravitational limit. Section 3.4 presents several sample calculations. Section 3.5 describes the construction of observables dressed to an observer whose values correspond to QFT vacuum correlators. Appendix B.1 shows how to find the gauge-invariant operator corresponding to a gauge-fixed cosmological correlator. Appendix B.2 provides further details of the relational construction and also includes a quick review of the corresponding construction in AdS/CFT that might be useful for some readers. Appendix B.3 explains the wavefunction formalism for principal-series fields, which is slightly subtle because these wavefunctions must be viewed as coherent-state wavefunctions.

## 3.2 Background

The papers [7,13] constrained the allowed states in a theory of quantum gravity in asymptotically de Sitter space by finding all possible solutions of the Wheeler-DeWitt equation on the asymptotically late-time slice. The solution space was lifted to a Hilbert space by means of an appropriate norm, and a version of the principle of holography of information was proved in terms of gauge-fixed observables termed cosmological correlators. In this section, we provide a self-contained review of the relevant results from [7,13].

In this chapter, our first objective is to explore the structure of the norm and gauge-fixed cosmological correlators defined in [7,13] within perturbation theory. For simplicity, we will focus on the  $G_N \rightarrow 0$  limit but allow nongravitational interactions to persist. This limit was studied in sections 5.2 and 5.3 of [13] and section 5.1 of [7], and our review summarizes those sections.

We focus on  $d + 1$  dimensional asymptotically de Sitter spacetimes where  $d \geq 2$ . However, interesting and related results have been obtained by studying the Wheeler-DeWitt equation in  $1 + 1$  dimensions [1,33–35].

**Matter fields and boundary values.** We use massive scalar fields to model the matter sector of the theory. Consider a late-time slice of de Sitter space where the spatial metric takes the form (we denote  $\kappa = \sqrt{8\pi G_N}$ )

$$g_{ij} \xrightarrow{\Omega \rightarrow \infty} \frac{\Omega^2}{(1 + |x|^2)^2} (\delta_{ij} + \kappa h_{ij}), \quad (3.1)$$

which is the metric of a sphere with small fluctuations scaled up by a large Weyl factor. Since we focus on the  $G_N \rightarrow 0$  limit, we will ignore the effect of the metric fluctuations  $h_{ij}$  in this chapter. A scalar field of mass  $m$  is expected to die off in the large  $\Omega$  limit as

$$\chi \xrightarrow{\Omega \rightarrow \infty} \left( \frac{\Omega}{1 + |x|^2} \right)^{-\Delta} \chi(x) + \left( \frac{\Omega}{1 + |x|^2} \right)^{-\bar{\Delta}} \bar{\chi}(x) + \dots, \quad (3.2)$$

where  $\Delta(d - \Delta) = m^2$  in the free-field limit. This is simply the usual “extrapolate limit” in de Sitter space.

We denote  $\bar{\Delta} \equiv d - \Delta$ . For light fields  $\Delta$  is real and can be chosen so that  $\Delta < \frac{d}{2}$ . In this case, the fields are said to belong to the “complementary series”; the first term displayed in (3.2) is the dominant term at late times, and  $\chi(x)$  can be taken to be real. Fields with larger values of  $m$  correspond to complex values of  $\Delta = \frac{d}{2} + i\mu$  and  $\bar{\Delta} = \frac{d}{2} - i\mu$ . In this case, the fields are said to belong to the “principal series.” In that case, the second term in (3.2) is also important.

We will present expressions below where we distinguish between  $\chi$  and  $\bar{\chi}$  so that they are valid when  $\chi$  is in the principal series. For complementary-series fields, the reader should simply identify  $\chi$  and  $\bar{\chi}$ .

In this chapter, we will confine our attention to the boundary values  $\chi(x)$  and  $\bar{\chi}(x)$  and not make reference to the bulk field. The description of the bulk origin of these operators above is provided only for intuition. Indeed, in the interacting theory, the map (3.2) between bulk and boundary fields might need to be improved as discussed in [36]. We will only require that the boundary values,  $\chi(x)$  and  $\bar{\chi}(x)$ , transform in a specified manner under de Sitter transformations, regardless of how they are obtained. Note also that these transformation properties do not allow for field redefinitions of the boundary values.

**States.** Let  $|0\rangle$  be the Euclidean vacuum. We use the term “Euclidean vacuum” interchangeably with “Hartle-Hawking state”.

In the nongravitational limit, it was shown in [13] that the states satisfying the Gauss

law take the form

$$|\Psi\rangle = \sum_m \int d\vec{x} \delta G_m(\vec{x}) \bar{\chi}(x_1) \dots \bar{\chi}(x_m) |0\rangle, \quad (3.3)$$

where  $\vec{x} \equiv (x_1, \dots, x_m)$  and  $\delta G_m(\vec{x})$  is a function that obeys the same Ward identities as the connected correlation function of  $m$  operators of dimension  $\Delta$ .

$$\delta G_m(\vec{x}) \sim \langle O(x_1) \dots O(x_m) \rangle_{\text{connected}}. \quad (3.4)$$

We omit the subscript ng on the states that was used in [7, 13] since we are always in the nongravitational limit.

We would like to make a few comments. First, as we explain in Appendix B.3, it is convenient to think of states produced by excitations of  $\bar{\chi}$  rather than  $\chi$ . Of course, when  $\chi$  is in the complementary series, this distinction is irrelevant.

Second, we use  $\sim$  rather than  $=$  to emphasize that the function on the left is only constrained by the Ward identities but need not necessarily obey the constraints of locality or unitarity that such correlators would obey in a physical CFT. The subscript “connected” also implies that  $\delta G_m(\vec{x})$  is not invariant under separate conformal transformations of subsets of its arguments, and this will be important later.

Third, on the late-time slice that we are studying, the action of the de Sitter group reduces to the conformal group. It is easy to see that the states (3.3) are invariant under conformal transformations: the transformation of  $\delta G_m$  under a conformal transformation is compensated by the transformation of  $\bar{\chi}(x_i)$ . It is also easy to see that these states correspond to the group-averaged Hilbert space described by Higuchi [19, 20]. Here, one starts with a QFT state of the form,

$$|\psi_{QFT}\rangle = \chi(\vec{y}_1) \chi(\vec{y}_2) \dots \chi(\vec{y}_m) |0\rangle,$$

and averages it over the de Sitter group to get a de Sitter invariant state.

$$|\hat{\psi}\rangle = \int dU U |\psi_{QFT}\rangle. \quad (3.5)$$

Here  $U$  is the unitary implementing de Sitter group transformations, and  $dU$  is the Haar measure of the group. One can perform the group averaging and show that (3.5) reduces to (3.3). We refer the reader to [13] for more details.

Finally, the  $\delta G_m$  are not fully fixed by de Sitter invariance for  $m > 3$  and different choices of  $\delta G_m$  yield different states. The reader may think of the states as being labeled by these choices.

The form (3.3) was derived in [7, 13] from a detailed analysis of a nongravitational limit of solutions of the Wheeler-DeWitt equation in asymptotically de Sitter space. Nevertheless, the invariance of states under the conformal group can be understood via a simple physical argument. When one studies a gauge theory in a noncompact space, the Hilbert space contains states that carry nonzero charge and therefore transform in nontrivial representations of the global symmetry group. In the same way, when one studies gravity in asymptotically flat space or anti-de Sitter space, it is natural to include states that have nonzero charges under the asymptotic symmetry group. However, when one places the gauge theory on a compact manifold, the Gauss law forces us to restrict attention to states that transform in the trivial representation of the symmetry group. For instance, this has interesting physical consequences when one studies  $SU(N)$  theories on a sphere [37]. This constraint must be preserved even as one takes the gauge coupling to zero. In global de Sitter space, the Cauchy slices are compact. Therefore, the gravitational Gauss law forces us to consider de Sitter invariant states. We comment more on the contrast between the classical and quantum cases below, after discussing the norm.

**Norm.** The states (3.3) are not normalizable in the standard QFT norm. This is clear because the smearing functions are not compactly supported. So a naive QFT computation of the norm of (3.3) receives an infinite contribution coming from the volume of the conformal group. In [7] it was shown that, starting with a natural norm in the gravitational case (obtained by squaring and integrating the wavefunction and dividing by the volume of the diff and Weyl redundancy), the nongravitational norm reduces to

$$(\Psi, \Psi) = \mathcal{K} \langle \Psi | \Psi \rangle, \quad (3.6)$$

where, on the right-hand side, we simply have the usual QFT norm of the state given formally by

$$\langle \Psi | \Psi \rangle = \sum_{m, m'} \int d\vec{x} d\vec{x}' \delta G_m(\vec{x}) \delta G_{m'}^*(\vec{x}') \langle 0 | \chi(x'_1) \dots \chi(x'_m) \bar{\chi}(x_1) \dots \bar{\chi}(x_{m'}) | 0 \rangle. \quad (3.7)$$

The usual angular bracket norm is used exclusively for the QFT norm, and we omit the subscript QFT that was used in [7, 13].

This expression has a divergence that is proportional to the volume of the conformal group that arises from the integrals over the coordinates that appear in the state above. This is cancelled by the constant

$$\mathcal{K} \equiv \frac{\text{vol}(\text{SO}(d-1))}{\text{vol}(\text{SO}(1, d+1))}. \quad (3.8)$$

This norm coincides with Higuchi's expression [19, 20] up to an unimportant normalization.<sup>1</sup>

One can make sense of the infinite volume of the conformal group simply by fixing three points in the integral above to 0, 1, and  $\infty$ .

$$(\Psi, \Psi) = \sum_{m, m'} \int d\vec{x} d\vec{x}' \delta(z_1) \delta(z_2 - 1) \delta(\tilde{z}_3) \delta G_m(\vec{x}) \delta G_{m'}^*(\vec{x}') \times \langle 0 | \chi(x'_1) \dots \chi(x'_{m'}) \bar{\chi}(x_1) \dots \bar{\chi}(x_m) | 0 \rangle, \quad (3.9)$$

where  $\tilde{z}_3^\mu = \frac{z_3^\mu}{|z_3|^2}$ , and  $z_1, z_2, z_3$  are *any* three coordinates from the set  $(x_1, \dots, x_m, x'_1, \dots, x'_{m'})$ . Since the norm requires us to fix at least three points, we note that the vacuum state itself is not a normalizable element of the Hilbert space. This issue has been noted earlier [20, 30] and we refer the reader to [38] for a recent discussion. Higher-genus effects are studied in [39, 40].

**An aside on dS invariance.** The construction of dS-invariant states and the norm described above might appear counterintuitive to some readers, particularly those who study classical de Sitter solutions. It is easy to find classical solutions that satisfy the constraints, but are not de Sitter invariant even asymptotically; so why do the constraints force de Sitter invariance on quantum states?

A simple analogy might help to explain the difference between the classical and quantum theory. Consider an ordinary quantum field theory in  $d+1$  dimensions, with a local field  $\phi(x)$ , and let us impose the constraint  $P = 0$ , where  $P$  is the momentum operator. (We have invented this toy model to illustrate some aspects of the de Sitter constraints, and we do not claim that it has any deep physical significance.)

In the classical theory, it is easy to find solutions that satisfy this constraint. One simply considers a solution to the equation of motion whose center-of-mass momentum is zero. For example, we take one lump of field moving to the left and another moving to the right.

On the other hand, the quantum constraint  $P|\Psi\rangle = 0$  forces states to be translationally invariant. The only normalizable state that satisfies this is the vacuum,  $|0\rangle$ . But it is possible to consider states

$$|\Psi\rangle = \int dx_1 \dots dx_n f(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n) |0\rangle, \quad (3.10)$$

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<sup>1</sup>One could immediately note that positivity of the inner-product follows from positivity of the QFT inner product. Also, the group averaging way of representing the QFT state, as given in (3.5) makes it clear that the group volume quotient will not be regulator dependent when one studies UV or IR divergences. We will not discuss these aspects further in this chapter.

for any smearing function that satisfies  $f(x_1, \dots, x_n) = f(x_1 + a, \dots, x_n + a)$  for all vectors  $a$ . These states satisfy the constraint. They are not normalizable in the original QFT norm, but one can define a new norm

$$(\Psi, \Psi) = \frac{\langle \Psi | \Psi \rangle}{\text{vol}(\mathbb{R}^d)}, \quad (3.11)$$

that makes these states normalizable subject to some additional constraints on  $f$ . The original vacuum is no longer normalizable, although the Hilbert space is again infinite-dimensional. We expect that carefully taking the classical limit in this new Hilbert space should lead us back to the set of classical solutions with zero center-of-mass momentum.

**Cosmological correlators.** The authors of [7] defined a set of gauge-fixed observables in the constrained de Sitter Hilbert space that were called “cosmological correlators.” The definition in the full gravitational theory is given in [7]. In the nongravitational limit, the definition simplifies significantly. The cosmological correlator of  $n$  insertions is simply defined by

$$\langle\langle \Psi | \mathcal{C}(y_1, \dots, y_n) | \Psi \rangle\rangle_{\text{CC}} = \langle \Psi | \chi(y_1) \dots \chi(y_n) | \Psi \rangle. \quad (3.12)$$

Note that in the expression above, the points  $y_i$  are fixed *by hand* and not integrated over. Of course, it is possible to study a linear combination of such cosmological correlators by smearing them with a sequence of functions  $g_q(\vec{y})$ .

$$\langle\langle \Psi | \mathcal{C}[g_1, \dots, g_n] | \Psi \rangle\rangle = \sum_{q=1}^n \int d\vec{y} g_q(\vec{y}) \langle \Psi | \chi(y_1) \dots \chi(y_q) | \Psi \rangle. \quad (3.13)$$

This is just the usual smearing that is often useful for correlation functions, and  $g_q(\vec{y})$  are not constrained by conformal invariance.

The expression (3.13) reduces to a linear combination of QFT vacuum expectation values.

$$\begin{aligned} \langle\langle \Psi | \mathcal{C}[g_1, \dots, g_n] | \Psi \rangle\rangle_{\text{CC}} &= \sum_{m, m', q} \int d\vec{x} d\vec{x}' d\vec{y} \delta G_{m'}^*(\vec{x}') \delta G_m(\vec{x}) g_q(\vec{y}) \\ &\times \langle 0 | \chi(x'_1) \dots \chi(x'_{m'}) \chi(y_1) \dots \chi(y_q) \bar{\chi}(x_1) \dots \bar{\chi}(x_m) | 0 \rangle. \end{aligned} \quad (3.14)$$

Two comments are in order. First, the reader might worry that the expectation values on the right-hand side of (3.13) suffer from group-volume divergences. Intuitively, 3 fixed points are sufficient to absorb the divergence of the group volume. A little more

thought shows that what we really need is that only *connected* diagrams with at least three fixed points should contribute to (3.14). Following this intuition, it will be shown below that for  $n \geq 3$ , provided we impose some constraints on the smearing functions  $g_q$ , the cosmological correlator is finite.

Second, it was shown in [7] that the cosmological correlators can be written as expectation values of a gauge-invariant operator

$$\langle\langle \Psi | \mathcal{C}(y_1, \dots, y_q) | \Psi \rangle\rangle_{\text{CC}} = (\Psi, O \Psi), \quad (3.15)$$

where, in the nongravitational limit,

$$O = \frac{1}{\text{vol}(\text{SO}(d-1))} \int dU U^\dagger \chi(y_1) \dots \chi(y_q) U, \quad (3.16)$$

and  $dU$  is the Haar measure on the conformal group. It is shown in the Appendix that this integral can be performed explicitly in some cases and leads to

$$O = \int d\vec{z} G(\vec{z}) \chi(z_1) \dots \chi(z_q). \quad (3.17)$$

Here,  $G$  is a function that obeys the Ward identities of CFT correlators in terms of the variables  $z_i$ . The function depends on the fixed points  $y_i$ , but this dependence is kept implicit in the expression above. Appendix B.1 provides explicit examples of this relation. Although our computations will not use the representation (3.16), it is important from a conceptual perspective since it tells us that cosmological correlators are secretly nonlocal observables even though they are labeled by points. This should not be surprising given that a theory of gravity can have no local gauge-invariant observables.

**Holography of information.** In [7], it was shown that specifying cosmological correlators in any open set of the late-time slice specifies them everywhere. This is indicative of the holographic properties of gravity: in a nongravitational field theory, correlators on different parts of the late-time slice carry separate pieces of quantum information; but, in gravity, this quantum information is squeezed down to correlators in an infinitesimal region. Incidentally, this is consistent with the conclusion reached in [41] from very different considerations.

As an aside, the phrase “dS/CFT” is sometimes used to simply refer to the fact that the coefficients of the Hartle-Hawking wavefunction are related to CFT correlators. However, this is not necessarily related to holography or even to gravity. The “CFT” that appears in this context is simply a machine for determining properties of the Euclidean

vacuum. Holography is always a statement about the unusual localization of quantum information in quantum gravity in all states, and not just about the properties of one state.

Our results for cosmological correlators in Section 3.4 serve to verify the holographic properties of gravity in de Sitter space in the perturbative limit. However, our construction of Section 3.5 also shows that, in an appropriate limit, holography can be obscured in gravity. This is important since it is by obscuring holography that we regain the QFT limit and mundane notions of locality.

**Generating function.** Before concluding this section, it is useful to introduce some notation that will be helpful below.

We see above that both the norm (3.9) and cosmological correlators (3.14) can be represented as integrated quantum-field-theory correlators in the Euclidean vacuum. So it is convenient to introduce a generating function for these QFT correlators

$$\begin{aligned} Z[J, \bar{J}] &= \langle 0 | e^{\int J(x') \chi(x') dx} e^{\int \bar{J}(x) \bar{\chi}(x) dx} | 0 \rangle \\ &= \sum_n \frac{1}{n! m!} \int d\vec{x} d\vec{x}' \bar{J}(x_1) \dots \bar{J}(x_m) J(x'_1) \dots J(x'_n) \langle 0 | \chi(x'_1) \dots \chi(x'_n) \bar{\chi}(x_1) \dots \bar{\chi}(x_m) | 0 \rangle \end{aligned} \quad (3.18)$$

One subtlety here is that, in the case of principal series,  $\chi(x)$  and  $\bar{\chi}(x)$  commute only up to a contact term. So we are careful to place  $\bar{\chi}$  on the right and  $\chi$  on the left — a convention that we have already tacitly adopted above. Appendix B.3 explains why this convention is superior to the opposite convention where the  $\chi$  are placed on the right.

As usual, this functional can be related to a generating function of “connected correlators”,  $W$ , via

$$Z[J, \bar{J}] = e^{W[J, \bar{J}]}. \quad (3.19)$$

The correlators we want can then be recovered via functional differentiation, and after integrating with the functions that define the external states.

We normalize the vacuum so that

$$Z[0] = 1 \quad W[0] = 0, \quad (3.20)$$

which ensures that vacuum bubbles, which are disconnected from all other pieces of the diagram, do not contribute to any correlator.

### 3.3 Perturbative observables

In this section, we formulate Feynman rules to compute the norm of perturbative states and gauge-fixed perturbative cosmological correlators as defined in (3.12).

As we have explained above, these computations reduce to integrated QFT vacuum correlators. Vacuum correlators in a QFT can be computed using the in-in formalism [42] or by computing the wavefunction by analytic continuation from AdS [43]. Here, we will simply assume that the wavefunction for the Euclidean vacuum is given to us, and we will remain agnostic about how it was computed.

The discussion of [7, 13] focused on wavefunctions of both the metric and of matter fluctuations,  $\chi$ . Since we are interested in the  $G_N \rightarrow 0$  limit, we will ignore the metric fluctuations. This limit is slightly subtle: one must first square the wavefunction and then perform the trivial integral over the metric fluctuations to obtain a wavefunction that depends on the matter fields. Moreover, [7, 13] carefully kept track of a leading phase-factor in the wavefunction. The correlators we study here are insensitive to the phase factor. So we will simply take the *effective* wavefunction of the matter fields to have the form

$$\Psi_0[\chi] = e^{\sum_{n=2}^{\infty} \int dx_1 \dots dx_n \lambda^{n-2} G_n(\vec{x}) \bar{\chi}(x_1) \dots \bar{\chi}(x_n)}, \quad (3.21)$$

where each  $G_n$  obeys the Ward identities of an  $n$ -point conformal correlator of operators with dimension  $\Delta$  [44, 45] and we have included a perturbative parameter  $\lambda$  for book-keeping.

A vacuum correlator is now computed via

$$\langle 0 | \chi(x_1) \dots \chi(x_n) \bar{\chi}(x'_1) \dots \bar{\chi}(x'_m) | 0 \rangle = \int [D\chi D\bar{\chi}] |\Psi_0[\chi]|^2 \chi(x_1) \dots \chi(x_n) \bar{\chi}(x'_1) \dots \bar{\chi}(x'_m). \quad (3.22)$$

As explained in Appendix B.3, this expression naturally computes correlators where the  $\bar{\chi}$  appear to the right of the  $\chi$ . Second, the measure of integration must be normalized carefully and, when  $\chi$  is in the principal series, gives rise to some contact-term contributions in the propagators that we describe below. Interactions can now be treated simply by expanding the non-Gaussian pieces in the wavefunction, and the norm (3.9) and (3.14) can be computed through smeared and integrated linear combinations of such correlators.

#### Feynman rules

We present Feynman rules in position space. In some cases, it is convenient to perform the integral by Fourier transforming to momentum space, and the momentum-space rules can be obtained easily from the rules given below.

**Propagators.** For principal-series fields we need to keep track of the following kinds of propagators, each of which corresponds to a free-field two-point function.

| Free – field expression                               | Symbol             | Value                                              |
|-------------------------------------------------------|--------------------|----------------------------------------------------|
| $\langle 0   \chi(x) \bar{\chi}(y)   0 \rangle$       | $x \text{-----} y$ | $c_{\chi\bar{\chi}} \delta(x-y)$                   |
| $\langle 0   \chi(x) \chi(y)   0 \rangle$             | $x \text{————} y$  | $\frac{c_{\chi\chi}}{ x-y ^{2\Delta}}$             |
| $\langle 0   \bar{\chi}(x) \bar{\chi}(y)   0 \rangle$ | $x \text{.....} y$ | $\frac{c_{\bar{\chi}\bar{\chi}}}{ x-y ^{2\Delta}}$ |

The first line above involves a contact term that allows the contraction of  $\chi$  and its conjugate [46]. It arises because if the wavefunction is written as a function of  $\chi(x)$ , then  $\bar{\chi}(x)$  can be thought of as its canonical conjugate. The explicit forms of the constant coefficients  $c_{\chi\bar{\chi}}$ ,  $c_{\chi\chi}$  and  $c_{\bar{\chi}\bar{\chi}}$  are given in Section 3.4. As mentioned above,  $\bar{\chi}(x)$  always appears to the right of  $\chi(x)$ .

For the complementary series, there is no conjugate field and no contact term. Therefore, only the  $\chi - \chi$  propagator displayed above is relevant.

**Vertices.** We also get interaction vertices by expanding the wavefunction in (3.22). However, it is important to note that the bra and the ket themselves contribute as effective vertices in the computation of a norm or a correlator in (3.9) or (3.14). We distinguish the vertices that come from the bra, the ket, the wavefunction, and its conjugate using different colors. This leads to the following Feynman rules.

| Origin         | Symbol | Value                                |
|----------------|--------|--------------------------------------|
| Ket            |        | $\delta G_n(x_1, x_2, \dots x_n)$    |
| Bra            |        | $\delta G_n^*(x_1, x_2, \dots x_n)$  |
| $\Psi[\chi]$   |        | $\lambda^{n-2} G_n(x_1 \dots x_n)$   |
| $\Psi^*[\chi]$ |        | $\lambda^{n-2} G_n^*(x_1 \dots x_n)$ |

The bra-ket and contributions from the wavefunction and its conjugate must all be integrated over the coordinates  $x_i$ .

**Fixed points.** Not all points in the diagram are integrated. Each diagram must have at least three fixed points. In the computation of a norm, there are exactly three fixed points that arise from the formula (3.9). The computation of a cosmological correlator has three or more fixed points that arise from the insertions of the operators in (3.14).



Figure 3.1: A diagram that displays a potential divergence due to the group volume.

These fixed points will be displayed explicitly in each diagram.

The presence of these fixed points is important. Each vertex involves a conformally covariant function. Therefore, before we perform the position integrals, the entire integrand obtained by joining the vertices with propagators is conformally covariant. If all the external points were integrated over, we would get a divergence proportional to the volume of the conformal group. The presence of at least three fixed points helps to avoid this divergence, subject to some additional constraints that we discuss next.

As is standard, the contribution of a diagram is obtained by simply joining the vertices with propagators and performing the position integrals associated with each vertex. We provide examples in the next section.

### 3.3.1 Group-volume divergences

For some diagrams, the presence of three fixed points might be insufficient to remove the divergence coming from the volume of the conformal group. This happens when the Feynman diagram breaks up into the product of two disconnected conformally covariant terms. In this section, we explain how to remove some simple group-volume divergences by restricting attention to states and observables that satisfy certain conditions. We divide the discussion into two parts: group-volume divergences in the norm and group-volume divergences in cosmological correlators. The Feynman diagrams discussed here follow from the rules given above. However, the reader might find some of them unfamiliar due to the presence of vertices associated with the bra and the ket. If so, it is possible to skip this subsection on a first reading, examine the examples in Section 3.4, and then return to this subsection.

#### Group-volume divergences in the norm

Recall that the norm is computed by multiplying the QFT norm of a state by the factor  $\mathcal{K}$ , which is, itself, inversely proportional to the volume of the de Sitter group. Equivalently, the norm may be computed by fixing any three points in the conformally invariant integral that appears in its computation.

The diagram shown in Figure 3.1 provides an example of how disconnected diagrams might retain a divergence even after fixing three points. This diagram appears in the computation of the norm of a three-particle state in the presence of an interaction that also contains a three-point term. The reader can consult the examples in the next

section if this is not apparent. For now, the origin of the diagram is not crucial; it is only necessary to note that although three points have been fixed in the contribution of the ket, the diagram retains a divergence because the contraction of the bra and the interaction yields a conformally invariant integral and a corresponding group-volume divergence.

Group-volume divergences that arise from such disconnected diagrams can be avoided by restricting attention to those states that satisfy

$$\langle 0|\Psi\rangle = 0, \quad (3.23)$$

and also

$$\langle 0|\chi(x)|\Psi\rangle = \langle 0|\bar{\chi}(x)|\Psi\rangle = 0. \quad (3.24)$$

Physically, these are very natural requirements to impose. The vacuum and one-particle states are non-normalizable. Therefore, it makes sense to subtract off their contribution from external states. Note that the overlap of a de Sitter-invariant state with a one-particle state can only be a constant, independent of  $x$ , since that is the only contribution allowed by conformal invariance [47].

We now prove that imposing these conditions leads to the removal of disconnected diagrams. In terms of the generating function of connected correlators the computation of a norm corresponds to

$$\begin{aligned} (\Psi, \Psi) &= \mathcal{K} \sum_{m,m'} \int d\vec{x} d\vec{x}' \delta G_m(\vec{x}) \delta G_{m'}^*(\vec{x}') \\ &\times \frac{\delta}{\delta J(x'_1)} \cdots \frac{\delta}{\delta J(x'_{m'})} \frac{\delta}{\delta \bar{J}(x_1)} \cdots \frac{\delta}{\delta \bar{J}(x_m)} e^{W[J, \bar{J}]} \Big|_{J=\bar{J}=0}. \end{aligned} \quad (3.25)$$

We define

$$\frac{\delta}{\delta \bar{J}(x_1)} \cdots \frac{\delta}{\delta \bar{J}(x_m)} e^{W[J, \bar{J}]} = \bar{P}[J, \bar{J}, x_1, \dots, x_m] e^{W[J, \bar{J}]}, \quad (3.26)$$

where  $\bar{P}[J, \bar{J}]$  is a functional of  $J$  and  $\bar{J}$  obtained by acting with  $m$  functional derivatives on a polynomial of  $W[J, \bar{J}]$ . It is related to vacuum correlators through

$$\bar{P}[J = 0, \bar{J} = 0, x_1, \dots, x_m] = \langle 0|\bar{\chi}(x_1) \cdots \bar{\chi}(x_m)|0\rangle. \quad (3.27)$$

If we had just been computing this vacuum correlator, then those terms in  $\bar{P}$  that contain multiple powers of  $W$  would correspond to disconnected diagrams where some of the points  $x_i$  are disconnected from each other. However, in the computation under consideration, all these diagrams are linked by the ket vertex. This is evident above in

the presence of the smearing function  $\delta G_m(\vec{x})$  that links all the external points.

Now consider acting with one of the remaining functional derivatives

$$\frac{\delta}{\delta J(x'_1)} \bar{P}[J, \bar{J}, x_1, \dots x_n] e^{W[J]} = \frac{\delta \bar{P}[J, \bar{J}, x_1, \dots x_n]}{\delta J(x'_1)} e^{W[J]} + \bar{P}[J, \bar{J}, y_1 \dots y_n] \frac{\delta}{\delta J(x'_1)} e^{W[J]}. \quad (3.28)$$

The first term on the right-hand side comprises diagrams where the point  $x'_1$  is connected to at least one of the  $x_i$  points. But the functional  $\delta G_m(\vec{x})$  connects the different  $x_i$  insertions together and therefore the first term comprises diagrams where  $x'_1$  is connected to the  $x_i$ . The second term generates a disconnected diagram.

Proceeding this way, we see that disconnected diagrams are only obtained if each  $\frac{\delta}{\delta J(x'_i)}$  functional derivative acts on the  $e^{W[J, \bar{J}]}$  factor. But, defining  $P$  analogously to  $\bar{P}$  above, this implies that the set of disconnected diagrams factorizes into

$$\begin{aligned} & \sum_{n,m} \delta G_n^*(\vec{x}') \delta G_m(\vec{x}) \bar{P}[J = 0, \bar{J} = 0, x_1, \dots x_n] P[J = 0, \bar{J} = 0, x'_1, \dots x'_m] \\ &= \sum_n \delta G_n^*(\vec{x}') P[J = 0, \bar{J} = 0, x'_1, \dots x'_n] \sum_m \delta G_m(\vec{x}) \bar{P}[J = 0, \bar{J} = 0, x_1, \dots x_m], \end{aligned} \quad (3.29)$$

but these terms correspond to the overlap of the state with the vacuum and its conjugate. Therefore, setting this to zero gets rid of diagrams where the bra and the ket are disconnected.

Even if the diagram is connected, a one-particle state is not normalizable since it is not possible to fix 3 points in the integral for the norm. Condition (3.24) disallows such states. Repeating the argument above, it can be seen that this condition also removes potential group-volume divergences from diagrams where the bra and the ket can be separated by cutting a single line.

### Group-volume divergences in cosmological correlators

We now turn to cosmological correlators. The formal analysis in this case is similar to the analysis performed above for the norm, and so we will simply state the results and provide some illustrative examples.

Some obvious group-volume divergences are removed by imposing

$$\langle 0 | \mathcal{C} | 0 \rangle = 0. \quad (3.30)$$

In addition the smearing functions that enter (3.13) must obey certain constraints. The

integral of these smearing functions over any one coordinate must vanish

$$\int dx_i g_q(\vec{x}) = 0 \quad \forall i. \quad (3.31)$$

This must hold for each  $g_q(\vec{x})$ . We emphasize that here the  $dx_i$  is a single integral and so the condition above can be expanded as

$$\int dx_1 g_q(\vec{x}) = \int dx_2 g_q(\vec{x}) \dots = \int dx_q g_q(\vec{x}) = 0. \quad (3.32)$$

Second, for any pair of coordinates  $i, j$  we impose

$$\int dx_i dx_j \frac{g_q(\vec{x})}{|x_i - x_j|^{2\Delta}} = 0. \quad (3.33)$$

The condition (3.30) may be satisfied by simply subtracting off a suitable multiple of the identity from  $\mathcal{C}$ . Equations (3.31) and (3.33) can be satisfied by appropriate choices of smearing functions.

Equations (3.31) and (3.33) can also be expressed in Fourier space. Defining

$$\tilde{g}_q(\vec{k}) = \int g_q(\vec{x}) e^{i\vec{k} \cdot \vec{x}} \prod_j dx_j, \quad (3.34)$$

the conditions (3.31) and (3.33) read

$$\int dk_i \tilde{g}_q(\vec{k}) \delta(k_i) = 0 \quad \int dk_i dk_j \delta(k_i + k_j) \tilde{g}_q(\vec{k}) |k_i - k_j|^{2\Delta-d} = 0. \quad (3.35)$$

These conditions can be satisfied by choosing  $\tilde{g}(\vec{k})$  so that it vanishes whenever any of the  $k_i$  vanish or when  $k_i + k_j = 0$  for any pair of arguments:

$$\tilde{g}(\vec{k}) \Big|_{k_i=0} = 0; \quad \tilde{g}(\vec{k}) \Big|_{k_i+k_j=0} = 0. \quad (3.36)$$

Repeating the arguments above now shows that some obvious group-volume divergences are removed by these conditions. Diagrams where the points in the cosmological correlator are completely disconnected from the bra and the ket vanish by virtue of (3.30). An example is shown in Figure 3.2a. Now consider those diagrams where only one or two external points from the cosmological correlator are connected to one part of the diagram, whereas the other points are connected to another part; an example is shown in Figure 3.2b and another example is shown in Figure 3.2c. Such diagrams vanish by virtue of (3.31) and (3.33). This leaves us with diagrams that are either completely connected or those that break up into two or more disconnected pieces, each of

which is connected to at least three points from the cosmological correlator. But these fixed points are enough to absorb the integral over the conformal group.

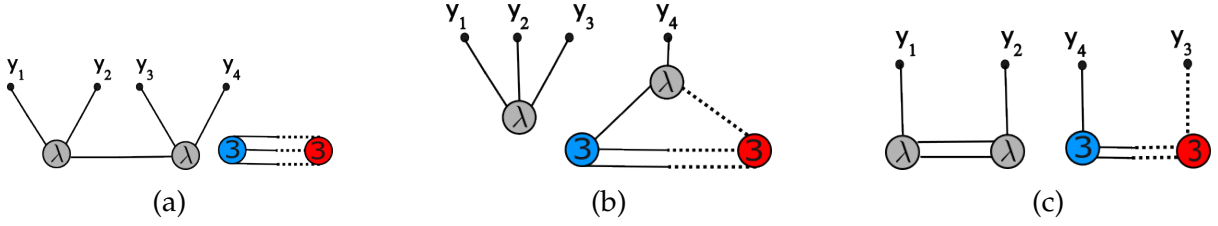


Figure 3.2: The three kinds of diagrams that are removed by the conditions (3.30), (3.31) and (3.33). All these represent contributions to a four-point cosmological correlator in a three-particle state, where the vacuum has three-point interactions.

Some other potential divergences in cosmological correlators, such as the one shown in Figure 3.3, are removed by the condition (3.23).

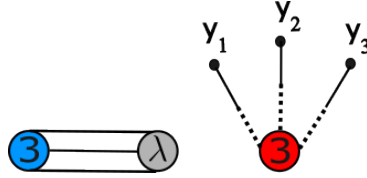


Figure 3.3: A divergent contribution to a 3-point cosmological correlator in a 3-particle state that is removed by condition (3.23).

**Additional divergences and renormalization.** We emphasize that the removal of simple group-volume divergences does not remove all divergences associated with the norm, or with cosmological correlators. For instance, at higher orders in perturbation theory, we see the usual UV divergences that appear in any effective field theory computation. In general, it is necessary to appropriately renormalize the wavefunction by adding counterterms to ensure that cosmological correlators are finite. We postpone an analysis of such effects to future work.

### 3.4 Sample computations

In this section, we provide some sample calculations that illustrate the use of the Feynman rules presented in the previous sections. We will compute norms and the values of cosmological correlators in a variety of states.

A striking feature of these computations is that every simple state yields cosmological correlators that differ from QFT expectation values in the Euclidean vacuum. We present several examples where the Euclidean vacuum is Gaussian and therefore vacuum expectation values trivially factorize into products of two-point functions. Nevertheless, a nontrivial excited state in such a theory gives rise to nontrivial conformally

invariant cosmological correlators. It is the conformal invariance that is special; even in a free nongravitational field theory in de Sitter space, excited states can give rise to nontrivial correlation functions for operator insertions on the late-time slice, but these correlators are never conformally invariant since the underlying excited state cannot be conformally invariant while being normalizable.

For technical simplicity, we will study fields in the principal-series representation. In this case, we are able to borrow various tools and orthogonality relations from harmonic analysis that make it possible for us to compute explicit position space correlators. To this end, we will use many of the results from [48]. We will quote mathematical identities relevant to us in the main text itself, but interested readers are encouraged to consult [48] to understand them from a representation theory point of view. Since these calculations are provided for the purpose of illustration, and they are performed at the lowest nontrivial order in perturbation theory, we set aside potential subtleties with principal-series fields that might arise at higher orders in perturbation theory [36].

This section is divided into two parts. In the first part, we study a Gaussian vacuum, and then we study an interacting vacuum. In both cases, we obtain beautiful closed-form answers for three- and four-point cosmological correlators in various states. The structural aspects of these answers deserve further investigation that we postpone to future work.

The norm of group-averaged states was previously studied from a Fock-space perspective in [21].

### 3.4.1 Cosmological correlators in excited states about a Gaussian vacuum

We start by considering the case of a Gaussian vacuum. We will consider the case where the vacuum has  $n$  different principal-series fields, denoted by  $\chi_i$ , having conformal weight  $\Delta_i = \frac{d}{2} + i\mu_i$ . The nonzero correlators in the vacuum are [46]

$$\begin{aligned}
\langle 0 | \chi_i(x) \chi_j(x') | 0 \rangle &= \frac{\mathfrak{c}_{\chi_i \chi_i}}{|x - x'|^{2\Delta_i}} \delta_{ij} \quad ; \mathfrak{c}_{\chi_i \chi_i} = \frac{\Gamma[\Delta_i] \Gamma[\frac{d}{2} - \Delta_i]}{4\pi^{\frac{d}{2}+1}} \\
\langle 0 | \bar{\chi}_i(x) \bar{\chi}_j(x') | 0 \rangle &= \frac{\mathfrak{c}_{\bar{\chi}_i \bar{\chi}_i}}{|x - x'|^{2\Delta_i}} \delta_{ij} \quad ; \mathfrak{c}_{\bar{\chi}_i \bar{\chi}_i} = \frac{\Gamma[\Delta_i] \Gamma[\Delta_i - \frac{d}{2}]}{4\pi^{\frac{d}{2}+1}} \\
\langle 0 | \chi_i(x) \bar{\chi}_j(x') | 0 \rangle &= \mathfrak{c}_{\chi_i \bar{\chi}_i} \delta(x - x') \delta_{ij} \quad ; \mathfrak{c}_{\chi_i \bar{\chi}_i} = \frac{\coth \pi \mu_i + 1}{4\mu_i}.
\end{aligned} \tag{3.37}$$

All higher-point correlators in the vacuum factorize into products of the two-point functions given above. In terms of the wavefunction (3.21), this means that it has Gaussian terms but no non-Gaussian terms.

### 3-particle state

We will start with the simplest example of a nontrivial state. Consider a theory with a single field in the principal-series,  $\chi(x)$  (and its conjugate field  $\bar{\chi}$ ), having conformal weight  $\Delta$  ( $\bar{\Delta}$  for the conjugate field). This theory has a single 3-particle state. We write it as

$$|3\rangle = \int dx_1 dx_2 dx_3 \delta G_3(x_1, x_2, x_3) : \bar{\chi}(x_1) \bar{\chi}(x_2) \bar{\chi}(x_3) : |0\rangle. \quad (3.38)$$

Here  $::$  denotes normal ordering. The form of  $\delta G_3(x_1, x_2, x_3)$  is fixed by conformal invariance to be

$$\delta G_3(x_1, x_2, x_3) = \frac{C_{\Delta\Delta\Delta}}{|x_1 - x_2|^\Delta |x_1 - x_3|^\Delta |x_2 - x_3|^\Delta}. \quad (3.39)$$

The constant  $C_{\Delta\Delta\Delta}$  fixes the overall normalization of the state.

The bra corresponding to the ket above is

$$\langle 3| = \int dx_1 dx_2 dx_3 \delta G_3^*(x_1, x_2, x_3) \langle 0| : \chi(x_1) \chi(x_2) \chi(x_3) :. \quad (3.40)$$

The norm  $\langle 3|3\rangle$  gets a contribution from the diagram displayed in Figure 3.4.

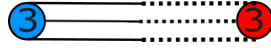


Figure 3.4: Diagram contributing to the norm of the 3-particle state

We find — either by using the Feynman rules discussed in Section 3.3 — or by directly contracting the bra and the ket,

$$\begin{aligned} (3,3) &= 3! c_{\chi\bar{\chi}}^3 \mathcal{K} \int dx_1 dx_2 dx_3 |\delta G_3(x_1, x_2, x_3)|^2 \\ &= 3! c_{\chi\bar{\chi}}^3 |C_{\Delta\Delta\Delta}|^2. \end{aligned} \quad (3.41)$$

To get the final equality, we gauge fixed the integral, which leaves behind only the constant coefficients. We refer the reader to section 5.3 of [13] for details.

Integrals over conformally invariant structures of the kind that appear above were also

studied extensively in [48]. To make contact with their notation, we define

$$\langle O^{\Delta_1}(x_1)O^{\Delta_2}(x_2)O^{\Delta_3}(x_3) \rangle \equiv \frac{C_{\Delta_1\Delta_2\Delta_3}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}}. \quad (3.42)$$

The notation is designed to be reminiscent of the correlation function of CFT operators although we reiterate that there is no physical CFT here; it is simply that the gravitational constraints force the states to be smeared with functions that obey the Ward identities of CFT correlators.

Using this notation, the result (3.41) can also be derived from the formula

$$\begin{aligned} & \int \frac{dx_1 dx_2 dx_3}{\text{vol}(\text{SO}(1, d+1))} \langle O^{\Delta_1}(x_1)O^{\Delta_2}(x_2)O^{\Delta_3}(x_3) \rangle \langle O^{\bar{\Delta}_1}(x_1)O^{\bar{\Delta}_2}(x_2)O^{\bar{\Delta}_3}(x_3) \rangle \\ &= \frac{C_{\Delta_1\Delta_2\Delta_3}C_{\bar{\Delta}_1\bar{\Delta}_2\bar{\Delta}_3}}{\text{vol}(\text{SO}(d-1))}. \end{aligned} \quad (3.43)$$

In our case, since we are considering principal-series fields  $C_{\bar{\Delta}\bar{\Delta}\bar{\Delta}} = C_{\Delta\Delta\Delta}^*$  which gives back (3.41). One can choose the value of  $|C_{\Delta\Delta\Delta}|$  to normalize the state to have norm 1, although we will keep it as it is.

Normal ordering the operators is important to ensure that additional diagrams, with self-contractions, such as Figure 3.5 do not contribute to the norm. Normal ordering also automatically ensures that the conditions of Section 3.3 are met for this state.

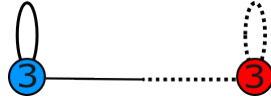


Figure 3.5: A potentially divergent contribution to the norm of the 3-particle state that is removed by normal ordering

### Cosmological correlators in the 3-particle state

We now turn to the computation of cosmological correlators. The simplest one to compute is a three-point cosmological correlator. However, we find that

$$\langle\langle 3|\chi(y_1)\chi(y_2)\chi(y_3)|3\rangle\rangle = 0. \quad (3.44)$$

This can be seen by expanding the expression above, which yields the expectation value of 9 fields in the vacuum. This expectation value vanishes since the number of insertions is odd (Recall that the vacuum is free).

So we now turn to the computation of a four-point cosmological correlator

$$\langle\langle 3 | : \chi(y_1)\chi(y_2)\chi(y_3)\chi(y_4) : | 3 \rangle\rangle.$$

Through normal ordering, we again ensure that the operator has no vacuum expectation value. In addition, it is to be understood that the result of this correlator is always smeared with a function of  $y_i$  that obeys the conditions (3.36).

One of the Feynman diagrams that contributes to this correlator is displayed in Figure 3.6. Denoting its value by  $\mathcal{F}_1$ , we find that

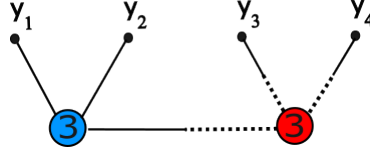


Figure 3.6: Diagram contributing to a 4-point function in a 3-particle state

$$\begin{aligned} \mathcal{F}_1 &= (3!)^2 \int \left( \prod_{i=1}^3 dx_i dx'_i \right) \delta G_3^*(x_1, x_2, x_3) \delta G_3(x'_1, x'_2, x'_3) \\ &\quad \times \overline{\chi(y_1)} \overline{\chi(x_1)} \overline{\chi(y_2)} \overline{\chi(x_2)} \overline{\chi(x_3)} \overline{\chi(x'_3)} \overline{\chi(y_3)} \overline{\chi(x'_1)} \overline{\chi(y_4)} \overline{\chi(x'_2)} \\ &= (3!)^2 c_{\chi\chi}^2 c_{\chi\bar{\chi}}^3 \int \left( \prod_{i=1}^3 dx_i dx'_i \right) \delta G_3^*(x_1, x_2, x_3) \delta G_3(x'_1, x'_2, x'_3) \\ &\quad \times \frac{1}{|y_1 - x_1|^{2\Delta}} \frac{1}{|y_2 - x_2|^{2\Delta}} \delta^d(x_3 - x'_3) \delta^d(y_3 - x'_1) \delta^d(y_4 - x'_2) \\ &= (3!)^2 c_{\chi\chi}^2 c_{\chi\bar{\chi}}^3 \int \left( \prod_{i=1}^3 dx_i \right) \delta G_3^*(x_1, x_2, x_3) \delta G_3(y_3, y_4, x_3) \frac{1}{|y_1 - x_1|^{2\Delta}} \frac{1}{|y_2 - x_2|^{2\Delta}}. \end{aligned} \quad (3.45)$$

The three remaining integrals are known conformal integrals. First, we have integrals that, in the CFT literature, are called “shadow transforms”. This converts a three-point function of operators with dimensions  $\Delta_1, \Delta_2, \Delta_3$  to a three-point function of operators with dimensions  $\bar{\Delta}_1, \Delta_2, \Delta_3$ . We write this as

$$\begin{aligned} \langle O^{S[\Delta_1]}(y_1) O^{\Delta_2}(x_2) O^{\Delta_3}(x_3) \rangle &\equiv \int dx_1 \langle O^{\Delta_1}(x_1) O^{\Delta_2}(x_2) O^{\Delta_3}(x_3) \rangle \frac{1}{|y_1 - x_1|^{2\bar{\Delta}_1}} \\ &= \frac{C_{\Delta_1 \Delta_2 \Delta_3} N(\Delta_1 + \Delta_2 - \Delta_3, \Delta_1 + \Delta_3 - \Delta_2, 2\bar{\Delta}_1)}{|y_1 - x_2|^{\bar{\Delta}_1 + \Delta_2 - \Delta_3} |y_1 - x_3|^{\bar{\Delta}_1 + \Delta_3 - \Delta_2} |x_2 - x_3|^{\Delta_2 + \Delta_3 - \bar{\Delta}_1}} \\ &\equiv \frac{C_{S[\Delta_1] \Delta_2 \Delta_3}}{|y_1 - x_2|^{\bar{\Delta}_1 + \Delta_2 - \Delta_3} |y_1 - x_3|^{\bar{\Delta}_1 + \Delta_3 - \Delta_2} |x_2 - x_3|^{\Delta_2 + \Delta_3 - \bar{\Delta}_1}}, \end{aligned} \quad (3.46)$$

where

$$N(a, b, c) = \frac{\pi^{\frac{d}{2}} \Gamma[\frac{d-a}{2}] \Gamma[\frac{d-b}{2}] \Gamma[\frac{d-c}{2}]}{\Gamma[\frac{a}{2}] \Gamma[\frac{b}{2}] \Gamma[\frac{c}{2}]}. \quad (3.47)$$

The notation  $S[\Delta]$  keeps track of the fact that shadow transformations have been performed and the resulting numerical factors are absorbed into the three-point function. Multiple shadow transforms can be performed in sequence. We will use the same notation below to indicate the shadow transformation of more-general expressions.

In the case at hand, we have two shadow transforms and obtain

$$\begin{aligned} \delta G^*(y_1^{S[\Delta]}, y_2^{S[\Delta]}, x_3) &\equiv \int dx_1 dx_2 \delta G_3^*(x_1, x_2, x_3) \frac{1}{|y_1 - x_1|^{2\Delta}} \frac{1}{|y_2 - x_2|^{2\Delta}} \\ &= \frac{C_{\Delta\Delta\Delta}^* N(\bar{\Delta}, \bar{\Delta}, 2\Delta) N(\Delta, 2\Delta, 2\Delta - \Delta)}{|x_3 - y_1|^{\bar{\Delta}} |x_3 - y_2|^{\bar{\Delta}} |y_1 - y_2|^{2\Delta - \bar{\Delta}}} \\ &\equiv \frac{C_{S[\Delta]S[\Delta]\Delta}^*}{|x_3 - y_1|^{\bar{\Delta}} |x_3 - y_2|^{\bar{\Delta}} |y_1 - y_2|^{2\Delta - \bar{\Delta}}}. \end{aligned} \quad (3.48)$$

Thus we arrive at the expression,

$$\mathcal{F}_1 = (3!)^2 \mathfrak{c}_{\chi\chi}^2 \mathfrak{c}_{\chi\bar{\chi}}^3 \int dx_3 \delta G_3^*(y_1^{S[\Delta]}, y_2^{S[\Delta]}, x_3) \delta G_3(x_3, y_3, y_4). \quad (3.49)$$

The last integral is an integral of the product of two three-point functions over a common point. This gives rise to a four-point partial wave using the formula [48]

$$\Theta_{\Delta}^{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_2, x_3, x_4) = \int dx \langle O^{\Delta_1}(x_1) O^{\Delta_2}(x_2) O^{\Delta}(x) \rangle \langle O^{\Delta_3}(x_3) O^{\Delta_4}(x_4) O^{\bar{\Delta}}(x) \rangle. \quad (3.50)$$

Hence,

$$\begin{aligned} \mathcal{F}_1 &= (3!)^2 \mathfrak{c}_{\chi\chi}^2 \mathfrak{c}_{\chi\bar{\chi}}^3 \int dx_3 \delta G_3^*(y_1^{S[\Delta]}, y_2^{S[\Delta]}, x_3) \delta G_3(x_3, y_3, y_4) \\ &= \mathcal{K}_1 \Theta_{\bar{\Delta}}^{\Delta\Delta\Delta\Delta}(y_1, y_2, y_3, y_4), \end{aligned} \quad (3.51)$$

where,

$$\mathcal{K}_1 = (3!)^2 \mathfrak{c}_{\chi\chi}^2 \mathfrak{c}_{\chi\bar{\chi}}^3 \frac{C_{S[\Delta]S[\Delta]\Delta}^*}{C_{\Delta\Delta\bar{\Delta}}}. \quad (3.52)$$

The full 4-point cosmological correlator can be easily written down, taking into ac-

count other Feynman diagrams obtained by simply permuting the external legs.

$$\begin{aligned} & \langle\langle 3 | : \chi(y_1)\chi(y_2)\chi(y_3)\chi(y_4) : | 3 \rangle\rangle \\ &= \mathcal{K}_1 \left( \Theta_{\Delta}^{\Delta\Delta\Delta\Delta}(y_1, y_2, y_3, y_4) + \Theta_{\Delta}^{\Delta\Delta\Delta\Delta}(y_1, y_3, y_2, y_4) + \Theta_{\Delta}^{\Delta\Delta\Delta\Delta}(y_1, y_4, y_2, y_3) \right). \end{aligned} \quad (3.53)$$

Note that the normal ordering prescription automatically removes the term in the four-point function that could have been the product of two two-point functions.

Explicit formulas for the four-point partial waves in terms of hypergeometric functions for  $d = 2, 4, 6$  can be found in [49].

So we see that the three-particle state gives rise to a beautiful and explicitly conformally invariant formula for the four-point cosmological correlator.

#### 4-particle state

For our next example, it is convenient to consider a theory with two scalar fields  $\chi_1$  and  $\chi_2$ , both in the principal-series. The reason for studying two fields rather than one is simply that it allows us to perform the relevant conformal integrals analytically as we explain at the end of the computation.

A four-particle state in the theory is,

$$|4\rangle = \int \left( \prod_{i=1}^4 dx_i \right) \delta G_4(x_1^{\Delta_1}, x_2^{\Delta_1}, x_3^{\Delta_2}, x_4^{\Delta_2}) : \bar{\chi}_1(x_1) \bar{\chi}_1(x_2) \bar{\chi}_2(x_3) \bar{\chi}_2(x_4) : |0\rangle, \quad (3.54)$$

where, for clarity, we have placed a superscript on the points to indicate the conformal weights of the function  $G_4$ . Conformal invariance restricts the form of  $\delta G_4(x_1^{\Delta_1}, x_2^{\Delta_1}, x_3^{\Delta_2}, x_4^{\Delta_2})$  up to arbitrary functions of the cross ratios. One convenient choice is to parameterize  $\delta G_4$  in terms of the conformal partial waves.

$$\delta G_4(x_1^{\Delta_1}, x_2^{\Delta_1}, x_3^{\Delta_2}, x_4^{\Delta_2}) = \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} I(\Delta) \Theta_{\Delta}^{\Delta_1\Delta_1\Delta_2\Delta_2}(x_1, x_2, x_3, x_4). \quad (3.55)$$

Here  $I(\Delta)$  is a function that parameterizes the state. We are allowed to choose  $I(\Delta)$  and different choices define different states. Since we defined the partial wave using (3.50), the normalization of  $\Theta_{\Delta}^{\Delta_1\Delta_1\Delta_2\Delta_2}$  also depends on the choice of some coefficients  $C_{\Delta_1\Delta_1\Delta}$  and  $C_{\bar{\Delta}\Delta_2\Delta_2}$ . These can always be reabsorbed into a redefinition of  $I(\Delta)$  since the only combination that can appear is the product  $I(\Delta)C_{\Delta_1\Delta_1\Delta}C_{\bar{\Delta}\Delta_2\Delta_2}$ .

Harmonic analysis on  $\text{SO}(1, d+1)$  tells us that the conformal partial waves satisfy an

orthogonality relation [48] (We denote  $\Delta = \frac{d}{2} + is$  and  $\bar{\Delta}' = \frac{d}{2} - is'$ ).

$$\begin{aligned} & \int \frac{\prod_{i=1}^4 dx_i}{\text{vol}(\text{SO}(1, d+1))} \Theta_{\Delta}^{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_2, x_3, x_4) \Theta_{\bar{\Delta}'}^{\bar{\Delta}_1 \bar{\Delta}_2 \bar{\Delta}_3 \bar{\Delta}_4}(x_1, x_2, x_3, x_4) \\ &= \frac{C_{\Delta_1 \Delta_2 \Delta} C_{\bar{\Delta}_1 \bar{\Delta}_2 \bar{\Delta}}}{\text{vol}(\text{SO}(d-1))} \frac{C_{\Delta_3 \Delta_4 \bar{\Delta}} C_{\bar{\Delta}_3 \bar{\Delta}_4 \Delta}}{\text{vol}(\text{SO}(d-1))} \frac{2\pi \delta(s-s')}{\mu(\Delta)}. \end{aligned} \quad (3.56)$$

Here  $\mu(\Delta)$  — the Plancherel measure of the representation — is given by

$$\mu(\Delta) = \frac{\Gamma[\Delta] \Gamma[\bar{\Delta}]}{\pi^d \Gamma[\frac{d-2\Delta}{2}] \Gamma[\frac{d-2\bar{\Delta}}{2}] \text{vol}(\text{SO}(d))}. \quad (3.57)$$

The norm of the state can now be easily computed using the orthogonality relation of the conformal partial waves. The relevant diagram is shown in Figure 3.7. (We use blue lines for  $\chi_1$  and red lines for  $\chi_2$ )



Figure 3.7: Norm of the 4-particle state

Applying the Feynman rules to this diagram, we find

$$\begin{aligned} (4, 4) &= (2!)^2 c_{\chi_1 \bar{\chi}_1}^2 c_{\chi_2 \bar{\chi}_2}^2 \mathcal{K} \int \left( \prod_{i=1}^4 dx_i \right) \delta G_4^*(x_1^{\Delta_1}, x_2^{\Delta_1}, x_3^{\Delta_2}, x_4^{\Delta_2}) \delta G_4(x_1^{\Delta_1}, x_2^{\Delta_1}, x_3^{\Delta_2}, x_4^{\Delta_2}) \\ &= (2!)^2 c_{\chi_1 \bar{\chi}_1}^2 c_{\chi_2 \bar{\chi}_2}^2 \text{vol}(\text{SO}(d-1)) \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} I(\Delta)^* \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta'}{2\pi i} I(\Delta') \\ &\times \int \frac{dx_1 dx_2 dx_3 dx_4}{\text{vol}(\text{SO}(1, d+1))} \left( \Theta_{\Delta}^{\Delta_1 \Delta_1 \Delta_2 \Delta_2}(x_1, x_2, x_3, x_4) \right)^* \times \left( \Theta_{\Delta'}^{\Delta_1 \Delta_1 \Delta_2 \Delta_2}(x_1, x_2, x_3, x_4) \right) \\ &= (2!)^2 \frac{c_{\chi_1 \bar{\chi}_1}^2 c_{\chi_2 \bar{\chi}_2}^2}{\text{vol}(\text{SO}(d-1))} \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{|I(\Delta)|^2}{\mu(\Delta)} |C_{\Delta_1 \Delta_1 \Delta}|^2 |C_{\bar{\Delta}_2 \Delta_2 \bar{\Delta}}|^2. \end{aligned}$$

Here, we have used the orthogonality relation of the conformal partial waves in the final step. As we can see from the final expression, one can appropriately choose  $I(\Delta) C_{\Delta_1 \Delta_1 \Delta} C_{\bar{\Delta}_2 \Delta_2 \bar{\Delta}}$  so as to get a finite norm for the state.

The Feynman diagrams for a single-field model are similar to those for the two-field model. However, these diagrams would lead to the integral of the product of two partial waves, where the points on the left are paired with the points on the right in all possible combinations. But the partial wave is only symmetric under the interchange of points  $1 \leftrightarrow 2$  and  $3 \leftrightarrow 4$ . Therefore, while such an integral can still be evaluated numerically, one cannot obtain a simple analytic answer using the orthogonality relation

above. This explains our choice of the two-field model for illustrative purposes. The same considerations guide us in the examples below.

### Cosmological correlators in the 4-particle state

The three-point cosmological correlator vanishes in the 4-particle state when the vacuum is Gaussian.

$$\langle\langle 4 | \chi_1(y_1) \chi_1(y_2) \chi_1(y_3) | 4 \rangle\rangle = 0. \quad (3.59)$$

Expanding the expression above leads to a vacuum correlator with 11 insertions which vanishes since it cannot be factorized into a product of 2-point functions.

So, we turn to the four-point function  $\langle\langle 4 | : \chi_1(y_1) \chi_1(y_2) \chi_1(y_3) \chi_1(y_4) : | 4 \rangle\rangle$ . The relevant Feynman diagram for the 4-point function is shown in Figure 3.8. Denoting its value by  $\mathcal{F}_2$ , we see that

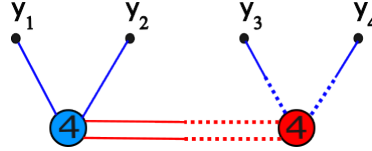


Figure 3.8: Four-point cosmological correlator in a four-particle state

$$\mathcal{F}_2 = (2!)^3 \mathfrak{c}_{\chi_1 \bar{\chi}_1}^2 \mathfrak{c}_{\chi_2 \bar{\chi}_2}^2 \mathfrak{c}_{\chi_1 \chi_1}^2 \int dx_1 dx_2 \delta G_4^*(y_1^{S[\Delta_1]}, y_2^{S[\Delta_1]}, x_1^{\Delta_2}, x_2^{\Delta_2}) \delta G_4(y_3^{\Delta_1}, y_4^{\Delta_1}, x_1^{\Delta_2}, x_2^{\Delta_2}), \quad (3.60)$$

where we have used  $S[\Delta_1], S[\Delta_2]$  to indicate that  $\delta G_4$  has been shadow transformed. Parsing this further, we find

$$\begin{aligned} \mathcal{F}_2 &= (2!)^3 \mathfrak{c}_{\chi_1 \bar{\chi}_1}^2 \mathfrak{c}_{\chi_2 \bar{\chi}_2}^2 \mathfrak{c}_{\chi_1 \chi_1}^2 \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} I(\Delta)^* \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta'}{2\pi i} I(\Delta') \\ &\quad \int dx_1 dx_2 \left( \left( \Theta_{\Delta}^{S[\Delta_1]S[\Delta_1]\Delta_2\Delta_2}(y_1, y_2, x_1, x_2) \right)^* \Theta_{\Delta'}^{\Delta_1\Delta_1\Delta_2\Delta_2}(y_3, y_4, x_1, x_2) \right) \\ &= (2!)^3 \mathfrak{c}_{\chi_1 \bar{\chi}_1}^2 \mathfrak{c}_{\chi_2 \bar{\chi}_2}^2 \mathfrak{c}_{\chi_1 \chi_1}^2 \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} |I(\Delta)|^2 \frac{|C_{\bar{\Delta}\Delta_2\Delta_2}|^2}{\text{vol}(\text{SO}(d-1))\mu(\Delta)} \\ &\quad \times \int dx \langle O^{S[\Delta_1]}(y_1) O^{S[\Delta_1]}(y_2) O^{\Delta}(x) \rangle^* \langle O^{\Delta}(x) O^{\Delta_1}(y_3) O^{\Delta_1}(y_4) \rangle \\ &= \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} |I(\Delta)|^2 \frac{|C_{\bar{\Delta}\Delta_2\Delta_2}|^2}{\text{vol}(\text{SO}(d-1))\mu(\Delta)} \mathcal{K}_2(\Delta) \Theta_{\Delta}^{\Delta_1\Delta_1\Delta_1\Delta_1}(y_1, y_2, y_3, y_4). \end{aligned} \quad (3.61)$$

Here,

$$\mathcal{K}_2(\Delta) = (2!)^3 \mathfrak{c}_{\chi_1 \bar{\chi}_1}^2 \mathfrak{c}_{\chi_2 \bar{\chi}_2}^2 \mathfrak{c}_{\chi_1 \chi_1}^2 \frac{C_{S[\Delta_1]S[\Delta_1]\Delta}^*}{C_{\Delta_1\Delta_1\bar{\Delta}}} \quad (3.62)$$

Here, we first write the conformal partial wave as an integral over the 3-point functions as given in (3.50). Then, we use the following result for integrating a product of two three-point structures, which is called a “bubble integral” [48].

$$\begin{aligned} & \int dx_1 dx_2 \langle O^{\Delta_1}(x_1) O^{\Delta_2}(x_2) O^{\Delta}(x) \rangle \langle O^{\bar{\Delta}_1}(x_1) O^{\bar{\Delta}_2}(x_2) O^{\bar{\Delta}'}(x') \rangle \\ &= \frac{C_{\Delta_1 \Delta_2 \Delta} C_{\bar{\Delta}_1 \bar{\Delta}_2 \bar{\Delta}}}{\text{vol}(\text{SO}(d-1)) \mu(\Delta)} \delta^d(x-x') 2\pi \delta(s-s'). \end{aligned} \quad (3.63)$$

The factor  $\mathcal{K}_2(\Delta)$  has various 3-point function coefficients, including those arising from shadow transforms. It ensures the proper normalisation of the conformal partial wave as given in (3.50). Also note that, once we account for the normalization of  $\Theta_{\frac{\Delta}{\Delta}}^{\Delta_1 \Delta_1 \Delta_1 \Delta_1}$ , it is only the product  $I(\Delta) C_{\Delta_1 \Delta_1 \Delta} C_{\bar{\Delta} \Delta_2 \Delta_2}$  that enters the expression as anticipated.

One can now write down the 4-point correlator, taking into account other diagrams obtained by permuting the external lines.

$$\begin{aligned} & \langle\langle 4 | : \chi_1(y_1) \chi_1(y_2) \chi_1(y_3) \chi_1(y_4) : | 4 \rangle\rangle \\ &= \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} I_4(\Delta) \left( \Theta_{\frac{\Delta}{\Delta}}^{\Delta_1 \Delta_1 \Delta_1 \Delta_1}(y_1, y_2, y_3, y_4) + y_2 \leftrightarrow y_3 + y_2 \leftrightarrow y_4 \right), \end{aligned} \quad (3.64)$$

where

$$I_4(\Delta) = |I(\Delta)|^2 \frac{|C_{\bar{\Delta} \Delta_2 \Delta_2}|^2}{\text{vol}(\text{SO}(d-1)) \mu(\Delta)} \mathcal{K}_2(\Delta). \quad (3.65)$$

So the four-point cosmological correlator is also given in terms of conformal partial waves, with the coefficient function  $I_4(\Delta)$  being determined in terms of the parameters defining the state.

### 3 + 4-particle state

In the previous sections, we computed four-point cosmological correlator in a simple 3-particle state and a 4-particle state. Both these states have a vanishing three-point correlator as the vacuum is Gaussian. It is possible to obtain a non-zero three-point function in a state that is a linear combination of 3- and 4-particle states.

$$\begin{aligned} |\psi\rangle &= a|3\rangle + b|4\rangle \\ |3\rangle &= \int \left( \prod_{i=1}^3 dx_i \right) \delta G_3(x_1^{\Delta_1}, x_2^{\Delta_2}, x_3^{\Delta_2}) : \bar{\chi}_1(x_1) \bar{\chi}_2(x_2) \bar{\chi}_2(x_3) : |0\rangle \\ |4\rangle &= \int \left( \prod_{i=1}^4 dx_i \right) \delta G_4(x_1^{\Delta_1}, x_2^{\Delta_1} x_3^{\Delta_2}, x_4^{\Delta_2}) : \bar{\chi}_1(x_1) \bar{\chi}_1(x_2) \bar{\chi}_2(x_3) \bar{\chi}_2(x_4) : |0\rangle. \end{aligned} \quad (3.66)$$

Here, we have taken a two-field model and defined states with some foresight so as to obtain an analytic answer. The form of the three-point function is fixed by conformal invariance, and its coefficient can be computed numerically in more general states, including a single-field model.

As in the previous two examples,  $\delta G_3$  and  $\delta G_4$  can be written as

$$\begin{aligned}\delta G_3(x_1^{\Delta_1}, x_2^{\Delta_2}, x_3^{\Delta_2}) &= \langle O^{\Delta_1}(x_1) O^{\Delta_2}(x_2) O^{\Delta_2}(x_3) \rangle = \frac{C_{\Delta_1 \Delta_2 \Delta_2}}{|x_1 - x_2|^{\Delta_1} |x_1 - x_3|^{\Delta_1} |x_2 - x_3|^{2\Delta_2 - \Delta_1}} \\ \delta G_4(x_1^{\Delta_1}, x_2^{\Delta_1} x_3^{\Delta_2}, x_4^{\Delta_2}) &= \int_{\frac{d}{2}}^{\frac{d}{2} + i\infty} \frac{d\Delta}{2\pi i} I(\Delta) \Theta_{\Delta}^{\Delta_1 \Delta_1 \Delta_2 \Delta_2}(x_1, x_2, x_3, x_4).\end{aligned}\quad (3.67)$$

We can first compute the norm of the state  $|\psi\rangle$ . Since the vacuum is Gaussian, there are no cross terms of the form (3,4), and we get

$$(\psi, \psi) = |a|^2 (3,3) + |b|^2 (4,4). \quad (3.68)$$

The computation of  $\langle 3|3\rangle$  and  $\langle 4|4\rangle$  follows in the exact same manner as we have done in the previous sections. We will quote the answer below.

$$\begin{aligned}(\psi, \psi) &= 2! |a|^2 c_{\chi_2 \bar{\chi}_2}^2 c_{\chi_1 \bar{\chi}_1} |C_{\Delta_1 \Delta_2 \Delta_2}|^2 \\ &\quad + |b|^2 \left( (2!)^2 \frac{c_{\chi_1 \bar{\chi}_1}^2 c_{\chi_2 \bar{\chi}_2}^2}{\text{vol}(\text{SO}(d-1))} \int_{\frac{d}{2}}^{\frac{d}{2} + i\infty} \frac{d\Delta}{2\pi i} \frac{|I(\Delta)|^2}{\mu(\Delta)} |C_{\Delta_1 \Delta_1 \Delta}|^2 |C_{\bar{\Delta} \Delta_2 \Delta_2}|^2 \right).\end{aligned}\quad (3.69)$$

We now move on to the computation of the 3-point function. This will get contributions only from the cross terms. More explicitly,

$$\langle\langle \psi | \chi(y_1) \chi(y_2) \chi(y_3) | \psi \rangle\rangle = a^* b \langle\langle 3 | \chi(y_1) \chi(y_2) \chi(y_3) | 4 \rangle\rangle + b^* a \langle\langle 4 | \chi(y_1) \chi(y_2) \chi(y_3) | 3 \rangle\rangle. \quad (3.70)$$

Second, conformal invariance forces

$$\langle\langle \psi | \chi(y_1) \chi(y_2) \chi(y_3) | \psi \rangle\rangle = \frac{c}{|y_1 - y_2|^{\Delta} |y_1 - y_3|^{\Delta} |y_2 - y_3|^{\Delta}}. \quad (3.71)$$

Our task is to determine  $c$  in terms of the parameters that define the state. As a check we see that only the combinations  $a C_{\Delta_1 \Delta_2 \Delta_2}$  and  $I(\Delta) C_{\Delta_1 \Delta_1 \Delta} C_{\bar{\Delta} \Delta_2 \Delta_2}$  can appear in the answer.

The diagram that captures the term  $\langle\langle 3 | \chi_1(y_1) \chi_1(y_2) \chi_1(y_3) | 4 \rangle\rangle$  is shown in Figure 3.9. Denoting its value by  $\mathcal{F}_3$  we have,

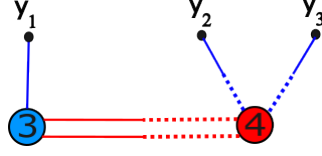


Figure 3.9: A contribution to the 3-point function in the 3+4-particle state

$$\begin{aligned}
\mathcal{F}_3 &= (2!)^2 \mathfrak{c}_{\chi_1 \bar{\chi}_1}^2 \mathfrak{c}_{\chi_2 \bar{\chi}_2}^2 \mathfrak{c}_{\chi_1 \chi_1} \int dx_1 dx_2 \delta G_3^*(y_1^{S[\Delta_1]}, x_1^{\Delta_2}, x_2^{\Delta_2}) \delta G_4(y_2^{\Delta_1}, y_3^{\Delta_1}, x_1^{\Delta_2}, x_2^{\Delta_2}) \\
&= (2!)^2 \mathfrak{c}_{\chi_1 \bar{\chi}_1}^2 \mathfrak{c}_{\chi_2 \bar{\chi}_2}^2 \mathfrak{c}_{\chi_1 \chi_1} \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} I(\Delta) \\
&\times \left( \int dx_1 dx_2 \langle O^{S[\Delta_1]}(y_1) O^{\Delta_2}(x_1) O^{\Delta_2}(x_2) \rangle^* \Theta_{\Delta}^{\Delta_1 \Delta_1 \Delta_2 \Delta_2}(y_2, y_3, x_1, x_2) \right) \\
&= \frac{\mathcal{K}_3}{|y_1 - y_2|^{\Delta_1} |y_1 - y_3|^{\Delta_1} |y_2 - y_3|^{\Delta_1}}.
\end{aligned} \tag{3.72}$$

Here, we have used the bubble integral formula (3.63) to integrate over  $x_1$  and  $x_2$  and arrive at the 3rd equality. In the final step, the numerical factor is given by

$$\mathcal{K}_3 = \frac{(2!)^2 \mathfrak{c}_{\chi_1 \bar{\chi}_1}^2 \mathfrak{c}_{\chi_2 \bar{\chi}_2}^2 \mathfrak{c}_{\chi_1 \chi_1} I(\Delta_1) C_{S[\Delta_1] \Delta_2 \Delta_2}^* C_{\bar{\Delta}_1 \Delta_2 \Delta_2} C_{\Delta_1 \Delta_1 \Delta_1}}{\text{vol}(\text{SO}(d-1)) \mu(\Delta_1)}. \tag{3.73}$$

The diagram that captures the term  $\langle 4 | \chi_1(y_1) \chi_1(y_2) \chi_1(y_3) | 3 \rangle$  is shown in Figure 3.10. Denoting its value by  $\mathcal{F}_4$ , we have

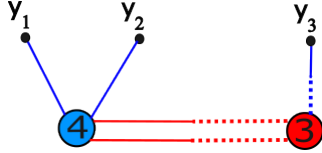


Figure 3.10: Another contribution to the 3-point function in the 3+4-particle state

$$\begin{aligned}
F_4 &= (2!)^2 \mathfrak{c}_{\chi_1 \chi_1}^2 \mathfrak{c}_{\chi_2 \bar{\chi}_2}^2 \mathfrak{c}_{\chi_1 \bar{\chi}_1} \int dx_1 dx_2 \delta G_4^*(y_1^{S[\Delta_1]}, y_2^{S[\Delta_1]}, x_1^{\Delta_2}, x_2^{\Delta_2}) \delta G_3(y_3^{\Delta_1}, x_1^{\Delta_2}, x_2^{\Delta_2}) \\
&= (2!)^2 \mathfrak{c}_{\chi_1 \chi_1}^2 \mathfrak{c}_{\chi_2 \bar{\chi}_2}^2 \mathfrak{c}_{\chi_1 \bar{\chi}_1} \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} (I(\Delta))^* \\
&\times \left( \int dx_1 dx_2 \left( \Theta_{\Delta}^{S[\Delta_1] S[\Delta_1] \Delta_2 \Delta_2}(y_1, y_2, x_1, x_2) \right)^* \delta G_3(y_3^{\Delta_1}, x_1^{\Delta_2}, x_2^{\Delta_2}) \right) \\
&= \frac{\mathcal{K}_4}{|y_1 - y_2|^{\Delta_1} |y_1 - y_3|^{\Delta_1} |y_2 - y_3|^{\Delta_1}}.
\end{aligned} \tag{3.74}$$

Here,

$$\mathcal{K}_4 = \frac{(2!)^2 \mathfrak{c}_{\chi_1 \chi_1}^2 \mathfrak{c}_{\chi_2 \bar{\chi}_2}^2 \mathfrak{c}_{\chi_1 \bar{\chi}_1} I(\bar{\Delta}_1)^* |C_{\Delta_1 \Delta_2 \Delta_2}|^2 C_{S[\Delta_1] S[\Delta_1] \bar{\Delta}_1}^*}{\text{vol}(\text{SO}(d-1)) \mu(\Delta_1)}. \tag{3.75}$$

Adding these two contributions and taking into account other diagrams which are related to these by permutation of the external legs we get,

$$\begin{aligned} \langle\langle\psi|\chi_1(y_1)\chi_1(y_2)\chi_1(y_3)|\psi\rangle\rangle &= \frac{c}{|y_1 - y_2|^{\Delta_1}|y_1 - y_3|^{\Delta_1}|y_2 - y_3|^{\Delta_1}} \\ c &= 3 \times (a^*b\mathcal{K}_3 + b^*a\mathcal{K}_4). \end{aligned} \quad (3.76)$$

Here again, we see that  $c$  is given purely in terms of the data defining the state. This example suggests that even if the vacuum has a vanishing 3-point function, an appropriately chosen ‘excited’ state can have large non-Gaussianity.

### 3.4.2 Cosmological correlators in the presence of a non-Gaussian vacuum

In the previous examples, we took the vacuum to be Gaussian. In this subsection, we study a case where the vacuum wavefunction has an interaction term, and the leading contribution to the correlator comes from this interaction.

We now introduce a third field  $\chi_3$ , which is again motivated by our desire to obtain explicit answers. Consider a state  $|\psi\rangle$  defined as follows.

$$\begin{aligned} |\psi\rangle &= a|3\rangle + b|4\rangle \\ |3\rangle &= \int dx_1 dx_2 dx_3 \delta G_3(x_1^{\Delta_1}, x_2^{\Delta_2}, x_3^{\Delta_2}) : \bar{\chi}_1(x_1) \bar{\chi}_2(x_2) \bar{\chi}_2(x_3) : |0\rangle \\ |4\rangle &= \int dx_1 dx_2 dx_3 dx_4 \delta G_4(x_1^{\Delta_1}, x_2^{\Delta_1}, x_3^{\Delta_3}, x_4^{\Delta_3}) : \bar{\chi}_1(x_1) \bar{\chi}_1(x_2) \bar{\chi}_3(x_3) \bar{\chi}_3(x_4) : |0\rangle \\ \delta G_3(x_1^{\Delta_1}, x_2^{\Delta_2}, x_3^{\Delta_2}) &= \frac{C_{\Delta_1 \Delta_2 \Delta_2}}{|x_1 - x_2|^{\Delta_1} |x_1 - x_3|^{\Delta_1} |x_2 - x_3|^{2\Delta_2 - \Delta_1}} \\ \delta G_4(x_1^{\Delta_1}, x_2^{\Delta_1}, x_3^{\Delta_3}, x_4^{\Delta_3}) &= \int_{\frac{d}{2}}^{\frac{d}{2} + i\infty} \frac{d\Delta}{2\pi i} I(\Delta) \Theta_{\Delta}^{\Delta_1 \Delta_1 \Delta_3 \Delta_3}(x_1, x_2, x_3, x_4). \end{aligned} \quad (3.77)$$

We take the vacuum to have a four-point interaction term as given below. In the notation of (3.21), this means that

$$\Psi_0[\chi] = \Psi_{\text{Gauss}} \exp \left\{ \lambda^2 \int \prod_{i=1}^4 dx_i G_4(x_1^{\Delta_2}, x_2^{\Delta_2}, x_3^{\Delta_3}, x_4^{\Delta_3}) \bar{\chi}_2(x_1) \bar{\chi}_2(x_2) \bar{\chi}_3(x_3) \bar{\chi}_3(x_4) \right\}, \quad (3.78)$$

where  $\Psi_{\text{Gauss}}$  comprises the Gaussian 2-point terms that give rise to the propagators above (see (B.68)), and we have shown the four-point term explicitly. We take  $G_4$  to be of the form

$$G_4(x_1^{\Delta_2}, x_2^{\Delta_2}, x_3^{\Delta_3}, x_4^{\Delta_3}) = \int_{\frac{d}{2}}^{\frac{d}{2} + i\infty} \frac{d\Delta}{2\pi i} L(\Delta) \Theta_{\Delta}^{\Delta_2 \Delta_2 \Delta_3 \Delta_3}(x_1, x_2, x_3, x_4), \quad (3.79)$$

where  $\lambda$  is a perturbative parameter. Admittedly, this non-Gaussianity is slightly contrived since we have chosen a term that couples  $\bar{\chi}_2$  to  $\bar{\chi}_3$  but set other non-Gaussianities to zero. This choice is made for the purpose of obtaining a simple analytic answer. We invite the reader to generalize this to other cases.

The normal ordering in (3.77) can be defined using the same subtractions as in the free-field case. This is sufficient to ensure the conditions of Section 3.3 at lowest order in perturbation theory.

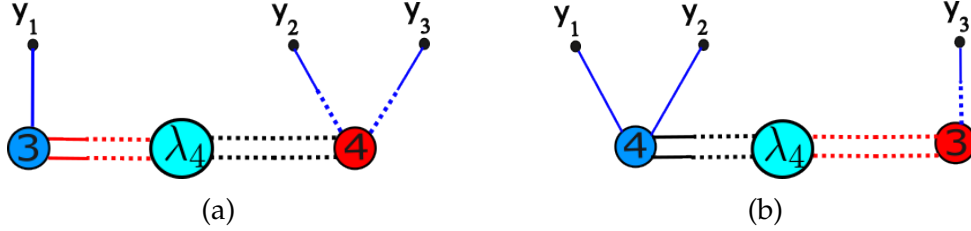


Figure 3.11: Feynman diagrams for the 3-point function in a 3+4-particle state built on a non-Gaussian vacuum.

We will compute  $\langle\langle\psi|\chi_1(y_1)\chi_1(y_2)\chi_1(y_3)|\psi\rangle\rangle$ . Without the non-Gaussianity shown above, this quantity would have vanished. In the presence of the non-Gaussianity, the two relevant diagrams are shown in Figure 3.11. (We use blue for  $\chi_1$ , red for  $\chi_2$  and black for  $\chi_3$ ). Denoting the value of the diagram in Figure 3.11a by  $\mathcal{F}_5$  we find

$$\begin{aligned}
\mathcal{F}_5 &= \lambda^2 (2!)^3 c_{\chi_1 \bar{\chi}_1}^2 c_{\chi_2 \bar{\chi}_2}^2 c_{\chi_3 \bar{\chi}_3}^2 c_{\chi_1 \chi_1} \int \prod_{i=1}^4 dx_i \delta G_3^*(y_1^{S[\Delta_1]}, x_1^{\Delta_2}, x_2^{\Delta_2}) G_4(x_1^{\Delta_2}, x_2^{\Delta_2}, x_3^{\Delta_3}, x_4^{\Delta_3}) \\
&\quad \times \delta G_4(y_2^{\Delta_1}, y_3^{\Delta_1}, x_3^{S[\Delta_3]}, x_4^{S[\Delta_3]}) \\
&= \lambda^2 (2!)^3 c_{\chi_1 \bar{\chi}_1}^2 c_{\chi_2 \bar{\chi}_2}^2 c_{\chi_3 \bar{\chi}_3}^2 c_{\chi_1 \chi_1} \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} L(\Delta) \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta'}{2\pi i} I(\Delta') \\
&\quad \times \int \prod_{i=1}^4 dx_i \delta G_3^*(y_1^{S[\Delta_1]}, x_1^{\Delta_2}, x_2^{\Delta_2}) \Theta_{\Delta}^{\Delta_2 \Delta_2 \Delta_3 \Delta_3}(x_1, x_2, x_3, x_4) \Theta_{\Delta'}^{\Delta_1 \Delta_1 S[\Delta_3] S[\Delta_3]}(y_2, y_3, x_3, x_4) \\
&= \lambda^2 \frac{\mathcal{K}_5}{|y_1 - y_2|^{\Delta_1} |y_1 - y_3|^{\Delta_1} |y_2 - y_3|^{\Delta_1}}.
\end{aligned} \tag{3.80}$$

The  $x_i$  integrals can be done using the usual techniques. First, we write the conformal partial waves in their integral representation (3.50) and then use the bubble integral formula (3.63). In the final step, the numerical factor is given by

$$\mathcal{K}_5 = \frac{(2!)^3 c_{\chi_1 \bar{\chi}_1}^2 c_{\chi_2 \bar{\chi}_2}^2 c_{\chi_3 \bar{\chi}_3}^2 c_{\chi_1 \chi_1} L(\bar{\Delta}_1) I(\Delta_1) C_{S[\Delta_1] \Delta_2 \Delta_2}^* C_{\Delta_2 \Delta_2 \bar{\Delta}_1} C_{\Delta_1 \Delta_3 \Delta_3} C_{\Delta_1 \Delta_1 \Delta_1} C_{\bar{\Delta}_1 S[\Delta_3] S[\Delta_3]}}{(\text{vol}(\text{SO}(d-1)) \mu(\Delta_1))^2}. \tag{3.81}$$

Note that the 3-point function has the required functional form. The multiplicative constant  $\mathcal{K}_5$  depends on the interaction vertex as well as the state.

Denoting the value of the diagram in Figure 3.11b by  $\mathcal{F}_6$  we have

$$\begin{aligned}
\mathcal{F}_6 &= \lambda^2 (2!)^3 \mathbf{c}_{\chi_1 \chi_1}^2 \mathbf{c}_{\bar{\chi}_2 \bar{\chi}_2}^2 \mathbf{c}_{\chi_3 \bar{\chi}_3}^2 \mathbf{c}_{\chi_1 \bar{\chi}_1} \int \prod_{i=1}^4 dx_i \delta G_3(y_3^{\Delta_1}, x_1^{S[\Delta_2]}, x_2^{S[\Delta_2]}) G_4(x_1^{\Delta_2}, x_2^{\Delta_2}, x_3^{\Delta_3}, x_4^{\Delta_3}) \\
&\times \delta G_4^*(y_1^{S[\Delta_1]}, y_2^{S[\Delta_1]}, x_4^{\Delta_3}) \\
&= \lambda^2 (2!)^3 \mathbf{c}_{\chi_1 \chi_1}^2 \mathbf{c}_{\bar{\chi}_2 \bar{\chi}_2}^2 \mathbf{c}_{\chi_3 \bar{\chi}_3}^2 \mathbf{c}_{\chi_1 \bar{\chi}_1} \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} L(\Delta) \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta'}{2\pi i} I(\Delta')^* \left[ \right. \\
&\left. \int \prod_{i=1}^4 dx_i \delta G_3(y_3^{\Delta_1}, x_1^{S[\Delta_2]}, x_2^{S[\Delta_2]}) \Theta_{\Delta}^{\Delta_2 \Delta_2 \Delta_3 \Delta_3}(x_1, x_2, x_3, x_4) \left( \Theta_{\Delta'}^{S[\Delta_1] S[\Delta_1] \Delta_3 \Delta_3}(y_1, y_2, x_3, x_4) \right)^* \right] \quad (3.82) \\
&= \lambda^2 \frac{\mathcal{K}_6}{|y_1 - y_2|^{\Delta_1} |y_1 - y_3|^{\Delta_1} |y_2 - y_3|^{\Delta_1}}.
\end{aligned}$$

Here,

$$\mathcal{K}_6 = \frac{(2!)^3 \mathbf{c}_{\chi_1 \chi_1}^2 \mathbf{c}_{\bar{\chi}_2 \bar{\chi}_2}^2 \mathbf{c}_{\chi_3 \bar{\chi}_3}^2 \mathbf{c}_{\chi_1 \bar{\chi}_1} L(\bar{\Delta}_1) I(\bar{\Delta}_1)^* C_{\Delta_1 S[\Delta_2] S[\Delta_2]} C_{\Delta_2 \Delta_2 \bar{\Delta}_1} |C_{\Delta_1 \Delta_3 \Delta_3}|^2 C_{S[\Delta_1] S[\Delta_1] \bar{\Delta}_1}^*}{(\text{vol}(\text{SO}(d-1)) \mu(\Delta_1))^2}. \quad (3.83)$$

One can multiply by a factor of 3 to take into account other diagrams obtained after permuting  $y_1, y_2$  and  $y_3$ . One also has to consider diagrams arising from the conjugate of the interaction vertex (contributed by  $\Psi^*[\bar{\chi}]$ ). These diagrams are shown in Figure 3.12.

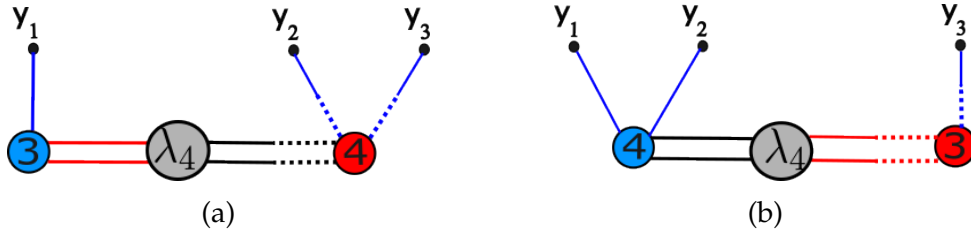


Figure 3.12: Additional Feynman diagrams for the 3-point cosmological correlator in a 3+4-particle state built on an interacting vacuum

The calculation of these Feynman diagrams is identical to the previous calculations, and we will not repeat it here. They evaluate to

$$\lambda^2 \frac{\mathcal{K}_7}{|y_1 - y_2|^{\Delta_1} |y_1 - y_3|^{\Delta_1} |y_2 - y_3|^{\Delta_1}},$$

and

$$\lambda^2 \frac{\mathcal{K}_8}{|y_1 - y_2|^{\Delta_1} |y_1 - y_3|^{\Delta_1} |y_2 - y_3|^{\Delta_1}}$$

respectively, with the constants given by

$$\mathcal{K}_7 = \frac{(2!)^3 \mathfrak{c}_{\chi_1 \bar{\chi}_1}^2 \mathfrak{c}_{\chi_3 \bar{\chi}_3}^2 \mathfrak{c}_{\chi_2 \chi_2}^2 \mathfrak{c}_{\chi_1 \chi_1} L(\Delta_1)^* I(\Delta_1) C_{S[\Delta_1]S[\Delta_2]S[\Delta_2]}^* C_{\Delta_2 \Delta_2 \Delta_1}^* \left| C_{\bar{\Delta}_1 \Delta_3 \Delta_3} \right|^2 C_{\Delta_1 \Delta_1 \Delta_1}}{(\text{vol}(\text{SO}(d-1))\mu(\Delta_1))^2}, \quad (3.84)$$

and

$$\mathcal{K}_8 = \frac{(2!)^3 \mathfrak{c}_{\chi_1 \chi_1}^2 \mathfrak{c}_{\chi_2 \bar{\chi}_2}^2 \mathfrak{c}_{\chi_3 \chi_3}^2 \mathfrak{c}_{\chi_1 \bar{\chi}_1} L(\Delta_1)^* I(\bar{\Delta}_1)^* |C_{\Delta_1 \Delta_2 \Delta_2}|^2 C_{\bar{\Delta}_1 \Delta_3 \Delta_3}^* C_{\Delta_1 S[\Delta_3]S[\Delta_3]}^* C_{S[\Delta_1]S[\Delta_1]\bar{\Delta}_1}^*}{(\text{vol}(\text{SO}(d-1))\mu(\Delta_1))^2} \quad (3.85)$$

So, the  $O(\lambda^2)$  result for the 3-point function is given by,

$$\langle\langle \psi | \chi_1(y_1) \chi_1(y_2) \chi_1(y_3) | \psi \rangle\rangle = 3\lambda^2 \frac{a^* b (\mathcal{K}_5 + \mathcal{K}_7) + b^* a (\mathcal{K}_6 + \mathcal{K}_8)}{|y_1 - y_2|^{\Delta_1} |y_1 - y_3|^{\Delta_1} |y_2 - y_3|^{\Delta_1}}. \quad (3.86)$$

## 3.5 Relational observables

In the previous sections, we have studied cosmological correlators of “simple operators” in simple states. We found that these correlators do not match the vacuum-expectation values of QFT operators. In this section, we show that in the presence of a heavy background state, which can be thought of as an observer, it is possible to construct relational observables whose correlators correspond to QFT vacuum-expectation values to good accuracy. Our construction utilizes the techniques initially developed in [25, 32] and recently utilized in [27–29]. In Appendix B.2.1, we provide a quick review of the construction of relational observables in AdS/CFT. We urge readers more familiar with that setting to scan Appendix B.2.1 to gain intuition about the construction below. The study of relational observables has a long history going back to DeWitt [50]. (See [51–53] and references in those papers for discussion from various perspectives.) Related recent work in the AdS context may be found in [54–57]. The dS construction of [58] is also related, although the objective there was to construct observables in a static patch, whereas here we are interested in the late-time slice.

### 3.5.1 Simple observables

We will construct operators whose expectation values and variances will match those of simple QFT observables in the vacuum. By simple QFT observables, we mean low-point polynomials in the light fields of the theory that are localized to a given patch of the late-time slice.

More explicitly, let  $\chi(x)$  be the late-time value of a matter field. As above, this field

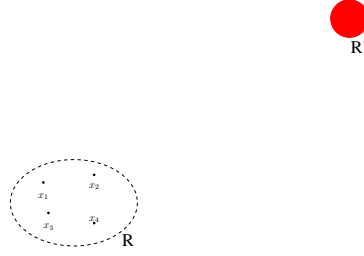


Figure 3.13: A heavy background excitation (shown as a red blob) in the region  $R'$  allows us to define relational observables that mimic QFT observables in the region  $R$  far from the background excitation.

can be in the principal series or the complementary series. Consider the set of QFT observables

$$\mathcal{A}_{\text{light}} = \text{span}\{\chi(x_1), \chi(x_1)\chi(x_2), \dots, \chi(x_1) \dots \chi(x_n)\}, \quad (3.87)$$

where we impose the cutoff  $n < n_c$ . The cutoff cannot be taken to be “too large” and we make this more precise below. Furthermore, we take all the points  $x_i$  to be localized in a given region  $R$  on the late-time slice. See Figure 3.13.

To the extent that we can ignore this cutoff, the set  $\mathcal{A}_{\text{light}}$  is also an algebra in that one can multiply elements of  $\mathcal{A}_{\text{light}}$  to obtain other elements in  $\mathcal{A}_{\text{light}}$ . This is precisely the sense in which local bulk algebras appear in the boundary theory — a discussion that was initiated in [59] and has later been extended in the language of von Neumann algebras following [60]. In this chapter, we do not explore this algebraic structure further.

We denote the QFT expectation values of these operators in the Euclidean vacuum state by

$$\langle a \rangle \equiv \langle 0|a|0 \rangle, \quad (3.88)$$

with  $a \in \mathcal{A}_{\text{light}}$ .

**Statement of objective.** The setup above allows us to precisely state the objective of our construction. We seek to construct a de Sitter invariant state,  $\Psi_{\text{obs}}$ , and a map,  $a \rightarrow \hat{a}$ , that maps each element  $a \in \mathcal{A}_{\text{light}}$  to a de Sitter-invariant operator  $\hat{a}$  with the property that

$$(\Psi_{\text{obs}}, \hat{a}\Psi_{\text{obs}}) = \langle a \rangle. \quad (3.89)$$

The relation above is expected to hold at leading order in perturbation theory in  $G_N$ . The map will be state-dependent in that it will depend on the background state  $\Psi_{\text{obs}}$ .

### 3.5.2 QFT background state

Next, we require a quantum field theory state that can serve as an appropriate background. Upon group averaging, this state will give rise to a dS invariant “observer”.

The QFT background state that we require must have two important properties. First, we require the state to have the property that

$$\langle \Psi_{\text{back}} | a | \Psi_{\text{back}} \rangle \approx \langle 0 | a | 0 \rangle, \quad (3.90)$$

where  $|0\rangle$  is the Hartle-Hawking state. Here, the  $\approx$  sign means that the correlators are equal to high accuracy. This accuracy will be one of the determinants of the accuracy of our construction.

Physically, this condition is easy to achieve. Recall that the elements of  $\mathcal{A}_{\text{light}}$  are all centered in a particular region  $R$ . If the background state has excitations in another part of the late-time slice, then we expect that the presence of the excitations will not affect probes in  $R$ .

Next, recall that, on the quantum field theory Hilbert space, the de Sitter group can be realized via unitary operators,  $U$ , that map a field operator on the late-time slice to another operator also on the late-time slice. We require the QFT background state to satisfy the property that if  $a \in \mathcal{A}_{\text{light}}$  then if  $U$  is outside of a small set,  $\Sigma \subset \text{SO}(1, d+1)$  near the identity, we have

$$|\langle \Psi_{\text{back}} | Ua | \Psi_{\text{back}} \rangle| \ll 1, \quad U \notin \Sigma. \quad (3.91)$$

The basic idea is to consider a state made up of a large number of well-localized elementary excitations. The action of any element of the de Sitter group (except for elements that are parametrically close to the identity) moves these elementary excitations around. This produces a state that is orthogonal to the original state.

The state must be “heavy” since it must have a large number of excitations to satisfy (3.91). This does not require the state to have enough localized energy density to back-react substantially. As explained in [29], even a single virion is heavy enough to meet the condition (3.91) but too small to warp the spacetime appreciably.

In Appendix B.2.2, we provide an explicit example of a state satisfying (3.91) and (3.90) in the free-field approximation. We do not see any obstacle to finding a state that satisfies these properties in the interacting theory. Going forward, we will simply assume that such a state has been found.

### 3.5.3 Auxiliary smeared quantities

The last technical step that is required is to define some auxiliary smeared versions of the little Hilbert space and operators. Their utility will be apparent below.

First, we choose a set  $\Sigma' \subset \text{SO}(1, d+1)$  which is also centered about the identity. We want  $\Sigma'$  to have the property that it is much larger than  $\Sigma$  but still corresponds to small elements of the conformal group in the sense that

$$\langle aUbU^\dagger \rangle \approx \langle ab \rangle, \quad U \in \Sigma'. \quad (3.92)$$

In what follows, we will neglect terms that are of size  $\frac{\text{vol}(\Sigma)}{\text{vol}(\Sigma')}$ .

We now define a smeared little Hilbert space about the original QFT state as

$$\mathcal{H}_{\text{sm}} = \text{span}\{aU|\Psi_{\text{back}}\rangle, \quad a \in \mathcal{A}_{\text{light}}, U \in \Sigma'\}. \quad (3.93)$$

Note that, by the Reeh-Schlieder property of ordinary quantum fields in de Sitter space [61], this space is dense in the space

$$\mathcal{H}'_{\text{sm}} = \text{span}\{Ua|\Psi_{\text{back}}\rangle, \quad a \in \mathcal{A}_{\text{light}}, U \in \Sigma'\}. \quad (3.94)$$

We now define the projector  $P$  to be the projector onto the space  $\mathcal{H}_{\text{sm}}$ .

Next, for every operator  $a$  we define a “smeared” operator

$$\bar{a} = \frac{1}{\text{vol}(\Sigma')} \int_{\Sigma'} UaU^\dagger dU, \quad (3.95)$$

If  $a$  contains some local operators, then the conformal transformations move these operators about by a small amount. To the extent that all elements of  $\Sigma'$  are close to the identity, we expect that insertions of  $\bar{a}$  are indistinguishable from insertions of  $a$  within expectation values, and we will assume this below.

Finally, we define a slightly smeared version of the state  $|\Psi_{\text{back}}\rangle$  as

$$|\psi_{\text{back}}\rangle = \frac{\mathcal{N}}{\sqrt{\text{vol}(\Sigma')}} \int_{\Sigma'} dUU|\Psi_{\text{back}}\rangle, \quad (3.96)$$

where the normalization constant  $\mathcal{N}$  is chosen as

$$\mathcal{N}^{-2} = \int \langle \Psi_{\text{back}} | U | \Psi_{\text{back}} \rangle dU. \quad (3.97)$$

This ensures that

$$\langle \psi_{\text{back}} | \psi_{\text{back}} \rangle = \frac{\mathcal{N}^2}{\text{vol}(\Sigma')} \int_{\Sigma'} dU \int_{\Sigma'} dV \langle \Psi_{\text{back}} | UV | \Psi_{\text{back}} \rangle = 1. \quad (3.98)$$

In the last line, we note that for any  $V \in \Sigma'$  we must have  $UV \in \Sigma$  for the inner product to be appreciable. Therefore, the inner product vanishes unless  $U$  is in a small open set about  $V^\dagger$ . Hence, up to edge-effects of size  $\frac{\text{vol}(\Sigma)}{\text{vol}(\Sigma')}$ , we can extend the integral over  $U$  to the entire group. We then use the invariance of the Haar measure to redefine  $U \rightarrow UV$  so that the integrand is independent of  $V$ . The integral over  $V$  cancels the factor of  $\text{vol}(\Sigma')$  in the denominator, and the remaining integral over  $U$  cancels the factor of  $\mathcal{N}^2$ .

Physically, we can think of the state  $|\psi_{\text{back}}\rangle$  as a superposition of states where the background density has been “moved around” a little by conformal transformations. Using the same steps as above, we see that

$$\langle \psi_{\text{back}} | a | \psi_{\text{back}} \rangle = \mathcal{N}^2 \int \langle \Psi_{\text{back}} | Ua | \Psi_{\text{back}} \rangle dU. \quad (3.99)$$

Since we are still focused on insertions far away from the background, it is natural to expect that

$$\langle \psi_{\text{back}} | a | \psi_{\text{back}} \rangle = \langle a \rangle. \quad (3.100)$$

This leads us to the conclusion that

$$\mathcal{N}^2 \int \langle \Psi_{\text{back}} | Ua | \Psi_{\text{back}} \rangle dU = \langle a \rangle \quad (3.101)$$

This shows that the normalization factor chosen to normalize the state in (3.97) is also the correct normalization factor to use in (3.101). The utility of defining  $|\psi_{\text{back}}\rangle$  is that it allows us to give a physical proof of this result. If one has a more macroscopic description for  $|\Psi_{\text{back}}\rangle$ , then it might be possible to prove (3.91) and (3.101) directly as, for instance, was done in section 7.6 of [26]. We will not need  $|\psi_{\text{back}}\rangle$  again.

### 3.5.4 Observer state and relational observables

We now have all the ingredients to construct “relational observables” whose correlators in an appropriate “observer state” are like correlators of simple QFT operators in the Hartle-Hawking state.

For any QFT operator  $a \in \mathcal{A}_{\text{light}}$ , we define

$$\hat{a} = \frac{1}{\text{vol}(\Sigma')} \int VaPV^\dagger dV. \quad (3.102)$$

We also define the state

$$|\Psi_{\text{obs}}\rangle = \mathcal{N}_1 \int dU U |\Psi_{\text{back}}\rangle. \quad (3.103)$$

with  $\mathcal{N}_1 = \mathcal{N}(\text{vol}(\text{SO}(d-1)))^{-\frac{1}{2}}$ . It is manifest that  $\hat{a}$  is a dS invariant operator and  $|\Psi_{\text{obs}}\rangle$  is a dS-invariant state since both the operator and the state are group averaged.

We also note that the state  $|\Psi_{\text{obs}}\rangle$  is normalized correctly since

$$\begin{aligned} (\Psi_{\text{obs}}, \Psi_{\text{obs}}) &= \mathcal{N}^2 \frac{1}{\text{vol}(\text{SO}(1, d+1))} \int \langle \Psi_{\text{back}} | U^\dagger V | \Psi_{\text{back}} \rangle dU dV \\ &= \mathcal{N}^2 \int \langle \Psi_{\text{back}} | V | \Psi_{\text{back}} \rangle dV = 1, \end{aligned} \quad (3.104)$$

In going from the first line to the second line, we use the invariance of the Haar measure to redefine  $V \rightarrow U^\dagger V$ . The integral over  $U$  then cancels the volume of  $\text{SO}(1, d+1)$ , leaving behind an integral that manifestly cancels the normalization factor.

We now show that expectation values of the operators  $\hat{a}$  on the state  $|\Psi_{\text{obs}}\rangle$  yield the vacuum expectation values  $\langle a \rangle$ . First note that

$$\begin{aligned} \hat{a} |\Psi_{\text{obs}}\rangle &= \frac{\mathcal{N}_1}{\text{vol}(\Sigma')} \int V a P V^\dagger U |\Psi_{\text{back}}\rangle dV dU \\ &= \frac{\mathcal{N}_1}{\text{vol}(\Sigma')} \int U W a P W^\dagger |\Psi_{\text{back}}\rangle dW dU \\ &= \mathcal{N}_1 \int U \bar{a} |\Psi_{\text{back}}\rangle dU \end{aligned} \quad (3.105)$$

Here we first defined  $V = UW$  and so  $V^\dagger = W^\dagger U^\dagger$ . There is no Jacobian factor due to the invariance of the Haar measure. Next, we note that the projector restricts the range of the  $W$  integral to  $W \in \Sigma'$ . This restriction is *not* exact but, by (3.91), it is valid up to terms of size  $\mathcal{O}\left(\frac{\text{vol}\Sigma}{\text{vol}\Sigma'}\right)$ . In words, acting with the hatted operator on the group-averaged state is the same as acting with the smeared operator on the original state and then group averaging the result.

Therefore,

$$\begin{aligned} (\Psi_{\text{obs}}, \hat{a} \Psi_{\text{obs}}) &= \frac{\mathcal{N}^2}{\text{vol}(\text{SO}(1, d+1))} \int dU dU' \langle \Psi_{\text{back}} | (U')^\dagger U \bar{a} | \Psi_{\text{back}} \rangle \\ &= \mathcal{N}^2 \int dU \langle \Psi_{\text{back}} | U \bar{a} | \Psi_{\text{back}} \rangle \\ &= \langle a \rangle. \end{aligned} \quad (3.106)$$

In the first step here, we redefined  $U \rightarrow (U')^\dagger U$ . There is no Jacobian due to the invariance of the Haar measure, and the integral over  $U'$  simply cancels the volume of the

conformal group. We then used the relation (3.99) and approximated the expectation value of the smeared operator by that of the original operator.

Repeating the manipulations above shows that the map  $a \rightarrow \hat{a}$  preserves the product structure of  $\mathcal{A}_{\text{light}}$ . Namely,

$$\hat{a}_1 \hat{a}_2 \dots \hat{a}_n |\Psi_{\text{obs}}\rangle = \mathcal{N}_1 \int U \bar{a}_1 \bar{a}_2 \dots \bar{a}_n |\Psi_{\text{back}}\rangle dU, \quad (3.107)$$

and therefore,

$$(\Psi_{\text{obs}}, \hat{a}_1 \dots \hat{a}_n \Psi_{\text{obs}}) = \langle a_1 \dots a_n \rangle, \quad (3.108)$$

As a special case, this shows that the variance of  $\hat{a}$  in the state  $|\Psi_{\text{obs}}\rangle$  coincides with the variance of  $a$  in the vacuum. This provides evidence that the quantum probabilities of various outcomes obtained by measuring  $\hat{a}$  in the state  $|\Psi_{\text{obs}}\rangle$  are close to those of outcomes obtained by measuring  $a$  in  $|0\rangle$ .

This completes the construction of relational observables whose expectation values in a particular state match the vacuum expectation values of simple operators.

The proof above also shows why the smeared quantities in Section 3.5.3 were necessary. The key relation (3.105) is valid up to terms of order  $\frac{\text{vol}(\Sigma)}{\text{vol}(\Sigma')}$ . If one tries to make the projectors too sharp, by reducing  $\text{vol}(\Sigma')$ , then this invalidates this relation. Some more comments may be found in Appendix B.2.

### 3.5.5 Comments on the construction

We would like to make some comments on the construction presented in the previous subsection.

1. A salient feature of the observables is that they are tailored to a specific background state. This state-dependence is manifest in the projectors that appear in (3.102). In particular, the same observables would not make sense in another state. This can be justified as follows. A physical observer finds themselves in a particular state of the Universe. In that state, they choose to describe the physics around them using observables whose correlators behave like those of simple local operators. On the other hand, state dependence raises broader questions and potential paradoxes that have been explored in the context of AdS/CFT [26, 62, 63]. (See [8] for a review.) So this feature of the construction deserves further attention.
2. This construction is valid to leading order in  $G_N$  perturbation theory. The validity of the construction relies on (3.91), but the size of  $\Sigma$  is typically perturbative

in  $G_N$ . (See Appendix B.2 for an example.) It is an open technical problem to extend this construction to higher orders in perturbation theory. We do not see any obstacle to this extension, although we would like to caution the reader against premature claims in the literature which suggest that such an extension is trivial. Such an extension is important to understand gravitational corrections to these relational observables. Some of the technical issues are explained in [29].

3. While it might be possible to improve (3.106) so that it is valid at higher orders in perturbation theory, it can never be made exact. The accuracy of the approximation is controlled by features of the background state. This tells us that if one makes precise-enough measurements to detect the background, then the nonlocal features of these cosmological observables eventually become evident. Therefore, the principle of holography of information is valid as an exact statement, even in the presence of a heavy background.

## 3.6 Discussion

In this chapter, we studied cosmological correlators in the Hilbert space of gravitationally constrained de Sitter states in the  $G_N \rightarrow 0$  limit. Even in this limit, it is necessary to impose the global part of the constraints and this forces all states to be de Sitter invariant. The Hilbert space is obtained by group averaging ordinary QFT states and defining the inner product to be the QFT inner product divided by the volume of the de Sitter group.

We utilized a convenient representation of such states that was obtained in [13] by taking the nongravitational limit of solutions to the WDW equation in asymptotically de Sitter space: de Sitter invariant states are obtained by smearing ordinary field excitations above the Euclidean vacuum with an appropriate conformally covariant function.

Gauge-invariant operators can also be obtained by group averaging QFT operators. Group averaging a product of ordinary quantum-field operators leads to a gauge-invariant operator that is represented by smearing the product of operators with a conformally covariant function. The gauge-fixed version of such an observable was termed a “cosmological correlator” in [7].

In Section 3.3, we formulated Feynman rules to compute cosmological correlators in de Sitter invariant states. We also explained how to remove a simple set of group-volume divergences that might appear in some observables. This leads to some necessary conditions for states to have finite norm, and for observables to have finite expectation

values.

In Section 3.4, we evaluated cosmological correlators in some simple examples. A striking feature of our result is that cosmological correlators take on nontrivial values even when the Euclidean vacuum is represented by a Gaussian wavefunction. We utilized techniques from harmonic analysis on the conformal group to obtain explicit expressions for some cosmological correlators. The final expressions are all conformally covariant, which is expected from the analysis of the symmetries of conformal correlators performed in [7].

The fact that cosmological correlators differ from QFT correlators in the vacuum, and can be nontrivial even when the vacuum is trivial, is to be expected. Any normalizable state must have excitations on top of the vacuum, and these excitations also contribute to the cosmological correlator. However, in the gravitational case, this contribution is conformally invariant. In a nongravitational QFT, cosmological correlators in normalizable excited states are never conformally invariant unless the excitation doesn't contribute to the correlator at all.

Nevertheless, one might ask whether it is possible to find any observables whose expectation values approximate quantum-field theory cosmological correlators in the Euclidean vacuum. We answer this question affirmatively in Section 3.5. The trick is to consider a heavy background state obtained by group averaging a QFT-state that is localized in one part of the late-time slice. We then define relational observables with respect to this background state. This construction utilizes techniques previously developed to study quasilocal observables in AdS/CFT in [25–29].

The construction of Section 3.5 does not violate the principle of holography of information. This is because while the expectation values of the observables constructed in 3.5 approximate expectation values in the Euclidean vacuum to leading order in perturbation theory, a sufficiently precise examination of a sufficiently high-point cosmological correlator would cause the construction to break down and would reveal the presence of the heavy background.

The holographic properties of gravity are somewhat analogous to unitarity in ordinary quantum field theories. Unitarity is an exact statement at the nonperturbative level and is important for the consistency of the theory. Unitarity can also be verified for light observables about empty space, such as the S-matrix of a few incoming and a few outgoing particles. Nevertheless, in the presence of a heavy background when one coarse-grains the set of observables, physics appears dissipative. This is a feature, not a bug, because it helps to reconcile the principle of unitarity with our mundane experience of dissipation.

The same regimes apply to the principle of holography of information. In principle, gauge-fixed correlators in any open set on the late-time slice provide complete information about the state. The calculations of 3.4 show that this result has perturbative signatures; in perturbative states with a small number of excitations, low-point cosmological correlators are sensitive to the excitations. Nevertheless, in the presence of a heavy background, when one coarse-grains the set of observables, several distinct states can resemble the Hartle-Hawking state. This is also a feature, and not a bug, because it helps to reconcile holography with our mundane experience of local physics.

The relational observables of Section 3.5 are related to the observables that can be defined in the presence of an observer [30] or in the presence of a timelike boundary [64]. A benefit of the relational perspective is that the observer arises naturally from the property of an excitation in a closed universe, and one does not need to puncture the spacetime; the cost that one must pay is that the construction is technically more elaborate.

This study raises several interesting issues that we hope to address in future work. An immediate extension is to improve the analysis of Section 3.4 to incorporate loops. This will require us to renormalize UV and IR divergences [65–68] and incorporate the effects of anomalous dimensions in the fields. From a more conceptual perspective, certain singularities of QFT correlators are fixed by a “flat space limit” [69–71]. This constrains the correlator everywhere [72]. Is there a similar structural constraint that the cosmological correlators in the gravitational Hilbert space must obey in the perturbative limit?

Another relevant technical problem is to extend the construction of Section 3.5 to higher orders in perturbation theory. The construction so far yields relational observables that approximate QFT correlators that are smeared by the action of elements of the conformal group drawn from a small open set near the identity. But this smearing is not nonperturbatively small. This means that the relational observables differ from QFT correlators at higher orders in perturbation theory. Can one systematically evaluate these effects? Does the short-distance structure of relational observables obey the Hadamard condition at higher orders in perturbation theory and yield a finite local stress-tensor using the usual methods of QFT in curved spacetime [73]?

Next, there has been considerable recent work on the formal properties of quantum field theory in de Sitter space [74–80]. Are these analyses modified if, instead of bare quantum-field observables, one uses the dressed observables presented in Section 3.5?

In this Chapter, our calculations were performed within an infinite-dimensional Hilbert space. Nonperturbative corrections to the inner product may cause the true Hilbert

space to be finite dimensional [81–89]. We do not expect the perturbative results presented here to be affected by such corrections.

There has also been recent discussion on the possibility that the Hilbert space of a closed Universe might be one-dimensional [90–92]. Since our analysis is perturbative, it does not shed much light on this issue. However, some of the arguments given for a one-dimensional Hilbert space are not very persuasive. For instance, it might naively appear that the Hartle-Hawking state is the only normalizable state that satisfies the constraints. But, as we have discussed, an alternative quantization gives rise to a rich Hilbert space. Moreover, the observation that the entanglement wedge of a point is the entire spacetime is consistent with the principle of holography of information: gauge-fixed observables in an infinitesimal open set of the late-time slice are sufficient to tell us about the entire spacetime when it is in a pure state. Arguments based on the gravitational path integral are somewhat inconclusive because the rules are ill defined.

On the other hand, the construction of [13] assumes the existence of continuous families of solutions to the Wheeler-DeWitt equation; the excited states that appear in this chapter are obtained by differentiating those solutions with respect to the continuous parameters. This assumption is valid in effective field theory but perhaps subtle UV effects disallow such families. Moreover, it is true that a closed Universe always remains in one state since, in the absence of external sources, it is impossible to transition to another state. So, perhaps an observer in one state can never learn about the presence of other states in the Hilbert space!

These considerations lead to deeper questions about the nature of observations in a cosmological setting. The standard von Neumann theory of measurements envisions a modification of the Hamiltonian of the observed system that couples it to a pointer. Clearly, this does not make sense in the context of gravity, where a local modification of the Hamiltonian is inconsistent with the constraints. So a complete description of measurements is only possible in an autonomous setup where the observer is part of the system. Presumably, this more-sophisticated theory will also elucidate how the state-dependent observables of Section 3.5 are automatically selected by such an observer in the semi-classical limit.

This completes our study of gravitational constraints. In Chapters 2 and 3, we saw how gravitational theories localise information in a radically different way compared to other gauge theories. In the next chapter, we consider a situation in string theory where a different kind of ‘unusual localisation of information’ could be seen.

# Chapter 4

## Tachyons

We have been primarily discussing low-energy effective theories of gravity and the importance of gravitational constraints in making quantum information holographic in nature. The notion of boundary degrees of freedom having access to information in the bulk is mainly motivated by the concrete scenario in string theory where we have a holographic gauge-gravity duality in the form of the AdS/CFT correspondence [93]. Here we see that a gravitational theory in AdS spacetime is dual to a lower-dimensional CFT living at the boundary. A lot of computations have been performed towards the successful verification of this conjecture [94, 95]. This motivates one to look for other examples in string theory where we can have similar dualities. However, till now, the AdS/CFT correspondence remains the most precise and useful holographic duality one could deduce from string theory.

*In this context, we would like to take motivation from our studies on low-energy effective theories of gravity and look for scenarios in string theory where there seems to be some ‘unusual’ localisation of information. It is in this spirit that we would like to study tachyon dynamics on unstable D-branes.*

It was shown in [96–98] that properties of the system resulting from the decay of an unstable D-brane, obtained from tree-level open string theory analysis, match with properties of closed string radiation produced by the same D-brane, computed from a closed string analysis [99] (in the  $g_s \rightarrow 0$  limit). This equivalence, at least classically, tells you that the open string degrees of freedom localise information in an ‘unusual’ way in the sense that they know about the final closed string states produced by the decaying D-brane.

In this chapter, we advance this idea further. We show that one can give a prescription to map the tachyon effective theory (obtained from a tree-level open string theory

analysis [100]) to a subsector of Klein-Gordon theory, namely coherent states and excitations on top. This sits nicely in agreement with the open-closed equivalence discussed above because the decaying D-brane acts as a time-dependent source for closed strings, and the final closed string state is indeed a coherent state [99]. However, care should be taken to interpret the results. Our results are only suggestive of a possible extension of open-closed equivalence beyond the classical regime. One has to do string theory computations at finite  $g_s$  to concretely prove such an equivalence. But we hope that this should be possible, in light of our results.

## 4.1 Brief summary

The dynamics of a tachyon on an unstable D-p-brane in string theory can be described by the following effective action [98, 100–102].

$$S = - \int d^p x V(T) \sqrt{1 + \eta^{\mu\nu} \partial_\mu T \partial_\nu T} \quad (4.1)$$

Here  $T$  is the tachyon field and  $V(T)$  is the potential governing its dynamics.  $V(T)$  has a maximum at  $T = 0$  and approaches zero in the limit  $T \rightarrow \infty$  (Figure 4.1). In this limit, the classical equations of motion obtained from (4.1) reduce to Euler and continuity equations for relativistic non-interacting dust. Given such a simple classical description, it is natural to ask whether this theory can be quantised. After all, at the classical level, the system behaves like a collection of particles. One might therefore attempt the usual perturbative quantisation methods used in scalar field theories. However, this approach fails because there are no plane wave solutions around the minimum of the potential. This suggests that a different approach to quantisation is needed. In this chapter, we suggest a quantisation procedure based on collective field theory methods. Collective field theory arises as a natural candidate because classically the system looks like non-interacting dust, and in collective field theory, one starts with a system of particles and trades the particle degrees of freedom for the collective field. Since we know the quantum description of a collection of free particles, we could understand the quantum mechanics of the collective theory obtained from it.

The collective theory analysis gives the simple result that the quantum theory of the tachyon effective action near the minimum of the potential describes a subsector of Klein-Gordon theory. More precisely, the Hilbert space consists of a coherent state  $|\mathcal{C}\rangle$  of particles at rest and excitations on top of it,  $a_{k_1}^\dagger a_{k_2}^\dagger \dots a_{k_n}^\dagger |\mathcal{C}\rangle$ . This is schematically shown in figure 4.2.

One would have naturally expected some relation with the Klein-Gordon theory be-

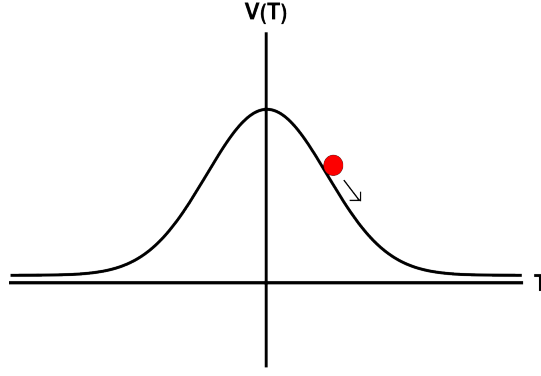


Figure 4.1: Qualitative form of the tachyon potential  $V(T)$ : A tachyon displaced away from the maximum will roll towards the minimum. However, the minimum is at  $T \rightarrow \infty$ , and the tachyon will reach the minimum only asymptotically.

cause its Fock space contains states having a large number of relativistic particles. However, a more precise understanding of the result involves concepts from string theory. There are two equivalent descriptions at play here. One is the open-string theoretic description [96] and another is a closed string description [99, 103, 104]. One gets the tachyon effective theory on the open-string side, whereas on the closed-string side, we get radiation of massive closed strings, which can be described using Klein-Gordon theory. Earlier works have established classical agreement between the physics of these two descriptions [97, 105]. Our results advance this map and point in the direction of a quantum equivalence between open and closed descriptions. We emphasise, however, that our result does not prove the open-closed quantum duality, and the appearance of coherent states is only suggestive of such an interpretation.

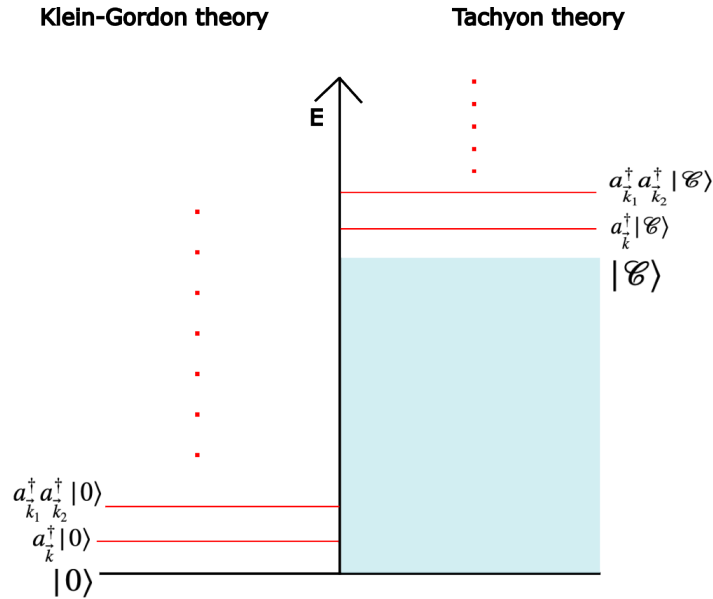


Figure 4.2: The spectrum in Klein-Gordon theory has a vacuum state  $|0\rangle$  and excitations on top. On the other hand, the Hilbert space of tachyon theory has a coherent state  $|C\rangle$  and excitations on top.

To be more specific, note that an unstable D-p brane in  $25 + 1$  dimensional bosonic string theory or  $9 + 1$  dimensional superstring theory has a tachyonic mode in the open-string spectrum [98]. This is due to the tachyon field ( $T$ ) being at the maximum of the potential  $V(T)$ . One can displace the tachyon from the maximum of the potential and let it *roll towards the minimum* of the potential <sup>1</sup>. In a tree-level analysis, such a scenario is captured by time-dependent solutions in open-string theory [96]. Here, one observes that the energy density of the system - which includes the D-brane tension as well as the kinetic and potential energies of the tachyon - stays constant during time evolution. However, the pressure goes to zero asymptotically ( $x^0 \rightarrow \infty$ ). Thus, in the late-time limit, the system behaves like pressureless dust.

It is important to note that the tachyon approaches the minimum of the potential only asymptotically. The vacuum of the theory corresponds to a zero-energy, zero-pressure *time-independent* configuration where the tachyon is at rest at the minimum of the potential. This corresponds to a system without any D-branes, and hence physical open-string degrees of freedom are absent [96]. This, in the effective field theory, translates to the statement that there are no plane wave solutions around the minimum of the potential [100]. The action given in (4.1) is not obtained from a direct string theory calculation; it is an effective description that captures the physics of a rolling tachyon [100–102, 106–109].

On the closed string side, one can turn on  $g_s$ , and study the closed strings resulting from the decay of the unstable D-brane [99, 104, 110–112]. It turns out that the D-brane completely decays away, transferring all its energy to closed strings, which are mostly localised along the plane of the original D-brane. Also, most of the energy of the D-brane gets transferred to closed string states of mass  $\sim g_s^{-1}$ . One can see that the decaying D-brane acts like a time-dependent source for closed strings, and the final state obtained in this process is a coherent state [110]. We note that these closed string states follow the relativistic dispersion relation  $E = \sqrt{p^2 + m^2}$ . So an effective field-theoretic description could be given by Klein-Gordon theories having different masses.

Classically, one could see similarities between the two descriptions. We get pressureless dust in the late-time limit from open-string theory and closed string radiation localised along the plane of the original D-brane from the closed string analysis. We are advancing this further in this work. The final closed string state obtained from a decaying D-brane is a coherent state of the closed string mode [99]. What we obtain from quantising the tachyon effective theory using collective theory methods is also a

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<sup>1</sup>In the case of bosonic string theory,  $V(T)$  is not symmetric about  $T = 0$ . Rolling it in the direction without the minimum will result in a singularity. Our analysis is only applicable to the case when the tachyon rolls towards the minimum

coherent state  $|\mathcal{C}\rangle$  (and excitations on top of it). *This suggests a duality between the open and closed string theory descriptions* [97, 105, 113]. Note that our analysis is only suggestive and not sufficient to establish such a duality, which can only be done by a proper string theory analysis.

This chapter is organised as follows:

- In section 4.2, we introduce the Tachyon effective action and describe its classical dynamics near the minimum of the potential. We see that, classically, the tachyon behaves like non-interacting, non-rotating dust. Analysing the classical equations of motion, we further establish the absence of plane wave solutions around the minimum.
- In section 4.3, we review the collective field theory methods for a collection of free particles. We first discuss them in the simple context of non-relativistic particles in 4.3.1 and then generalise the results to the relativistic case in 4.3.2.
- In section 4.4 we use collective theory methods to quantise the tachyon effective field theory. The Hilbert space description of the Tachyon theory is obtained and is compared with states in a Klein-Gordon theory.
- We conclude with discussions and future directions in 4.5.

## 4.2 Tachyon effective theory

In this section, we briefly review the tachyon effective action proposed in [114], which describes the classical dynamics of the tachyon field. We will also explain the absence of plane wave solutions around the vacuum.

### 4.2.1 Classical dynamics

Following [114], the tachyon field is described by the effective action,

$$S = - \int d^{p+1}x V(T) \sqrt{1 + \eta^{\mu\nu} \partial_\mu T \partial_\nu T} \quad (4.2)$$

The potential  $V(T)$  has a maximum at  $T = 0$  and decays exponentially for large  $T$ . Near the minimum of the potential at  $T \rightarrow \infty$ , it behaves as  $V(T) = e^{-\frac{\alpha T}{2}}$  where  $\alpha = 1$  for bosonic string theory and  $\alpha = \sqrt{2}$  for superstring theory. The exponential form of the potential is chosen so that the late-time dynamics of the tachyon field reproduce the expected behaviour of a rolling tachyon [96, 100, 115, 116]. In particular, as the tachyon rolls towards the minimum of the potential, the pressure approaches

zero while the energy density remains finite. Due to this constant energy density, one can see that this asymptotic limit is different from the minimum energy configuration where the tachyon is at rest at the minimum of the potential with zero energy and pressure. Therefore, in all the analysis that follows, even though we work in the large  $T$  limit, we never truly reach the vacuum solution.

To study the classical dynamics in the  $T \rightarrow \infty$  limit, it is convenient to use the Hamiltonian formalism. Since the Lagrangian is proportional to the potential  $V(T)$ , which vanishes in this limit, the Lagrangian description becomes less useful. The Hamiltonian formulation, however, remains well defined and allows us to analyze the dynamics more easily.

The momentum conjugate to  $T$  is given by

$$\Pi(x) = \frac{\delta S}{\delta \partial_0 T(x)} = \frac{V(T) \partial_0 T(x)}{\sqrt{1 + (\partial T)^2}}, \quad (4.3)$$

which leads to the Hamiltonian

$$H = \int d^p x \sqrt{\Pi^2 + V(T)^2} \sqrt{1 + (\nabla T)^2}. \quad (4.4)$$

In the  $T \rightarrow \infty$  limit, near the minimum of the potential, the  $V(T)^2$  term can be neglected, and the Hamiltonian reduces to

$$H = \int d^p x |\Pi(x)| \sqrt{1 + (\nabla T)^2}. \quad (4.5)$$

In what follows, we drop the absolute value sign and assume  $\Pi > 0$ . Hamilton's equations give

$$\begin{aligned} \partial_0 T &= \sqrt{1 + (\vec{\nabla} T)^2}, \\ \partial_0 \Pi(x) &= \vec{\nabla} \cdot \left( \frac{\Pi(x) \vec{\nabla} T}{\sqrt{1 + (\vec{\nabla} T)^2}} \right). \end{aligned} \quad (4.6)$$

Now, let us define  $u_\mu = -\partial_\mu T$  and  $\epsilon(x) = \frac{\Pi(x)}{\sqrt{1 + (\nabla T)^2}}$ . In terms of these variables, the equations of motion take the form

$$u^2 = -1, \quad \partial_\mu (\epsilon(x) u^\mu) = 0. \quad (4.7)$$

These are precisely the equations governing relativistic, non-rotating, non-interacting dust with four-velocity  $u^\mu$  and energy density  $\epsilon(x)$ . This implies that the classical

dynamics of the tachyon field near the minimum of the potential is that of pressureless dust, where  $\partial_\mu T$  plays the role of the velocity field, and  $\Pi(x)$  corresponds to the local energy density<sup>2</sup>. Although the classical dynamics becomes simple in this limit, the analysis of quantum fluctuations around this background is more subtle. In particular, the usual perturbative quantisation breaks down, as we now discuss.

#### 4.2.2 Failure of perturbative quantisation

Given the tachyon effective action and its simple classical dynamics, one might expect that the usual perturbative quantization can be carried out. However, as was shown in [100], the absence of plane wave solutions around the minima of the potential implies that such a perturbative quantisation is not possible. We summarize this property below. Expanding the square root in (4.2), we get

$$S = - \int d^{p+1}x e^{-\frac{\alpha T}{2}} \left( 1 + \frac{\eta^{\mu\nu}}{2} \partial_\mu T \partial_\nu T + \dots \right) \quad (4.8)$$

Now, defining a new field  $\phi = e^{-\frac{\alpha T}{4}}$ , the above expression reduces to

$$S = \int d^{p+1}x \frac{16}{\alpha^2} \left( -\frac{\eta^{\mu\nu}}{2} \partial_\mu \phi \partial_\nu \phi - \frac{\alpha^2}{16} \phi^2 + \dots \right) \quad (4.9)$$

The minimum of the potential is at  $\phi = 0$ . By ignoring the higher derivative terms, we see that  $\phi = a e^{ik \cdot x}$  is a solution of the equation of motion, with  $k^2 = -\frac{\alpha^2}{8}$ . However, this ceases to be a solution when one takes into account all the higher derivative corrections. To see this, we rewrite the action in terms of  $\phi$  without expanding the square root,

$$S = - \int d^{p+1}x \phi^2 \sqrt{1 + \frac{16}{\alpha^2} \phi^{-2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi} \quad (4.10)$$

The corresponding equation of motion is

$$\partial_\mu \left( \frac{\partial^\mu \phi}{\sqrt{1 + \frac{16}{\alpha^2} \phi^{-2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}} \right) = \frac{\alpha^2}{8} \left( \frac{\phi + \frac{8}{\alpha^2} \phi^{-1} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}{\sqrt{1 + \frac{16}{\alpha^2} \phi^{-2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}} \right) \quad (4.11)$$

Substituting  $\phi = a e^{ik \cdot x}$  gives

$$\frac{a}{\sqrt{1 - \frac{16}{\alpha^2} k^2}} = 0 \quad (4.12)$$

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<sup>2</sup>This physical interpretation of the field  $\Pi$  will become important later when we try to quantise the system using collective theory methods

This is not satisfied by any choice of  $k^\mu$  and therefore we must have  $a = 0$ . Thus, there are no plane-wave solutions around the minimum of the potential. This result agrees with the expected properties of the tachyon vacuum. The vacuum corresponds to the configuration where the tachyon is at rest at the minima of the potential, and the system has zero energy and zero pressure. This means the absence of D-branes and physical open string degrees of freedom.

Since perturbative quantization relies on expanding the field in terms of plane-wave modes, the usual perturbative approach cannot be applied. This motivates the search for an alternative quantization method. In the following, we will use insights from collective field theory to quantise the tachyon effective theory.

### 4.3 Collective field theory

The basic idea of collective field theory is to trade discrete quantum mechanical degrees of freedom for a continuous field degree of freedom. Works by B. Sakita, A. Jevicki, and collaborators [117–119] are good resources for collective field theory concepts relevant to the context we are working on. We will see that collectivising a system of free particles gives an alternative quantum description in terms of field variables, which in the  $\hbar \rightarrow 0$  limit will also give the classical equations of motion governing pressureless dust. We will briefly review the collective field description for a collection of  $N_*$  free non-relativistic particles, as this will make some of the analysis conceptually easier to grasp. We will then generalise it to relativistic particles.

#### 4.3.1 Non-relativistic free particles

Consider a system of  $N_*$  free non-interacting non-relativistic particles in  $p$  spatial dimensions, with the Hamiltonian,

$$H = \sum_{i=1}^{N_*} \frac{\vec{p}_i^2}{2m} \quad (4.13)$$

In the usual particle description, the dynamics of this system is written in terms of the phase space coordinates  $\{\vec{x}_i, \vec{p}_i\}$ , where  $\{\vec{x}_i; i = 1, 2, \dots, N_*\}$  denotes the positions of the particles and  $\vec{p}_i = -i\hbar \vec{\nabla}_{\vec{x}_i}$  is the conjugate momentum in the position space. In the collective description, instead of working with the particle coordinates  $\{\vec{x}_i, \vec{p}_i\}$  we describe the system using a field  $\phi(\vec{y})$  and its conjugate momentum  $\pi(\vec{y})$ , where  $\pi(\vec{y}) = -i\hbar \delta / \delta \phi(\vec{y})$ . The quantum theory is then formulated in terms of these fields. In particular, the Hamiltonian and the Hilbert space will be expressed in terms of these

variables. Following [117–119], the field is related to the particle positions through

$$\phi(\vec{y}) = m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i) \quad (4.14)$$

However, this relation should not be viewed as a strict definition of  $\phi(\vec{y})$ . In the collective description,  $\phi(\vec{y})$  is treated as an independent field variable. The relation above is instead implemented through a non-trivial Jacobian that appears in the inner product of the field theory. We will see how this arises below.

The Jacobian is defined such that the quantum mechanical inner product between two QM states  $\psi_1$  and  $\psi_2$  is equal to the field theory inner product between corresponding wavefunctionals  $\Phi_1[\phi]$  and  $\Phi_2[\phi]$ .

$$\begin{aligned} (\psi_1, \psi_2) &= \int \left( \prod_{i=1}^N d^p \vec{x}_i \right) \psi_1^* (\vec{x}_1, \dots, \vec{x}_{N_*}) \psi_2 (\vec{x}_1, \dots, \vec{x}_{N_*}) \\ &= \int \left( \prod_{i=1}^N d^p \vec{x}_i \right) \left( \int D\phi \delta \left[ \phi(\vec{y}) - m \sum_i \delta(\vec{y} - \vec{x}_i) \right] \right) \Phi_1^*[\phi] \Phi_2[\phi] \\ &= \int D\phi J_{N_*}[\phi] \Phi_1^*[\phi] \Phi_2[\phi] \end{aligned} \quad (4.15)$$

The steps performed above require some explanation. The first equality is the usual QM inner-product for an  $N_*$  particle system. In the second line, we inserted an identity

$$1 = \int D\phi \delta \left[ \phi(\vec{y}) - m \sum_i \delta(\vec{y} - \vec{x}_i) \right]. \quad (4.16)$$

The wavefunctionals  $\Phi_1$  and  $\Phi_2$  are defined by demanding that enforcing (4.14) on  $\Phi_1$  and  $\Phi_2$  should give  $\psi_1$  and  $\psi_2$ .

$$\begin{aligned} \Phi_1[\phi] \Big|_{\phi(\vec{y})=m \sum_{i=1}^{N_*} \delta(\vec{y}-\vec{x}_i)} &= \psi_1(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*}) \\ \Phi_2[\phi] \Big|_{\phi(\vec{y})=m \sum_{i=1}^{N_*} \delta(\vec{y}-\vec{x}_i)} &= \psi_2(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*}) \end{aligned} \quad (4.17)$$

Note that there might be different wavefunctionals ( $\Phi$ ) which will reduce to the same QM wavefunction ( $\psi$ ) under (4.14). So, one could say that the map from the collective theory state to the state in quantum mechanics is many-to-one. We will comment more on this later.

The final equality in (4.15) defines the Jacobian as,

$$J_{N_*}[\phi] = \int \left( \prod_{i=1}^N d^p x_i \right) \delta \left[ \phi[\vec{y}] - m \sum_{i=1}^{N_*} (\vec{y} - \vec{x}_i) \right]. \quad (4.18)$$

For computations, it is convenient to write this Jacobian as,

$$J_{N_*}[\phi] = \int D\lambda e^{i \int d^p y \lambda(\vec{y}) \phi(\vec{y})} \left( \int d^p x e^{-im\lambda(\vec{x})} \right)^{N_*} \quad (4.19)$$

by introducing an auxiliary field  $\lambda(y)$ , which allows the functional delta function to be written in Fourier form.

Thus, in the collective description, the quantum states are represented by wavefunctionals  $\Phi[\phi]$ , and the inner product between these states is defined using the Jacobian that encodes the relation between  $\phi$  and the particle coordinates. With this setup, we now analyze the Hamiltonian in terms of the collective fields and study the resulting eigenstates. We will also compute the inner product between these eigenstates and check that it agrees with the result obtained in the original quantum mechanical description.

### Hamiltonian and eigenstates

In this section, we rewrite the Hamiltonian (4.13) in terms of the collective fields  $\phi$  and  $\pi$ , and study the corresponding eigenstates and the spectrum of the theory.

It can be shown, using (4.14) and application of the chain rule of differentiation, that operators invariant under exchanging any two particles could be written in terms of the collective field and its conjugate. So, the Hamiltonian could be written in terms of  $\phi$  and  $\pi$ . We will first show how to write  $\sum_{i=1}^{N_*} \vec{p}_i$  in terms of  $\phi$  and  $\pi$ . Generalising this method to  $H = \sum_{i=1}^{N_*} \frac{\vec{p}_i^2}{2m}$  is straightforward and we derive this in the Appendix C.1.2.

$$\begin{aligned} \sum_i \vec{p}_i &= -i\hbar \sum_i \vec{\nabla}_{\vec{x}_i} \\ &= -i\hbar \sum_i \int d^p y \left( \vec{\nabla}_{\vec{x}_i} \phi(\vec{y}) \right) \frac{\delta}{\delta \phi(\vec{y})} \\ &= -i\hbar \sum_i \int d^p y \vec{\nabla}_{\vec{x}_i} m \delta(\vec{y} - \vec{x}_i) \frac{\delta}{\delta \phi(\vec{y})} \\ &= -i\hbar \int d^p y \left( \sum_i m \delta(\vec{y} - \vec{x}_i) \right) \vec{\nabla}_{\vec{y}} \frac{\delta}{\delta \phi(\vec{y})} \\ &= \int d^p y \phi(\vec{y}) \vec{\nabla}_{\vec{y}} \pi(\vec{y}) \end{aligned} \quad (4.20)$$

In going to the second line, we applied the chain rule to rewrite derivatives with respect to  $x_i$  in terms of the collective variable  $\phi(\vec{y})$ . The third line follows from the definition of  $\phi(\vec{y})$  given in (4.14), which will be imposed by the Jacobian while computing any matrix element. In the last step, we again used the defining expressions for  $\phi(\vec{y})$  and  $\pi(\vec{y})$ .

One should be cautious about the derivation given in (4.20). Here we used the chain rule to go from  $\{\vec{x}_i; i = 1, 2, \dots, N_*\}$ , having a finite number of degrees of freedom, to  $\phi(\vec{y})$ , which has infinitely many degrees of freedom. Strictly speaking, such a change of variables is ill-defined. A more precise derivation is given in Appendix C.1.1, where we start from the quantum mechanical expression for  $\left(\psi_1, \left(\sum_{i=1}^{N_*} \vec{p}_i\right) \psi_2\right)$  and then use (4.16) to get to a field theoretic expression. The final answer will be the same as what we got in (4.20) using the naive chain rule for differentiation.

Similar manipulations can be performed to get the expression for  $\sum_i p_i^2$ , in terms of  $\phi$  and  $\pi$ . So, the Hamiltonian is obtained as,

$$H = \sum_{i=1}^{N_*} \frac{p_i^2}{2m} = \frac{1}{2} \int d^p y \phi(\vec{y}) \left( \vec{\nabla} \pi(\vec{y}) \right)^2 - \frac{i\hbar}{2m} \int d^p y \phi(\vec{y}) \vec{\nabla}^2 \pi(\vec{y}) \quad (4.21)$$

Note that the second term is linear in  $\pi$ , and therefore carries an explicit factor of  $\hbar$ . It may thus be interpreted as a quantum correction. In particular, if this term is neglected, the resulting equations of motion reduce to those of a fluid. We will show this explicitly in section 4.3.1.

Given the Hamiltonian in the collective description, we now determine its energy eigenfunctionals and the corresponding spectrum. Using the relation (4.17), which relates the quantum mechanical wavefunctions  $\psi$  to the collective theory wavefunctionals  $\Phi$ , we construct the corresponding states in the collective description. The simplest case to consider is  $N_* = 1$ , i.e., the one-particle quantum mechanical system. (From now on, we set  $\hbar = 1$ , restoring it whenever necessary.)

For single-particle QM, the energy eigenstate is just a plane wave,

$$\begin{aligned} \psi_{\vec{k}}(\vec{x}) &= e^{i\vec{k} \cdot \vec{x}} \\ H \psi_{\vec{k}}(\vec{x}) &= \frac{\vec{k}^2}{2m} \psi_{\vec{k}}(\vec{x}) \end{aligned} \quad (4.22)$$

Following the reasoning explained earlier, the corresponding wavefunctional is obtained by requiring that substituting the definition of the collective field (4.14) repro-

duces the original quantum-mechanical wavefunction. This leads to

$$\Phi_{\vec{k}}[\phi] = \frac{1}{m} \int d^p y e^{i\vec{k} \cdot \vec{y}} \phi(\vec{y}) \quad (4.23)$$

One can see that substituting  $\phi(\vec{y}) = m\delta(\vec{y} - \vec{x})$  will give the QM energy eigenstate (4.22) as required. One can also check that (4.23) is an eigenstate of the Hamiltonian.

$$\begin{aligned} H \Phi_{\vec{k}}[\phi] &= -\frac{1}{2m} \int d^p y' \phi(\vec{y}') \vec{\nabla}_{\vec{y}'}^2, \frac{\delta}{\delta \phi(\vec{y}')} \frac{1}{m} \int d^p y e^{i\vec{k} \cdot \vec{y}} \phi(\vec{y}) \\ &= -\frac{1}{2m^2} \int d^p y' d^p y e^{i\vec{k} \cdot \vec{y}} \phi(\vec{y}') \vec{\nabla}_{\vec{y}'}^2 \delta(\vec{y}' - \vec{y}) \\ &= \frac{\vec{k}^2}{2m} \Phi_{\vec{k}}[\phi] \end{aligned} \quad (4.24)$$

Here, we did an integration by parts to get the last step. This simple example makes the key idea involved in finding the eigenstates clear. We will later introduce a systematic way to write down energy eigenstates by means of a creation operator. For the purposes of this section, we will work with a small value for  $N_*$ , and this is enough to understand the main concepts involved. We now turn to the case  $N_* = 2$ . The two-particle eigenfunction is

$$\psi_{(\vec{k}_1, \vec{k}_2)}(\vec{x}_1, \vec{x}_2) = e^{i\vec{k}_1 \cdot \vec{x}_1} e^{i\vec{k}_2 \cdot \vec{x}_2} + \vec{x}_1 \leftrightarrow \vec{x}_2 \quad (4.25)$$

with,

$$H \psi_{(\vec{k}_1, \vec{k}_2)} = \frac{\vec{k}_1^2 + \vec{k}_2^2}{2m} \psi_{(\vec{k}_1, \vec{k}_2)} \quad (4.26)$$

Using the same prescription, the corresponding collective wavefunctional becomes

$$\Phi_{(\vec{k}_1, \vec{k}_2)}[\phi] = \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1} e^{i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \phi(\vec{y}_2) - \frac{1}{m} \int d^p y e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \phi(\vec{y}) \quad (4.27)$$

and one can verify that

$$H \Phi_{(\vec{k}_1, \vec{k}_2)}[\phi] = \frac{\vec{k}_1^2 + \vec{k}_2^2}{2m} \Phi_{(\vec{k}_1, \vec{k}_2)}[\phi] \quad (4.28)$$

For simplicity, we have omitted the conventional normalization factor  $\frac{1}{\sqrt{2}}$ . One can verify the last equation given above by the usual techniques of functional differentiation. Have a look at Appendix C.1.2 for detailed calculations.

Alternatively, one could directly solve the time-independent Schrödinger equation for the collective theory Hamiltonian. This is analogous to the Schrödinger picture treat-

ment of the Klein-Gordon theory, and it leads to the same energy eigenstates. The details are given in Appendix C.1.3.

### Computing the inner product

It remains to check that the inner product in the collective theory reproduces the usual quantum-mechanical result. One can verify that this is indeed the case. We will see that (4.15) holds in general as expected. We will start with the simple case of  $N_* = 1$ , where the QM inner product gives

$$\left(\psi_{\vec{k}_1}, \psi_{\vec{k}_2}\right) = (2\pi)^p \delta(\vec{k}_1 - \vec{k}_2) \quad (4.29)$$

One can use the explicit form of the Jacobian given in (4.19) to compute the field theory inner product as follows,

$$\begin{aligned} \left(\Phi_{\vec{k}_1}, \Phi_{\vec{k}_2}\right)_1 &= \int D\phi J_1[\phi] \Phi_{\vec{k}_1}^*[\phi] \Phi_{\vec{k}_2}[\phi] \\ &= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{-i\vec{k}_1 \cdot \vec{y}_1} e^{i\vec{k}_2 \cdot \vec{y}_2} \left( \int D\phi J_1[\phi] \phi(\vec{y}_1) \phi(\vec{y}_2) \right) \end{aligned} \quad (4.30)$$

The subscript “1” on the LHS is to denote that we are in  $N_* = 1$ . The term in the bracket can be computed easily. We will do it for general  $N_*$  as it will be useful later.

$$\begin{aligned} \int D\phi J_{N_*}[\phi] \phi(y_1) \phi(y_2) &= \int D\phi D\lambda e^{i \int \lambda \phi} \left( \int d^p x e^{-im\lambda(\vec{x})} \right)^{N_*} \phi(\vec{y}_1) \phi(\vec{y}_2) \\ &= \int D\lambda D\phi \left( -\frac{\delta^2}{\delta\lambda(\vec{y}_1) \delta\lambda(\vec{y}_2)} e^{i \int d^p y \lambda(\vec{y}) \phi(\vec{y})} \right) \left( \int d^p x e^{-im\lambda(\vec{x})} \right)^{N_*} \\ &= - \int D\lambda \frac{\delta^2 \delta[\lambda]}{\delta\lambda(\vec{y}_1) \delta\lambda(\vec{y}_2)} \left( \int d^p x e^{-im\lambda(\vec{x})} \right)^{N_*} \\ &= m^2 N_* (N_* - 1) V^{N_*-2} + m^2 N_* V^{N_*-1} \delta^p(\vec{y}_1 - \vec{y}_2) \end{aligned} \quad (4.31)$$

In the last step, we did an integration by parts, followed by performing the  $\lambda$  integral using the  $\delta$  functional.  $V$  is the spatial volume of the  $D - p$  brane ( $V = \int d^p y$ ). Substituting (4.31) for  $N_* = 1$  in (4.30) gives,

$$\left(\Phi_{\vec{k}_1}, \Phi_{\vec{k}_2}\right)_1 = (2\pi)^p \delta(\vec{k}_1 - \vec{k}_2) \quad (4.32)$$

We see that this is exactly the QM inner product. This is expected from the equality of inner products in collective theory and quantum mechanics (4.15) which arises from the definition of the Jacobian.

One can also perform similar computations for  $N_* = 2$  states given in (4.27) and arrive

at the same conclusion (See C.1.7 for calculations).

$$\begin{aligned} \left( \psi_{(\vec{k}'_1, \vec{k}'_2)}, \psi_{(\vec{k}_1, \vec{k}_2)} \right) &= 2 \{ 2\pi\delta(\vec{k}_1 - \vec{k}'_1) 2\pi\delta(\vec{k}_2 - \vec{k}'_2) + \vec{k}_1 \leftrightarrow \vec{k}_2 \} \\ \left( \Phi_{(\vec{k}'_1, \vec{k}'_2)}, \Phi_{(\vec{k}_1, \vec{k}_2)} \right)_2 &= 2 \{ 2\pi\delta(\vec{k}_1 - \vec{k}'_1) 2\pi\delta(\vec{k}_2 - \vec{k}'_2) + \vec{k}_1 \leftrightarrow \vec{k}_2 \} \end{aligned} \quad (4.33)$$

The results we have discussed so far suggest that collective field theory is effectively capturing  $N_*$ -particle quantum mechanics. However, trading finitely many degrees of freedom in QM for infinitely many degrees of freedom of a field in collective theory means we expect some sort of many-to-one mapping. This is indeed the case, but the Jacobian constrains the field by imposing (4.14) such that the Hilbert space structure is the same as that of  $N_*$ -particle quantum mechanics.

### Hilbert space structure

Having discussed the Hamiltonian, the eigenstates, and the inner product, we now describe the Hilbert space structure of the collective theory.

Consider collective field theory with  $N_* = 1$ . We saw in the previous section that the one-particle energy eigenstates in quantum mechanics can be mapped to the eigenfunctionals  $\Phi_{\vec{k}}$  defined in (4.23). However, the collective theory also admits states such as  $\Phi_{\vec{k}_1, \vec{k}_2}$ . Since the Hamiltonian does not depend on  $N_*$ , even in the  $N_* = 1$  sector the state  $\Phi_{\vec{k}_1, \vec{k}_2}$  is an eigenstate of the Hamiltonian with eigenvalue  $(\vec{k}_1^2 + \vec{k}_2^2)/2m$ . Thus, the interpretation of such states within the  $N_* = 1$  theory requires some clarification. These states are in fact null states of the  $N_* = 1$  theory. To see this, note that the Jacobian  $J_1$  imposes  $\phi(\vec{y}) = m\delta(\vec{y} - \vec{x})$ , and substituting this in the expression for  $\Phi_{\vec{k}_1, \vec{k}_2}$  evaluates to zero.

$$\begin{aligned} \Phi_{\vec{k}_1, \vec{k}_2} &= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \phi(\vec{y}_2) - \frac{1}{m} \int d^p y e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \phi(\vec{y}) \\ &= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \left( \phi(\vec{y}_2) - m\delta(\vec{y}_2 - \vec{y}_1) \right) \\ &\stackrel{N_*=1}{=} \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} m\delta(\vec{y}_1 - \vec{x}) \left( m\delta(\vec{y}_2 - \vec{x}) - m\delta(\vec{y}_2 - \vec{y}_1) \right) \\ &= 0 \end{aligned} \quad (4.34)$$

Here the superscript  $N_* = 1$  above the 3rd equality means that we are imposing  $\phi(\vec{y}) = m\delta(\vec{y} - \vec{x})$ . One can also show this using the Jacobian explicitly, and this is given in the Appendix C.1.6. The statement on the field theory side is that the inner product of  $\Phi_{\vec{k}_1, \vec{k}_2}$  with any other state is zero, and hence the state is null.

$$\left( \Phi, \Phi_{\vec{k}_1, \vec{k}_2} \right)_1 = 0 \quad \forall \Phi \Rightarrow \Phi_{\vec{k}_1, \vec{k}_2} = 0 \quad (4.35)$$

Similarly, one can show that the 3-insertion energy eigenstate  $\Phi_{\vec{k}_1, \vec{k}_2, \vec{k}_3}$  is also a null state. The result can be generalized. Thus, all higher-insertion states are null in the  $N_* = 1$  theory. The only remaining possibility is the zero-insertion state. Such a state will just be a constant, independent of  $\phi$ . We choose this constant to be unity, as it is just a matter of normalisation. *This state will have important applications later, so we denote it by  $\mathcal{C}$ , i.e.,  $\mathcal{C}[\phi] = 1$ .*

It turns out that the state  $\mathcal{C}$  in the  $N_* = 1$  collective theory could be interpreted as a 1-particle state in QM with zero momentum. This also follows from the fact that the Jacobian enforces  $\phi(\vec{y}) = m\delta(\vec{y} - x)$ . One can arrive at this result from a field theory perspective by noting that the Jacobian makes sure that the operator  $\int d^p y \phi(\vec{y})$  is equivalent to  $N_* m$  (See C.1.5).

$$\Phi_{\vec{k}=0} = \frac{1}{m} \int d^p y \phi(\vec{y}) \stackrel{N_*=1}{=} 1 \quad (4.36)$$

This tells you that the state  $\Phi_{\vec{k}=0}$  is the same as  $\mathcal{C}$ ! This is an instance of the many-to-one mapping we mentioned in earlier sections.

These results point in the direction that the Hilbert space of  $N_* = 1$  collective field theory is the same as the 1-particle QM Hilbert space denoted by  $\mathcal{H}_1$ . The energy eigenstates in QM are in one-to-one correspondence with the states  $\Phi_{\vec{k}}$ . Naively, one might think that the collective theory has more (and different) states than QM, but the Jacobian constrains the field  $\phi$  such that we get effectively a one-particle Hilbert space.

Similar results go through for  $N_* = 2$  collective theory also. The states  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}$  for  $n > 2$  are all null states, whereas the state  $\Phi_{\vec{k}}$  should be identified with  $\Phi_{\vec{k}, 0}$  and the state  $\mathcal{C}$  should be identified with  $\Phi_{\vec{k}_1=0, \vec{k}_2=0}$ . These results can be generalised to any value of  $N_*$ .

We conclude this section by summarizing the main results:

- Quantum Mechanics of  $N_*$  number of particles can be reformulated in terms of the collective field  $\phi$  and its conjugate momentum  $\pi$ .
- Although the field  $\phi$  appears to have infinitely many degrees of freedom, the nontrivial Jacobian in the inner product effectively constrains it so that the theory remains equivalent to ordinary quantum mechanics.
- The Hilbert space of collective theory is the same as that of  $N_*$ -particle QM, i.e.,  $\mathcal{H}_{N_*}$ . However, the map from collective theory to QM is many-to-one. The Hilbert space is spanned by states  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*}}$  which are  $N_*$ -particle energy eigenstates with momenta  $\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*}$ . Energy eigenstates having  $N_* - s$  insertions

$(\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*-s}})$  are equivalent to an  $N_*$ -insertion state  $(\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*}})$  with  $s$  of the momenta put to zero whereas states having more than  $N_*$  insertions  $(\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}; n > N_*)$  are null.

Hence, the collective description provides an alternative quantum formulation of the system in terms of field variables. We will now show that, in the  $\hbar \rightarrow 0$  limit, this description also gives the classical equations of motion of non-relativistic pressureless dust.

### Classical limit

We now briefly examine the classical limit of the non-relativistic collective Hamiltonian given in (4.21). In the limit  $\hbar \rightarrow 0$ , the Hamiltonian reduces to,

$$H = \frac{1}{2} \int d^p y \phi(\vec{y}) \left( \vec{\nabla} \pi(\vec{y}) \right)^2 \quad (4.37)$$

The corresponding Hamilton's equations are,

$$\begin{aligned} \frac{\delta H}{\delta \phi(\vec{y})} &= -\partial_t \pi \Rightarrow \partial_t \pi + \frac{(\vec{\nabla} \pi)^2}{2} = 0 \xrightarrow{\vec{\nabla}} \partial_t(\vec{\nabla} \pi) + \vec{\nabla} \pi \cdot \vec{\nabla}(\vec{\nabla} \pi) = 0 \\ \frac{\delta H}{\delta \pi(\vec{y})} &= \partial_t \phi \Rightarrow \partial_t \phi + \vec{\nabla} \cdot (\phi \vec{\nabla} \pi) = 0 \end{aligned} \quad (4.38)$$

Defining the velocity field  $\vec{v} = \vec{\nabla} \pi$ , we get,

$$\partial_t \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{v} = 0, \quad \partial_t \phi + \vec{\nabla} \cdot (\phi \vec{v}) = 0 \quad (4.39)$$

These are precisely the Euler and the continuity equations for a pressureless (dust-like) non-relativistic fluid. Thus, in the classical limit, the collective field theory reproduces the dynamics of pressureless matter. This result is important because, in the large  $T$  limit, the tachyon effective action also describes pressureless dust. Thus, we expect that the relativistic collective theory will reproduce the correct classical tachyon dynamics and, moreover, provide a natural way to include quantum effects through systematic  $\hbar$ -corrections.

### 4.3.2 Relativistic free particles

In this section, we analyze the relativistic free  $N_*$ -particle system in the collective description and study the corresponding Hilbert space structure. We will explicitly show how the classical part of the collective Hamiltonian maps to the tachyon effective Hamiltonian. This will also provide insights into how the tachyon effective action

can be quantised by including the order  $\hbar$  terms that arise from the collective theory.

For the system of relativistic free particles, a collective field theory description can be obtained in an identical way to the non-relativistic case we discussed in previous sections. The steps involved are the same, and the conclusions we could draw from them are also analogous. The only difference is that the relativistic Hamiltonian is different from the non-relativistic Hamiltonian. This will be reflected in the observation that the form of the Hamiltonian in collective theory, as well as energy eigenvalues, will be different.

### Collective Hamiltonian

We start with a system of  $N_*$  relativistic free particles of mass  $m$ , given by the Hamiltonian,

$$H = \sum_{i=1}^{N_*} \sqrt{\vec{p}_i^2 + m^2} \quad (4.40)$$

Let us introduce the collective field  $\phi(x)$  in the same way as in the non-relativistic case (see (4.14)).

$$\phi(\vec{y}) = m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i)$$

As discussed in the previous section, one can use the chain rule of differentiation to write terms of the form  $\sum_i \vec{p}_i^n$  in terms of the collective field  $\phi$  and its conjugate momentum  $\pi$ . By expanding the square root in (4.40) and replacing each term in the expansion with the collective variables, we obtain the first few terms as, (See Appendix C.1.4 for details),

$$\begin{aligned} N_* m &= \int d^p y \phi(\vec{y}) \\ \sum_{i=1}^{N_*} \frac{\vec{p}_i^2}{2m} &= \frac{1}{2} \int d^p y \phi(y) \left( \vec{\nabla} \pi(y) \right)^2 - \frac{i\hbar}{2m} \int d^p y \phi(\vec{y}) \vec{\nabla}^2 \pi(\vec{y}) \\ - \sum_{i=1}^{N_*} \frac{\vec{p}_i^4}{8m^3} &= -\frac{1}{8} \int d^p y \phi(\vec{y}) \left( \vec{\nabla} \pi(\vec{y}) \right)^4 + \frac{i\hbar}{4m} \int d^p y \phi(\vec{y}) \vec{\nabla} \pi(\vec{y}) \cdot \vec{\nabla} \left( \vec{\nabla} \pi(\vec{y}) \right)^2 \\ &\quad + \frac{i\hbar}{4m} \int d^p y \phi(\vec{y}) \vec{\nabla}^2 \pi \left( \vec{\nabla} \pi(\vec{y}) \right)^2 + \frac{\hbar^2}{8m^2} \int d^p y \phi(\vec{y}) \vec{\nabla} \cdot \vec{\nabla} \left( \vec{\nabla} \pi(\vec{y}) \right)^2 \\ &\quad + \frac{\hbar^2}{8m^2} \int d^p y \phi(\vec{y}) \vec{\nabla}^2 \pi(\vec{y}) \vec{\nabla}^2 \pi(\vec{y}) + \frac{\hbar^2}{4m^2} \int d^p y \phi(\vec{y}) \vec{\nabla} \pi(\vec{y}) \cdot \vec{\nabla} \left( \vec{\nabla}^2 \pi(\vec{y}) \right) \\ &\quad - \frac{i\hbar^3}{8m^3} \int d^p y \phi(\vec{y}) \vec{\nabla}^4 \pi(\vec{y}) \end{aligned} \quad (4.41)$$

One can see a general pattern here.  $\sum_i \vec{p}_i^n$  will have terms up to  $O(\hbar^{n-1})$ . Examining the  $O(\hbar^0)$  terms, we see that they can be resummed to get the classical part of the

Hamiltonian.

$$\begin{aligned}
H &= \int d^p y \phi(\vec{y}) + \frac{1}{2} \int d^p y \phi(\vec{y}) \left( \vec{\nabla} \pi(\vec{y}) \right)^2 - \frac{1}{8} \int d^p y \phi(\vec{y}) \left( \vec{\nabla} \pi(\vec{y}) \right)^4 + \dots + O(\hbar) \\
&= \int d^p y \phi(\vec{y}) \left( 1 + \frac{1}{2} \left( \vec{\nabla} \pi(\vec{y}) \right)^2 - \frac{1}{8} \left( \vec{\nabla} \pi(\vec{y}) \right)^4 + \dots \right) + O(\hbar) \\
&= \int d^p y \phi(\vec{y}) \sqrt{1 + \left( \vec{\nabla} \pi(\vec{y}) \right)^2} + O(\hbar)
\end{aligned} \tag{4.42}$$

If we identify  $\phi = \Pi$ ,  $\pi = -T$ , then the  $O(\hbar^0)$  term (classical part) matches precisely the  $T \rightarrow \infty$  limit of the Tachyon effective hamiltonian in (4.5). Therefore, the equations of motion obtained from the leading term in (4.42) will describe the dynamics of a non-interacting, non-rotating fluid. And we can say that the classical parts of collective theory and tachyon effective theory are related by a canonical transformation. The higher-order terms in  $(\hbar)$  can be considered as quantum corrections to the tachyon effective theory Hamiltonian.

### Hilbert space structure

Having obtained the Hamiltonian, one can now write down the energy eigenfunctionals and study the Hilbert space structure. It turns out that the eigenfunctionals have the same form as in the non-relativistic theory, except that their energy eigenvalues change owing to the relativistic dispersion relation. The Hilbert space will be,  $\mathcal{H}_{N_*}$ , the relativistic  $N_*$ -particle Hilbert space.

To see that the form of the wavefunctionals does not change, note that in quantum mechanics, plane waves are eigenstates of the relativistic Hamiltonian, having  $\omega_k = \sqrt{\vec{k}^2 + m^2}$  as the eigenvalue.

$$\begin{aligned}
\sqrt{\vec{p}^2 + m^2} e^{i\vec{k} \cdot \vec{x}} &= \sqrt{-\vec{\nabla}_{\vec{x}}^2 + m^2} e^{i\vec{k} \cdot \vec{x}} \\
&= \sqrt{\vec{k}^2 + m^2} e^{i\vec{k} \cdot \vec{x}} \\
&= \omega_{\vec{k}} e^{i\vec{k} \cdot \vec{x}}
\end{aligned} \tag{4.43}$$

So a general  $N_*$ -particle energy eigenstate is given as,

$$\psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*}) = e^{i\vec{k}_1 \cdot \vec{x}_1} e^{i\vec{k}_2 \cdot \vec{x}_2} \dots e^{i\vec{k}_{N_*} \cdot \vec{x}_{N_*}} + \text{permutations} \tag{4.44}$$

Just like we did in the non-relativistic case, one can now write down the corresponding energy eigenfunctional  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*}}$  of the collective theory Hamiltonian. The guiding idea is that, when the relation (4.14) is imposed, the collective theory wavefunctionals should reproduce the corresponding quantum mechanical wavefunctions. This im-

plies that the form of the eigenfunctionals remains the same as in the non-relativistic theory. Thus, the structure of the Hilbert space remains unchanged going from the non-relativistic to the relativistic case. The Hamiltonian and its eigenvalue, however, now follow the relativistic dispersion relation.

For example, the  $N_* = 1$  case will have eigenstates,

$$\begin{aligned}
\psi_{\vec{k}}(\vec{x}) &= e^{i\vec{k}\cdot\vec{x}} \\
H\psi_{\vec{k}}(\vec{x}) &= \omega_{\vec{k}}\psi_{\vec{k}}(\vec{x}) \\
\Rightarrow \Phi_{\vec{k}} &= \frac{1}{m} \int d^p y e^{i\vec{k}\cdot\vec{y}} \phi(\vec{y}) \\
H\Phi_{\vec{k}} &= \left( m + \frac{\vec{k}^2}{2m} - \frac{\vec{k}^4}{8m^3} + \dots \right) \Phi_{\vec{k}} = \omega_{\vec{k}}\Phi_{\vec{k}}
\end{aligned} \tag{4.45}$$

Similarly, the  $N_* = 2$  case will have eigenstates,

$$\begin{aligned}
\psi_{(\vec{k}_1, \vec{k}_2)}(\vec{x}_1, \vec{x}_2) &= e^{i\vec{k}_1\cdot\vec{x}_1} e^{i\vec{k}_2\cdot\vec{x}_2} + \vec{x}_1 \leftrightarrow \vec{x}_2 \\
H\psi_{(\vec{k}_1, \vec{k}_2)} &= (\omega_{\vec{k}_1} + \omega_{\vec{k}_2}) \psi_{(\vec{k}_1, \vec{k}_2)} \\
\Rightarrow \Phi_{(\vec{k}_1, \vec{k}_2)}[\phi] &= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1\cdot\vec{y}_1} e^{i\vec{k}_2\cdot\vec{y}_2} \phi(\vec{y}_1) \phi(\vec{y}_2) - m \int d^p y e^{i(\vec{k}_1 + \vec{k}_2)\cdot\vec{y}} \phi(\vec{y}) \\
H\Phi_{(\vec{k}_1, \vec{k}_2)}[\phi] &= \left( 2m + \frac{\vec{k}_1^2 + \vec{k}_2^2}{2m} - \frac{\vec{k}_1^4 + \vec{k}_2^4}{8m^3} + \dots \right) \Phi_{(\vec{k}_1, \vec{k}_2)}[\phi] = (\omega_{\vec{k}_1} + \omega_{\vec{k}_2}) \Phi_{(\vec{k}_1, \vec{k}_2)}[\phi]
\end{aligned} \tag{4.46}$$

Note that the collective theory Hamiltonian was obtained by expanding the square root in the quantum mechanical Hamiltonian and then writing each term in terms of collective variables. Therefore, one must check term by term that the states  $\Phi_{\vec{k}}$ ,  $\Phi_{\vec{k}_1, \vec{k}_2}$  defined above are indeed eigenstates (see appendix C.1.4). There is a subtlety here regarding the first term in the Hamiltonian (4.41). Its action on a general state cannot be further simplified a priori.

$$\int d^p y \phi(\vec{y}) \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}$$

This is in contrast to the other terms in the Hamiltonian. For example, acting with the field theory expression for  $\sum_{i=1}^{N_*} \frac{\vec{p}_i^2}{2m}$ , we directly get the correct eigenvalue.

$$\left( \frac{1}{2} \int d^p y \phi(y) \left( \vec{\nabla} \pi(y) \right)^2 - \frac{i\hbar}{2m} \int d^p y \phi(\vec{y}) \vec{\nabla}^2 \pi(\vec{y}) \right) \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} = \left( \sum_{i=1}^n \frac{\vec{k}_i^2}{2m} \right) \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}$$

So, we would expect the action of  $\int d^p y \phi(\vec{y})$  to give an eigenvalue equation. This is ensured by the Jacobian. One can show that working with collective field theory at

fixed  $N_*$ , the operator  $\int d^p y \phi(\vec{y})$  is equivalent to  $N_* m$ . (We discuss the proof in C.1.5).

$$\begin{aligned} \left( \Phi_1, \int d^p y \phi(\vec{y}) \Phi_2 \right) &= N_* m \quad \forall \Phi_1, \Phi_2 \\ \Rightarrow \int d^p y \phi(\vec{y}) \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} &= N_* m \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} \end{aligned} \quad (4.47)$$

So, the Jacobian makes sure of the fact that the operator  $\int d^p y \phi(\vec{y})$  is equivalent to  $N_* m$ , i.e.,  $\int d^p y \phi(\vec{y}) = N_* m$ . This result will have implications for understanding the Hilbert space of collective theory.

Once we have the states, we can now compute the inner product between the states to understand the Hilbert-space structure. Note that the form of the Jacobian remains unchanged when going from the non-relativistic case to the relativistic case. Since the Jacobian determines which states are null and which states are equivalent to each other, the conclusions we obtained in the non-relativistic case continue to hold here:

For a fixed  $N_*$  relativistic collective theory,

1. The Hilbert space  $\mathcal{H}_{N_*}$  is spanned by  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*}}$  which are  $N_*$  particle energy eigenstates with momenta  $\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*}$ .
2. The state  $\mathcal{C}$  is to be understood as an  $N_*$  particle state ( $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*}}$ ) with all particles at rest and the state  $\Phi_{\vec{k}}$  is an  $N_*$  particle state with one particle having momentum  $\vec{k}$  and the remaining  $N_* - 1$  particles at rest. Generalising this, one can say  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}$ ;  $n < N_*$  corresponds to an  $N_*$ -particle state having  $n$  of the particles with momenta  $\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n$  and the remaining  $N_* - n$  particles at rest.
3. The states  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}$  for  $n > N_*$  are all null states.

We conclude the section by writing down different energy eigenstates. They are all  $N_*$ -particle states with different numbers of particles at rest.

$$\begin{aligned} \mathcal{C} &= 1 \\ \Phi_{\vec{k}} &= \frac{1}{m} \int d^p y e^{i\vec{k} \cdot \vec{y}} \phi(\vec{y}) \\ \Phi_{\vec{k}_1, \vec{k}_2} &= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1} e^{i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \phi(\vec{y}_2) - \frac{1}{m} \int d^p y e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \phi(\vec{y}) \\ &\dots\dots\dots \\ \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} &= \frac{1}{m^n} \int \prod_{i=1}^n \left( d^p y_i e^{i\vec{k}_i \cdot \vec{y}_i} \right) \phi(\vec{y}_1) \prod_{j=2}^n \left( \phi(\vec{y}_j) - m \sum_{i=1}^{j-1} \delta(\vec{y}_j - \vec{y}_i) \right) \end{aligned} \quad (4.48)$$

## 4.4 From Collective theory to Tachyon effective theory

In the previous sections, we showed that the collective Hamiltonian for relativistic particles agrees with the tachyon effective Hamiltonian at the classical level. This suggests that the collective description may provide a useful way to study the quantum structure of the tachyon theory. In particular, the matching of the classical Hamiltonians indicates that the collective variables  $(\phi, \pi)$  and the tachyon variables  $(\Pi, T)$  are related by a canonical transformation,

$$\phi = \Pi, \quad \pi = -T. \quad (4.49)$$

In this section, we study this map in more detail. As we will see, the situation is more subtle in tachyon theory. Imposing physical requirements, we see that the Jacobian appearing in the inner product is modified, the naive collective field theory eigenstates are not eigenstates of the tachyon Hamiltonian, and the interpretation of the states is different. To understand these differences more clearly, we begin by examining an important distinction between the collective description and the tachyon effective theory.

### 4.4.1 Difference between tachyon theory and collective theory

There is an important difference between the  $N_*$ -particle collective theory and the tachyon effective theory. In the collective description, the field  $\phi$  satisfies the constraint

$$\int d^p y \phi(\vec{y}) = N_* m.$$

In contrast, the corresponding quantity

$$\int d^p y \Pi(\vec{y})$$

in the tachyon theory represents the total rest mass of the system and is not fixed; it can take different values depending on the energy of the original  $Dp$ -brane configuration. Therefore, in order to pass from the collective theory to the tachyon theory, we must relax the constraint  $\int d^p y \phi(\vec{y}) = N_* m$ . Relaxing this condition effectively allows  $N_*$  to vary. This has the following consequences:

- The condition  $\int d^p y \phi(\vec{y}) = N_* m$  is enforced by the Jacobian  $J_{N_*}$ . If this condition is relaxed, the Jacobian must also be modified.
- In collective theory, the states listed in (4.48) are  $N_*$ -particle energy eigenstates. However, these states no longer remain eigenstates of the tachyon Hamiltonian.

This also changes the way we interpret these states: they can now be viewed as states with different particle numbers rather than having a fixed  $N_*$ -number of particles. In ordinary quantum field theory, different particle-number sectors are connected by creation and annihilation operators. These operators change the eigenvalue of the number operator and allow us to move from one particle-number sector to another. As we will see, an analogous set of well-defined operators also emerges in tachyon theory once this condition is relaxed.

To understand how such a structure may arise in the present case, let us first recall how this works in the Klein–Gordon theory. In the Klein–Gordon theory, the number operator is given by

$$N = \int \frac{d^p k}{(2\pi)^p} a_{\vec{k}}^\dagger a_{\vec{k}}.$$

It satisfies the commutation relations

$$[N, a_{\vec{k}}] = -a_{\vec{k}}, \quad [N, a_{\vec{k}}^\dagger] = a_{\vec{k}}^\dagger. \quad (4.50)$$

These relations show that  $a_{\vec{k}}$  and  $a_{\vec{k}}^\dagger$  change the eigenvalue of  $N$ : the operator  $a_{\vec{k}}$  lowers the particle number, while  $a_{\vec{k}}^\dagger$  raises it. In this way, they connect different particle-number sectors.

In terms of these operators, the Hamiltonian takes the simple form

$$H = \int \frac{d^p k}{(2\pi)^p} \omega_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}}. \quad (4.51)$$

We now ask whether an analogue of  $N$ ,  $a_{\vec{k}}$  and  $a_{\vec{k}}^\dagger$  exists in the tachyon theory. The answer is yes. Moreover, it will allow us to write the tachyon Hamiltonian in a compact form similar to (4.51). We now construct these operators explicitly.

Motivated by the relation  $\int d^p y \phi(\vec{y}) = N_* m$  in collective theory, we define the number operator in tachyon theory as

$$N = \frac{1}{m} \int d^p y \Pi(\vec{y}) \quad (4.52)$$

We now look for operators  $a_{\vec{k}}$  and  $a_{\vec{k}}^\dagger$  satisfying (4.50). With some effort, one can verify that the following operators satisfy these relations:

$$\begin{aligned} a_{\vec{k}} &= \int d^p y e^{-i\vec{k} \cdot \vec{y}} e^{m \frac{\delta}{\delta \Pi(\vec{y})}} \\ a_{\vec{k}}^\dagger &= \frac{1}{m} \int d^p y e^{i\vec{k} \cdot \vec{y}} \Pi(\vec{y}) e^{-m \frac{\delta}{\delta \Pi(\vec{y})}} \end{aligned} \quad (4.53)$$

<sup>3</sup> For example, one finds that,

$$\begin{aligned}
[N, a_{\vec{k}}] &= \left[ \frac{1}{m} \int d^p y' \Pi(\vec{y}'), \int d^p y e^{-i\vec{k} \cdot \vec{y}} e^{m \frac{\delta}{\delta \Pi(\vec{y})}} \right] \\
&= \frac{1}{m} \int d^p y' d^p y e^{-i\vec{k} \cdot \vec{y}} \left[ \Pi(\vec{y}'), e^{m \frac{\delta}{\delta \Pi(\vec{y})}} \right] \\
&= - \int d^p y' d^p y e^{-i\vec{k} \cdot \vec{y}} \delta(\vec{y} - \vec{y}') e^{m \frac{\delta}{\delta \Pi(\vec{y})}} \\
&= -a_{\vec{k}}
\end{aligned} \tag{4.54}$$

In the 3rd equality, we used the result,

$$\left[ \Pi(\vec{y}'), e^{m \frac{\delta}{\delta \Pi(\vec{y})}} \right] = m \delta(\vec{y} - \vec{y}') e^{m \frac{\delta}{\delta \Pi(\vec{y})}}, \tag{4.55}$$

which can be easily derived from the BCH formula,

$$e^{m \frac{\delta}{\delta \Pi(\vec{y})}} \Pi(\vec{y}') e^{-m \frac{\delta}{\delta \Pi(\vec{y})}} = \Pi(\vec{y}') + m \delta^p(\vec{y} - \vec{y}'). \tag{4.56}$$

Similarly, one can check that  $a_{\vec{k}}^\dagger$  satisfies the required commutation relation. Moreover, these operators obey

$$[a_{\vec{k}_1}, a_{\vec{k}_2}^\dagger] = (2\pi)^p \delta^p(\vec{k}_1 - \vec{k}_2). \tag{4.57}$$

We have therefore relaxed the constraint  $\int d^p y \phi(\vec{y}) = N_* m$  when going from collective theory to tachyon theory. In the tachyon theory, the corresponding quantity  $\int d^p y \Pi(y)$  is proportional to the number operator and is unrestricted. This allows us to introduce creation and annihilation operators. As we mentioned earlier, this also means that the Jacobian in tachyon theory must differ from the one in collective theory. It turns out that demanding consistency of the structure we introduced so far uniquely determines the Jacobian. We will see this explicitly in the next subsection.

#### 4.4.2 Jacobian of the tachyon theory

Recall that in collective theory, the inner product, for a fixed  $N_*$ , is defined as

$$(\Phi_1, \Phi_2) = \int D\phi J_{N_*}[\phi] \Phi_1^* \Phi_2. \tag{4.58}$$

In tachyon theory, however, we are not working within a fixed  $N_*$  sector. As we have seen, the operators  $a_{\vec{k}}$  and  $a_{\vec{k}}^\dagger$  do not commute with the number operator  $\int d^p y \Pi(y)$ , and therefore they allow us to move between different particle sectors. For this reason,

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<sup>3</sup>Restoring  $\hbar$ , we get,  $a_{\vec{k}} = \frac{1}{\hbar} \int d^p y e^{-\frac{i\vec{k} \cdot \vec{y}}{\hbar}} e^{m \frac{\delta}{\delta \Pi(\vec{y})}}$  and  $a_{\vec{k}}^\dagger = \frac{1}{m} \int d^p y e^{\frac{i\vec{k} \cdot \vec{y}}{\hbar}} \Pi(\vec{y}) e^{-m \frac{\delta}{\delta \Pi(\vec{y})}}$ . The expressions for  $H$  and  $N$  remains unchanged in terms of  $a_{\vec{k}}$  and  $a_{\vec{k}}^\dagger$ .

we cannot use the Jacobian  $J_{N^*}$  that appears in the collective theory inner product. Instead, we define the tachyon theory inner product with an arbitrary Jacobian and then determine it uniquely by requiring that the creation and annihilation operators are Hermitian conjugates of each other. We explain this construction in this subsection.

Consider a general inner product of the form,

$$(\Phi_1[\Pi], \Phi_2[\Pi]) = \int D\Pi J[\Pi] \Phi_1[\Pi]^* \Phi_2[\Pi]. \quad (4.59)$$

Note that under this inner product, the Hermitian conjugate of  $\frac{\delta}{\delta\Pi(\vec{y})}$  is given by,

$$\left(\frac{\delta}{\delta\Pi(\vec{y})}\right)^\dagger = -\frac{\delta}{\delta\Pi(\vec{y})} - \frac{1}{J[\Pi]} \frac{\delta J[\Pi]}{\delta\Pi(\vec{y})}. \quad (4.60)$$

So, the condition that  $a_k^\dagger$  and  $a_{\vec{k}}$  are Hermitian conjugates of each other can be written as,

$$\begin{aligned} (a_k^\dagger)^\dagger &= a_{\vec{k}} \\ \Rightarrow \frac{1}{m} \int d^p \vec{y} e^{-i\vec{k} \cdot \vec{y}} e^{m\left(\frac{\delta}{\delta\Pi(\vec{y})} + \frac{1}{J} \frac{\delta J[\Pi]}{\delta\Pi(\vec{y})}\right)} \Pi(\vec{y}) &= \int d^p y e^{-i\vec{k} \cdot \vec{y}} e^{m\frac{\delta}{\delta\Pi(\vec{y})}} \Pi(\vec{y}) \\ \Rightarrow \frac{1}{m} e^{-m\frac{\delta}{\delta\Pi(\vec{y})}} e^{m\left(\frac{\delta}{\delta\Pi(\vec{y})} + \frac{\delta \ln J[\Pi]}{\delta\Pi(\vec{y})}\right)} \Pi(\vec{y}) &= 1 \end{aligned} \quad (4.61)$$

One can simplify this expression further. First, note that the argument of the second exponential can be written as,

$$m\frac{\delta}{\delta\Pi(\vec{y})} + m\frac{\delta \ln J[\Pi]}{\delta\Pi(\vec{y})} = e^{-\ln J[\Pi]} m\frac{\delta}{\delta\Pi(\vec{y})} e^{\ln J[\Pi]} \quad (4.62)$$

This is of the form  $BAB^{-1}$ , and it can be exponentiated using  $e^{BAB^{-1}} = Be^AB^{-1}$ . This leads to,

$$e^{m\left(\frac{\delta}{\delta\Pi(\vec{y})} + \frac{\delta \ln J[\Pi]}{\delta\Pi(\vec{y})}\right)} = e^{-\ln J[\Pi]} e^{m\frac{\delta}{\delta\Pi(\vec{y})}} e^{\ln J[\Pi]} \quad (4.63)$$

Substituting this in (4.61) will simplify it further if one uses the fact that the adjoint action of  $e^{-m\frac{\delta}{\delta\Pi(\vec{y})}}$  on a functional  $F[\Pi]$  will shift the argument of  $F$  by a delta function peaked at  $\vec{y}$  having strength  $m$ .

$$e^{-m\frac{\delta}{\delta\Pi(\vec{y})}} F[\Pi(\vec{y}')] e^{m\frac{\delta}{\delta\Pi(\vec{y})}} = F[\Pi(\vec{y}') - m\delta(\vec{y}' - \vec{y})] \equiv F[\Pi - m\delta_y] \quad (4.64)$$

We introduced a compact notation in the last step. One can apply (4.63) and (4.64) in (4.61) to obtain,

$$\begin{aligned}
& \frac{1}{m} e^{-m \frac{\delta}{\delta \Pi(\vec{y})}} e^{-\ln J[\Pi]} e^{m \frac{\delta}{\delta \Pi(\vec{y})}} e^{\ln J[\Pi]} \Pi(\vec{y}) = 1 \\
& \Rightarrow \frac{1}{m} e^{-\ln J[\Pi - m\delta_y]} e^{\ln J[\Pi]} \Pi(\vec{y}) = 1 \\
& \Rightarrow \frac{1}{m} e^{\ln J[\Pi] - \ln J[\Pi - m\delta_y]} \Pi(\vec{y}) = 1 \\
& \Rightarrow J[\Pi - m\delta_y] = \frac{\Pi(\vec{y})}{m} J[\Pi]
\end{aligned} \tag{4.65}$$

To solve for the final expression, go to functional momentum space and write,

$$J[\Pi] = \int D\lambda e^{i \int d^p y \lambda(\vec{y}) \Pi(\vec{y})} J[\lambda] \tag{4.66}$$

In momentum space, the condition (4.65) reduces to,

$$\frac{i}{m} \frac{\delta J[\lambda]}{\delta \lambda(\vec{y})} = e^{-im\lambda(\vec{y})} J[\lambda] \tag{4.67}$$

This can be solved to get,

$$J[\lambda] = \exp \left\{ \int d^p y e^{-im\lambda(\vec{y})} \right\} \tag{4.68}$$

Writing down the series expansion of the exponential puts the Jacobian in a form familiar from the collective field theory discussions.

$$\begin{aligned}
J[\Pi] &= \int D\lambda e^{i \int d^p y \lambda(\vec{y}) \Pi(\vec{y})} \exp \left\{ \int d^p x e^{-im\lambda(\vec{x})} \right\} \\
&= \int D\lambda e^{i \int d^p y \lambda(\vec{y}) \Pi(\vec{y})} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int d^p x e^{-im\lambda(\vec{x})} \right)^n \\
&= \sum_{n=0}^{\infty} \frac{J_n[\Pi]}{n!}
\end{aligned} \tag{4.69}$$

where  $J_n[\Pi]$  corresponds precisely to the Jacobian associated with the  $N_* = n$  sector of the collective theory. Thus, the Jacobian in tachyon theory is naturally a sum over all  $N_*$ -sector Jacobians. This is consistent with the operator structure: the operators  $a_{\vec{k}}$  and  $a_{\vec{k}}^\dagger$  connect different  $N_*$  sectors.

The Jacobian will play an important role in the Hilbert space structure of the tachyon theory. On a technical note, one can immediately see that using the Jacobian (4.69), the states listed in (4.48) (under the exchange of  $\phi$  with  $\Pi$ ) will have inner products different from those in collective theory. Here, one has to compute the collective field theory inner product using  $J_n$  and then perform the sum over  $n$  weighted by  $\frac{1}{n!}$ . We

will address the implications of these changes in section 4.4.5.

### 4.4.3 Tachyon theory Hamiltonian

We now discuss the Hamiltonian in tachyon theory. We know from the classical limit of collective field theory that the tachyon theory variables and collective theory variables are related by a canonical transformation (4.49). We further suggest that the Hamiltonian obtained from the collective theory Hamiltonian under this transformation is the tachyon theory Hamiltonian.

$$H_{tachyon} = \int d^p y \Pi(\vec{y}) + \frac{1}{2} \int d^p y \Pi(\vec{y}) \left( \vec{\nabla} T(y) \right)^2 + \frac{i\hbar}{2m} \int d^p y \Pi(\vec{y}) \vec{\nabla}^2 T(y) + \dots \quad (4.70)$$

The first term describes the correct classical dynamics of a rolling tachyon near the minimum of the potential, while the  $O(\hbar)$  and higher order terms can be considered as quantum corrections. In other words, we are adding  $O(\hbar)$  and higher order terms to the Hamiltonian given in Section 4.2, to incorporate quantum corrections. To show the effectiveness of these  $\hbar$ -corrections, we first show, using the creation and annihilation operators introduced in the last section (4.53), that the Hamiltonian (4.70) has a compact form analogous to the Klein-Gordon Hamiltonian given in (4.51)<sup>4</sup>.

$$\begin{aligned} H_{tachyon} &= \int \frac{d^p k}{(2\pi)^p} \omega_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}} \\ &= \int \frac{d^p k}{(2\pi)^p} \left\{ m + \frac{\vec{k}^2}{2m} - \frac{\vec{k}^4}{8m^3} + \dots \right\} \left( \frac{1}{m} \int d^p y e^{i\vec{k} \cdot \vec{y}} \Pi(\vec{y}) e^{-m \frac{\delta}{\delta \Pi(\vec{y})}} \right) \left( \int d^p y' e^{-i\vec{k} \cdot \vec{y}'} e^{m \frac{\delta}{\delta \Pi(\vec{y}')}} \right) \end{aligned}$$

Consider the first term in the curly brackets. It is easy to show that this matches precisely with the first term in (4.70).

$$\begin{aligned} &\int \frac{d^p k}{(2\pi)^p} m \left( \frac{1}{m} \int d^p y e^{i\vec{k} \cdot \vec{y}} \Pi(\vec{y}) e^{-m \frac{\delta}{\delta \Pi(\vec{y})}} \right) \left( \int d^p y' e^{-i\vec{k} \cdot \vec{y}'} e^{m \frac{\delta}{\delta \Pi(\vec{y}')}} \right) \\ &= \int d^p y d^p y' (2\pi)^p \delta(\vec{y} - \vec{y}') \left( \frac{1}{m} \Pi(\vec{y}) e^{-m \frac{\delta}{\delta \Pi(\vec{y})}} \right) \left( e^{m \frac{\delta}{\delta \Pi(\vec{y}')}} \right) \\ &= \int d^p y \Pi(\vec{y}) \end{aligned} \quad (4.71)$$

By similar manipulations, one can show that the other terms will also match (see Appendix C.1.4 for details). It is then trivial to check that the Hamiltonian will have the required commutation relations.

$$[H_{tachyon}, a_{\vec{k}}] = -\omega_{\vec{k}} a_{\vec{k}}, \quad [H_{tachyon}, a_{\vec{k}}^\dagger] = \omega_{\vec{k}} a_{\vec{k}}^\dagger \quad (4.72)$$

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<sup>4</sup>One has to restore  $\hbar$  factors in  $a_{\vec{k}}$  and  $a_{\vec{k}}^\dagger$  to get the precise matching with (4.70); however, we will let  $\hbar = 1$  in the derivations for simplicity.

Thus, we find a close similarity between the tachyon theory and the Klein–Gordon theory. In particular, one can introduce creation and annihilation operators in terms of which both the number operator and the Hamiltonian can be written. These operators satisfy the usual commutation relations, just as in the Klein–Gordon theory.

#### 4.4.4 Theory A and Theory B

A lot has happened in the last few sections. It might be a good point to pause and reflect on the developments.

1. In the  $\hbar \rightarrow 0$  limit, the collective theory Hamiltonian (4.42) agrees with the tachyon effective Hamiltonian (4.5) (with the identification:  $\phi = \Pi$ ,  $\pi = -T$ ). *One can say that this is the main motivation to study collective field theory, that the classical dynamics looks the same.*
2. When one considers all the  $\hbar$ -terms in the collective theory Hamiltonian, and takes into account the non-trivial Jacobian  $J_{N_*}$ , the theory becomes a consistent quantum theory describing  $N_*$  relativistic free particles. The Hilbert space of the theory is  $\mathcal{H}_{N_*}$ .
3. One would've then suggested adding identical  $\hbar$ -terms to the tachyon effective theory Hamiltonian to get  $H_{tachyon}$  (4.70), defining the Jacobian of the theory to be  $J_{N_*}$  and claiming that the tachyon effective theory is also describing quantum mechanics of  $N_*$  relativistic particles. *But this is not what we are doing.*
4. What we are doing instead is to take  $H_{tachyon}$  as the Hamiltonian, but change the Jacobian to  $J[\phi] = \sum_{n=0}^{\infty} \frac{J_n[\phi]}{n!}$ . The change in Jacobian results from imposing the physical condition that  $\int d^p y \Pi(\vec{y})$  should be unconstrained.

So we have two consistent quantum theories, Theory A and Theory B.

| Name     | Canonical variables | Hamiltonian   | Jacobian                                             |
|----------|---------------------|---------------|------------------------------------------------------|
| Theory A | $\Pi, T$            | $H_{tachyon}$ | $J_{N_*}[\phi]$                                      |
| Theory B | $\Pi, T$            | $H_{tachyon}$ | $J[\phi] = \sum_{n=0}^{\infty} \frac{J_n[\phi]}{n!}$ |

Theory A describes the quantum mechanics of  $N_*$  relativistic free particles. In this theory, due to the Jacobian  $J_{N_*}$ , the value of  $\int d^p y \Pi(\vec{y})$  is fixed to be  $N_* m$ .

Theory B, on the other hand, has  $\int d^p y \Pi(\vec{y}) = Nm$  where  $N = \int \frac{d^p k}{(2\pi)^p} a_k^\dagger a_{\vec{k}}$  is a number operator, having an unconstrained spectrum. The creation and annihilation operators take you from one eigensector of the number operator to another. We will study the Hilbert space of Theory B in Section 4.4.5.

At the level of classical equations of motion, obtained in the  $\hbar \rightarrow 0$  limit, Theory A and Theory B are the same. However, quantum mechanically, they are different. Physical properties of Theory B are more similar to what we expect in a quantum version of tachyon effective theory. *Thus, we suggest that Theory B should be equated with tachyon effective theory. This is what we mean by quantising tachyon effective theory.* In other words, by adding the  $\hbar$ -terms to the Hamiltonian and introducing the Jacobian, we map the tachyon effective theory to Theory B, which is a consistent quantum theory as it stands.

Since we spent a considerable amount of time studying operators, states, inner products, etc., in Collective theory (Theory A), it will be important to point out which of the results derived there will be valid/useful for Theory B (Tachyon theory).

- Collective theory was used to get to Theory A. After that, we make a departure from collective theory by imposing physical requirements on the theory. This introduces more structure (like creation/annihilation operators), which in some sense "deforms" the theory to get to Theory B. This theory has the desired properties that one would expect to have in a quantum tachyon theory.
- The form of the operators (Hamiltonian, Momentum, etc) in Theory A and Theory B will be the same
- As the Hamiltonian is the same for both Theory A and Theory B, one would expect the list of states given in (4.48) to be the eigenstates (under,  $\phi = \Pi$ ,  $\pi = -T$ ) of both Theory A and Theory B. This is almost true, except for the first term in  $H_{tachyon}$ . Given an energy eigenstate  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}$  from (4.48),

$$\begin{aligned} \text{Theory A: } H_{tachyon} \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} &= \left( N_* m + \sum_{i=1}^n \frac{\vec{k}_i^2}{2m} + \dots \right) \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} \\ \text{Theory B: } H_{tachyon} \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} &= \left( \int d^p y \Pi(\vec{y}) + \sum_{i=1}^n \frac{\vec{k}_i^2}{2m} + \dots \right) \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} \end{aligned}$$

We used the fact that  $\int d^p y \Pi(\vec{y}) = N_* m$  in Theory A, but no such restriction is there in B. This difference is crucial, and it means that the states  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}$  will be  $N_*$ -particle energy eigenstates in Theory A, but in Theory B, they are no longer energy eigenstates and their physical interpretation will be different.

- The inner products will be different in Theory A and Theory B. However, since the Jacobian in Theory B is  $J[\phi] = \sum_{n=0}^{\infty} \frac{J_n[\phi]}{n!}$ , we can use methods developed in collective theory to compute the inner product between any two states using  $J_n$ , and then sum over  $n$  with the weight  $\frac{1}{n!}$  to get the inner product in Theory B (See Appendix C.2). This, along with the last point, implies that the Hilbert space

structure of Theory A is different from that of Theory B.

In the next section, we will study the Hilbert space structure of Theory B (Or tachyon theory), and compare how it differs from Theory A (Or collective field theory). We will also enquire about the similarities between Theory B and Klein-Gordon theory, because Theory B has creation and annihilation operators in terms of which the Hamiltonian looks exactly like the Klein-Gordon Hamiltonian.

#### 4.4.5 Hilbert space of tachyon theory

The tachyon theory has similarities with both collective theory and Klein-Gordon theory. Our starting point was the assertion that the tachyon theory Hamiltonian is the same as the collective theory Hamiltonian up to a canonical transformation. We then further imposed the physical requirement of the existence of different particle number sectors on the tachyon theory. This led to the introduction of creation and annihilation operators, which expressed the number operator and Hamiltonian in a form analogous to Klein-Gordon theory. However, in this section, we will see that the Hilbert-space structure of tachyon theory is different from those of both collective theory and Klein-Gordon theory.

In Klein-Gordon theory, the Hilbert space is spanned by states of the form,

$$|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\rangle = \left( \prod_{i=1}^n a_{\vec{k}_i}^\dagger \right) |0\rangle \quad (4.73)$$

Here  $|0\rangle$  denotes the vacuum state. These states form the usual Fock space and are eigenstates of the Hamiltonian (4.51). Since the tachyon Hamiltonian also has the form (4.51), one might expect the eigenstates of tachyon theory to be of the same form (4.73). However, this is not the case. In particular, the vacuum state  $|0\rangle$  is not accessible in the tachyon theory description.

Recall that the definition of the vacuum state is that it is annihilated by the operator  $a_{\vec{k}}$ . Suppose  $\Psi[\Pi]$  is the vacuum state in tachyon theory. Then,

$$\begin{aligned} a_{\vec{k}} \Psi[\Pi] &= 0 \\ \int d^p y \, e^{-i\vec{k} \cdot \vec{y}} e^{m \frac{\delta}{\delta \Pi(\vec{y})}} \Psi[\Pi] &= 0 \\ \Rightarrow \int d^p y \, e^{-i\vec{k} \cdot \vec{y}} \Psi[\Pi(\vec{y}') + m\delta(\vec{y} - \vec{y}')] &= 0 \\ \Rightarrow \Psi[\Pi(\vec{y}') + m\delta(\vec{y} - \vec{y}')] &= 0 \\ \Rightarrow \Psi[\Pi] &= 0 \end{aligned} \quad (4.74)$$

Thus, the only solution is the trivial one. This means that one cannot express the vacuum state in the tachyon theory variables. This can be explained using the fact that the rolling tachyon solution in string theory is distinct from the vacuum configuration where the tachyon is placed at the minimum of the potential, having zero energy. This is reflected on the tachyon theory side in the fact that the vacuum is inaccessible to the theory. We will comment more on this in 4.4.5.

The discussion above shows that even though the operators and their commutation relations are similar, the Hilbert space of tachyon theory is different from that of Klein–Gordon theory. In particular, the vacuum state is not a part of the tachyon theory Hilbert space.

This leaves us with the set of states given in (4.48), which are the energy eigenstates in collective theory. Since the Hamiltonians of collective theory and tachyon theory have the same form, one would expect that the eigenstates of collective theory (under the substitution  $\phi = \Pi$  and  $\pi = -T$ ) will be eigenstates of the tachyon theory. However, this is not the case as we will show below.<sup>5</sup>

First, note that the repeated action of creation operators on  $\mathcal{C}$  generates precisely the set of states given in (4.48). This can be verified as follows,

$$\begin{aligned} a_{\vec{k}}^{\dagger} \mathcal{C} &= \frac{1}{m} \int d^p y \, e^{i\vec{k} \cdot \vec{y}} \Pi(\vec{y}) e^{-m \frac{\delta}{\delta \Pi(\vec{y})}} \cdot 1 \\ &= \frac{1}{m} \int d^p y \, e^{i\vec{k} \cdot \vec{y}} \Pi(\vec{y}) \\ &= \Phi_{\vec{k}} \end{aligned} \tag{4.75}$$

And, in general, an  $n$ -particle energy eigenstate of collective theory is compactly written as,

$$\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} = \prod_{i=1}^n a_{\vec{k}_i}^{\dagger} \mathcal{C} \tag{4.76}$$

It is good to re-emphasise here that in collective theory, the state given above is an  $N_*$ -particle energy eigenstate having  $n$  non-zero momenta, labelled by  $\{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\}$ . However, in tachyon theory, they cannot be interpreted as such because the form of the Jacobian is different.<sup>6</sup>

From the form of these states, it is clear that, if the states  $\prod_{i=1}^n a_{\vec{k}_i}^{\dagger} \mathcal{C}$  are to be eigenstates of  $H$ , then the state  $\mathcal{C}$  itself must be an eigenstate of  $H$  (or equivalently of  $N$ ). But this

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<sup>5</sup>The collective theory states in (4.48) are given in the  $\phi$  basis. When we talk about the same states in tachyon theory, it is implicit that the  $\phi$  is replaced with  $\Pi$ .

<sup>6</sup>For example, see (C.42). Here, it is important that the Jacobian  $J_{N_*}$  imposes  $\int d^p y \, \phi(\vec{y}) = N_* m$ . This is not the case in tachyon theory as the Jacobian is  $J[\phi] = \sum_{n=0}^{\infty} \frac{I_n}{n!}$ , leaving  $\int d^p y \, \Pi(\vec{y})$  unconstrained.

is not the case. This can be easily shown using the action of  $a_{\vec{k}}$  on  $\mathcal{C}$ .

$$\begin{aligned} a_{\vec{k}} \mathcal{C} &= \int d^p y e^{-i\vec{k}\cdot\vec{y}} e^{m \frac{\delta}{\delta\Pi(\vec{y})}} \cdot 1 \\ &= \int d^p y e^{-i\vec{k}\cdot\vec{y}} \\ &= (2\pi)^p \delta^p(\vec{k}) \cdot \mathcal{C}. \end{aligned} \tag{4.77}$$

This shows that  $\mathcal{C}$  is an eigenstate of the annihilation operator. Hence, it will not be an eigenstate of  $N$  or  $H^7$ . This implies that the states  $\prod_{i=1}^n a_{\vec{k}_i}^\dagger \mathcal{C}$  are also not eigenstates of the tachyon Hamiltonian. On the other hand, (4.77) tells you that  $\mathcal{C}$  is a coherent state. So the states  $\prod_{i=1}^n a_{\vec{k}_i}^\dagger \mathcal{C}$  should be interpreted as excitations built on top of this coherent state.

Formally, coherent states can be written as exponentials of creation operators acting on the vacuum. In the present case, the above relations imply that  $\mathcal{C}$  may be written as

$$|\mathcal{C}\rangle = e^{a_0^\dagger} |0\rangle \tag{4.78}$$

where the creation operator carries zero momentum. The state  $|0\rangle$  here denotes the tachyon vacuum. Although we have argued earlier that the tachyon vacuum cannot be accessed within the effective theory, it is still useful to introduce it formally in order to express the coherent state in this way. Thus, the lowest energy state accessible to tachyon theory is  $\mathcal{C}$ , a coherent state created by the zero-momentum creation operator acting on the tachyon vacuum, i.e., a coherent state of particles at rest.

Let us now summarise the main points of this section. We started with the Hamiltonian of the tachyon theory (4.70). This Hamiltonian is obtained from the collective theory Hamiltonian (4.41) through the canonical transformations (4.49). Moreover, it has a form similar to the Klein–Gordon Hamiltonian (4.51), upon using creation and annihilation operators (4.53). Despite these similarities, the Hilbert space of tachyon theory is different from those of both Klein–Gordon theory and collective theory. Comparing with Klein–Gordon, we see that the tachyon theory does not have an analogue of the vacuum state. So, one does not have the usual Fock space states of the form (4.73) in tachyon theory. On the other hand, comparing with collective theory, we see that, owing to the change in the form of the Jacobian (4.69), the lowest energy state  $\mathcal{C}$  becomes a coherent state of particles at rest and therefore not an eigenstate of the Hamiltonian. These results suggest that the Hilbert space of tachyon theory consists

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<sup>7</sup>As  $\mathcal{C}$  is an eigenstate of  $a_{\vec{k}}$ , it should also be an eigenstate of  $a_{\vec{k}}^\dagger$  in order to be an eigenstate of  $N$ . But this is not possible as  $a_{\vec{k}}^\dagger$  and  $a_{\vec{k}}$  do not commute with each other and hence cannot have simultaneous eigenstates.

of a coherent state  $\mathcal{C}$  of particles at rest, and then excitations on top of it <sup>8</sup>.

$$\mathcal{H}_{tachyon} = \text{span} \left\{ \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} = a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger \dots a_{\vec{k}_n}^\dagger \mathcal{C}; n \in \mathbb{Z}_{\geq 0} \right\} \quad (4.79)$$

The state  $\mathcal{C}$  is the lowest energy state as all the particles are at rest. One can then increase the energy and particle number by acting with creation operators on  $\mathcal{C}$ . The inner product between any two states in (4.79) can be obtained either by directly using the Jacobian (4.69), or by using commutation relations between creation and annihilation operators and the norm of  $\mathcal{C}$ . Relevant computations are compiled in C.2.

This concludes our discussion of the tachyon theory Hilbert space. We have a coherent state  $\mathcal{C}$  and excitations on top of this. *Thus, the tachyon effective theory captures a subsector of Klein-Gordon theory.* To see this more clearly, recall that in Klein-Gordon theory, one can also construct coherent states. In particular, a coherent state can be produced by acting with the creation operator with zero momentum. The excitations of the field can then be viewed as fluctuations on top of this coherent state. From this point of view, the tachyon theory can be understood as describing this sector of the Klein-Gordon theory: a coherent state corresponding to the zero-momentum mode together with the excitations built on top of it.

One can understand the physical meaning of this coherent state from the closed string analysis presented in [99]. In string theory, the decaying D-brane acts as a time-dependent source for closed string radiation. In the case of homogeneous decay, where the tachyon field depends only on time, the final closed string state is a coherent state of closed string modes  $e^{\alpha a^\dagger} |0\rangle$ . The coherent state  $\mathcal{C}$  that appears in the tachyon analysis can be understood in a similar way. Excitations on top of this coherent state take into account inhomogeneities in the initial D-brane configuration, corresponding to small spatial variations of the tachyon field on the brane. Given this interpretation, the energy of the coherent state can be related to the energy of the initial D-brane configuration. Moreover, one would expect the quantity  $\int d^p y \Pi(\vec{y})$  to have a continuous spectrum, whereas the number operator  $N$  has a discrete spectrum. So, one would expect the quantity  $\langle \int d^p y \Pi(\vec{y}) \rangle$  to be related to the continuous parameter  $\alpha$ . In the next section, we will see how to go about this.

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<sup>8</sup>There is a distinction between the case of a D-brane having finite volume ( $V$ ) and the  $V \rightarrow \infty$  case. We are primarily interested in a D-brane of infinite volume. More details on this distinction are given in Appendix C.2.4

## Coherent state and initial D-brane system

In this section, we study the energy associated with the coherent state  $\mathcal{C}$ . We begin by writing down the expectation value of the Hamiltonian  $H$  in this state (C.2),

$$\frac{(\mathcal{C}, H_{tachyon} \mathcal{C})}{(\mathcal{C}, \mathcal{C})} = mV \quad (4.80)$$

One would like the state  $|\mathcal{C}\rangle$  to have energy parameterised by a real number. This is because the original D-brane system can have different values of energy, which is a sum of the D-brane tension and the energy (kinetic+potential) of the rolling tachyon. This should be reflected in the energy of the coherent state in the tachyon theory. Also, the closed string analysis performed in [99] suggests the coherent state of a closed string mode has the form  $e^{\alpha a^\dagger}|0\rangle$  where  $\alpha$  is the one-point disk amplitude of the closed string mode. So the coherent state in tachyon theory should also admit a similar parametrization. This can be achieved by redefining the creation and annihilation operators in tachyon theory (4.53) by multiplying  $a_{\vec{k}}$  by a real number  $\mathfrak{C}$  and  $a_{\vec{k}}^\dagger$  by  $\frac{1}{\mathfrak{C}}$  such that the commutation relations and expressions for  $H, N$  are unchanged.

$$\begin{aligned} a_{\vec{k}} &= \mathfrak{C} \int d^p y e^{-i\vec{k}\cdot\vec{y}} e^{m \frac{\delta}{\delta \Pi(\vec{y})}} \\ a_{\vec{k}}^\dagger &= \frac{1}{\mathfrak{C}} \frac{1}{m} \int d^p y e^{i\vec{k}\cdot\vec{y}} \Pi(\vec{y}) e^{-m \frac{\delta}{\delta \Pi(\vec{y})}} \end{aligned} \quad (4.81)$$

With this definition, the action of the annihilation operator on the state  $\mathcal{C}$  becomes,

$$a_{\vec{k}} \mathcal{C} = (2\pi)^p \delta(\vec{k}) \mathfrak{C} \mathcal{C} \quad (4.82)$$

In Klein–Gordon notation, the state can therefore be written as a coherent state of the zero- momentum mode.

$$|\mathcal{C}\rangle = e^{\mathfrak{C} a_0^\dagger} |0\rangle \quad (4.83)$$

However, one should note that, unlike in a Klein-Gordon theory, here  $a_{\vec{k}}$  and  $a_{\vec{k}}^\dagger$  depend on a real number  $\mathfrak{C}$ . What this essentially means is that the Jacobian will also have a  $\mathfrak{C}$ -dependence given by the following equation,

$$J^\mathfrak{C}[\Pi] = \int D\lambda e^{\int d^p y \lambda(\vec{y}) \Pi(\vec{y})} \exp \left\{ \int d^p x \mathfrak{C}^2 e^{-im\lambda(\vec{y})} \right\}. \quad (4.84)$$

This result can easily be seen from a slight modification of the calculations given in 4.4.2. What we have done essentially is to change the Jacobian such that the wave-functional  $\mathcal{C}[\Pi]$  now represents the state  $e^{\mathfrak{C} a_0^\dagger}$  instead of  $e^{a_0^\dagger}$ . Alternatively, one could've stayed with the Jacobian  $J[\Pi]$ , in which case the state  $\mathcal{C}$  will represent  $e^{a_0^\dagger}|0\rangle$ , such that to get to another coherent state  $e^{\mathfrak{C} a_0^\dagger}$  one has to act with the operator  $e^{(\mathfrak{C}-1)a_0^\dagger}$  on  $\mathcal{C}$ . The

technical aspects of this fact are discussed in Appendix C.2.3.

The norm and expectation value of  $H$  can now be computed to be,

$$\begin{aligned} (\mathcal{C}, \mathcal{C}) &= e^{\mathfrak{E}^2 V} \\ \frac{(\mathcal{C}, H_{tachyon} \mathcal{C})}{(\mathcal{C}, \mathcal{C})} &= \mathfrak{E}^2 m V \end{aligned} \quad (4.85)$$

Since the coherent state  $\mathcal{C}$  has particles at rest, the expectation value of the number operator ( $N$ ) times  $m$  also coincides with the expectation value of energy given above.

$$\langle \int d^p y \Pi(\vec{y}) \rangle \equiv \frac{(\mathcal{C}, Nm \mathcal{C})}{(\mathcal{C}, \mathcal{C})} = \mathfrak{E}^2 m V \quad (4.86)$$

So, the quantity  $\langle \int d^p y \Pi(\vec{y}) \rangle$  is parametrised by a real number  $\mathfrak{E}^2$  and physically it can be used to encode the energy of the original D-brane configuration so that different D-brane decays will result in different coherent states. Under such an interpretation, the quantity  $\mathfrak{E}^2$  is proportional to the initial energy of the rolling tachyon configuration. Note that  $\mathfrak{E} \rightarrow 0$  is a singular limit: the annihilation operator vanishes in this limit, and the creation operator blows up. One would've expected the  $\mathfrak{E}^2 \rightarrow 0$  limit to describe the vacuum state where the tachyon is placed at the minimum of the potential. However, this is not the case, and this is in accordance with the fact that time-dependent rolling solutions are distinct from the vacuum solution [96, 120, 121]. Even if the initial configuration has a slight positive energy  $\epsilon > 0$ , the tachyon will roll towards the minimum, and it will not reach the vacuum in finite time. This translates into the fact that the vacuum state is not accessible in the tachyon effective theory, as we see from our analysis.

## 4.5 Discussion

In this chapter, we studied the quantization of the tachyon effective action proposed in [114] in the late-time limit. At the classical level, the tachyon system behaves like non-interacting, non-rotating dust. This motivated us to ask whether a similar picture exists at the quantum level, and in particular, whether the spectrum resembles that of relativistic free particles. Because plane wave solutions are absent in this theory, the usual perturbative quantization cannot be applied. We therefore had to look for an alternative way to quantize the system. We thus turned to collective field theory, motivated by the fact that pressureless dust - described in terms of continuous variables such as energy density and velocity- microscopically corresponds to a collection of free relativistic particles. Collective field theory provides precisely this type of con-

struction, where a system having discrete degrees of freedom is described in terms of continuous field variables. Furthermore, we found that the classical part of relativistic collective field theory gives the dynamics of free relativistic dust, matching the tachyon effective theory in the large  $T$  limit. This relation provides a natural way to incorporate systematic  $\hbar$  corrections to the tachyon Hamiltonian so as to map it to a quantum theory of free particles. Using this framework, we could determine the spectrum of the tachyon theory. We found that the lowest energy state of the theory is a coherent state  $\mathcal{C}$  of particles at rest, and the excited states are built on top of this coherent state. So, the Hilbert-space structure of tachyon theory consists of the coherent state  $\mathcal{C}$  and excitations on top of this, i.e., a subsector of Klein-Gordon theory.

This structure admits a natural interpretation from the point of view of string theory. In string theory, tachyons are associated with unstable D-branes, and the rolling of tachyons on an unstable D-brane is described by classical time-dependent solutions in open string theory [96]<sup>9</sup>. Studying the energy-momentum tensor of this system, one observes that in the late time limit, it behaves like pressureless dust localised on the plane of the original D-brane. On the other hand, in a closed string description, the decaying D-brane acts as a time-dependent source producing closed string radiation. In the  $g_s \rightarrow 0$  limit, one can see that the closed string radiation produced from the decay is localised along the plane of the initial D-brane [98, 99, 124–128]. So, in both open string analysis and closed string analysis, the classical properties of the final system resulting from the decay of a D-brane are the same. In this chapter, we advance this equality to a quantum level. The analysis performed in [99] suggests that the final closed string state produced from a homogeneous decay of a D-brane is a coherent state of the closed string mode. The lowest energy state we obtain in the tachyon effective theory is also a coherent state. This is *suggestive* of a duality between the open string description and closed string description. In our description, apart from the coherent state, we also have excitations on top of the coherent state. This could be interpreted as the fact that there can be inhomogeneities due to open-string excitations on the initial D-brane.

Another interesting result is that the vacuum state is not accessible in tachyon theory. This can also be understood from known results in string theory [96, 98, 129]. The vacuum configuration is a time-independent solution where the tachyon sits exactly at the minimum of its potential, and the system has zero energy and pressure [120, 121]. The rolling tachyon solution, on the other hand, is a time-dependent solution in open string theory where the tachyon is displaced from the maximum of the potential and rolls towards the minimum. In an ordinary scalar field theory, one would expect the

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<sup>9</sup>Also see [122, 123] for recent discussions on rolling tachyon solutions from the point of view of OSFT.

field to oscillate around the minimum of the potential. However, this does not happen for the tachyon. Instead, the tachyon rolls down the potential while keeping its energy density constant, and it never reaches the minimum in finite time. Even if the tachyon has a small positive energy, it continues to roll towards the minimum [96]. In our analysis, the energy of the coherent state  $|\mathcal{C}\rangle$  is proportional to  $|\mathfrak{C}|^2 > 0$ . This means that our description can only describe time-dependent solutions of string theory where the initial system has at least a small positive energy to begin with. Hence, the vacuum state is not accessible in this description.

The results we obtained in this analysis are amenable to an interpretation in terms of an ‘unusual localisation of information’. The key point to note here is that the open string description of a decaying D-brane matches the closed string description of the same. This is different from the usual equivalence between the open string description and the closed string description in perturbative string theory for stable D-branes. The situation we have is more interesting as we are dealing with unstable D-branes.

*Even after the D-brane has completely decayed away into closed string radiation, the original open string degrees of freedom know some things about the closed string states resulting from the decay. This is the precise sense in which we attribute a notion of ‘unusual localisation of information’ to this scenario.*

However, we emphasise that our results are only suggestive of such an interpretation, and one has to do a string theory computation at finite  $g_s$  to prove the equivalence. We hope to pursue this in the future.

# Chapter 5

## Outlook

We tried to understand how, in theories of gravity, information is encoded in a peculiar way compared to local QFTs. In Chapters 2 and 3, we looked at this from an effective field theory perspective. For flat spacetime, working in perturbation theory in  $G_N$  and keeping only the leading-order contribution, we saw that gravitational Gauss' law will lead to holography of information, even for finite regions. We showed it explicitly for unitary excitations on top of the vacuum in a fixed coordinate system, and it would be useful to frame it in a more general way, using gravitationally dressed gauge-invariant observables, and by using algebraic QFT techniques. However, even at the simplistic level in which we approached the problem, it is clear that gravitational Gauss' law is more powerful than its counterpart in other gauge theories, such that quantum information gets stored holographically.

In the case of de Sitter spacetime, the implications of the gravitational constraints are much more constraining, owing to the compactness of the Cauchy slice. In flat spacetime, any general state of the form of excitations on top of the vacuum will continue to be an allowed state in the gravitational theory. Here, the effects of gravity will be to holographically store information regarding this state at the boundary of space-time. This is not the case in de Sitter, in the sense that only  $SO(1, d + 1)$  invariant states in a de Sitter QFT would be uplifted to allowed states (solutions of the Wheeler-DeWitt equation) in a gravitational theory. This has serious implications as we saw in Chapter 3, where we computed cosmological correlators in gravitationally constrained de Sitter states. The lower-point correlators in perturbative states with a small number of excitations show signatures of holography, which makes them different from the usual de Sitter QFT correlations in the vacuum. One has to introduce a heavy observer state and coarse-grain the related observables to obtain QFT-like correlations from this gravitationally constrained Hilbert space. This analysis sheds light on the peculiar properties of gravitational theories and also teaches us that even within the constrained theory,

there is room to maneuver, so as to get back to our notion of locality and so on.

Finally, in Chapter 4, we studied a scenario different from the ones in the previous chapters. Here, we studied decaying D-branes in string theory, and quantised the tachyon effective theory using collective field theory methods. The resulting Hilbert space consists of coherent states and excitations on top, such that it is a sub-sector of the Klein-Gordon theory. We suggested an interpretation of this result along the lines of [97, 98], that the original open string degrees of freedom seem to know something about the closed strings into which the D-brane decays. Thus, we see an ‘unusual localisation of information’ in this context also, although it is different to the notion of holography of information that we saw in the previous chapters. However, the key point to note here is that all these are theories of gravity, and information seems to have unusual localisation properties.

From a broader perspective, we can think of it in the following way. From String Theory, we got to the AdS/CFT correspondence, which gave a concrete example of holographic duality. Inspired by this, we could look for signatures of holography even in low-energy effective theories of gravity, and this led to an understanding of the holography of information. We could now go back to String Theory and look for scenarios, like the one we have discussed, where an unusual localisation of information could be seen. This seems to be an interesting direction to explore, and we hope to pursue this in the future.

# Appendix A

## Technical details for Chapter 2

In this section, we compile various computations related to Chapter 2.

- Section [A.1](#) computes the quadratic fluctuations of the finite-radius Hamiltonian operator  $H_R$  in the vacuum. This quantity diverges in the  $R \rightarrow \infty$  limit. This means that one has to be careful while taking the large  $R$  limit. The correct procedure is to define a smeared version of  $H_R$  and then take the limit.
- The gravitational Gauss' law used in [\(2.3\)](#) is derived in Section [A.2](#).
- Technical details involved in inverting the integral equations to extract information about the functions  $f_n$  are discussed in Section [A.3](#).

### A.1 Quadratic fluctuations of the Hamiltonian

Consider the mode expansion of a massive scalar field in asymptotically flat space in 3+1 dimensions with mostly plus signature

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \left( a_{\vec{k}} e^{ik \cdot x} + a_{\vec{k}}^\dagger e^{-ik \cdot x} \right) \quad \text{with} \quad [a_{\vec{k}}, a_{\vec{k}'}^\dagger] = \delta^{(3)}(\vec{k} - \vec{k}') . \quad (\text{A.1})$$

The corresponding Hamiltonian density is then given by,

$$\begin{aligned}
\mathcal{H}(t, \vec{x}) &:= \frac{1}{2} : \left( \dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2 \right) : \\
&= -\frac{1}{2} \int \frac{d^3 k d^3 q \left( \omega_{\vec{k}} \omega_{\vec{q}} + \vec{k} \cdot \vec{q} \right)}{2 (2\pi)^3 \sqrt{\omega_{\vec{k}} \omega_{\vec{q}}}} \left[ a_{\vec{k}} a_{\vec{q}} e^{-i(\omega_{\vec{k}} + \omega_{\vec{q}})t + i(\vec{k} + \vec{q}) \cdot \vec{x}} \right. \\
&\quad \left. + a_{\vec{k}}^\dagger a_{\vec{q}}^\dagger e^{i(\omega_{\vec{k}} + \omega_{\vec{q}})t - i(\vec{k} + \vec{q}) \cdot \vec{x}} - a_{\vec{k}}^\dagger a_{\vec{q}} e^{-i(\omega_{\vec{k}} + \omega_{\vec{q}})t + i(\vec{k} - \vec{q}) \cdot \vec{x}} \right. \\
&\quad \left. - a_{\vec{q}}^\dagger a_{\vec{k}} e^{i(\omega_{\vec{k}} + \omega_{\vec{q}})t - i(\vec{k} - \vec{q}) \cdot \vec{x}} \right], \tag{A.2}
\end{aligned}$$

Integrating this density over a finite region of space from, say,  $r = 0$  to  $r = R$  allows us to define

$$\begin{aligned}
H(R, t) &:= \int_{|\vec{x}| \leq R} d^3 x \mathcal{H}(t, \vec{x}) = \int_0^R r^2 dr \int_0^\pi \sin(\theta) d\theta \int_0^{2\pi} d\phi \mathcal{H}(t, r, \theta, \phi) \\
&= \frac{1}{2} \int \frac{d^3 k d^3 q \left( \omega_{\vec{k}} \omega_{\vec{q}} + \vec{k} \cdot \vec{q} \right)}{(2\pi)^2 \sqrt{\omega_{\vec{k}} \omega_{\vec{q}}}} \left[ a_{\vec{k}}^\dagger a_{\vec{q}} \mathcal{D}(k - q, R, t) - a_{\vec{k}} a_{\vec{q}} \mathcal{D}(k + q, R, t) + \text{h.c.} \right], \tag{A.3}
\end{aligned}$$

where we defined

$$\mathcal{D}(q, R, t) := \mathcal{D}(q, R) e^{-iq^0 t} := \frac{\sin(|\vec{q}| R) - |\vec{q}| R \cos(|\vec{q}| R)}{|\vec{q}|^3} e^{-iq^0 t}. \tag{A.4}$$

We now smear the above operator (A.3) in time in terms of a small length scale  $\eta$ , say about the  $t = 0$  slice, as

$$H_{\text{sm}} := \int_{-\infty}^{+\infty} dt \mathfrak{g}(t) H(R, t) \quad \text{with} \quad \mathfrak{g}(t) := \frac{1}{\eta \sqrt{\pi}} \exp\left(-\frac{t^2}{\eta^2}\right). \tag{A.5}$$

Carrying out the time-smeared integral in (A.5), we find

$$H_{\text{sm}} = \frac{1}{2} \int \frac{d^3 k d^3 q \left( \omega_{\vec{k}} \omega_{\vec{q}} + \vec{k} \cdot \vec{q} \right)}{(2\pi)^2 \sqrt{\omega_{\vec{k}} \omega_{\vec{q}}}} \left[ a_{\vec{k}}^\dagger a_{\vec{q}} \tilde{\mathcal{D}}(k - q, R, \eta) - a_{\vec{k}} a_{\vec{q}} \tilde{\mathcal{D}}(k + q, R, \eta) + \text{h.c.} \right], \tag{A.6}$$

where we now defined

$$\tilde{\mathcal{D}}(q, R, \eta) := \mathcal{D}(q, R) \exp\left(-\frac{(q^0)^2 \eta^2}{4}\right). \tag{A.7}$$

Therefore, we see that the vacuum expectation value of the quadratic fluctuations of this smeared operator takes the form

$$\langle \Omega | H_{\text{sm}}^2 | \Omega \rangle = \int dt_1 dt_2 \mathfrak{g}(t_1) \mathfrak{g}(t_2) H(R, t_1) H(R, t_2) . \quad (\text{A.8})$$

Since

$$\int_{-\infty}^{+\infty} dt \mathfrak{g}(t) e^{\pm i b t} = \exp\left(-\frac{b^2 \eta^2}{4}\right) , \quad (\text{A.9})$$

we see that the quadratic fluctuations of the momentum integrals in the smeared Hamiltonian at finite radius converge owing to the exponential suppression.

In the large  $R$  limit, however, the operator is still divergent. To deal with this, in [6], an additional radial smearing was proposed to ensure that the  $R \rightarrow \infty$  limit does not result in divergent quadratic fluctuations for massless scalar fields. A refined smearing can be employed to deal with the massive case. In this thesis, we only concern ourselves with the determination of a localised state inside a ball of the fixed radius  $R$  in terms of correlation functions on the boundary of the said ball, with no large  $R$  limit. To this end, a time-smeared Hamiltonian suffices.

## A.2 Finite-radius Hamiltonian as a boundary operator

In this section, we review the standard result that the Hamiltonian defined in (2.4) is indeed a boundary term on the surface of the ball. To this end, let us consider Einstein's equations in the presence of matter sources:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} . \quad (\text{A.10})$$

Perturbing the metric about flat space as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \quad (\text{A.11})$$

we have the following first-order variations of its derivatives

$$\begin{aligned} \Gamma_{\nu\lambda}^{\mu(1)} &= \frac{1}{2} \eta^{\mu\rho} (\partial_\nu h_{\rho\lambda} + \partial_\lambda h_{\rho\nu} - \partial_\rho h_{\nu\lambda}) \\ R_{\mu\nu}^{(1)} &= \frac{1}{2} (\partial^\lambda \partial_\mu h_{\lambda\nu} + \partial^\lambda \partial_\nu h_{\lambda\mu} - \partial^2 h_{\mu\nu} - \partial_\mu \partial_\nu h) \\ R^{(1)} &= \partial^\lambda \partial^\rho h_{\lambda\rho} - \partial^2 h . \end{aligned} \quad (\text{A.12})$$

As is common practice, we defined the quantities  $h := \eta^{\mu\nu} h_{\mu\nu}$  and  $\partial^2 := \partial^\mu \partial_\mu$ . Considering higher order variations, we may in general write

$$R_{\mu\nu} = R_{\mu\nu}^{(1)} + \sum_n R_{\mu\nu}^{(n)} \quad \text{and} \quad R = R^{(1)} + \sum_n R^{(n)}. \quad (\text{A.13})$$

The summation here includes all the non-linear terms in  $h_{\mu\nu}$ . In this notation, Einstein's equations can now be written as

$$\begin{aligned} R_{\mu\nu}^{(1)} - \frac{1}{2} \eta_{\mu\nu} R^{(1)} &= 8\pi G T_{\mu\nu} + \frac{1}{2} h_{\mu\nu} R + \frac{1}{2} \eta_{\mu\nu} \sum_n R^{(n)} - \sum_n R_{\mu\nu}^{(n)} \\ &=: 8\pi G \left( T_{\mu\nu} + T_{\mu\nu}^h \right), \end{aligned} \quad (\text{A.14})$$

where we have defined all the non-linear terms in metric perturbations suggestively as  $T_{\mu\nu}^h$ . The reason for doing so is the simple observation that

$$\eta^{\mu\rho} \partial_\rho \left( R_{\mu\nu}^{(1)} - \frac{1}{2} \eta_{\mu\nu} R^{(1)} \right) = 0 \quad \text{which implies that} \quad \eta^{\mu\rho} \partial_\rho \left( T_{\mu\nu} + T_{\mu\nu}^h \right) = 0. \quad (\text{A.15})$$

Therefore, we may consider  $T_{\mu\nu}^h$  as the energy-momentum tensor associated with the fluctuations  $h_{\mu\nu}$ . This allows us to now integrate both sides of the equations (A.14) inside the ball of radius  $R$ , using the unit vector in the radial direction  $n_i$ , as

$$\int_{|\vec{x}| \leq R} d^3 \vec{x} \left( T_{00} + T_{00}^h \right) = \frac{1}{8\pi G} \int_{|\vec{x}| \leq R} d^3 \vec{x} \left( R_{00}^{(1)} - \frac{1}{2} \eta_{00} R^{(1)} \right) \quad (\text{A.16})$$

$$= \frac{1}{16\pi G} \int_{|\vec{x}| \leq R} d^3 \vec{x} \partial_i (\partial_j h_{ij} - \partial_i h_{jj}) \quad (\text{A.17})$$

$$= \frac{1}{16\pi G} \int_{|\vec{x}|=R} d^2 x n_i (\partial_j h_{ij} - \partial_i h_{jj}). \quad (\text{A.18})$$

In this thesis, we focus on linearised gravity<sup>1</sup> and therefore ignore the energy-momentum tensor associated with the gravitons. With the knowledge that the energy-momentum tensor of a free Klein-Gordon scalar field is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left( g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi + m^2 \phi^2 \right), \quad (\text{A.19})$$

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<sup>1</sup>It is an interesting exercise to extend our results to the non-linear gravitational theory, which we expect to be straightforward at quadratic order.

we see that

$$\begin{aligned}
\int_{|\vec{x}| \leq R} d^3 \vec{x} T_{00} &= \frac{1}{2} \int_{|\vec{x}| \leq R} d^3 \vec{x} \left( \dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2 \right) \\
&= \int_{|\vec{x}| \leq R} d^3 \vec{x} \mathcal{H}(t, \vec{x}) \\
&= \frac{1}{16\pi G} \int_{|\vec{x}|=R} d^2 x n_i (\partial_j h_{ij} - \partial_i h_{jj}) .
\end{aligned} \tag{A.20}$$

Therefore, we see that the finite-radius Hamiltonian is indeed a boundary term in terms of metric fluctuations in linearised quantum gravity. Of course, we may include corrections to this by including non-linear terms in the metric fluctuations.

### A.3 Inversion for a correlator from its moments

In this appendix, we describe what information we may infer about the functions  $f_p(\{\vec{x}_p\})$  given measurements of the kind discussed in eq. (2.22) and eq. (2.23).

#### A.3.1 Inversion for bulk correlators

We begin with the result for the generic bulk correlator of interest (2.22):

$$\begin{aligned}
\langle \psi | \pi(\vec{y}) \phi^m(\{\vec{y}_m\}) | \psi \rangle - \langle 0 | \pi(\vec{y}) \phi^m(\{\vec{y}_m\}) | 0 \rangle &= \\
= -i\gamma \int \sum_{i=1}^{\infty} \left( \prod_{j=1}^i d^3 \vec{x}_k \right) f_i(\{\vec{x}_i\}) \sum_{k=1}^i \langle 0 | : \phi^{i-1}(\{\vec{x}_{i-1}\}) : \phi^m(\{\vec{y}_m\}) | 0 \rangle &\left[ i\delta^3(|\vec{x}_k - \vec{y}|) \right] .
\end{aligned} \tag{A.21}$$

As we saw in section 2.3, the one-point function allows us to determine the function  $f_1(\vec{x}_1)$  defining a one-particle state. To illustrate how we may invert this integral equation for the function  $f_i(\{\vec{x}_i\})$  in general, let us now specialise to the case of the two-point function where  $m = 1$ :

$$\begin{aligned}
\mathcal{O}_2(\vec{y}_1, \vec{y}_2) &:= \langle \psi | \pi(\vec{y}_1) \phi(\vec{y}_2) | \psi \rangle - \langle 0 | \pi(\vec{y}_1) \phi(\vec{y}_2) | 0 \rangle \\
&= \frac{m\gamma}{4\pi^2} \int d^3 \vec{x}_1 \left( \frac{K_1(m|\vec{y}_2 - \vec{x}_1|)}{|\vec{y}_2 - \vec{x}_1|} \right) \left[ f_2(\vec{x}_1, \vec{y}_1) + f_2(\vec{y}_1, \vec{x}_1) \right] \\
&=: \frac{m\gamma}{4\pi^2} \int d^3 \vec{x}_1 \left( \frac{K_1(m|\vec{y}_2 - \vec{x}_1|)}{|\vec{y}_2 - \vec{x}_1|} \right) \tilde{f}_2(\vec{y}_1, \vec{x}_1) .
\end{aligned} \tag{A.22}$$

We first observe that the dependence on one of the arguments,  $\vec{y}_1$ , is explicit in this equation. Therefore, given a fixed  $\vec{y}_2$ , we may simply make various measurements everywhere inside the ball to determine the dependence of the function  $\tilde{f}_2$  on  $\vec{y}_1$ .

Moreover, in order for this correlator to be well-behaved, we assume that the function  $f_2(\vec{x}_1, \vec{y}_1)$  is such that the correlator does not blow up when  $\vec{x}_1$  approaches  $\vec{y}_2$ . In what follows, we will use the identities:

$$\frac{1}{\Gamma(n+1)} \int_0^\infty ds s^n e^{-s|\vec{y}_2 - \vec{x}_1|^m} = \frac{1}{|\vec{y}_2 - \vec{x}_1|^{m(n+1)}} \quad (\text{A.23})$$

$$K_n(z) = \Gamma\left[n + \frac{1}{2}\right] \frac{(2z)^n}{\sqrt{\pi}} \int_0^\infty du \frac{\cos(u)}{(u^2 + z^2)^{n+\frac{1}{2}}}, \quad (\text{A.24})$$

combining which we have

$$\begin{aligned} \frac{K_q(m|\vec{y}_2 - \vec{x}_1|)}{|\vec{y}_2 - \vec{x}_1|^l} &= \Gamma\left[q + \frac{1}{2}\right] \frac{(2m)^q}{\sqrt{\pi}} |\vec{y}_2 - \vec{x}_1|^{q-l} \int_0^\infty du \frac{\cos(u)}{(u^2 + m^2|\vec{y}_2 - \vec{x}_1|^2)^{q+\frac{1}{2}}} \\ &= \frac{(2m)^q}{\sqrt{\pi}} |\vec{y}_2 - \vec{x}_1|^{q-l} \\ &\quad \times \int_0^\infty du \cos(u) \int_0^\infty ds s^{q-\frac{1}{2}} \exp\left[-s(u^2 + m^2|\vec{y}_2 - \vec{x}_1|^2)\right]. \end{aligned} \quad (\text{A.25})$$

Therefore, we may write

$$\begin{aligned} \mathcal{O}_2(\vec{y}_1, \vec{y}_2) &= \frac{m^2\gamma}{2\pi^2\sqrt{\pi}} \int d^3\vec{x} \left[ \int_0^\infty du \cos(u) \right. \\ &\quad \times \left. \int_0^\infty ds s^{\frac{1}{2}} \exp\left[-s(u^2 + m^2|\vec{y}_2 - \vec{x}|^2)\right] \right] \tilde{f}_2(\vec{y}_1, \vec{x}) \\ &= \frac{m^2\gamma}{2\pi^2\sqrt{\pi}} \sum_{\ell m} \int_0^\infty |\vec{x}|^2 d|\vec{x}| \left[ \int_0^\infty du \cos(u) \int_0^\infty ds s^{\frac{1}{2}} e^{-s(u^2 + m^2|\vec{y}_2|^2 + m^2|\vec{x}|^2)} \right. \\ &\quad \left. \int d\Omega_{\vec{x}}^{(2)} e^{2sm^2\vec{y}_2 \cdot \vec{x}} \right] \tilde{f}_2^{\ell m}(\vec{y}_1, |\vec{x}|) Y_\ell^m(\Omega_{\vec{x}}^{(2)}) \\ &= \frac{2m^2\gamma}{\pi\sqrt{\pi}} \sum_{\ell m} \int_0^\infty |\vec{x}|^2 d|\vec{x}| \left[ \int_0^\infty du \cos(u) \int_0^\infty ds s^{\frac{1}{2}} e^{-s(u^2 + m^2|\vec{y}_2|^2 + m^2|\vec{x}|^2)} \right. \\ &\quad \left. (2sm^2|\vec{y}_2|)^\ell i_\ell(2sm^2|\vec{y}_2||\vec{x}|) \right] \tilde{f}_2^{\ell m}(\vec{y}_1, |\vec{x}|) Y_\ell^m(\Omega_{\vec{y}_2}), \end{aligned} \quad (\text{A.26})$$

where  $i_\ell(x)$  is the spherical Bessel function that respects the identity

$$\int d\Omega_{\vec{x}}^{(2)} e^{2sm^2\vec{y}_2 \cdot \vec{x}} Y_\ell^m(\Omega) = 4\pi (2sm^2|\vec{y}_2|)^\ell Y_\ell^m(\Omega_{\vec{y}_2}) i_\ell(2sm^2|\vec{y}_2||\vec{x}|). \quad (\text{A.27})$$

The spherical Bessel I function can also be written in terms of reverse Bessel functions as

$$i_\ell(x) = \frac{e^{-x}\theta_\ell(x) - e^x\theta_\ell(-x)}{2(-x)^{\ell+1}} \quad \text{with} \quad \theta_\ell(x) := \sum_{n=0}^{\ell} \frac{x^{\ell-n}(\ell+n)!}{2^n n! (\ell-n)!}. \quad (\text{A.28})$$

This implies that

$$i_\ell(2sm^2|\vec{y}_2||\vec{x}|) = \sum_{n=0}^{\ell} \left[ \frac{e^{-(2sm^2|\vec{y}_2||\vec{x}|)} - e^{(2sm^2|\vec{y}_2||\vec{x}|)}(-1)^{\ell-n}}{(2sm^2|\vec{y}_2||\vec{x}|)^{n+1}} \right] \frac{(\ell+n)!}{2^{n+1}n!(-1)^{\ell+1}(\ell-n)!}. \quad (\text{A.29})$$

Therefore, we have that

$$\begin{aligned} \int_0^\infty ds s^{\frac{1}{2}} e^{-s(u^2+m^2|\vec{y}_2|^2+m^2|\vec{x}|^2)} (2sm^2|\vec{y}_2|)^\ell i_\ell(2sm^2|\vec{y}_2||\vec{x}|) &= \\ &= \sum_{n=0}^{\ell} \left( \frac{(-1)^{-n} (u^2 + m^2(|\vec{x}| - |\vec{y}_2|)^2)^{-\ell+n-\frac{1}{2}} - (-1)^{-\ell} (u^2 + m^2(|\vec{x}| + |\vec{y}_2|)^2)^{-\ell+n-\frac{1}{2}}}{2^{2n+2-\ell} m^{2n+2-2\ell} |\vec{y}_2|^{-\ell} (|\vec{x}||\vec{y}_2|)^{n+1}} \right. \\ &\quad \left. \times \Gamma\left[\ell - n + \frac{1}{2}\right] \frac{(\ell+n)!}{n! (\ell-n)!} \right) \quad \text{with} \quad n < \frac{1}{2} + \ell. \end{aligned} \quad (\text{A.30})$$

We may now easily carry out the  $u$ -integral in (A.26) to find:

$$\begin{aligned} \int_0^\infty du \cos(u) \int_0^\infty ds s^{\frac{1}{2}} e^{-s(u^2+m^2|\vec{y}_2|^2+m^2|\vec{x}|^2)} (2sm^2|\vec{y}_2|)^\ell i_\ell(2sm^2|\vec{y}_2||\vec{x}|) &= \\ &= \sum_{n=0}^{\ell} \frac{\sqrt{\pi} m^{\ell-n} (\ell+n)!}{2^{n+2} m^2 n! (\ell-n)!} \frac{|\vec{y}_2|^\ell}{|\vec{x}|^{n+1} |\vec{y}_2|^{n+1}} \\ &\quad \times \left[ (-1)^{-n} \frac{K_{\ell-n}(m||\vec{x}| - |\vec{y}_2||)}{||\vec{x}| - |\vec{y}_2||^{\ell-n}} - (-1)^{-\ell} \frac{K_{\ell-n}(m|\vec{x}| + |\vec{y}_2|)}{|\vec{x}| + |\vec{y}_2|^{\ell-n}} \right]. \end{aligned} \quad (\text{A.31})$$

By making many measurements at various angular positions of  $\vec{y}_2$ , we may expand them in spherical harmonics. Then, from the orthogonality of the spherical harmonics, (A.26) implies

$$\begin{aligned} \mathcal{O}_2^{\ell m}(\vec{y}_1, |\vec{y}_2|) &= \frac{\gamma}{\pi} \sum_{n=0}^{\ell} \int_0^\infty \frac{|\vec{x}|^2 d|\vec{x}| |\vec{y}_2|^\ell}{|\vec{x}|^{n+1} |\vec{y}_2|^{n+1}} \frac{m^{\ell-n} (\ell+n)!}{2^{n+1} n! (\ell-n)!} \tilde{f}_2^{\ell m}(\vec{y}_1, |\vec{x}|) \\ &\quad \times \left[ (-1)^{-n} \frac{K_{\ell-n}(m||\vec{x}| - |\vec{y}_2||)}{||\vec{x}| - |\vec{y}_2||^{\ell-n}} - (-1)^{-\ell} \frac{K_{\ell-n}(m|\vec{x}| + |\vec{y}_2|)}{||\vec{x}| + |\vec{y}_2||^{\ell-n}} \right]. \end{aligned} \quad (\text{A.32})$$

This integral can now be split into a region where  $|\vec{x}| < |\vec{y}_2|$  and another where  $|\vec{x}| > |\vec{y}_2|$ . In the former, we may expand the integrand in a power series in  $\frac{|\vec{x}|}{|\vec{y}_2|}$  whereas in the latter, the expansion may be done in  $\frac{|\vec{y}_2|}{|\vec{x}|}$ . This expansion takes the following form:

$$\begin{aligned} \mathcal{O}_2^{\ell m}(\vec{y}_1, |\vec{y}_2|) &= \frac{\gamma}{\pi} \sum_{n=0}^{\ell} \int_0^{\infty} \frac{|\vec{x}|^2 d|\vec{x}| |\vec{y}_2|^{\ell}}{|\vec{x}|^{n+1} |\vec{y}_2|^{n+1}} \frac{m^{\ell-n} (\ell+n)!}{2^{n+1} n! (\ell-n)!} \tilde{f}_2^{\ell m}(\vec{y}_1, |\vec{x}|) \\ &\times \begin{cases} \sum_{q=0}^{\infty} |\vec{x}|^q c_{\ell n}^{(q)}(|\vec{y}_2|) & \text{when } \frac{|\vec{x}|}{|\vec{y}_2|} < 1 \\ \sum_{q=0}^{\infty} |\vec{y}_2|^q c_{\ell n}^{(q)}(|\vec{x}|) & \text{when } \frac{|\vec{x}|}{|\vec{y}_2|} > 1, \end{cases} \end{aligned} \quad (\text{A.33})$$

where the kernels can be worked out systematically

$$\begin{aligned} c_{\ell n}^{(0)}(r) &= \left[ (-1)^n - (-1)^{\ell} \right] \left( \frac{1}{r} \right)^{\ell-n} K_{\ell-n}(mr) \\ c_{\ell n}^{(1)}(r) &= \left[ (-1)^n + (-1)^{\ell} \right] m \left( \frac{1}{r} \right)^{\ell-n} K_{\ell-n+1}(mr) \\ c_{\ell n}^{(2)}(r) &= \frac{1}{2} \left[ (-1)^{\ell} - (-1)^n \right] m \left( \frac{1}{r} \right)^{\ell-n+1} \left[ mr K_{\ell-n}(mr) + (2\ell - 2n + 1) K_{\ell-n+1}(mr) \right] \\ &\vdots \end{aligned}$$

This expansion results in a system of linear equations for different points of measurement  $|\vec{y}_2|$  with coefficients being the various moments of the function  $\tilde{f}_2$  that we aim to determine. At this stage, massive and massless cases need to be treated separately.

It is rather subtle to take the massless limit in (A.32) because in that case, the  $u$ -integral results in divergent contributions in the  $\ell = n$  sector. The cleanest way to address this is to simply start with the massless kernel in (A.22). Then, we may perform the spherical averaging of the exponential to obtain the spherical Bessel function  $i_{\ell}$ . Thereafter, we may separate out each partial wave independently to split it into internal and external moments. Alternatively, we may write the result of the spherical average in the form of reverse Bessel polynomials as we did in the massive case. In the latter case, however, the  $\ell = n$  sector has to be treated separately before performing the  $s$  and  $u$  integrals. In that case, it turns out that the expansion in the region  $|\vec{x}| > |\vec{y}_2|$  comes with negative powers of  $|\vec{x}|$  instead of the combinations of Bessel functions as we found above. Then, a simple redefinition as  $|\vec{x}| = \frac{1}{w}$  turns them into positive moments in terms of the new parameter  $w$ . Then we may split the integral into two parts from 0 to  $|\vec{y}_2|$  and then from  $|\vec{y}_2|$  to  $\infty$ . This way, solving the system of linear equations for various measurements at various radial positions  $|\vec{y}_2|$  allows us to determine the dependence of the function everywhere in space, in terms of its internal (within the ball)

and external (outside the ball) moments. Clearly, this procedure works for all functions that have a well-defined inverse Laplace transform, which is what we demand of the functions defining the states. The conclusion is then that we have knowledge of the function  $\tilde{f}_2$  everywhere in space on one argument, whereas our knowledge of its dependence on the other argument is limited to within the ball.

In the massive case, however, the situation is slightly more complicated. The internal moments are identical, and we may determine the function inside the ball entirely using the same method. However, the expansion outside the ball contains kernels which are combinations of the modified Bessel functions, as can be seen in (A.33). This makes it harder to determine exactly what information about the function  $\tilde{f}_2$  we have access to outside the ball.

**Basis of functions:** To address this concern of external moments in the case of massive fields, we may approach the problem in an alternative fashion. We may pick an appropriate basis of functions to expand  $\tilde{f}_2^{\ell m}(\vec{y}_1, |\vec{x}|)$  in. This gives us undetermined coefficients that govern the linear combination of the basis functions that yield our function of interest  $\tilde{f}_2^{\ell m}(\vec{y}_1, |\vec{x}|)$ . Now, by making many different measurements for each value of  $|\vec{y}_2|$ , we may determine these coefficients by solving a system of linear equations. Therefore, by measuring the correlator for all values of  $|\vec{y}_2|$  in a small region (which is still an infinite number of measurements), we may determine the function exactly. A finite number of measurements yields the function within a certain error that depends on various factors, such as the appropriateness of the choice of basis functions, etc. This, of course, requires us to have knowledge of a good basis to begin with that is practical and appropriate for the physical system at hand. Moreover, the basis functions must be analytic everywhere in space to prevent the coefficients of the basis expansion from being dependent on space. This implies that sufficiently precise measurements inside the ball will give us all information about the function in both arguments everywhere in space. However, this is an artefact of our prior knowledge of a good basis and the approximation that the function defining the state  $f_2$  can be expanded in a linear combination of analytic functions.<sup>2</sup> This is a counter-intuitive result, as we expect that correlators inside the ball should have no access to unitary local excitations far away from the ball. In a practical sense, this is still true in that given some experimental precision, two states may very well differ dramatically far outside the ball, provided that their difference within the ball falls within that experimental precision that the low-energy observer has no access to.

On the other hand, this method of expanding into a basis of functions can be done

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<sup>2</sup>Generically, the functions defining the state were not assumed to be analytic. However, this expansion into a basis approximates the functions in a superposition of analytic basis elements.

for both the massive and massless cases alike. Notwithstanding this, there is still an important difference between the two cases in practice. Since the kernels in the massive case are modified Bessel functions outside the ball, they fall off exponentially. By contrast, the massless case exhibits power-law fall-offs. This means that extracting the coefficients corresponding to modes beyond a Compton wavelength (of the massive field) away from the ball is increasingly harder. By contrast, the massless case has long-range effects that are easier to detect.<sup>3</sup>

In the following sub-section, we will see that when there are multiple variables in the function to be inverted for, there is a general multivariate moment expansion from which we will be able to invert for the corresponding function  $f_n$  where one of its arguments is restricted to be inside the sphere of radius  $R$ .

### A.3.2 Inversion for boundary correlators

In the main text, we found the boundary correlator that was relevant for the bulk one-point function to be

$$\begin{aligned}\mathcal{O}_1(\vec{z}) &:= \langle \psi | H_R \pi(\vec{z}) | \psi \rangle - \langle 0 | \pi(\vec{z}) | 0 \rangle \\ &= \frac{-m^2 \gamma}{4\pi^2} \int d^3 \vec{x} f_1(\vec{x}) \frac{K_2(m|\vec{x} - \vec{z}|)}{|\vec{x} - \vec{z}|^2} \Theta(R - |\vec{x}|) .\end{aligned}\tag{A.34}$$

Now, using (A.25) and the following steps, we may write this as

$$\mathcal{O}_1(\vec{z}) = \frac{-m^4 \gamma}{\pi^2 \sqrt{\pi}} \int d^3 \vec{x} f_1(\vec{x}) \int_0^\infty du \cos(u) \int_0^\infty ds s^{\frac{3}{2}} e^{-s(u^2 + |\vec{y}_2 - \vec{x}_1|^2)} \Theta(R - |\vec{x}|) .\tag{A.35}$$

The only difference between the  $s$ -integral in this expression and the corresponding integral in (A.30) is an extra power of  $s$  in front of the exponential. Therefore, while the resulting expression after carrying out the  $s$  and  $u$  integrals change, the conclusion can explicitly be checked to be the same that we may invert the above correlator for the function  $f_1(\vec{x})$  using all its moments. As is evident, the explicit appearance of the  $\Theta$ -function restricts our knowledge of the one-particle exponent to within the sphere of radius  $R$ .

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<sup>3</sup>We are grateful to R. Loganayagam for discussions on this point and several other subtleties discussed in this appendix.

Similarly, the boundary correlator that corresponds to the bulk two-point function is

$$\begin{aligned}\mathcal{O}(\vec{z}, \vec{z}_1) &:= \langle \psi | H_R \pi(\vec{z}) \phi(\vec{z}_1) | \psi \rangle - \langle 0 | H_R \pi(\vec{z}) \phi(\{\vec{z}_m\}) | 0 \rangle \\ &= \frac{-m^3 \gamma}{16\pi^4} \int d^3 \vec{x}_1 d^3 \vec{x}_2 \tilde{f}_2(\vec{x}_1, \vec{x}_2) \left[ \frac{K_2(m|\vec{x}_2 - \vec{z}|)}{|\vec{x}_2 - \vec{z}|^2} \frac{K_1(m|\vec{x}_1 - \vec{z}_1|)}{|\vec{x}_1 - \vec{z}_1|} \Theta(R - |\vec{x}_1|) \right].\end{aligned}\tag{A.36}$$

This integral contains two fractions, both of which have appeared earlier in this appendix, resulting in a product of expressions of the kind (A.25). Therefore, we may proceed as before to carry out the integrals arising from this product. Furthermore, we may expand the left-hand side of the correlator in partial waves by moving around both along the angular directions of  $\vec{z}$  and  $\vec{z}_1$ . Consequently, we will find internal and external moments as in the bulk. Therefore, from this bi-variate moment expansion, we may invert for the symmetrised function  $\tilde{f}_2(\vec{x}_1, \vec{x}_2)$  everywhere provided one of the arguments (labelled  $\vec{x}_1$  in this case) is inside the sphere of radius  $R$ . The dependence on the other argument is identical to the bulk correlator because the kernel is identical. Therefore, without further inverting the functions, we already see that all information that we have of the function  $f_2$  from the bulk correlators is also available to the boundary correlator in a small radial band. Of course, we may proceed to invert for the function either in its moments or alternatively, again, by expanding the function  $\tilde{f}_2$  in an appropriate basis to turn the problem into a system of linear equations. This is precisely the same information as contained in the bulk two-point function in (2.19).

All higher-point functions will similarly involve multi-variate moment expansions or systems of linear equations arising from expanding them into a basis, from which we will be able to invert for all information about the corresponding symmetrised support function  $\tilde{f}_n$ , where one of the arguments will be restricted to be inside the sphere of radius  $R$ .

# Appendix B

## Technical details for Chapter 3

In this section, we compile various computations related to Chapter 3.

- Section [B.1](#) explains how to represent cosmological correlators as expectation values of gauge-invariant operators.
- A brief review of relational observables in the AdS/CFT context is given in Section [B.2.1](#). This is followed by an explicit construction of a heavy observer state in de Sitter in Section [B.2.2](#).
- Subtleties involved in the wavefunctional representation of gravitationally constrained states in the late-time limit of de Sitter are discussed in Section [B.3](#).

### B.1 Cosmological correlators as gauge-invariant expectation values

In this Appendix, we explain how to represent cosmological correlators as expectation values of gauge-invariant operators. It is sufficient to study cosmological correlators corresponding to simple monomials of the field operators since more general correlators can be written as linear combinations of these. In the nongravitational limit, we use the simple relation

$$\langle \Psi | \chi(z_1) \dots \chi(z_n) | \Psi \rangle = (\Psi, O \Psi), \quad (\text{B.1})$$

where

$$O = \frac{1}{\text{vol}(\text{SO}(d-1))} \int U \chi(z_1) \dots \chi(z_n) U^\dagger dU, \quad (\text{B.2})$$

where  $dU$  is the Haar measure on the set of unitary operators,  $U$ , that implement conformal transformations in the QFT Hilbert space.

We recall that if  $g$  is an element of the conformal group, these operators act on fields via

$$U\chi(\tilde{x})U^\dagger = \Lambda(\tilde{x}, g)^\Delta \chi(x), \quad (\text{B.3})$$

where

$$x = g\tilde{x}; \quad \Lambda(\tilde{x}, g) = \left| \frac{\partial x}{\partial \tilde{x}} \right|^{\frac{1}{d}}. \quad (\text{B.4})$$

Here, we use  $g\tilde{x}$  to denote the action of the conformal transformation  $g$  on  $\tilde{x}$ . We caution the reader that the notation is slightly different from [13].

Therefore, we can write

$$O = \frac{1}{\text{vol}(\text{SO}(d-1))} \int dg \prod_i \Lambda(g, z_i)^\Delta \chi(gz_i), \quad (\text{B.5})$$

where  $dg$  is the Haar measure on the conformal group. Inserting an identity in the form

$$1 = \int \prod_i \delta(x_i - gz_i) dx_i. \quad (\text{B.6})$$

We see that we can write

$$O = \int \mathcal{G}(\vec{x}, \vec{z}) \prod_i \chi(x_i) dx_i, \quad (\text{B.7})$$

where

$$\mathcal{G}(\vec{x}, \vec{z}) = \frac{1}{\text{vol}(\text{SO}(d-1))} \int \prod_i \delta(x_i - gz_i) \Lambda(g, z_i)^\Delta dg. \quad (\text{B.8})$$

While this provides an explicit expression for  $\mathcal{G}$  in terms of an integral over the conformal group, it is difficult to perform this integral by brute force. The objective of the calculations below is to identify the smearing function  $\mathcal{G}$  in various cases. We proceed in three steps. First, we identify the conformal transformation properties of  $\mathcal{G}$  under transformations of  $x_i$  and  $z_i$ . Next, we use this to constrain the form that  $\mathcal{G}$  can take. Finally, we provide explicit expressions for  $\mathcal{G}$  in the case of 3-point and 4-point cosmological correlators and explain how these calculations can be generalized to higher-point correlators.

**Transformation of the smearing function under conformal transformations.** We first show that under a conformal transformation of the points  $x_i$

$$\mathcal{G}(\gamma\vec{x}, \vec{z}) = \Lambda(\gamma, x_i)^{-\bar{\Delta}} \mathcal{G}(\vec{x}, \vec{z}). \quad (\text{B.9})$$

This is also necessary to ensure that the operator, as written in the form (B.7), is invariant under conformal transformations so that the transformation of  $\mathcal{G}$  compensates for the transformation of the fields.

We see that

$$\delta(\gamma x_i - gz) = \delta(x_i - \gamma^{-1}gz_i)\Lambda(\gamma, x_i)^{-d}, \quad (\text{B.10})$$

and, in the presence of the delta functions,

$$\Lambda(g, z_i) = \Lambda(\gamma, x_i)\Lambda(\gamma^{-1}g, z_i). \quad (\text{B.11})$$

The required property (B.9) then follows by substituting the relations above into (B.8) and using the invariance of the Haar measure under left multiplication.

Using a similar argument, one may also check that under a conformal transformation of the points  $z_i$ , we have

$$\mathcal{G}(\vec{x}, \gamma\vec{z}) = \Lambda(\gamma, x_i)^{-\Delta}\mathcal{G}(\vec{x}, \vec{z}). \quad (\text{B.12})$$

**General form of the smearing function.** The delta functions in (B.8) imply that the cross ratios of the  $x_i$  must necessarily be the same as the cross ratios of the  $z_i$ . This, together with the transformation of  $\mathcal{G}$  under independent conformal transformations of  $x_i$  and  $z_i$  derived above, leads to the conclusion that one can write

$$\mathcal{G}(\vec{x}, \vec{z}) = C_1(x_i)C_2(z_i)\mathcal{C}(\vec{u}^z)\mathcal{D}(\vec{u}^x, \vec{u}^z). \quad (\text{B.13})$$

Here  $C_1(x_i)$  transforms as a conformal correlator of operators of dimension  $\bar{\Delta}$ ,  $C_2(z_i)$  transforms as a conformal correlator of operators of dimension  $\Delta$ , and  $\mathcal{C}$  is an arbitrary function of the cross ratios made of the  $z_i$ . We denote the list of cross ratios by  $\vec{u}^z$ . Finally,  $\mathcal{D}$  is a product of delta functions that sets  $\vec{u}^z$  to the cross ratios made out of the  $x_i$ , which we denote by  $\vec{u}^x$ .  $\mathcal{D}$  only appears when  $n > 3$ .

Provided that the number of points satisfies,  $3 < n \leq d + 2$ , we can simply set

$$\mathcal{D} = \prod_{\alpha} \delta(u_{\alpha}^z - u_{\alpha}^x), \quad (\text{B.14})$$

where the product runs over all cross ratios indexed by  $\alpha$ . However, when  $n > d + 2$ , some of the cross ratios are redundant, in which case only a smaller number of delta functions can be imposed explicitly.

The form (B.13) has some redundancy since  $\mathcal{C}$  can be absorbed into  $C_2$ . We have written it as above so that  $C_1$  and  $C_2$  can be chosen as per convenience, and the task then

reduces to finding  $\mathcal{C}$ .

Before we explain how to fix  $\mathcal{G}$  explicitly, we note that the relation (B.2) defines a many-to-one map between gauge-fixed operators and gauge-invariant operators, i.e., multiple distinct monomials on the right-hand side of (B.2) might get mapped to the same gauge-invariant operator. For instance, if one acts on a monomial with any unitary from the conformal group that transforms the fields according to (B.3), one obtains a distinct monomial. However, the image of this new monomial under the map (B.2) is the same as that of the previous monomial.

Next, the following formula, derived in [13], is helpful. Let us use the notation  $\vec{x}'$  to denote the list of  $x$  positions with the first three fixed  $x_1 = z_1, x_2 = z_2, x_3 = z_3$ :  $\vec{x}' \equiv (x_1 = z_1, x_2 = z_2, x_3 = z_3, x_4 \dots x_n)$ . Then

$$\frac{1}{f} \int dU d\vec{x}' U \mathcal{G}(\vec{x}', \vec{z}) \prod_i \chi(x'_i) U^\dagger = \int d\vec{x} \mathcal{G}(\vec{x}, \vec{z}) \prod_i \chi(x_i), \quad (\text{B.15})$$

with

$$f = \text{vol}(\text{SO}(d-1)) (|x'_{12}| |x'_{23}| |x'_{13}|)^{-d}. \quad (\text{B.16})$$

The task at hand is to deduce a smearing function  $\mathcal{G}$  consistent with (B.13) so that the left-hand side reduces either to the desired monomial *or* one of its images under the action of a unitary. This is most conveniently done by choosing an appropriate ansatz for  $\mathcal{G}$  and fixing parameters in the ansatz. We now do this explicitly for the case of 3-point and 4-point cosmological correlators and indicate the generalization to higher points.

### B.1.1 3-point cosmological correlator

For the case of the 3-point cosmological correlator, we see that (B.13) forces a specific form on  $\mathcal{G}$

$$\mathcal{G}(\vec{x}, \vec{z}) = \frac{1}{|x_{12}|^{\bar{\Delta}} |x_{13}|^{\bar{\Delta}} |x_{23}|^{\bar{\Delta}} |z_{12}|^{\Delta} |z_{13}|^{\Delta} |z_{23}|^{\Delta}}, \quad (\text{B.17})$$

and the overall constant is fixed using (B.15). This case is simple because there are no cross ratios, and so  $\mathcal{C}$  and  $\mathcal{D}$  cannot appear at all.

### B.1.2 4-point cosmological correlator

For the case of the 4-point correlator, we make an ansatz of the form

$$\mathcal{G}(\vec{x}, \vec{z}) = \frac{1}{\prod_{i < j} |x_{ij}|^{\frac{2\bar{\Delta}}{3}} |z_{ij}|^{\frac{2\Delta}{3}}} \delta(u_x - u_z) \delta(v_x - v_z) h(u_z, v_z), \quad (\text{B.18})$$

where  $h$  is a smooth function of

$$u_z = \frac{|z_{41}||z_{23}|}{|z_{43}||z_{12}|}, \quad v_z = \frac{|z_{42}||z_{13}|}{|z_{43}||z_{12}|}. \quad (\text{B.19})$$

$h$  can be fixed using (B.15) by demanding that the smearing function reduce the restricted integral over  $x_4$  to the product  $\chi(z_1) \dots \chi(z_4)$  or one of its conformal images. Moreover, since  $h$  depends only on the cross ratios, we can fix it in the special case where  $z_1 = 0, z_2 = 1, z_3 = \infty$ . In this setting  $u_z = |z_4|$  and  $v_z = |1 - z_4|$ .

For this choice with  $x_1, x_2, x_3$  set to  $z_1, z_2, z_3$  the prefactors above (including the contribution of  $f$ ) become

$$(|z_{12}||z_{23}||z_{13}|)^d \frac{1}{\prod_{i<j} |x_{ij}|^{\frac{2\Delta}{3}} |z_{ij}|^{\frac{2\Delta}{3}}} = \frac{1}{u_z^{\frac{2d}{3}}} \frac{1}{v_z^{\frac{2d}{3}}}. \quad (\text{B.20})$$

We find that  $h$  must be

$$\begin{aligned} h &= \frac{u_z^{\frac{2d}{3}} v_z^{\frac{2d}{3}}}{\int dx_4 \delta(|x_4| - u_z) \delta(|1 - x_4| - v_z)} \\ &= \frac{2^{d-3} u_z^{\frac{2d}{3}} v_z^{\frac{2d}{3}}}{u_z v_z \Omega_{d-2} [(u_z + v_z + 1)(u_z + v_z - 1)(-u_z + v_z + 1)(u_z - v_z + 1)]^{\frac{d-3}{2}}}. \end{aligned} \quad (\text{B.21})$$

Substituting this value of  $h$  into (B.18) yields the correct smearing function for the four-point cosmological correlator. This answer holds for arbitrary values of  $z_i$  and not just for the special values that we chose above to fix  $h$ .

### B.1.3 Higher-point cosmological correlators

In principle, the procedure outlined above works for an arbitrary-point correlation function. However, when the number of points is high enough, some of the cross ratios become redundant. In this case, the delta functions that appear in the smearing function must be restricted to the independent cross ratios.

We naively obtain  $\frac{n(n-3)}{2}$  cross ratios from  $n$  points. But for large  $n$ , this is an overestimate, and not all cross ratios are independent. Each point has  $d$  coordinates and therefore  $n$  points have  $nd$  degrees of freedom. Of these, conformal transformations can at most remove  $\dim(SO(d+1, 1)) = \frac{(d+1)(d+2)}{2}$  degrees of freedom. Therefore, for large  $n$  the number of independent cross ratios is given by  $nd - \dim(SO(d+1, 1))$ . The two formulas cross over at  $n = d + 2$  and, in general, the number of independent

cross ratios is given by [130]

$$N_{\text{ind}} = \begin{cases} \frac{n(n-3)}{2} & n \leq d+2 \\ nd - \frac{(d+1)(d+2)}{2} & n > d+2 \end{cases}. \quad (\text{B.22})$$

In general, one must choose an ansatz with delta functions that set  $N_{\text{ind}}$  cross ratios of  $x$  equal to those of  $z$ . The precise smearing function can then be obtained, as above, after computing the volume of the compact space left undetermined by these delta functions.

## B.2 More details on relational observables

### B.2.1 Review of relational observables in AdS/CFT

In this Appendix, we provide a lightning review of the construction of relational observables in AdS/CFT. The construction we present below was first presented in [25, 26] to obtain operators in the interior of the eternal black-hole that had a specified commutator with the boundary Hamiltonian. It is closely related to the construction of “mirror operators” in the single-sided black hole [31, 32]. The same construction was recently applied to non-black hole states in [27, 28] to obtain operators that commute with the boundary Hamiltonian. The construction was slightly refined in [29] where it was shown that the relational observables obtained via this method take on the form of a crossed product that has been studied recently in [30, 131]. The appeal of this approach is that one does not need to introduce an “observer”, and the properties of the background states naturally give rise to an autonomous notion of time. Second, this approach shows that relational observables confined to a compact region are only approximate; their existence does not contradict the principle of holography of information, and from a fine-grained perspective, observables in a bounded region are also accessible in its complement.

Let  $|\psi\rangle$  be a state in the bulk theory, with energy that is parametrically larger than the AdS scale

$$E = \langle \psi | H | \psi \rangle \gg 1, \quad (\text{B.23})$$

where the AdS scale is set to unity. In addition, we require the *spread* of energies in the state to be large

$$\delta E = \sqrt{\langle \psi | H^2 | \psi \rangle - \langle \psi | H | \psi \rangle^2} \gg 1, \quad (\text{B.24})$$

while  $\frac{\delta E}{E} \ll 1$ .

In a typical state with these properties, it is possible to “dress” operators so that they approximately commute with the ADM Hamiltonian. Let  $\mathcal{A}_{\text{light}}$  be the set of simple bulk operators that we are interested in. For instance, this set might correspond to products of bulk field operators in some region

$$\mathcal{A}_{\text{light}} = \text{span}\{\phi(x_1), \phi(x_1)\phi(x_2), \dots, \phi(x_1)\phi(x_2)\dots\phi(x_n)\}. \quad (\text{B.25})$$

Two points must be kept in mind.

1. The Hamiltonian itself is not included in  $\mathcal{A}_{\text{light}}$ . The distinction between ordinary operators and the Hamiltonian is clear at lowest order in perturbation theory, but subtle beyond lowest order since the OPE of ordinary operators can generate the Hamiltonian.
2. This set is not quite an algebra since the order of the highest monomial that appears in (B.25) must be cut off to prevent the construction from breaking down. We refer the reader to [25, 26, 29, 32] for more discussion of the cutoff.

We may now consider a family of “little Hilbert spaces” [32] produced by first time-translating  $|\psi\rangle$  and then acting with elements of  $\mathcal{A}_{\text{light}}$  on  $|\psi\rangle$

$$\mathcal{H}_t = \mathcal{A}_{\text{light}} e^{-iHt} |\psi\rangle. \quad (\text{B.26})$$

Under reasonable conditions (see [29] for a more detailed discussion) it may be shown that  $\mathcal{H}_t$  is dense in the space

$$\mathcal{H}'_t = e^{-iHt} \mathcal{A}_{\text{light}} |\psi\rangle. \quad (\text{B.27})$$

This is a limited version of the Reeh-Schlieder property, and we will use this property below.

For a typical heavy state, we expect that correlators of simple operators are time translationally invariant.

$$\langle \psi | e^{iHt} a e^{-iHt} | \psi \rangle = \langle \psi | a | \psi \rangle. \quad (\text{B.28})$$

For a typical state,  $|\psi\rangle$ , one may also show that these little Hilbert spaces are orthogonal to each other if the difference in time-parameters is larger than  $\frac{1}{\delta E}$ .<sup>1</sup>

$$|\langle \psi | a e^{-iHt} | \psi \rangle| \ll 1, \quad \forall a \in \mathcal{A}_{\text{light}}, \text{ if } |t| \gg \frac{1}{\delta E}. \quad (\text{B.29})$$

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<sup>1</sup>As discussed in [29], this assumes that functions of  $H$  are not elements of  $\mathcal{A}_{\text{light}}$ . For instance  $\langle \psi | \frac{e^{-iHt}}{H} | \psi \rangle = \langle \psi | 1 - i \int_0^t e^{-iHs} ds | \psi \rangle \neq 0$ . Therefore, at higher orders in perturbation theory, where  $H$  can be generated through OPEs of other operators, the construction needs to be refined.

Now, let  $\delta T$  be a time interval with the property that

$$1 \gg \delta T \gg \frac{1}{\delta E}. \quad (\text{B.30})$$

We now define a *smeared* Hilbert space

$$\mathcal{H}_t^{\text{sm}} = \text{span}\{|v\rangle : |v\rangle \in \mathcal{H}_s \text{ for } s \in (t - \frac{\delta T}{2}, t + \frac{\delta T}{2})\}, \quad (\text{B.31})$$

and we define  $P_t$  to be the projector onto  $\mathcal{H}_t^{\text{sm}}$ . Alternately, we can think of  $P_t$  as the smallest projector that contains projectors onto all the spaces  $\mathcal{H}_s$  for  $s \in (t - \frac{\delta T}{2}, t + \frac{\delta T}{2})$ .

The smearing above is physically motivated and was first introduced in [29]. It causes the relational operators to be defined below to be spread out over a time interval of size  $\delta T$ . Such a spread is to be expected since an observer with energy spread  $\delta E$  can only measure time intervals larger than  $\frac{1}{\delta E}$ . This smearing was not utilized in the original construction of [25, 26], where projectors onto a specific  $\mathcal{H}_t$  were used instead; this oversight was carried over to [27, 28]. However, projectors onto a specific  $\mathcal{H}_t$  are “too sharp” and, as we will see below, they cause difficulty when one studies the action of relational operators on the Hilbert space or higher-point functions of relational operators.

The original elements of  $\mathcal{A}_{\text{light}}$ , in general, do not commute with  $H$ . However, one can dress them to obtain operators that do commute with  $H$  to high precision.

$$\hat{a} = \frac{1}{\delta T} \int_{-T_{\text{cut}}}^{T_{\text{cut}}} e^{iHt} a P_0 e^{-iHt} dt. \quad (\text{B.32})$$

We refer the reader to [26] for extensive discussion of the cutoff  $T_{\text{cut}}$  and its relation to state dependence.

However, within low-point correlators about the state  $|\psi\rangle$  and nearby time-translated states the precise value of  $T_{\text{cut}}$  is unimportant since the state  $e^{-iHt}|\psi\rangle$  is orthogonal to  $\mathcal{H}_0^{\text{sm}}$ . For any  $b, c \in \mathcal{A}_{\text{light}}$ ,

$$\langle \psi | c \frac{\partial \hat{a}}{\partial T_{\text{cut}}} b | \psi \rangle \approx 0. \quad (\text{B.33})$$

Hence, for the purpose of the current discussion, we can simply assume that  $T_{\text{cut}}$  is effectively infinite.

Up to the approximations enumerated explicitly above, these operators commute with the Hamiltonian to excellent accuracy

$$e^{iHs} \hat{a} e^{-iHs} = \hat{a}, \quad (\text{B.34})$$

since the only effect of the time translation is to change the limit of the integral over  $t$  in (B.32), which, as explained above, is irrelevant in low-point correlators.

The action of these operators on any element of the original little Hilbert space is given by

$$\hat{a}b|\psi\rangle = \frac{1}{\delta T} \int_{-\frac{\delta T}{2}}^{\frac{\delta T}{2}} e^{iHt} a e^{-iHt} dt b|\psi\rangle + \mathcal{O}\left(\frac{1}{\delta E \delta T}\right) \approx \bar{a}b|\psi\rangle, \quad (\text{B.35})$$

where  $\bar{a}$  is a smeared version of the original operator obtained by time averaging it over an interval of size  $\delta T$ .

$$\bar{a} \equiv \frac{1}{\delta T} \int_{-\frac{\delta T}{2}}^{\frac{\delta T}{2}} e^{iHt} a e^{-iHt} dt. \quad (\text{B.36})$$

The error term above arises because we recall that projectors onto different little Hilbert spaces are not exactly orthogonal. So the  $P_0$  projector also picks up contributions from the  $t$  integral in the range  $|t| \in (\frac{\delta T}{2}, \frac{\delta T}{2} + \frac{1}{\delta E})$ . However, in the regime where  $\delta T \gg \frac{1}{\delta E}$ , this error is negligible. Conversely, if one chooses  $\delta T$  to be too small, it leads to large errors above and the action of the hatted operators is no longer close to the action of the original operators.<sup>2</sup>

Note that  $\bar{a}b|\psi\rangle$  is also an element of  $\mathcal{H}^0$  by the Reeh-Schlieder property alluded to above. A short calculation then shows that the correlators of the hatted operators in the original state reduce to correlators of averaged operators

$$\langle \psi | \hat{a}_1 \dots \hat{a}_n | \psi \rangle = \langle \psi | \bar{a}_1 \dots \bar{a}_n | \psi \rangle. \quad (\text{B.37})$$

To the extent that  $\delta T$  is small compared to other time intervals in the problem, the distinction between  $\bar{a}$  and  $a$  is physically irrelevant. Therefore, in this regime, the hatted operators approximately commute with the Hamiltonian and act approximately like the original operators within low-point correlators. They may be thought of as relational operators that are measured by an “observer” with an autonomous clock derived from the properties of the state  $|\psi\rangle$  that is distinct from the clock on the boundary.

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<sup>2</sup>As noted, the construction (B.32) is a slight variant of the mirror-operator construction. Mirror operators are usually constructed [31,32] by specifying their action on a state through relations of the form (B.35). The projector construction was presented in [26] as an alternative to a parallel construction that defined the mirror operators in the eternal black hole in terms of their action on states; this parallel construction is also described in [28]. But unless one smears the projectors appropriately, the two constructions are not equivalent. Even without smearing, the one-point function of the operators works out correctly as shown in section 7.6 of [26]; however, the action on the Hilbert space is not accurate and this affects higher-point functions as was also recently noted in [132]. However, [132] propose a fine-tuned modification of the projector, whereas we feel that the smearing, as earlier described in [29], provides a more physical solution to the issue.

### B.2.2 Observer states

In this appendix, we provide an explicit example of a free-field state that satisfies the conditions required of a background QFT state in Section 3.5.

Let  $\phi(x)$  be the late-time limit of a principal-series field. We use this notation to indicate that the background state might comprise a field that is distinct from the observable field,  $\chi(x)$  although the construction below allows  $\phi = \chi$ .

Now consider,

$$|\Psi_{\text{back}}\rangle = \int \left( \prod_{i=1}^S dx_i f_i(x_i) \right) : \phi(x_1) \dots \phi(x_S) : |0\rangle, \quad (\text{B.38})$$

where the functions  $f_1(x_1) \dots f_S(x_S)$  are some  $L^2$  normalizable smearing functions that have compact support in the region  $R'$  and the product of fields is normal ordered. We will consider the limit where  $S \gg 1$ , so that the state contains a “large number” of particles.

We want these smearing functions to be “random”. The space of  $L^2$  normalizable functions is infinite-dimensional, but we can regularize it to avoid functions that vary too rapidly and then use the Haar measure on this space to select the smearing functions  $f_i$ . We then expect these smearing functions to be uncorrelated under the standard  $L^2$  inner product. For  $i \neq j$ ,

$$\langle f_i, f_j \rangle = \int f_i^*(x) f_j(x) dx = 0, \quad (\text{B.39})$$

to a good approximation. More precisely if we regularize the space of  $L^2$ -normalizable functions to have dimension  $D$ , and each smearing function  $f_i$  to have unit norm, the typical size of the inner product is expected to be  $\frac{1}{\sqrt{D}}$

The normalization constant  $\mathcal{N}_2$  can then be chosen as

$$\mathcal{N}_2^{-2} = \langle \Psi_{\text{back}} | \Psi_{\text{back}} \rangle = \prod_i \mathfrak{c}_{\phi\bar{\phi}} \langle f_i, f_i \rangle, \quad (\text{B.40})$$

so the state is unit normalized. Here, the normalization of the propagator is the same as (3.37)

Now consider an element of the dS group that maps point  $x_i \rightarrow \tilde{x}_i$ . This acts on the state as

$$U |\Psi_{\text{back}}\rangle = \mathcal{N}_2 \int \left( \prod_{i=1}^S dx_i \tilde{f}_i(x_i) \right) : \phi(x_1) \dots \phi(x_S) : |0\rangle, \quad (\text{B.41})$$

where

$$\tilde{f}(x_i) = \left| \frac{\partial \tilde{x}_i}{\partial x_i} \right|^{\frac{\bar{\Delta}}{d}} f(\tilde{x}_i). \quad (\text{B.42})$$

When  $U$  is close to the identity so that the points are moved by only a small amount, we see that

$$\langle \Psi_{\text{back}} | U | \Psi_{\text{back}} \rangle = \prod_i \frac{\langle f_i, \tilde{f}_i \rangle}{\sqrt{\langle f_i, f_i \rangle \langle \tilde{f}_i, \tilde{f}_i \rangle}} \equiv \prod_i (1 - \delta_i), \quad (\text{B.43})$$

where the numbers  $\delta_i$  are defined by the relation above. Here, we have neglected overlaps  $\langle f_i, \tilde{f}_j \rangle$  that come from other Wick contractions since we do not expect a generic conformal transformation to generate correlations between these random smearing functions.

In the case where we have a large number of field insertions, we can approximate

$$\prod_i (1 - \delta_i) \approx e^{-S\delta} \ll 1, \quad (\text{B.44})$$

where  $\delta$  is the “mean value” of the  $\delta_i$ . So if the transformation  $U$  moves points by an amount that is larger than the typical scale of variation of the  $f_i$ , we expect that  $\delta = \mathcal{O}(1)$  and the inner product then drops off rapidly if  $S$  is large.

We see that this state satisfies both the necessary conditions outlined in the main text. Since the insertions of  $\phi$  are localized in a region that is well separated from the region  $R$ , we see that

$$\langle \Psi_{\text{back}} | a | \Psi_{\text{back}} \rangle = \langle a \rangle. \quad (\text{B.45})$$

Second,

$$\langle \Psi_{\text{back}} | Ua | \Psi_{\text{back}} \rangle \ll 1 \quad (\text{B.46})$$

for  $U$  outside of a small neighbourhood of the identity.

While an explicit construction is difficult to present in the interacting theory, we expect that a state that satisfies the properties above will continue to exist in the interacting theory. First, (B.45) is expected to hold on the basis of cluster decomposition if the excitation and the observables are located on very different parts of the late-time slice. Second, we used the free-field two-point function to show (B.46). In the interacting theory, the norm does not factorize into a product of two-point functions. While perturbative corrections can remove some of the factors of  $(1 - \delta_i)$  in (B.43), this does not alter (B.46) unless we go to  $S^{\text{th}}$  order in perturbation theory. But such a contribution is highly suppressed in any case.

## B.3 Wavefunctions for the principal series

In this Appendix, we quickly review how principal-series matter fields in de Sitter can be treated in a wavefunction formalism. This issue was clarified in [133]. Principal-series wavefunctions in de Sitter space are essentially coherent-state wavefunctions. Before we proceed to the de Sitter case, we remind the reader of such wavefunctions in ordinary quantum mechanics; the discussion for de Sitter field theory will be parallel to this.

### B.3.1 Coherent state wavefunctions in quantum mechanics

Consider a single particle with a Hamiltonian

$$H_{1p} = \frac{p^2}{2} + \frac{x^2}{2}. \quad (\text{B.47})$$

The vacuum is annihilated by the standard annihilation operator

$$A = (x + ip)/\sqrt{2}. \quad (\text{B.48})$$

To bring out the analogy with de Sitter space, we define new creation and annihilation operators

$$a = \alpha A + \beta A^\dagger; a^\dagger = \beta^* A + \alpha^* A^\dagger, \quad (\text{B.49})$$

where  $\alpha, \beta$  are complex numbers that satisfy  $|\alpha|^2 - |\beta|^2 = 1$ . These operators do not annihilate the usual vacuum. Instead we can define a different state  $|\tilde{0}\rangle$  that satisfies  $a|\tilde{0}\rangle = 0$  and is normalized so that  $\langle\tilde{0}|\tilde{0}\rangle = 1$ .

We now define the state  $|z\rangle$  to satisfy

$$a|z\rangle = |z\rangle, \quad (\text{B.50})$$

which fixes the state up to normalization to be

$$|z\rangle = e^{za^\dagger} |\tilde{0}\rangle. \quad (\text{B.51})$$

It is not difficult to check that these states obey the completeness relation

$$\int [d^2z] |z\rangle\langle z| = 1, \quad (\text{B.52})$$

where  $[d^2z] \equiv e^{-|z|^2} \frac{d^2z}{\pi}$ . This identity is proved by expanding the states (B.51) in terms

of normalized eigenstates of the number operator  $a^\dagger a |n\rangle = n |n\rangle$ .

$$|z\rangle = \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle, \quad (\text{B.53})$$

and then noting that

$$\int \frac{d^2 z}{\pi} \sum_{n,m} e^{-|z|^2} z^n \bar{z}^m |n\rangle \langle m| = \sum_n |n\rangle \langle n| = 1. \quad (\text{B.54})$$

Now, for any state,  $|\Psi\rangle$ , we can define the wavefunction

$$\Psi(z) = \langle z | \Psi \rangle. \quad (\text{B.55})$$

This notation is slightly deceptive since  $\Psi(z)$  is, in general, not a holomorphic function of  $z$ . We can compute expectation values of polynomials of  $z$  and  $\bar{z}$  by integrating the polynomial against  $|\Psi(z)|^2$ . In the operator formalism, such an integral yields expectation values where the  $a^\dagger$  appears to the right of the  $a$ .

$$\int [d^2 z] |\Psi(z)|^2 z^n \bar{z}^m = \int [d^2 z] \langle \Psi | a^n | z \rangle \langle z | (a^\dagger)^m | \Psi \rangle = \langle \Psi | a^n (a^\dagger)^m | \Psi \rangle. \quad (\text{B.56})$$

Here, we first used the fact that  $|z\rangle$  is a right eigenvector of  $a$  and  $\langle z|$  is a left eigenvector of  $a^\dagger$ , followed by the use of the completeness relation.

The analysis so far was kinematic and independent of the Hamiltonian. Now, we turn to the wavefunction of the true ground state

$$\Psi_0(z) = \langle z | 0 \rangle. \quad (\text{B.57})$$

Since  $A = \alpha^* a - \beta a^\dagger$ , and  $A|0\rangle = 0$ , one can write

$$|0\rangle = N e^{\frac{\beta}{2\alpha^*} (a^\dagger)^2} |\tilde{0}\rangle, \quad (\text{B.58})$$

where  $N$  is a normalization we will fix below. Since  $\langle z|$  is a left eigenvector of  $a^\dagger$  we find

$$\langle z | 0 \rangle = N e^{\frac{\beta}{2\alpha^*} \bar{z}^2}. \quad (\text{B.59})$$

The normalization can be fixed by demanding

$$\int |\Psi_0(z)|^2 [d^2 z] = 1, \quad (\text{B.60})$$

which fixes

$$N = \frac{1}{|\alpha|^{\frac{1}{2}}}. \quad (\text{B.61})$$

The wavefunction for states obtained by acting with powers of  $a^\dagger$  on the true vacuum is

$$\langle z | (a^\dagger)^m | 0 \rangle = \bar{z}^m \Psi_0(z) = \frac{1}{|\alpha|^{\frac{1}{2}}} \bar{z}^m e^{\bar{z}^2 \frac{\beta}{2\alpha^*}}. \quad (\text{B.62})$$

We see that these wavefunctions are all antiholomorphic. This set of antiholomorphic wavefunctions already yields a complete basis of states.

There is nothing wrong with wavefunctions that depend on  $z$  rather than  $\bar{z}$ . For instance, the norm of the wavefunction  $z\Psi_0(z)$  is the same as the norm of  $\bar{z}\Psi_0(z)$ . However, the states corresponding to these wavefunctions do not have a simple operator representation in terms of the action of  $a$  or  $a^\dagger$  on  $|0\rangle$ . Since we already have a complete basis above, it is redundant to study such wavefunctions.

### B.3.2 Wavefunctions for principal-series fields

In [13], the states were described as joint wavefunctions of metric and matter fluctuations. As explained in the main text, strictly speaking, one must mod-square this joint wavefunction and then integrate out the metric fluctuations in order to focus on the matter fluctuations. In this Appendix, we provide a simplified treatment starting with quantum-field theory in de Sitter space, where we simply neglect the metric fluctuations from the start. We then group average these wavefunctions to obtain effective wavefunctions that can be used to compute gravitational correlators in the  $G_N \rightarrow 0$  limit.

The canonical commutation relation of the bulk field with its conjugate momentum implies that the boundary values,  $\chi(x)$  and  $\bar{\chi}(x)$ , satisfy

$$[\chi(x), \bar{\chi}(x')] = \frac{1}{2\mu} \delta(x - x'), \quad (\text{B.63})$$

where  $\Delta = \frac{d}{2} + i\mu$ .

Precisely as in the quantum-mechanical example above, one can define coherent states that satisfy

$$\chi(x)|\chi\rangle = \chi(x)|\chi\rangle. \quad (\text{B.64})$$

Here we have abused notation since the left-hand side displays the operator  $\chi(x)$  acting on a state and the right-hand side displays an ordinary c-number function  $\chi(x)$  acting on the same state. We hope that the context will always make it clear when

$\chi(x)$  is used as an operator and when it is used as an ordinary function. We normalize these states, just as in the quantum-mechanical example, using

$$\langle \chi | \chi \rangle = e^{2\mu \int dx |\chi(x)|^2}, \quad (\text{B.65})$$

so that they satisfy the completeness relation

$$\int [D\chi D\bar{\chi}] |\chi\rangle \langle \chi| = 1, \quad (\text{B.66})$$

where

$$[D\chi D\bar{\chi}] \equiv D\chi D\bar{\chi} e^{-2\mu \int dx |\chi(x)|^2}, \quad (\text{B.67})$$

and  $D\chi D\bar{\chi}$  is the usual functional measure.

As is well known, the Euclidean vacuum itself is not annihilated by  $\chi(x)$  [134]. Instead, the correct positive-frequency modes are those that are well behaved when one continues global de Sitter time,  $\tau$ , to the complex plane and studies the limit  $\tau \rightarrow -i\frac{\pi}{2}$ ; the negative-frequency modes are those that are well behaved in the limit  $\tau \rightarrow +i\frac{\pi}{2}$ . This is precisely the analogue of the quantum-mechanical problem studied above. The wavefunction of the free Euclidean vacuum is given by [133]

$$\Psi_0[\chi] := \langle \chi | 0 \rangle = \exp \left\{ \frac{\mu \beta^*}{\alpha^* C} \int dx_1 dx_2 \frac{\bar{\chi}(x_1) \bar{\chi}(x_2)}{|x_1 - x_2|^{2\Delta}} \right\}. \quad (\text{B.68})$$

$$\frac{\beta}{\alpha} = \frac{2^{2i\mu} \mu \Gamma(i\mu)^2}{\pi (1 + \coth(\pi\mu))}, \quad C = \frac{2^{d-2\Delta} \pi^{d/2} \Gamma(d/2 - \Delta)}{\Gamma(\Delta)}. \quad (\text{B.69})$$

This is consistent with the momentum space expression in (5.21) of [133], after we recall the relation

$$|x|^{-2\Delta} = C \int \frac{dp}{(2\pi)^d} e^{-ix \cdot p} |p|^{2\Delta-d}, \quad (\text{B.70})$$

with  $C$  as above. The expression in [133] is valid for all times and also includes an additional term in the exponent that is proportional to  $(-\eta)^{i\mu}$  where  $\eta$  is the Poincare time. Here, it is removed by the choice of normalization (B.65) and (B.66) and by discarding a phase that is not relevant for our computations.

In the presence of interactions, the Euclidean vacuum is no longer expected to be Gaussian in the fields, and instead, we have

$$\Psi_0[\chi] = \exp \left\{ \sum_{n=2}^{\infty} \int dx_1 \dots dx_n G_n(x_1, \dots, x_n) \bar{\chi}(x_1) \dots \bar{\chi}(x_n) \right\}, \quad (\text{B.71})$$

where  $G_n$  satisfy the Ward identities of connected conformal correlators of operators

with dimension  $\overline{\Delta}$ . Excited dS-invariant states take on the form

$$\Psi[\chi] = \int dx_1 \dots dx_m \delta G_m \bar{\chi}(x_1) \dots \bar{\chi}(x_m) \Psi_0[\chi], \quad (\text{B.72})$$

where  $\delta G_m$  is the difference of two possible  $G_m$ . In ket notation, this is

$$|\Psi\rangle = \int dx_1 \dots dx_m \delta G_m(x_1, \dots, x_m) \bar{\chi}(x_1) \dots \bar{\chi}(x_m) |0\rangle. \quad (\text{B.73})$$

As in the quantum-mechanical example studied above, the correspondence between the operator notation and wavefunctions is simplest when we study excitations produced by  $\bar{\chi}$  that are represented by anti-holomorphic wavefunctions.

Moreover, using the same arguments as in the quantum-mechanical case, integrating the squared wavefunction with a polynomial of  $\bar{\chi}$  and  $\chi$  produces an expectation value where insertions of  $\bar{\chi}$  are ordered to the right of insertions of  $\chi$ .

$$\langle \Psi | \chi(x_1) \dots \chi(x_m) \bar{\chi}(y_1) \dots \bar{\chi}(y_n) | \Psi \rangle = \int [D\chi D\bar{\chi}] |\Psi[\chi]|^2 \chi(x_1) \dots \chi(x_m) \bar{\chi}(y_1) \dots \bar{\chi}(y_n). \quad (\text{B.74})$$

If  $\Psi[\chi]$  has non-Gaussian terms in the exponent apart from excitations, these can simply be perturbatively expanded within the functional integral in the usual manner, reducing the entire expression to a sum of Gaussian expectation values.

The contact term in (3.37) can be understood as coming from the measure  $[D\chi D\bar{\chi}]$ . This naturally yields the propagator with  $\bar{\chi}$  on the right, which is all we need in this thesis. If needed, the propagator with the other ordering can be obtained by adding the commutator (B.63).

# Appendix C

## Technical details for Chapter 4

In this section, we compile various computations related to Chapter 4. In particular, calculations related to collective theory are given in Section C.1, and tachyon theory calculations in Section C.2.

### C.1 Computations in collective theory

Section C.1.1 explains how to obtain a field-theoretic expression for general quantum-mechanical operators, using  $\sum_{i=1}^{N_*} \vec{p}_i$  as an example. The next two sections (C.1.2, C.1.3) discuss the collective theory Hamiltonian and its eigenstates for the non-relativistic case, and in section C.1.4 we discuss that for the relativistic case. In C.1.5, we prove  $\int d^p y \phi(\vec{y}) = N_* m$  from a field theory perspective. This is complementary to discussions in the main text where we motivated this result by using  $\phi(\vec{y}) = m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i)$ . Also, we use this identity to show that energy eigenstates having  $N_* - s$  insertions ( $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*-s}}$ ) are equivalent to an  $N_*$ -insertion state ( $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*}}$ ) with  $s$  of the momenta put to zero. By similar methods, in C.1.6, we argue that the states  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}$  for  $n > N_*$  are all null states in collective theory at fixed  $N_*$ . Inner product computation for the  $N_* = 2$  energy eigenstate is given in C.1.7.

#### C.1.1 Deriving field-theoretic expressions for QM operators

We will derive the collective theory Hamiltonian for the non-relativistic system of free particles and discuss its eigenstates. However, we will start with finding the field-theoretic equivalent of the operator  $\sum_{i=1}^{N_*} \vec{p}_i$ . This is complementary to the derivation given in (4.20) and also provides a justification for the same.

Consider the matrix element  $\left(\psi_1, \left(\sum_{i=1}^{N_*} \vec{p}_i\right) \psi_2\right)$  in QM.

$$\begin{aligned}
\left(\psi_1, \left(\sum_{i=1}^{N_*} \vec{p}_i\right) \psi_2\right) &= \left(\left(\sum_{i=1}^{N_*} \vec{p}_i\right) \psi_1, \psi_2\right) \\
&= \int \left(\prod_{i=1}^{N_*} d^p x_i\right) \left(i\hbar \sum_{i=1}^{N_*} \vec{\nabla}_{x_i} \psi_1^*(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*})\right) \psi_2(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*}) \\
&= \int \left(\prod_{i=1}^{N_*} d^p x_i\right) \int D\phi \delta\left[\phi(\vec{y}) - m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i)\right] \left(i\hbar \sum_{i=1}^{N_*} \vec{\nabla}_{x_i} \psi_1^*(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*})\right) \Phi_2[\phi] \\
&= \int \left(\prod_{i=1}^{N_*} d^p x_i\right) \int D\phi \left(-i\hbar \sum_{i=1}^{N_*} \vec{\nabla}_{x_i} \delta\left[\phi(\vec{y}) - m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i)\right]\right) \psi_1^*(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*}) \Phi_2[\phi]
\end{aligned}$$

In the 3rd equality we inserted the identity (4.16), and replaced  $\psi_2$  with  $\Phi_2$ . We cannot replace  $\psi_1$  with  $\Phi_1$  at this point because  $\psi_1$  is being differentiated with  $\vec{x}_i$ <sup>1</sup>. In the last equality, we did an integration by parts in the  $\vec{x}_i$  variable such that the derivative acts on the delta functional. This can be further simplified as follows,

$$\begin{aligned}
\left(-i\hbar \sum_{i=1}^{N_*} \vec{\nabla}_{x_i} \delta\left[\phi(\vec{y}) - m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i)\right]\right) &= -i\hbar \sum_{i=1}^{N_*} \vec{\nabla}_{x_i} \int D\lambda e^{i \int d^p y \lambda(\vec{y}) \phi(\vec{y})} e^{-im \sum_{j=1}^{N_*} \lambda(\vec{x}_j)} \\
&= \int D\lambda e^{i \int d^p y \lambda(\vec{y}) \phi(\vec{y})} e^{-im \sum_{j=1}^{N_*} \lambda(\vec{x}_j)} \left(-m\hbar \sum_{i=1}^{N_*} \vec{\nabla}_{\vec{x}_i} \lambda(\vec{x}_i)\right) \\
&= \int D\lambda e^{i \int d^p y \lambda(\vec{y}) \phi(\vec{y})} e^{-im \sum_{j=1}^{N_*} \lambda(\vec{x}_j)} \left(-\hbar \int d^p y \vec{\nabla}_y \lambda(\vec{y}) \sum_{i=1}^{N_*} m \delta(\vec{y} - \vec{x}_i)\right) \\
&= i\hbar \int d^p y \sum_{i=1}^{N_*} m \vec{\nabla}_{x_i} \delta(\vec{y} - \vec{x}_i) \frac{\delta}{\delta \phi(\vec{y})} \int D\lambda e^{i \int d^p y \lambda(\vec{y}) \phi(\vec{y})} e^{-im \sum_{j=1}^{N_*} \lambda(\vec{x}_j)} \\
&= i\hbar \int d^p y \sum_{i=1}^{N_*} m \vec{\nabla}_{x_i} \delta(\vec{y} - \vec{x}_i) \frac{\delta}{\delta \phi(\vec{y})} \delta\left[\phi(\vec{y}) - m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i)\right]
\end{aligned} \tag{C.2}$$

Here, in the first equality, we used the integral representation of the delta functional. To get the 4th equality, we did an integration by parts in the  $\vec{y}$  variable. One can

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<sup>1</sup>We tacitly chose the differential operator to act on  $\psi_1$  as this will give the field theory operator that acts on  $\Phi_2$ .

substitute this expression into (C.1) to get,

$$\begin{aligned}
\left( \psi_1, \left( \sum_{i=1}^{N_*} \vec{p}_i \right) \psi_2 \right) &= \int \left( \prod_{i=1}^{N_*} d^p x_i \right) \int D\phi \psi_1^*(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*}) \Phi_2[\phi] \\
&\quad \left( i\hbar \int d^p y \sum_{i=1}^{N_*} m \vec{\nabla}_{x_i} \delta(\vec{y} - \vec{x}_i) \frac{\delta}{\delta \phi(\vec{y})} \delta \left[ \phi(\vec{y}) - m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i) \right] \right) \\
&= \int \left( \prod_{i=1}^{N_*} d^p x_i \right) \int D\phi \psi_1^*(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*}) \delta \left[ \phi(\vec{y}) - m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i) \right] \\
&\quad \left( -i\hbar \int d^p y \sum_{i=1}^{N_*} m \vec{\nabla}_{x_i} \delta(\vec{y} - \vec{x}_i) \frac{\delta}{\delta \phi(\vec{y})} \Phi_2[\phi] \right) \tag{C.3} \\
&= \int D\phi \Phi_1^*[\phi] \left( \int \left( \prod_{i=1}^{N_*} d^p x_i \right) \delta \left[ \phi(\vec{y}) - m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i) \right] \right) \\
&\quad \left( -i\hbar \int d^p y \sum_{i=1}^{N_*} m \delta(\vec{y} - \vec{x}_i) \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \Phi_2[\phi] \right) \\
&= \int D\phi J[\phi] \Phi_1^*[\phi] \int d^p y \phi(\vec{y}) \vec{\nabla}_y \pi(\vec{y}) \Phi_2[\phi]
\end{aligned}$$

In the second equality, we performed an integration by parts in the  $\phi$  variable. The final equation is obtained using the defining equations of the conjugate momenta and the Jacobian. The final result implies that the operator  $\left( \sum_{i=1}^{N_*} \vec{p}_i \right)$  is equivalent to,

$$\left( \sum_{i=1}^{N_*} \vec{p}_i \right) \Rightarrow \int d^p y \phi(\vec{y}) \vec{\nabla}_y \pi(\vec{y}) \tag{C.4}$$

This is precisely what we got in (4.20) by naive application of the chain rule. However, a more precise derivation, like the one shown here, involves starting with the QM matrix element and inserting (4.16). Since the final answer is the same, we will only use the chain rule method to get to field-theoretic expressions for relevant operators.

### C.1.2 Collective theory Hamiltonian and eigenstates; non-relativistic case

We will derive the expression for the non-relativistic Hamiltonian  $H = \sum_{i=1}^{N_*} \frac{\vec{p}_i^2}{2m}$  in terms of collective variables  $\phi$  and  $\pi$ .

$$\begin{aligned}
\sum_{i=1}^{N_*} \frac{\vec{p}_i^2}{2m} &= -\frac{\hbar^2}{2m} \sum_{i=1}^{N_*} \vec{\nabla}_{\vec{x}_i}^2 \\
&= -\frac{\hbar^2}{2m} \sum_{i=1}^{N_*} \vec{\nabla}_{\vec{x}_i} \cdot \int d^p y \left( \vec{\nabla}_{\vec{x}_i} \phi(\vec{y}) \right) \frac{\delta}{\delta \phi(\vec{y})} \\
&= -\frac{\hbar^2}{2m} \sum_{i=1}^{N_*} \left( \int d^p y \vec{\nabla}_{\vec{x}_i}^2 \phi(\vec{y}) \frac{\delta}{\delta \phi(\vec{y})} + \int d^p y_1 d^p y_2 \vec{\nabla}_{\vec{x}_i} \phi(\vec{y}_1) \cdot \vec{\nabla}_{\vec{x}_i} \phi(\vec{y}_2) \frac{\delta^2}{\delta \phi(\vec{y}_1) \delta \phi(\vec{y}_2)} \right) \\
&= -\frac{\hbar^2}{2m} \sum_{i=1}^{N_*} \left( \int d^p y \vec{\nabla}_{\vec{x}_i}^2 m \delta(\vec{y} - \vec{x}_i) \frac{\delta}{\delta \phi(\vec{y})} \right. \\
&\quad \left. + \int d^p y_1 d^p y_2 \vec{\nabla}_{\vec{x}_i} m \delta(\vec{y}_1 - \vec{x}_i) \vec{\nabla}_{\vec{x}_i} \cdot m \delta(\vec{y}_2 - \vec{x}_i) \frac{\delta^2}{\delta \phi(\vec{y}_1) \delta \phi(\vec{y}_2)} \right) \\
&= -\frac{\hbar^2}{2m} \sum_{i=1}^{N_*} \left( \int d^p y m \delta(\vec{y} - \vec{x}_i) \vec{\nabla}_{\vec{y}}^2 \frac{\delta}{\delta \phi(\vec{y})} \right. \\
&\quad \left. + \int d^p y_1 d^p y_2 m \delta(\vec{y}_1 - \vec{y}_2) m \delta(\vec{y}_2 - \vec{x}_i) \vec{\nabla}_{\vec{y}_1} \frac{\delta}{\delta \phi(\vec{y}_1)} \cdot \vec{\nabla}_{\vec{y}_2} \frac{\delta}{\delta \phi(\vec{y}_2)} \right) \\
&= -\frac{\hbar^2}{2m} \left( \int d^p y \phi(\vec{y}) \vec{\nabla}_{\vec{y}}^2 \frac{\delta}{\delta \phi(\vec{y})} + m \int d^p y \phi(\vec{y}) \left( \vec{\nabla}_{\vec{y}} \frac{\delta}{\delta \phi(\vec{y})} \right)^2 \right) \\
&= \frac{1}{2} \int d^p y \phi(\vec{y}) \left( \vec{\nabla} \pi(\vec{y}) \right)^2 - \frac{i\hbar}{2m} \int d^p y \phi(\vec{y}) \vec{\nabla}^2 \pi(\vec{y})
\end{aligned}
\tag{C.5}$$

Here we have used the chain rule in the 2nd and 3rd equalities, i.e.,  $\vec{\nabla}_{\vec{x}_i} = \int d^p y \vec{\nabla}_{\vec{x}_i} \phi(\vec{y}) \frac{\delta}{\delta \phi(\vec{y})}$ . This tacitly assumes that the Jacobian enforces  $\phi(\vec{y}) = m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i)$ . Thus,  $\vec{\nabla}_{\vec{x}_i} \phi(\vec{y})$  is a proxy for  $\vec{\nabla}_{\vec{x}_i} \sum_{j=1}^{N_*} m \delta(\vec{y} - \vec{x}_j)$ . This is made explicit in the 4th equality. Integration by parts is used in the 5th equality, such that derivatives are made to act on  $\frac{\delta}{\delta \phi}$ . Then, we again used the definition of  $\phi$  and  $\pi$  to get to the final expression.

Next, we will show that the state  $\Phi_{\vec{k}_1, \vec{k}_2}$  is an eigenstate of this Hamiltonian. We put  $\hbar = 1$  from now on. One can compute relevant terms one by one.

$$\begin{aligned}
\vec{\nabla}_{\vec{y}}^2 \frac{\delta}{\delta \phi(\vec{y})} \Phi_{\vec{k}_1, \vec{k}_2} &= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{x}_1 + i\vec{k}_2 \cdot \vec{x}_2} \left( \phi(\vec{y}_2) \vec{\nabla}_{\vec{y}}^2 \delta(\vec{y} - \vec{y}_1) + \phi(\vec{y}_1) \vec{\nabla}_{\vec{y}}^2 \delta(\vec{y} - \vec{y}_2) \right) \\
&\quad - \frac{1}{m} \int d^p y' e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}'} \vec{\nabla}_{\vec{y}}^2 \delta(\vec{y} - \vec{y}') \\
&= \frac{1}{m^2} \int d^p y' \phi(\vec{y}') \left( -\vec{k}_1^2 e^{i\vec{k}_1 \cdot \vec{y} + i\vec{k}_2 \cdot \vec{y}'} + \vec{k}_1 \leftrightarrow \vec{k}_2 \right) + \frac{(\vec{k}_1 + \vec{k}_2)^2}{m} e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}}
\end{aligned}
\tag{C.6}$$

Using this, we get the action of the 2nd term in the Hamiltonian as,

$$\begin{aligned} -\frac{1}{2m} \int d^p y \phi(\vec{y}) \vec{\nabla}_y^2 \frac{\delta \Phi_{\vec{k}_1, \vec{k}_2}}{\delta \phi(\vec{y})} &= \frac{\vec{k}_1^2 + \vec{k}_2^2}{2m} \int \frac{d^p y_1 d^p y_2}{m^2} \phi(\vec{y}_1) \phi(\vec{y}_2) e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \\ &+ \frac{(\vec{k}_1 + \vec{k}_2)^2}{2m} \left( \frac{-1}{m} \right) \int d^p y e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \phi(\vec{y}) \end{aligned} \quad (\text{C.7})$$

To compute the first term in the Hamiltonian, we first have to compute

$\vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \cdot \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \Phi_{\vec{k}_1, \vec{k}_2}$ . This is given below,

$$\begin{aligned} &\left( \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \right)^2 \Phi_{\vec{k}_1, \vec{k}_2} \\ &= \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \cdot \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \left( \phi(\vec{y}_2) \vec{\nabla}_y \delta(\vec{y} - \vec{y}_1) + \phi(\vec{y}_1) \vec{\nabla}_y \delta(\vec{y} - \vec{y}_2) \right) \\ &\quad - \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \cdot \frac{1}{m} \int d^p y' e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}'} \vec{\nabla}_y \delta(\vec{y} - \vec{y}') \\ &= \frac{2}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \left( \vec{\nabla}_y \delta(\vec{y}_1 - \vec{y}) \cdot \vec{\nabla}_y \delta(\vec{y}_2 - \vec{y}) \right) \\ &= \frac{1}{m^2} \left( -2\vec{k}_1 \cdot \vec{k}_2 \right) e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \end{aligned} \quad (\text{C.8})$$

So, the first term in the Hamiltonian becomes,

$$-\frac{1}{2} \int d^p y \phi(\vec{y}) \left( \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \right)^2 \Phi_{\vec{k}_1, \vec{k}_2} = \frac{\vec{k}_1 \cdot \vec{k}_2}{m^2} \int d^p y \phi(\vec{y}) e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \phi(\vec{y}) \quad (\text{C.9})$$

Adding this to (C.7) we get the result that  $\Phi_{\vec{k}_1, \vec{k}_2}$  is indeed an eigenstate of the Hamiltonian,

$$\begin{aligned} H \Phi_{\vec{k}_1, \vec{k}_2} &= \frac{\vec{k}_1^2 + \vec{k}_2^2}{2m} \left( \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \phi(\vec{y}_2) - \frac{1}{m} \int d^p y e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \phi(\vec{y}) \right) \\ &= \frac{\vec{k}_1^2 + \vec{k}_2^2}{2m} \Phi_{\vec{k}_1, \vec{k}_2} \end{aligned} \quad (\text{C.10})$$

### C.1.3 Directly solving for eigenfunctionals of the Hamiltonian

In collective theory, we used the map (4.14) to obtain the energy eigenfunctionals in the collective theory, knowing their counterpart in quantum mechanics. Alternatively, one could directly try solving the time-independent Schrodinger equation for the collective theory Hamiltonian. This is analogous to what we do in usual Klein-Gordon theory when working in the Schrodinger picture [135]. In this section, we will explicitly do this for the non-relativistic Hamiltonian (4.21). Generalising this to the relativistic Hamiltonian is straightforward in principle, but technically tedious.

The non-relativistic collective Hamiltonian in  $\hbar = 1$  units is,

$$H = -\frac{1}{2} \int d^p y \phi(\vec{y}) \left( \vec{\nabla} \frac{\delta}{\delta \phi(\vec{y})} \right)^2 - \frac{1}{2m} \int d^p y \phi(\vec{y}) \vec{\nabla}^2 \frac{\delta}{\delta \phi(\vec{y})} \quad (\text{C.11})$$

If  $\Phi_E$  is an energy eigenstate of energy  $E$ , the time-independent Schrodinger equation is given by,

$$H\Phi[\phi] = E\Phi[\phi] \quad (\text{C.12})$$

Inspired by the Gaussian ground state wavefunctional in Klein-Gordon theory, we can also consider a Gaussian ansatz to begin with.

$$\Phi[\phi] = \exp \left\{ - \int d^p y_1 d^p y_2 G_2(\vec{y}_1, \vec{y}_2) \phi(\vec{y}_1) \phi(\vec{y}_2) \right\} \quad (\text{C.13})$$

The time-independent Schrodinger equation will reduce to the following equation for  $G_2$ ,

$$\begin{aligned} & \int d^p y_1 d^p y_2 d^p y_3 g_3(\vec{y}_1, \vec{y}_2, \vec{y}_3) \phi(\vec{y}_1) \phi(\vec{y}_2) \phi(\vec{y}_3) + \int d^p y_1 d^p y_2 g_2(\vec{y}_1, \vec{y}_2) \phi(\vec{y}_1) \phi(\vec{y}_2) \\ & + \int d^p y g_1(\vec{y}) \phi(\vec{y}) = E \end{aligned}$$

$$\text{where } g_1(\vec{y}) = \vec{\nabla}_{\vec{y}_1} \cdot \vec{\nabla}_{\vec{y}_2} G_2(\vec{y}_1, \vec{y}_2) \Big|_{\vec{y}_1, \vec{y}_2 = \vec{y}}, \quad g_2(\vec{y}_1, \vec{y}_2) = \vec{\nabla}_{\vec{y}_1}^2 \tilde{G}(\vec{y}_1, \vec{y}_2) \quad (\text{C.14})$$

$$\text{and } g_3(\vec{y}_1, \vec{y}_2, \vec{y}_3) = \vec{\nabla}_{\vec{y}_1} \tilde{G}_2(\vec{y}_1, \vec{y}_2) \cdot \vec{\nabla}_{\vec{y}_1} \tilde{G}_2(\vec{y}_1, \vec{y}_3) \text{ with } \tilde{G}_2(\vec{y}_1, \vec{y}_2) = G_2(\vec{y}_1, \vec{y}_2) + G_2(\vec{y}_2, \vec{y}_1)$$

One can compare the number of  $\phi$ s on both sides of the equation to conclude that only a trivial solution exists,  $G(\vec{y}_1, \vec{y}_2) = \mathfrak{s} \in \mathbb{R}$ . So, the ground state having  $E = 0$  seems to be infinitely degenerate, parametrised by a real number  $\mathfrak{s}$ . However, all of these are the same state  $\Phi = 1$ . Notice that the Jacobian  $J_{N_*}$  enforces  $\phi(\vec{y}) = m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i)$ , which implies  $\int d^p y \phi(y) = N_* m$  (See C.1.5). Using this, we get,

$$\Phi[\phi] = e^{\mathfrak{s} \int d^p y_1 d^p y_2 \phi(\vec{y}_1) \phi(\vec{y}_2)} = e^{\mathfrak{s} N_*^2 m^2} \quad (\text{C.15})$$

As the *RHS* is a number, we can identify this with the  $\Phi = 1$  state. This is also an instance where the many-to-one map from Collective theory to Quantum Mechanics is evident.

Having obtained the ground state as  $\Phi = 1$ , we can consider excitations on top of this state. We propose the following ansatz for an energy eigenstate with non-zero energy.

$$\Phi_f = \int d^p y f(y) \phi(y) \quad (\text{C.16})$$

The time-independent Schrodinger equation reduces to,

$$\int d^p y \phi(\vec{y}) \left( -\frac{\vec{\nabla}^2}{2m} \right) f(y) = E \int d^p y \phi(\vec{y}) f(\vec{y}) \quad (\text{C.17})$$

This gives the usual plane wave solutions for  $f$ ,

$$\begin{aligned} (\vec{\nabla}^2 + 2mE) f(\vec{y}) &= 0 \\ \Rightarrow f(\vec{y}) &= e^{i\vec{k}\cdot\vec{y}}; \quad E = \frac{\vec{k}^2}{2m} \end{aligned} \quad (\text{C.18})$$

These are exactly the states  $\Phi_{\vec{k}}$ , obtained in the main text. Note that  $H\Phi_{\vec{k}} = \frac{\vec{k}^2}{2m}\Phi_{\vec{k}}$ , and each of these states is to be interpreted as an  $N_*$ -particle state with one of them moving with momentum  $\vec{k}$  and the remaining at rest. However, if  $\vec{k} = 0$ , then owing to the Jacobian,  $\Phi_{\vec{k}=0} = \int d^p y \phi(\vec{y}) = N_* m$ , such that it should be identified with the ground state.

Next, we will consider a general 2-insertion state.

$$\Phi[\phi] = \int d^p y_1 d^p y_2 f_2(\vec{y}_1, \vec{y}_2) \phi(\vec{y}_1, \vec{y}_2) + \int d^p y f_1(\vec{y}) \phi(\vec{y}) \quad (\text{C.19})$$

Acting with the Hamiltonian, we get,

$$\begin{aligned} H\Phi[\phi] &= \int d^p y_1 d^p y_2 g_2(\vec{y}_1, \vec{y}_2) \phi(\vec{y}_1) \phi(\vec{y}_2) + \int d^p y g_1(\vec{y}) \phi(\vec{y}) \\ g_2(\vec{y}_1, \vec{y}_2) &= -\frac{1}{2m} (\vec{\nabla}_{\vec{y}_1}^2 + \vec{\nabla}_{\vec{y}_2}^2), \quad g_1(\vec{y}) = -\frac{1}{2m} \vec{\nabla}_{\vec{y}}^2 f_1(\vec{y}) - \vec{\nabla}_{\vec{y}_1} \cdot \vec{\nabla}_{\vec{y}_2} f_2(\vec{y}_1, \vec{y}_2) \Big|_{\vec{y}_1, \vec{y}_2 = \vec{y}} \end{aligned} \quad (\text{C.20})$$

Using the Schrodinger equation  $H\Phi = E\Phi$  we get the following conditions for  $f_1$  and  $f_2$ ,

$$\begin{aligned} (i) \quad & -\frac{1}{2m} (\vec{\nabla}_{\vec{y}_1}^2 + \vec{\nabla}_{\vec{y}_2}^2) f_2(\vec{y}_1, \vec{y}_2) = E f_2(\vec{y}_1, \vec{y}_2) \\ (ii) \quad & -\frac{1}{2m} \vec{\nabla}_{\vec{y}}^2 f_1(\vec{y}) - \vec{\nabla}_{\vec{y}_1} \cdot \vec{\nabla}_{\vec{y}_2} f_2(\vec{y}_1, \vec{y}_2) \Big|_{\vec{y}_1, \vec{y}_2 = \vec{y}} = E f_1(\vec{y}) \end{aligned} \quad (\text{C.21})$$

(i) has a unique solution in terms of plane waves.

$$f_2(\vec{y}_1, \vec{y}_2) = e^{i\vec{k}_1 \cdot \vec{y}_1} e^{i\vec{k}_2 \cdot \vec{y}_2}; \quad E = \frac{\vec{k}_1^2 + \vec{k}_2^2}{2m} \quad (\text{C.22})$$

Substituting this in (ii) gives a differential equation for  $f_1$  which can also be uniquely

solved.

$$\begin{aligned}
-\frac{1}{2m} \vec{\nabla}_{\vec{y}}^2 f_1(\vec{y}) &= \left( \frac{\vec{k}_1^2 + \vec{k}_2^2}{2m} \right) f_1(\vec{y}) - \vec{k}_1 \cdot \vec{k}_2 e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \\
\Rightarrow f_1(\vec{y}) &= m e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}}
\end{aligned} \tag{C.23}$$

So the state having energy eigenvalue  $\frac{\vec{k}_1^2 + \vec{k}_2^2}{2m}$  can be written as,

$$\Phi = \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1} e^{i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \phi(\vec{y}_2) - m \int d^p y e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \phi(\vec{y}) \tag{C.24}$$

This state is the same as  $\Phi_{\vec{k}_1, \vec{k}_2}$  in (4.48), except for an overall  $\frac{1}{m^2}$  normalisation factor.

One can generalise this procedure to obtain other energy eigenfunctionals. Start with the most general  $N$ -insertion state,

$$\begin{aligned}
\Phi &= \int \left( \prod_{i=1}^N d^p y_i \phi(\vec{y}_i) \right) f_N(\vec{y}_1, \dots, \vec{y}_N) + \int \left( \prod_{i=1}^{N-1} d^p y_i \phi(\vec{y}_i) \right) f_{N-1}(\vec{y}_1, \dots, \vec{y}_{N-1}) + \dots \\
&\quad + \int d^p y f_1(\vec{y}) \phi(\vec{y})
\end{aligned} \tag{C.25}$$

Writing down the time-independent Schrodinger equation, one can immediately solve for  $f_N$  in terms of plane waves, just like the 2-insertion case discussed above. Once  $f_N$  is obtained, substituting it back in the equations will give a differential equation for  $f_{N-1}$  that can be solved. Once  $f_{N-1}$  is fixed, we can solve for  $f_{N-2}$  and so on. One key point to remember is that the many-to-one mapping of collective theory to quantum mechanics also results in degeneracies of eigenstates. This we saw explicitly in the case of the zero energy ground state. However, one takes into account the Jacobian  $J_{N^*}$  and the fact that it implements  $\phi(\vec{y}) = m \sum_{i=1}^{N^*} \delta(\vec{y} - \vec{x}_i)$  to make sense of the degeneracy.

### C.1.4 Collective theory Hamiltonian and eigenstates; relativistic case

The Hamiltonian for relativistic free particles is given as,

$$H = \sum_{i=1}^{N^*} \sqrt{\vec{p}_i^2 + m^2} = N_* m + \sum_{i=1}^{N^*} \frac{\vec{p}_i^2}{2m} - \sum_{i=1}^{N^*} \frac{\vec{p}_i^4}{8m^3} + \dots \tag{C.26}$$

Each term in this expansion can be expressed in terms of collective theory variables  $\phi$  and  $\pi$ . The first term is easy, as it follows from the definition of  $\phi$  (4.14).

$$\int d^p y \phi(\vec{y}) = N_* m \tag{C.27}$$

We already computed the second term in (C.5). The  $\sum_i \vec{p}_i^4$  term can be calculated in a similar way. A straightforward integration by parts gives,

$$\begin{aligned}
\vec{\nabla}_{x_i}^4 = & \int d^p y \vec{\nabla}_{x_i}^4 \phi(\vec{y}) \frac{\delta}{\delta \phi(\vec{y})} + \int d^p y_1 d^p y_2 \left( 2 \vec{\nabla}_{x_i} \phi(\vec{y}_1) \cdot \vec{\nabla}_{x_i} \left( \vec{\nabla}_{x_i}^2 \phi(\vec{y}_2) \right) \right. \\
& + \left. \vec{\nabla}_{x_i}^2 \phi(\vec{y}_1) \vec{\nabla}_{x_i} \phi(\vec{y}_2) + \vec{\nabla}_{x_i} \cdot \vec{\nabla}_{x_i} \left( \vec{\nabla}_{x_i} \phi(\vec{y}_1) \cdot \vec{\nabla}_{x_i} \phi(\vec{y}_2) \right) \right) \frac{\delta^2}{\delta \phi(\vec{y}_1) \delta \phi(\vec{y}_2)} \\
& + \int d^p y_1 d^p y_2 d^p y_3 \left( 2 \vec{\nabla}_{x_i} \phi(\vec{y}_1) \cdot \vec{\nabla}_{x_i} \phi(\vec{y}_2) \vec{\nabla}_{x_i}^2 \phi(\vec{y}_3) \right. \\
& + \left. 2 \vec{\nabla}_{x_i} \phi(\vec{y}_1) \cdot \vec{\nabla}_{x_i} \left( \vec{\nabla}_{x_i} \phi(\vec{y}_2) \cdot \vec{\nabla}_{x_i} \phi(\vec{y}_3) \right) \right) \frac{\delta^2}{\delta(\vec{y}_1) \delta(\vec{y}_2) \delta(\vec{y}_3)} \\
& + \int d^p y_1 d^p y_2 d^p y_3 d^p y_4 \vec{\nabla}_{x_i} \phi(\vec{y}_1) \cdot \vec{\nabla}_{x_i} \phi(\vec{y}_2) \vec{\nabla}_{x_i} \phi(\vec{y}_3) \cdot \vec{\nabla}_{x_i} \phi(\vec{y}_4) \frac{\delta^2}{\delta(\vec{y}_1) \delta(\vec{y}_2) \delta(\vec{y}_3) \delta \phi(\vec{y}_4)}
\end{aligned} \tag{C.28}$$

Recall that  $\vec{\nabla}_{\vec{x}_i} \phi(y)$  is a proxy for  $\vec{\nabla}_{\vec{x}_i} \sum_{j=1}^{N_*} m \delta(\vec{y} - \vec{x}_j)$  just like in (C.5). One can then use the delta function property  $\vec{\nabla}_{\vec{x}_i} \delta(\vec{y} - \vec{x}_i) \frac{\delta}{\delta \phi(\vec{y})} = \delta(\vec{y} - \vec{x}_i) \vec{\nabla}_{\vec{y}} \frac{\delta}{\delta \phi(\vec{y})}$  such that all the derivatives now act on  $\frac{\delta}{\delta \phi}$ . The resulting term can be summed over from  $i = 1$  to  $N_*$  to get the following answer.

$$\begin{aligned}
-\frac{\hbar^4}{8m^3} \sum_{i=1}^{N_*} \vec{\nabla}_{\vec{x}_i}^4 = & -\frac{\hbar^4}{8m^3} \int d^p y \phi(\vec{y}) \left\{ \vec{\nabla}_y^4 \frac{\delta}{\delta \phi(\vec{y})} + 2m \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \cdot \vec{\nabla} \left( \vec{\nabla}_y^2 \frac{\delta}{\delta \phi(\vec{y})} \right) \right. \\
& + m \vec{\nabla}_y^2 \frac{\delta}{\delta \phi(\vec{y})} \vec{\nabla}_y^2 \frac{\delta}{\delta \phi(\vec{y})} + m \vec{\nabla}_y \cdot \vec{\nabla} \left( \vec{\nabla} \frac{\delta}{\delta \phi(\vec{y})} \right)^2 \\
& + 2m^2 \left( \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \right)^2 \vec{\nabla}_y^2 \frac{\delta}{\delta \phi(\vec{y})} \\
& \left. + 2m^2 \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \cdot \vec{\nabla}_y \left( \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \right)^2 + m^3 \left( \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \right)^4 \right\}
\end{aligned} \tag{C.29}$$

Upon the substitution  $\pi = -i\hbar \frac{\delta}{\delta \phi(\vec{y})}$ , one can verify that this is the same as what we wrote in (4.41). One can generalise this procedure to arbitrary higher orders in  $\vec{p}_i^2$ . One would naturally ask the question, "Is there a compact form for the relativistic Hamiltonian?" As we saw in 4.4, the answer is yes, given that you introduce creation and annihilation operators. But we know that for collective field theory we work with a fixed value of  $N_*$ , such that the creation and annihilation operators cannot be introduced without changing the Jacobian. However, we can still use the compact form of the Hamiltonian in terms of  $\phi$  and  $\frac{\delta}{\delta \phi}$ , given below.

$$\begin{aligned}
H &= \int \frac{d^p k}{(2\pi)^p} \omega_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}} \\
&= \int \frac{d^p k}{(2\pi)^p} \sqrt{\vec{k}^2 + m^2} \frac{1}{m} \int d^p y_1 e^{i\vec{k} \cdot \vec{y}_1} \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \int d^p y_2 e^{-i\vec{k} \cdot \vec{y}_2} e^{m \frac{\delta}{\delta \phi(\vec{y})}} \\
&= \int d^p y_1 d^p y_2 \frac{1}{m} \phi(\vec{y}_1) e^{-m \frac{\delta}{\delta \phi(\vec{y}_1)}} e^{m \frac{\delta}{\delta \phi(\vec{y}_2)}} \int \frac{d^p k}{(2\pi)^p} \sqrt{\vec{k}^2 + m^2} e^{i\vec{k} \cdot (\vec{y}_1 - \vec{y}_2)} \quad (C.30) \\
&= \int d^p y_1 d^p y_2 \frac{1}{m} \phi(\vec{y}_1) e^{-m \frac{\delta}{\delta \phi(\vec{y}_1)}} e^{m \frac{\delta}{\delta \phi(\vec{y}_2)}} \sqrt{-\vec{\nabla}_{y_2}^2 + m^2} \delta(\vec{y}_1 - \vec{y}_2) \\
&= \int d^p y \frac{1}{m} \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \sqrt{-\vec{\nabla}_y^2 + m^2} e^{m \frac{\delta}{\delta \phi(\vec{y})}}
\end{aligned}$$

The term  $\sqrt{-\vec{\nabla}_y^2 + m^2}$  is to be understood as the series,

$$\sqrt{-\vec{\nabla}_y^2 + m^2} = m - \frac{\vec{\nabla}_y^2}{2m} - \frac{(\vec{\nabla}_y^2)^2}{8m^3} + \dots \quad (C.31)$$

We will compute the expression for the Hamiltonian term by term. Note that,

$$\begin{aligned}
\vec{\nabla}_y^2 e^{f(\vec{y})} &= e^{f(\vec{y})} \left\{ \left( \vec{\nabla}_y f(\vec{y}) \right)^2 + \vec{\nabla}_y^2 f(\vec{y}) \right\} \\
\vec{\nabla}_y^4 e^{f(\vec{y})} &= e^{f(\vec{y})} \left\{ \left( \vec{\nabla}_y f(\vec{y}) \right)^4 + 2 \vec{\nabla}_y^2 f(\vec{y}) \left( \vec{\nabla}_y f(\vec{y}) \right)^2 + 2 \vec{\nabla}_y f(\vec{y}) \cdot \vec{\nabla} \left( \vec{\nabla}_y f(\vec{y}) \right)^2 \right. \\
&\quad \left. + 2 \vec{\nabla}_y f(\vec{y}) \cdot \vec{\nabla} \left( \vec{\nabla}_y^2 f \right) + \vec{\nabla}_y^2 f(\vec{y}) \vec{\nabla}_y^2 f(\vec{y}) + \vec{\nabla}_y \cdot \left( \vec{\nabla}_y f(\vec{y}) \right)^2 + \vec{\nabla}_y^4 f(\vec{y}) \right\} \quad (C.32)
\end{aligned}$$

We used compact notations such as  $(\vec{\nabla}_y f)^2 = \vec{\nabla}_y f \cdot \vec{\nabla}_y f$ . One can use  $f(\vec{y}) = m \frac{\delta}{\delta \phi(\vec{y})}$  and evaluate the expression for  $H$ . The first three terms in the series expansion give,

$$\begin{aligned}
\int d^p y \frac{1}{m} \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} m e^{m \frac{\delta}{\delta \phi(\vec{y})}} &= \int d^p y \phi(\vec{y}) \\
\int d^p y \frac{1}{m} \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \left( -\frac{\vec{\nabla}_y^2}{2m} \right) e^{m \frac{\delta}{\delta \phi(\vec{y})}} &= - \int d^p y \phi(\vec{y}) \left( \frac{1}{2} \left( \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \right)^2 + \frac{1}{2m} \vec{\nabla}_y^2 \frac{\delta}{\delta \phi(\vec{y})} \right) \\
\int d^p y \frac{1}{m} \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \left( -\frac{\vec{\nabla}_y^4}{8m^3} \right) e^{m \frac{\delta}{\delta \phi(\vec{y})}} &= - \int d^p y \phi(\vec{y}) \left\{ \frac{1}{8} \left( \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \right)^4 \right. \\
&\quad + \frac{1}{4m} \vec{\nabla}_y^2 \frac{\delta}{\delta \phi(\vec{y})} \left( \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \right)^2 + \frac{1}{4m} \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \cdot \vec{\nabla} \left( \frac{\delta}{\delta \phi(\vec{y})} \right)^2 \\
&\quad + \frac{1}{4m^2} \vec{\nabla} \frac{\delta}{\delta \phi(\vec{y})} \cdot \vec{\nabla} \left( \vec{\nabla}_y^2 \frac{\delta}{\delta \phi(\vec{y})} \right) + \frac{1}{8m^2} \vec{\nabla}_y^2 \frac{\delta}{\delta \phi(\vec{y})} \vec{\nabla}_y^2 \frac{\delta}{\delta \phi(\vec{y})} \\
&\quad \left. + \frac{1}{8m^2} \vec{\nabla} \cdot \vec{\nabla} \left( \vec{\nabla}_y \frac{\delta}{\delta \phi(\vec{y})} \right)^2 + \frac{1}{8m^3} \vec{\nabla}_y^4 \frac{\delta}{\delta \phi(\vec{y})} \right\}
\end{aligned}$$

These precisely match the expressions (C.5), (C.29). It is clear that all terms in the Hamiltonian will match with those obtained from collective field theory analysis. However, an all-orders proof looks tedious, and we will not discuss it here (See [4]).

Let us now show that the energy eigenstates given in (4.48) are indeed eigenstates of the relativistic Hamiltonian. We will use the compact form of the Hamiltonian,

$$H = \int d^p y \frac{1}{m} \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \sqrt{-\vec{\nabla}_y^2 + m^2} e^{m \frac{\delta}{\delta \phi(\vec{y})}} \quad (\text{C.33})$$

Let us start with the simple state  $\mathcal{C}$ .

$$\begin{aligned}
H \mathcal{C} &= \int d^p y \frac{1}{m} \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \sqrt{-\vec{\nabla}_y^2 + m^2} e^{m \frac{\delta}{\delta \phi(\vec{y})}} \cdot 1 \\
&= \int d^p y \phi(\vec{y}) \\
&= N_* m \mathcal{C}
\end{aligned} \quad (\text{C.34})$$

In the last step, we input the fact that we are in a collective theory with fixed  $N_*$  such that  $\int d^p y \phi(y) = N_* m^2$ .

Next, consider the 1-insertion states  $\Phi_{\vec{k}}$ .

$$\begin{aligned}
H \Phi_{\vec{k}} &= \int d^p y \frac{1}{m} \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \sqrt{-\vec{\nabla}_y^2 + m^2} e^{m \frac{\delta}{\delta \phi(\vec{y})}} \left( \frac{1}{m} \int d^p y' e^{i\vec{k} \cdot \vec{y}'} \phi(\vec{y}') \right) \\
&= \int d^p y \frac{1}{m} \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \sqrt{-\vec{\nabla}_y^2 + m^2} \left( \frac{1}{m} \int d^p y' e^{i\vec{k} \cdot \vec{y}'} (\phi(\vec{y}') + m \delta(\vec{y} - \vec{y}')) \right)
\end{aligned} \quad (\text{C.35})$$

---

<sup>2</sup>Note that this is not true if you are in Tachyon theory. There it is just the number operator times the mass,  $\int d^p y \Pi(\vec{y}) = N m$

where the action of  $e^{m\frac{\delta}{\delta\phi(\vec{y})}}$  shifts the field  $\phi(\vec{y}')$  by  $m\delta(\vec{y} - \vec{y}')$ . We will now simplify the terms in (C.35) one by one. The first term gives,

$$\begin{aligned} & \int d^p y \frac{1}{m} \phi(\vec{y}) e^{-m\frac{\delta}{\delta\phi(\vec{y})}} \sqrt{-\vec{\nabla}_y^2 + m^2} \left( \frac{1}{m} \int d^p y' e^{i\vec{k}\cdot\vec{y}'} \phi(\vec{y}') \right) \\ &= \int d^p y \phi(\vec{y}) \left( \frac{1}{m} \int d^p y' e^{i\vec{k}\cdot\vec{y}'} (\phi(\vec{y}') - m\delta(\vec{y} - \vec{y}')) \right) \\ &= \left( \int d^p y \phi(\vec{y}) - m \right) \Phi_{\vec{k}} \end{aligned} \quad (\text{C.36})$$

Here also we used  $e^{-m\frac{\delta}{\delta\phi(\vec{y})}}$  to shift  $\phi(\vec{y}')$ . As for the second term in (C.35) we get,

$$\begin{aligned} & \int d^p y \frac{1}{m} \phi(\vec{y}) e^{-m\frac{\delta}{\delta\phi(\vec{y})}} \sqrt{-\vec{\nabla}_y^2 + m^2} \left( \frac{1}{m} \int d^p y' e^{i\vec{k}\cdot\vec{y}'} m\delta(\vec{y} - \vec{y}') \right) \\ &= \int d^p y \frac{1}{m} \phi(\vec{y}) e^{-m\frac{\delta}{\delta\phi(\vec{y})}} \sqrt{-\vec{\nabla}_y^2 + m^2} e^{i\vec{k}\cdot\vec{y}} \\ &= \int d^p y \frac{1}{m} \phi(\vec{y}) \sqrt{\vec{k}^2 + m^2} e^{i\vec{k}\cdot\vec{y}} \\ &= \omega_{\vec{k}} \Phi_{\vec{k}} \end{aligned} \quad (\text{C.37})$$

Putting these together, we get,

$$\begin{aligned} H \Phi_{\vec{k}} &= \left( \int d^p y \phi(\vec{y}) + \omega_{\vec{k}} - m \right) \Phi_{\vec{k}} \\ &= (\omega_{\vec{k}} + (N_* - 1)m) \Phi_{\vec{k}} \end{aligned} \quad (\text{C.38})$$

here we expanded the square root  $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$  and also used  $\int d^p y \phi(\vec{y}) = N_* m$ . This makes it clear that  $\Phi_{\vec{k}}$  is an eigenstate of the collective theory Hamiltonian, and it physically means an  $N_*$ -particle state with one particle moving with momentum  $\vec{k}$  and the remaining particles at rest.

We will show that the 2-insertion state is also an eigenstate of the Hamiltonian. The calculation is straightforward and is very similar to the case we discussed above.

$$\begin{aligned} \Phi_{\vec{k}_1, \vec{k}_2} &= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1\cdot\vec{y}_1} e^{i\vec{k}_2\cdot\vec{y}_2} \phi(\vec{y}_1) \phi(\vec{y}_2) - \frac{1}{m} \int d^p y e^{i(\vec{k}_1 + \vec{k}_2)\cdot\vec{y}} \phi(\vec{y}) \\ &= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1\cdot\vec{y}_1 + i\vec{k}_2\cdot\vec{y}_2} \phi(\vec{y}_1) \left( \phi(\vec{y}_2) - m\delta(\vec{y}_2 - \vec{y}_1) \right) \end{aligned} \quad (\text{C.39})$$

In the second line, we wrote it in a form that will be useful for generalisation. Acting

with  $H$  on this, we get,

$$\begin{aligned}
H \Phi_{\vec{k}_1, \vec{k}_2} &= \frac{1}{m} \int d^p y \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \sqrt{-\vec{\nabla}_{\vec{y}}^2 + m^2} e^{m \frac{\delta}{\delta \phi(\vec{y})}} \\
&\quad \times \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \left( \phi(\vec{y}_2) - m\delta(\vec{y}_2 - \vec{y}_1) \right) \\
&= \frac{1}{m} \int d^p y \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \sqrt{-\vec{\nabla}_{\vec{y}}^2 + m^2} \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \\
&\quad \times \left( \underline{\phi(\vec{y}_1)} + m\delta(\vec{y} - \vec{y}_1) \right) \left( \underline{\phi(\vec{y}_2) - m\delta(\vec{y}_2 - \vec{y}_1)} + m\delta(\vec{y} - \vec{y}_2) \right)
\end{aligned} \tag{C.40}$$

The underlined terms are independent of  $\vec{y}$ . We will denote the term having only the products of the underlined terms as  $I_1$  and the rest of the terms as  $I_2$ . Let us first evaluate  $I_1$ .

$$\begin{aligned}
I_1 &\equiv \frac{1}{m} \int d^p y \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \sqrt{-\vec{\nabla}_{\vec{y}}^2 + m^2} \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \left( \phi(\vec{y}_2) - m\delta(\vec{y}_2 - \vec{y}_1) \right) \\
&= \int d^p \vec{y} \phi(\vec{y}) \int \frac{d^p y_1 d^p y_2}{m^2} e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \left( \phi(\vec{y}_1) - m\delta(\vec{y}_1 - \vec{y}) \right) \left( \phi(\vec{y}_2) - m\delta(\vec{y}_2 - \vec{y}_1) - m\delta(\vec{y}_2 - \vec{y}) \right) \\
&= \left( \int d^p y \phi(\vec{y}) - 2m \right) \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \left( \phi(\vec{y}_2) - m\delta(\vec{y}_2 - \vec{y}_1) \right) \\
&= \left( \int d^p y \phi(\vec{y}) - 2m \right) \Phi_{\vec{k}_1, \vec{k}_2}
\end{aligned}$$

Here we have made use of  $e^{\pm m \frac{\delta}{\delta \phi(\vec{y})}} \phi(\vec{y}') = \phi(\vec{y}') \pm m\delta(\vec{y}' - \vec{y})$ . Similarly, one can easily evaluate  $I_2$ . One just has to keep track of the fact that here there are terms that depend on  $\vec{y}$  on which the differential operator  $\sqrt{-\vec{\nabla}_{\vec{y}}^2 + m^2}$  can act.

$$\begin{aligned}
I_2 &= \int \frac{d^p y}{m} \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \sqrt{-\vec{\nabla}_{\vec{y}}^2 + m^2} \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \left( \phi(\vec{y}_1) m\delta(\vec{y} - \vec{y}_2) + \vec{y}_1 \leftrightarrow \vec{y}_2 \right) \\
&= \int d^p y \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \int \frac{d^p y_1 d^p y_2}{m^2} e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \left( \phi(\vec{y}_1) \sqrt{-\vec{\nabla}_{\vec{y}}^2 + m^2} \delta(\vec{y} - \vec{y}_2) + \vec{y}_1 \leftrightarrow \vec{y}_2 \right) \\
&= \int d^p y \phi(\vec{y}) e^{-m \frac{\delta}{\delta \phi(\vec{y})}} \int \frac{d^p y_1 d^p y_2}{m^2} e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \left( \omega_{\vec{k}_1} \phi(\vec{y}_1) \delta(\vec{y} - \vec{y}_2) + \omega_{\vec{k}_2} \phi(\vec{y}_2) \delta(\vec{y} - \vec{y}_1) \right) \\
&= \int d^p y \phi(\vec{y}) \int \frac{d^p y_1 d^p y_2}{m^2} e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \left( \omega_{\vec{k}_1} (\phi(\vec{y}_1) - m\delta(\vec{y} - \vec{y}_1)) \delta(\vec{y} - \vec{y}_2) + 1 \leftrightarrow 2 \right) \\
&= (\omega_{\vec{k}_1} + \omega_{\vec{k}_2}) \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \left( \phi(\vec{y}_2) - m\delta(\vec{y}_2 - \vec{y}_1) \right) \\
&= (\omega_{\vec{k}_1} + \omega_{\vec{k}_2}) \Phi_{\vec{k}_1, \vec{k}_2}
\end{aligned}$$

Adding  $I_1$  and  $I_2$ , we get,

$$\begin{aligned} H \Phi_{\vec{k}_1, \vec{k}_2} &= \left( \int d^p y \phi(\vec{y}) - 2m + \omega_{\vec{k}_1} + \omega_{\vec{k}_2} \right) \Phi_{\vec{k}_1, \vec{k}_2} \\ &= \left( \omega_{\vec{k}_1} + \omega_{\vec{k}_2} + (N_* - 2)m \right) \Phi_{\vec{k}_1, \vec{k}_2} \end{aligned} \quad (\text{C.41})$$

This is also in accordance with our interpretation that  $\Phi_{\vec{k}_1, \vec{k}_2}$  is an  $N_*$  particle eigenstate of the collective Hamiltonian with two particles having momenta  $\vec{k}_1, \vec{k}_2$  and the remaining at rest. We close this section by generalising the results to an  $n$ -insertion state  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}$ , with  $n \leq N_*$ .

$$\begin{aligned} H \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} &= \left( \int d^p y \phi(\vec{y}) + \sum_{i=1}^n \omega_{\vec{k}_i} - nm \right) \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} \\ &= \left( \sum_{i=1}^n \omega_{\vec{k}_i} + (N_* - n)m \right) \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} \end{aligned} \quad (\text{C.42})$$

This can be interpreted as an  $N_*$ -particle state with  $n$  of the particles moving with momenta  $\{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\}$  and the remaining  $N_* - n$  particles at rest.

### C.1.5 Proof of $\int d^p y \phi(\vec{y}) = N_* m$

In this section, we will prove that  $\int d^p y \phi(\vec{y}) = N_* m$ . This is essentially a result of the definition of the Jacobian.

$$J_{N_*}[\phi] = \int \left( \prod_{i=1}^{N_*} d^p x_i \right) \delta \left[ \phi(\vec{y}) - m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i) \right] \quad (\text{C.43})$$

Consider two arbitrary states  $\Phi_1$  and  $\Phi_2$  in collective theory. The matrix element of  $\int d^p y \phi(\vec{y})$  between these two states can be shown to be equal to  $N_* m$  as follows.

$$\begin{aligned} \left( \Phi_1, \int d^p y \phi(\vec{y}) \Phi_2 \right) &= \int D\phi J_{N_*}[\phi] \Phi_1^*[\phi] \left( \int d^p y \phi(\vec{y}) \right) \Phi_2[\phi] \\ &= \int D\phi \int \left( \prod_{i=1}^{N_*} d^p x_i \right) \delta \left[ \phi(\vec{y}) - m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i) \right] \Phi_1^*[\phi] \left( \int d^p y \phi(\vec{y}) \right) \Phi_2[\phi] \\ &= N_* m \int \left( \prod_{i=1}^{N_*} d^p x_i \right) \psi_1^*(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*}) \psi_2(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*}) \quad (\text{C.44}) \\ &= N_* m (\Phi_1, \Phi_2) \end{aligned}$$

We have used the definition of the Jacobian in the 2nd equality, followed by defining the quantum mechanical wavefunctions  $\psi_1$  and  $\psi_2$  in the 3rd equality as

$$\begin{aligned}\Phi_1[\phi] \Big|_{\phi(y)=m \sum_{i=1}^{N_*} \delta(\vec{y}-\vec{x}_i)} &= \psi_1(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*}) \\ \Phi_2[\phi] \Big|_{\phi(y)=m \sum_{i=1}^{N_*} \delta(\vec{y}-\vec{x}_i)} &= \psi_2(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{N_*}).\end{aligned}$$

The final equality follows from the equivalence between the quantum mechanics inner product and the collective theory inner product owing to the Jacobian (C.43).

One can also prove these results without invoking any quantum mechanics variables and staying purely on the field theory side. This requires the use of the alternative form of the Jacobian given in (4.19). But the key idea is just the fact that the definition of  $J_{N_*}$  implements  $\phi(\vec{y}) = m \sum_{i=1}^{N_*} \delta(\vec{y} - \vec{x}_i)$ .

$$\begin{aligned}J_{N_*}[\phi] \int d^p y \phi(\vec{y}) &= \int D\lambda e^{i \int d^p y' \lambda(\vec{y}') \phi(\vec{y}')} \left( \int dx e^{-im \lambda(\vec{x})} \right)^{N_*} \int d^p y \phi(\vec{y}) \\ &= \int d^p y \int D\lambda \left( \int dx e^{-im \lambda(\vec{x})} \right)^{N_*} \left( -i \frac{\delta}{\delta \lambda(\vec{y})} \right) \left( e^{i \int d^p y' \lambda(\vec{y}') \phi(\vec{y}')} \right) \\ &= \int d^p y \int D\lambda \left( e^{i \int d^p y' \lambda(\vec{y}') \phi(\vec{y}')} \right) \left( i \frac{\delta}{\delta \lambda(\vec{y})} \right) \left( \int dx e^{-im \lambda(\vec{x})} \right)^{N_*} \\ &= \int D\lambda \left( e^{i \int d^p y' \lambda(\vec{y}') \phi(\vec{y}')} \right) \left( \int dx e^{-im \lambda(\vec{x})} \right)^{N_*-1} N_* m \int d^p y e^{-im \lambda(\vec{y})} \\ &= \int D\lambda \left( e^{i \int d^p y' \lambda(\vec{y}') \phi(\vec{y}')} \right) \left( \int dx e^{-im \lambda(\vec{x})} \right)^{N_*} N_* m \\ &= J_{N_*}[\phi] N_* m\end{aligned}$$

This makes it clear that working in collective field theory at fixed  $N_*$  we get,  $\int d^p y \phi(\vec{y}) = N_* m$ .

This result can be put to use for proving the statement: Energy eigenstates having  $N_* - s$  insertions ( $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*-s}}$ ) are equivalent to an  $N_*$ -insertion state ( $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_*}}$ ) with  $s$  of the momenta put to zero.

We will show this explicitly for  $N_* = 2$ . We saw, in this case, that the state  $\Phi_{\vec{k},0}$  should be identified with  $\Phi_{\vec{k}}$ . This follows immediately from using  $\int d^p y \phi(\vec{y}) = N_* m$ , in the definition of  $\Phi_{\vec{k},0}$ .

$$\begin{aligned}
\Phi_{\vec{k},0} &= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k} \cdot \vec{y}_2} \phi(\vec{y}_1) \phi(\vec{y}_2) - \frac{1}{m} \int d^p y e^{i\vec{k} \cdot \vec{y}} \phi(\vec{y}) \\
&= \frac{2}{m} \int d^p y_2 e^{i\vec{k} \cdot \vec{y}_2} \phi(\vec{y}_2) - \frac{1}{m} \int d^p y e^{i\vec{k} \cdot \vec{y}} \phi(\vec{y}) \\
&= \frac{1}{m} \int d^p y e^{i\vec{k} \cdot \vec{y}} \phi(\vec{y}) \\
&= \Phi_{\vec{k}}
\end{aligned} \tag{C.45}$$

In the main text, we motivated this result by directly using  $\phi(\vec{y}) = m \sum_{i=1}^2 \delta(\vec{y} - \vec{x}_i)$ . Here we see that one can independently derive this in a more field-theoretic way, although the idea is the same, that the Jacobian enforces  $\phi(\vec{y}) = m \sum_{i=1}^2 \delta(\vec{y} - \vec{x}_i)$ . The proof can be generalised for any value of  $N_*$ .

### C.1.6 Higher-insertion energy eigenstates are null

We will prove that when you work in collective theory at fixed  $N_*$ , an energy eigenstate  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}$  with  $n > N_*$  is a null state in the theory. This is also a direct consequence of the definition of the Jacobian (C.43).

The two-insertion state is given by,

$$\begin{aligned}
\Phi_{\vec{k}_1, \vec{k}_2} &= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \phi(\vec{y}_2) - \frac{1}{m} \int d^p y e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \phi(\vec{y}) \\
&= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \left( \phi(\vec{y}_2) - m \delta(\vec{y}_2 - \vec{y}_1) \right)
\end{aligned} \tag{C.46}$$

In the last line, we wrote it in a form that will be useful for future generalisation. Working with  $N_* = 1$ , the Jacobian  $J_{N_*=1}$  will enforce  $\phi(\vec{y}) = m \delta(\vec{y} - \vec{x})$ . Substituting this in the above expression gives,

$$\begin{aligned}
\Phi_{\vec{k}_1, \vec{k}_2} \Big|_{\phi(\vec{y})=m\delta(\vec{y}-\vec{x})} &= \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} m \delta(\vec{y}_1 - \vec{x}) \left( m \delta(\vec{y}_2 - \vec{x}) - m \delta(\vec{y}_2 - \vec{y}_1) \right) \\
&= 0
\end{aligned} \tag{C.47}$$

This means that the state  $\Phi_{\vec{k}_1, \vec{k}_2}$  is a null state for  $N_* = 1$ . One can also arrive at this result from a field theory computation by using (4.19). We will quickly sketch this

below.

$$\begin{aligned}
J_1[\phi]\Phi_{\vec{k}_1, \vec{k}_2} &= \int D\lambda e^{i \int d^p y' \lambda(\vec{y}') \phi(\vec{y}')} \left( \int d^p x e^{-im\lambda(\vec{x})} \right) \\
&\times \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \left( \phi(\vec{y}_2) - m\delta(\vec{y}_2 - \vec{y}_1) \right) \\
&= \int \frac{d^p y_1 d^p y_2}{m^2} e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2} \left\{ \int D\lambda \left( \int d^p x e^{-im\lambda(\vec{x})} \right) \right. \\
&\times \left( -\frac{\delta^2}{\delta\lambda(\vec{y}_1)\delta\lambda(\vec{y}_2)} + im\delta(\vec{y}_1 - \vec{y}_2) \frac{\delta}{\delta\lambda(\vec{y}_1)} \right) e^{i \int d^p y' \lambda(\vec{y}') \phi(\vec{y}')} \left. \right\}
\end{aligned} \tag{C.48}$$

Here in the 2nd equality we used  $\phi(\vec{y})e^{i \int \lambda \phi} = -i \frac{\delta}{\delta\lambda(\vec{y})} e^{i \int \lambda \phi}$ . One can do integration by parts and show that the terms in the curly bracket evaluate to zero.

$$\begin{aligned}
&\int D\lambda \left( \int d^p x e^{-im\lambda(\vec{x})} \right) \left( -\frac{\delta^2}{\delta\lambda(\vec{y}_1)\delta\lambda(\vec{y}_2)} + im\delta(\vec{y}_1 - \vec{y}_2) \frac{\delta}{\delta\lambda(\vec{y}_1)} \right) e^{i \int d^p y' \lambda(\vec{y}') \phi(\vec{y}')} \\
&= \int D\lambda e^{i \int d^p y' \lambda(\vec{y}') \phi(\vec{y}')} \left( -\frac{\delta^2}{\delta\lambda(\vec{y}_1)\delta\lambda(\vec{y}_2)} - im\delta(\vec{y}_1 - \vec{y}_2) \frac{\delta}{\delta\lambda(\vec{y}_1)} \right) \left( \int d^p x e^{-im\lambda(\vec{x})} \right) \\
&= \int D\lambda e^{i \int d^p y' \lambda(\vec{y}') \phi(\vec{y}')} \left( m^2\delta(\vec{y}_1 - \vec{y}_2)e^{-im\lambda(\vec{y}_1)} - m^2\delta(\vec{y}_1 - \vec{y}_2)e^{-im\lambda(\vec{y}_1)} \right) \\
&= 0
\end{aligned} \tag{C.49}$$

This proves,

$$J_1[\phi]\Phi_{\vec{k}_1, \vec{k}_2} = 0, \tag{C.50}$$

which implies that for  $N_* = 1$ , the state  $\Phi_{\vec{k}_1, \vec{k}_2}$  is null.

$$\left( \Phi, \Phi_{\vec{k}_1, \vec{k}_2} \right)^{N_*=1} \stackrel{N_*=1}{=} 0 \quad \forall \Phi \tag{C.51}$$

This argument can be generalised to any value of  $N_*$ . We will briefly show it for  $\Phi_{\vec{k}_1, \vec{k}_2, \vec{k}_3}$  and then generalise for arbitrary  $N_*$ . The 3-insertion energy eigenstate is given by,

$$\begin{aligned}
\Phi_{\vec{k}_1, \vec{k}_2, \vec{k}_3} &= \frac{1}{m^3} \int d^p y_1 d^p y_2 d^p y_3 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2 + i\vec{k}_3 \cdot \vec{y}_3} \phi(\vec{y}_1) \left( \phi(\vec{y}_2) - m\delta(\vec{y}_2 - \vec{y}_1) \right) \\
&\times \left( \phi(\vec{y}_3) - m\delta(\vec{y}_3 - \vec{y}_2) - m\delta(\vec{y}_3 - \vec{y}_1) \right)
\end{aligned} \tag{C.52}$$

If you put  $N_* = 3$ , you will get the 3-particle energy eigenstate in QM, labelled by momenta  $\{\vec{k}_1, \vec{k}_2, \vec{k}_3\}$ . If you put  $N_* = 2, 1$ , you can easily see that the state reduces to

zero as claimed. For example, putting  $N_* = 2$  in the above equation gives,

$$\begin{aligned}
\Phi_{\vec{k}_1, \vec{k}_2, \vec{k}_3} &= \frac{1}{m^3} \int d^p y_1 d^p y_2 d^p y_3 e^{i\vec{k}_1 \cdot \vec{y}_1 + i\vec{k}_2 \cdot \vec{y}_2 + i\vec{k}_3 \cdot \vec{y}_3} \left( m\delta(\vec{y}_1 - \vec{x}_1) + m\delta(\vec{y}_1 - \vec{x}_2) \right) \\
&\times \left( m\delta(\vec{y}_2 - \vec{x}_1) + m\delta(\vec{y}_2 - \vec{x}_2) - m\delta(\vec{y}_2 - \vec{y}_1) \right) \\
&\times \left( m\delta(\vec{y}_3 - \vec{x}_1) + m\delta(\vec{y}_3 - \vec{x}_2) - m\delta(\vec{y}_3 - \vec{y}_2) - m\delta(\vec{y}_3 - \vec{y}_1) \right) \\
&= 0
\end{aligned} \tag{C.53}$$

This result implies that in  $N_* = 2$  collective theory  $\Phi_{\vec{k}_1, \vec{k}_2, \vec{k}_3}$  is a null state.

It is clear from here that the general  $N_*$  result will hold. We will briefly argue for this below. The key equation needed here is the expression for an  $n$ -insertion energy eigenstate. This is given as,

$$\begin{aligned}
\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} &= \int \left( \prod_{i=1}^n d^p \vec{y}_i \right) e^{i \sum_{i=1}^n \vec{k}_i \cdot \vec{y}_i} \phi(\vec{y}_1) \left( \phi(\vec{y}_2) - m\delta(\vec{y}_2 - \vec{y}_1) \right) \\
&\times \left( \phi(\vec{y}_3) - m\delta(\vec{y}_3 - \vec{y}_2) - m\delta(\vec{y}_3 - \vec{y}_1) \right) \dots \\
&\times \left( \phi(\vec{y}_n) - m\delta(\vec{y}_n - \vec{y}_{n-1}) - m\delta(\vec{y}_n - \vec{y}_{n-2}) \dots - m\delta(\vec{y}_n - \vec{y}_1) \right)
\end{aligned} \tag{C.54}$$

One can easily convince oneself that this expression for  $\Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n}$ ;  $n > N_*$  reduces to zero if one substitutes  $\phi(\vec{y}) = \sum_{i=1}^{N_*} m\delta(\vec{y} - \vec{x}_i)$ .

### C.1.7 Inner-product computation

We want to compute the inner product

$$\left( \Phi_{(\vec{k}'_1, \vec{k}'_2)}, \Phi_{(\vec{k}_1, \vec{k}_2)} \right) = \int D\phi \Phi_{(\vec{k}'_1, \vec{k}'_2)}^*[\phi] \int D\lambda e^{i \int \lambda \phi} \left( \int d^p x e^{-im\lambda(x)} \right)^2 \Phi_{(\vec{k}_1, \vec{k}_2)}[\phi], \tag{C.55}$$

for the case  $N_* = 2$ . The two-particle state is given by

$$\Phi_{(\vec{k}_1, \vec{k}_2)}[\phi] = \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1} e^{i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \phi(\vec{y}_2) - \frac{1}{m} \int d^p y e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \phi(\vec{y}). \tag{C.56}$$

Then we need to compute the following,

$$\begin{aligned}
&\left( \frac{1}{m^2} \int d^p y'_1 d^p y'_2 e^{-i\vec{k}'_1 \cdot \vec{y}'_1} e^{-i\vec{k}'_2 \cdot \vec{y}'_2} \phi(\vec{y}'_1) \phi(\vec{y}'_2) - \frac{1}{m} \int d^p y' e^{-i(\vec{k}'_1 + \vec{k}'_2) \cdot \vec{y}'} \phi(\vec{y}') \right) \\
&\int D\lambda e^{i \int \lambda \phi} \left( \int d^p x e^{-im\lambda(x)} \right)^2 \left( \frac{1}{m^2} \int d^p y_1 d^p y_2 e^{i\vec{k}_1 \cdot \vec{y}_1} e^{i\vec{k}_2 \cdot \vec{y}_2} \phi(\vec{y}_1) \phi(\vec{y}_2) - \frac{1}{m} \int d^p y e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}} \phi(\vec{y}) \right)
\end{aligned}$$

Thus, the inner product contains three types of contributions:

- quartic in  $\phi$ ,
- cubic in  $\phi$ ,
- quadratic in  $\phi$ .

To compute the quartic term, we use the following trick,

$$\begin{aligned}
& \int D\phi \int D\lambda \phi(\vec{y}') \phi(\vec{y}) e^{i \int \lambda \phi} \left( \int d^p x e^{-im\lambda(\vec{x})} \right)^2 \\
&= \int D\lambda D\phi \left( \frac{1}{i^2} \frac{\delta^2}{\delta\lambda(\vec{y}) \delta\lambda(\vec{y}')} e^{i \int d^p y \lambda(\vec{y}) \phi(\vec{y})} \right) \left( \int d^p x e^{-im\lambda(\vec{x})} \right)^2 \\
&= \frac{1}{i^2} \int D\lambda \frac{\delta^2 \delta[\lambda]}{\delta\lambda(\vec{y}) \delta\lambda(\vec{y}')} \left( \int d^p x e^{-im\lambda(\vec{x})} \right)^2 \\
&= 2m^2 + 2m^2 V \delta^p(\vec{y} - \vec{y}')
\end{aligned} \tag{C.57}$$

In the last step, we performed an integration by parts and then carried out the  $\lambda$  integral using the  $\delta$  function. Here,  $V = \int d^p y$  denotes the spatial volume of the  $D - p$  brane. By using this trick, we can compute the following general identities,

$$\int D\lambda D\phi e^{i \int \lambda \phi} \left( \int dx e^{-im\lambda(x)} \right)_*^N = V_*^N, \tag{C.58}$$

$$\int D\lambda D\phi \phi(y_1) \phi(y_2) e^{i \int \lambda \phi} \left( \int dx e^{-im\lambda(x)} \right)_*^N = m^2 \left[ N_* V_*^{N_*-1} \delta(y_1 - y_2) + N_*(N_* - 1) V_*^{N_*-2} \right], \tag{C.59}$$

$$\int D\lambda D\phi \phi(y_1) \phi(y_2) \phi(y_3) e^{i \int \lambda \phi} \left( \int dx e^{-im\lambda(x)} \right)_*^N \tag{C.60}$$

$$\begin{aligned}
&= m^3 \left[ N_* V_*^{N_*-1} \delta(y_1 - y_2) \delta(y_1 - y_3) + \right. \\
&\quad \left. N_*(N_* - 1) V_*^{N_*-2} \sum_{\text{pairs}} \delta(y_i - y_j) + N_*(N_* - 1)(N_* - 2) V_*^{N_*-3} \right],
\end{aligned} \tag{C.61}$$

$$\int D\lambda D\phi \phi(y_1) \phi(y_2) \phi(y_3) \phi(y_4) e^{i \int \lambda \phi} \left( \int dx e^{-im\lambda(x)} \right)_*^N \tag{C.62}$$

$$= m^4 \left[ N_* V_*^{N_*-1} \delta_{12} \delta_{34} \delta_{14} \right. \tag{C.63}$$

$$\left. + N_*(N_* - 1) V_*^{N_*-2} (\delta_{13} \delta_{24} + \delta_{14} \delta_{23} + \delta_{14} \delta_{13} + \delta_{24} \delta_{23} + \delta_{12} \delta_{14} + \delta_{12} \delta_{13} + \delta_{12} \delta_{34}) \right] \tag{C.64}$$

$$\left. + N_*(N_* - 1)(N_* - 2)(V)^{N_*-3} \sum_{\text{pairs}} \delta(y_i - y_j) + N_*(N_* - 1)(N_* - 2)(N_* - 3)(V)^{N_*-4} \right].$$

Here  $\delta_{ij} \equiv \delta(y_i - y_j)$ . For our problem, we set  $N_* = 2$ . Now we can compute each term in the inner product.

## Quadratic Term

The quadratic contribution gives

$$2 \left[ (2\pi)^2 \delta(\vec{k}_1 + \vec{k}_2) \delta(\vec{k}'_1 + \vec{k}'_2) + V(2\pi) \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}'_1 - \vec{k}'_2) \right], \quad (\text{C.65})$$

## Cubic Terms

The cubic contributions combine to give

$$\begin{aligned} & -4 \left[ (2\pi) V \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}'_1 - \vec{k}'_2) + (2\pi)^2 \delta(\vec{k}_1 + \vec{k}_2) \delta(\vec{k}'_1 + \vec{k}'_2) \right] \\ & -2(2\pi)^2 \left[ \delta(\vec{k}_1) \delta(\vec{k}_2 - \vec{k}'_1 - \vec{k}'_2) + \delta(\vec{k}_2) \delta(\vec{k}_1 - \vec{k}'_1 - \vec{k}'_2) \right. \\ & \quad \left. + \delta(\vec{k}'_1) \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}'_2) + \delta(\vec{k}'_2) \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}'_1) \right]. \end{aligned} \quad (\text{C.66})$$

## Quartic Term

The quartic contribution produces

$$\begin{aligned} & 2 \left[ (2\pi) V \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}'_1 - \vec{k}'_2) + (2\pi)^2 (\delta(\vec{k}_1 - \vec{k}'_1) \delta(\vec{k}_2 - \vec{k}'_2) \right. \\ & \quad + \delta(\vec{k}_1 - \vec{k}'_2) \delta(\vec{k}_2 - \vec{k}'_1) + \delta(\vec{k}_1 + \vec{k}_2) \delta(\vec{k}'_1 + \vec{k}'_2) \\ & \quad + \delta(\vec{k}_1 - \vec{k}'_1 - \vec{k}'_2) \delta(\vec{k}_2) + \delta(\vec{k}_2 - \vec{k}'_1 - \vec{k}'_2) \delta(\vec{k}_1) \\ & \quad \left. + \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}'_2) \delta(\vec{k}'_1) + \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}'_1) \delta(\vec{k}'_2) \right) \end{aligned} \quad (\text{C.67})$$

## Final Result

After adding all quadratic, cubic, and quartic pieces, all volume-dependent terms cancel.

We are left with the expected symmetric result

$$\boxed{\left( \Phi_{(\vec{k}'_1, \vec{k}'_2)}, \Phi_{(\vec{k}_1, \vec{k}_2)} \right) = 2 \left[ (2\pi) \delta(\vec{k}_1 - \vec{k}'_1) (2\pi) \delta(\vec{k}_2 - \vec{k}'_2) + (1 \leftrightarrow 2) \right]}. \quad (\text{C.68})$$

This is precisely the correctly normalized two-particle inner product.

## C.2 Tachyon theory computations

In this section, we perform computations in tachyon theory, using the Jacobian derived in 4.4.2. The techniques are similar to those used in computations of the collective theory inner product in C.1.7, with the only difference being that the tachyon theory Jacobian is a sum over collective theory Jacobians for different particle numbers. Also, unlike in collective theory, one can use the commutator algebra of creation and annihilation operators to further simplify the computations.

### C.2.1 Inner product between states

We will start by computing the norm of the state  $\mathcal{C}$ .

$$\begin{aligned} (\mathcal{C}, \mathcal{C}) &= \int D\Pi J[\Pi] 1 \cdot 1 \\ &= \int D\Pi \sum_{n=0}^{\infty} \frac{J_n[\Pi]}{n!} \end{aligned} \quad (\text{C.69})$$

Note that the  $\Pi$  integral can be performed easily,

$$\begin{aligned} \int D\Pi J_n[\Pi] &= \int D\Pi \int D\lambda e^{i \int d^p y \lambda(\vec{y}) \Pi(\vec{y})} \left( \int d^p x e^{-im\lambda(\vec{x})} \right)^n \\ &= \int D\lambda \delta[\lambda] \left( \int d^p x e^{-im\lambda(\vec{x})} \right)^n \\ &= \left( \int d^p x \right)^n \\ &= V^n \end{aligned} \quad (\text{C.70})$$

Here  $V = \int d^p x$  is the spatial volume of the D-brane. Substituting this result in (C.69), we get,

$$(\mathcal{C}, \mathcal{C}) = \sum_{n=0}^{\infty} \frac{V^n}{n!} = e^V \quad (\text{C.71})$$

Using this, we can compute the inner product between any two states in the tachyon theory Hilbert space. For example, consider the excited states  $\Phi_{\vec{k}_1}$  and  $\Phi_{\vec{k}_2}$ . Their inner product is computed as,

$$\begin{aligned} (\Phi_{\vec{k}_1}, \Phi_{\vec{k}_2}) &= (a_{\vec{k}_1}^\dagger \mathcal{C}, a_{\vec{k}_2}^\dagger \mathcal{C}) \\ &= (\mathcal{C}, a_{\vec{k}_1} a_{\vec{k}_2}^\dagger \mathcal{C}) \\ &= (\mathcal{C}, ((2\pi)^p \delta^p(\vec{k}_1 - \vec{k}_2) + a_{\vec{k}_2}^\dagger a_{\vec{k}_1}) \mathcal{C}) \\ &= (\mathcal{C}, \mathcal{C}) \left( (2\pi)^p \delta^p(\vec{k}_1 - \vec{k}_2) + (2\pi)^p \delta^p(\vec{k}_1) (2\pi)^p \delta^p(\vec{k}_2) \right). \end{aligned} \quad (\text{C.72})$$

Here we used the commutation relation between the creation and annihilation operators, as well as the action of  $a_{\vec{k}}$  on  $\mathcal{C}$  given in (4.77). Similarly, one can compute the inner product between any two states in the tachyon theory Hilbert space.

As an aside, recall that in the main text, we obtained the result that  $\mathcal{C}$  is to be interpreted as a coherent state of particles at rest.

$$|\mathcal{C}\rangle = e^{a_0^\dagger} |0\rangle \quad (\text{C.73})$$

As a sanity check, we can compute the norm of this state in a Klein-Gordon-like analysis, using properties of the vacuum state  $|0\rangle$ .

$$\begin{aligned} \langle \mathcal{C} | \mathcal{C} \rangle &= \langle 0 | e^{a_0} e^{a_0^\dagger} | 0 \rangle \\ &= \sum_{n,n'} \frac{1}{n!n'!} \langle 0 | a_0^n (a_0^\dagger)^{n'} | 0 \rangle \\ &= e^V \end{aligned} \quad (\text{C.74})$$

Here in the intermediate step, we have used  $\langle 0 | a_0^n (a_0^\dagger)^{n'} | 0 \rangle = \delta_{n,n'} n! V^n$ . This precisely matches the computation (C.69) performed using the Jacobian.

## C.2.2 Energy of the coherent state

In the last section, we discussed how to compute the inner product between different states in the tachyon theory. Using similar methods, we can also compute matrix elements of operators. In this section, we will briefly discuss the computation of the expectation value of the Hamiltonian in the coherent state. <sup>3</sup>

$$\begin{aligned} (\mathcal{C}, H\mathcal{C}) &= \left( \mathcal{C}, \int \frac{d^p k}{(2\pi)^p} \omega_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}} \mathcal{C} \right) \\ &= \int \frac{d^p k}{(2\pi)^p} \omega_{\vec{k}} (a_{\vec{k}} \mathcal{C}, a_{\vec{k}} \mathcal{C}) \\ &= \int \frac{d^p k}{(2\pi)^p} \omega_{\vec{k}} (2\pi)^p \delta(\vec{k}) (2\pi)^p \delta(\vec{k}) (\mathcal{C}, \mathcal{C}) \\ &= mV (\mathcal{C}, \mathcal{C}) \end{aligned} \quad (\text{C.75})$$

Here in the last step, we used the fact that  $(2\pi)^p \delta(0) = V$ . Notice that here we used only the algebraic properties of the creation operator and its action on the coherent state. One can generalise this computation to compute the matrix element of the Hamiltonian between any two states.

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<sup>3</sup>We are working with the Jacobian  $J[\Pi]$ , so the state  $|\mathcal{C}\rangle = e^{a_0^\dagger} |0\rangle$ , and  $a_{\vec{k}} \mathcal{C} = (2\pi)^p \delta(\vec{k}) \mathcal{C}$ . One can easily generalise this computation to the case when the Jacobian is changed to  $J^c[\Pi]$ .

### C.2.3 Change of basis between $J[\Pi]$ and $J^{\mathfrak{C}}[\Pi]$

In this section, we will explain the change in the wavefunctional representation of states when we go from the Jacobian  $J[\Pi]$  to  $J^{\mathfrak{C}}[\Pi]$ . Since the case of  $J^{\mathfrak{C}}[\Pi]$  is more general and putting  $\mathfrak{C} = 1$  will give you  $J[\Pi]$ , we will start with the coherent state  $e^{\mathfrak{C}a_0^\dagger}|0\rangle$  represented as  $\mathcal{C}[\Pi] = 1$ . The creation/annihilation operators and the Jacobian have the following explicit form.

$$\begin{aligned} a_{\vec{k}} &= \mathfrak{C} \int d^p y e^{-i\vec{k}\cdot\vec{y}} e^{m \frac{\delta}{\delta\Pi(\vec{y})}} \\ a_{\vec{k}}^\dagger &= \frac{1}{\mathfrak{C}} \frac{1}{m} \int d^p y e^{i\vec{k}\cdot\vec{y}} \Pi(\vec{y}) e^{-m \frac{\delta}{\delta\Pi(\vec{y})}} \\ J^{\mathfrak{C}}[\Pi] &= \int D\lambda e^{\int d^p y \lambda(\vec{y}) \Pi(\vec{y})} \exp \left\{ \int d^p x \mathfrak{C}^2 e^{-im\lambda(\vec{y})} \right\}. \end{aligned} \quad (\text{C.76})$$

With this definition, the action of the annihilation operator on the state  $\mathcal{C}$  becomes,

$$a_{\vec{k}} \mathcal{C} = 2\pi\delta(\vec{k}) \mathfrak{C} \mathcal{C} \quad (\text{C.77})$$

This tells you that the state  $\mathcal{C}$  is the coherent state of particles at rest,  $|\mathcal{C}\rangle = e^{\mathfrak{C}a_0^\dagger}|0\rangle$ . One can immediately ask about the other coherent states in the theory. These can be obtained by the action of the exponential of creation operators on  $\mathcal{C}$ , or by solving a differential equation analogous to (C.77). We will follow the latter route, and one can show that it is equivalent to the former.

A coherent state  $\beta[\Pi]$  (of particles at rest) is defined as the eigenstate of the annihilation operator having eigenvalue  $\beta$ .

$$a_{\vec{k}} \beta[\Pi] = \beta (2\pi)^p \delta(\vec{k}) \beta[\Pi] \quad (\text{C.78})$$

Using the definition of the annihilation operator in (C.77), this can be written as,

$$\begin{aligned} \mathfrak{C} \int d^p y e^{-i\vec{k}\cdot\vec{y}} e^{m \frac{\delta}{\delta\Pi(\vec{y})}} \beta[\Pi] &= \beta (2\pi)^p \delta(\vec{k}) \beta[\Pi] \\ \Rightarrow \mathfrak{C} \int d^p y e^{-i\vec{k}\cdot\vec{y}} \beta[\Pi + m\delta_y] &= \beta (2\pi)^p \delta(\vec{k}) \beta[\Pi] \\ \Rightarrow \beta[\Pi + m\delta_y] &= \frac{\beta}{\mathfrak{C}} \beta[\Pi] \end{aligned} \quad (\text{C.79})$$

Here we have used the notation  $\beta[\Pi + m\delta_y] = \beta[\Pi(\vec{y}') + m\delta(\vec{y}' - \vec{y})]$ .

One can now go to the Fourier space to solve for  $\beta[\Pi]$ .

$$\begin{aligned}\beta[\Pi] &= \int D\lambda e^{i \int d^p y \lambda(\vec{y}) \Pi(\vec{y})} \beta[\lambda] \\ \beta[\Pi + m\delta_y] &= \frac{\beta}{\mathfrak{C}} \beta[\Pi] \Rightarrow e^{im\lambda(\vec{y})} \beta[\lambda] = \frac{\beta}{\mathfrak{C}} \beta[\lambda] \\ \Rightarrow \beta[\lambda] &= \delta \left[ \lambda(\vec{y}) - \frac{1}{im} \ln \left( \frac{\beta}{\mathfrak{C}} \right) \right]\end{aligned}\tag{C.80}$$

So the solution of the functional difference equation is that  $\beta[\lambda]$  is a delta functional. One can put this back into the Fourier transform representation to get  $\beta[\Pi]$ . One gets,

$$\beta[\Pi] = \exp \left\{ \ln \left( \frac{\beta}{\mathfrak{C}} \right) \frac{1}{m} \int d^p y \Pi(\vec{y}) \right\} \tag{C.81}$$

As a consistency check, note that putting  $\beta = \mathfrak{C}$ , one gets  $\mathcal{C}[\Pi] = 1$  which is the coherent state having eigenvalue  $\mathfrak{C}$ , as claimed. One can further see that the logarithm diverges when  $\beta \rightarrow 0$ , which is in accordance with our conclusion in the main text that the vacuum is not accessible in tachyon theory.

Let us now come to the question of equivalence between the states in wavefunctional representations corresponding to  $J[\Pi]$  and  $J^\mathfrak{C}[\Pi]$ . Using the analysis given above, we note that the state  $|\beta\rangle = e^{\beta a_0^\dagger} |0\rangle$  is given the following two representations, which are related. (We put a subscript to distinguish the representations)

$$\begin{aligned}\beta_\mathfrak{C}[\Pi] &= \exp \left\{ \ln \left( \frac{\beta}{\mathfrak{C}} \right) \frac{1}{m} \int d^p y \Pi(\vec{y}) \right\}; \text{ with Jacobian } J^\mathfrak{C}[\Pi] \\ \beta_1[\Pi] &= \exp \left\{ \ln \left( \frac{\beta}{1} \right) \frac{1}{m} \int d^p y \Pi(\vec{y}) \right\}; \text{ with Jacobian } J[\Pi] \\ \Rightarrow \beta_\mathfrak{C}[\Pi] &= \exp \left\{ \ln \left( \frac{1}{\mathfrak{C}} \right) \frac{1}{m} \int d^p y \Pi(\vec{y}) \right\} \beta_1[\Pi]\end{aligned}\tag{C.82}$$

Since both representations correspond to the same state  $|\beta\rangle = e^{\beta a_0^\dagger} |0\rangle$ , the inner product computations should match<sup>4</sup>.

$$\begin{aligned}\langle \beta | \beta \rangle &= \int D\Pi J[\Pi] \beta_1[\Pi] \beta_1[\Pi] = \int D\Pi J^\mathfrak{C}[\Pi] \beta_\mathfrak{C}[\Pi] \beta_\mathfrak{C}[\Pi] \\ \Rightarrow \int D\Pi J[\Pi] \beta_1[\Pi] \beta_1[\Pi] &= \int D\Pi J^\mathfrak{C}[\Pi] \beta_\mathfrak{C}[\Pi] \beta_\mathfrak{C}[\Pi] \\ \Rightarrow \int D\Pi J[\Pi] \beta_1[\Pi] \beta_1[\Pi] &= \int D\Pi J^\mathfrak{C}[\Pi] \exp \left\{ 2 \ln \left( \frac{1}{\mathfrak{C}} \right) \frac{1}{m} \int d^p y \Pi(\vec{y}) \right\} \beta_1[\Pi] \beta_1[\Pi] \tag{C.83}\end{aligned}$$

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<sup>4</sup>Since we chose the constants  $\beta$  and  $\mathfrak{C}$  as real, we can avoid complex conjugation

This implies that the two Jacobians,  $J[\Pi]$  and  $J^{\mathfrak{C}}[\Pi]$ , should be related by a multiplicative factor. One can check that this indeed is the case.

$$J^{\mathfrak{C}}[\Pi] \exp \left\{ 2 \ln \left( \frac{1}{\mathfrak{C}} \right) \frac{1}{m} \int d^p y \Pi(\vec{y}) \right\} = J[\Pi] \quad (\text{C.84})$$

*This makes it clear that the theory remains the same even if we change the Jacobian to  $J^{\mathfrak{C}}$ ; only the wavefunctional representation of the state changes.* This is precisely why the wavefunctional  $\mathcal{C}[\Pi] = 1$  corresponds to  $e^{a_0^\dagger}|0\rangle$  when we use the Jacobian  $J[\Pi]$  whereas if we change the Jacobian to  $J^{\mathfrak{C}}[\Pi]$ , the wavefunctional  $\mathcal{C}[\Pi] = 1$  corresponds to  $e^{\mathfrak{C}a_0^\dagger}|0\rangle$ .

## C.2.4 Allowed states in tachyon theory for finite $V$ and $V \rightarrow \infty$ limit

In this section we will discuss the different states that have an admissible wavefunctional representation in the tachyon theory. We will also see that these states have a finite norm for the case of finite  $V$ , which is the volume of the D-brane. However, in the  $V \rightarrow \infty$  limit, different states become orthogonal to each other and the Hilbert space structure is that of a coherent state and excitations on top of it.

A general zero-momentum coherent state  $|\beta\rangle = e^{\beta a_0^\dagger}|0\rangle$  will have the following wavefunctional representation (We choose  $\mathfrak{C} = 1$  in (C.81)) in tachyon theory.

$$\beta[\Pi] = e^{\frac{\ln \beta}{m} \int d^p y \Pi(\vec{y})} \quad (\text{C.85})$$

As we remarked earlier, the  $\beta \rightarrow 0$  limit is singular, implying that the vacuum does not have an admissible wavefunctional representation. The norm of this state is given by<sup>5</sup>,

$$(\beta, \beta) = e^{\beta^2 V} \quad (\text{C.86})$$

So, a normalised state  $|\hat{\beta}\rangle$  will have the form,

$$|\hat{\beta}\rangle = e^{-\frac{\beta^2 V}{2}} |\beta\rangle \quad (\text{C.87})$$

The overlap between two such coherent states is given by,

$$\langle \hat{\beta}_1 | \hat{\beta}_2 \rangle = e^{-(\beta_1 - \beta_2)^2 \frac{V}{2}} \quad (\text{C.88})$$

So, at finite  $V$ , all these different coherent states have finite norm, and have a non-zero overlap with each other. However, in the  $V \rightarrow \infty$  limit, different coherent states become orthogonal to each other. This means that in the large volume limit, the tachyon theory seems to describe a coherent state background and excitations on top of it, with

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<sup>5</sup>We choose  $\beta$  to be real for simplicity

different coherent state backgrounds being orthogonal to each other.

In other words, if we define,

$$\mathcal{H}_\beta = \text{span} \left\{ \Phi_{\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n} = a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger \dots a_{\vec{k}_n}^\dagger \beta[\Pi]; n \in \mathbb{Z}_{\geq 0} \right\} \quad (\text{C.89})$$

Then, the tachyon theory admits orthogonal subspaces of the form  $\mathcal{H}_\beta$ , with  $\beta \neq 0$ .

One can also note that these coherent states are to be thought of as backgrounds with fixed energy density. This can be seen from the expectation value of energy,

$$(\hat{\beta}, H\hat{\beta}) = \beta^2 mV. \quad (\text{C.90})$$

The quantity which remains finite in the large  $V$  limit is the energy density  $\frac{(\hat{\beta}, H\hat{\beta})}{V}$ . Thus the subspace  $\mathcal{H}_\beta$  describes Fock-space fluctuations about a background  $|\beta\rangle$  with constant energy density.

We close this section by noting that not only the zero-momentum coherent state but a more general coherent state will also have an admissible wavefunctional representation. Following the methods used in deriving (C.81), we get,<sup>6</sup> for a general state,

$$\begin{aligned} |f\rangle &= e^{\int \frac{d^p k}{(2\pi)^p} f(\vec{k}) a_{\vec{k}}^\dagger} |0\rangle, \quad a_{\vec{k}} |f\rangle = f(\vec{k}) |f\rangle \\ \Rightarrow f[\Pi] &= e^{\frac{1}{m} \int d^p y \ln(\tilde{f}(\vec{y})) \Pi(\vec{y})} \\ \text{with } \tilde{f}(\vec{y}) &= \int \frac{d^p k}{(2\pi)^p} e^{i\vec{k} \cdot \vec{y}} f(\vec{k}), \quad \text{the Fourier transform of } f(\vec{k}) \end{aligned} \quad (\text{C.91})$$

Here also note that if  $\tilde{f}(\vec{y}) = 0$  at some point, we will encounter a singularity. So, only an appropriate class of functions will have an admissible representation.

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<sup>6</sup>We work with the Jacobian  $J[\Pi]$ , having  $\mathfrak{C} = 1$ .

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